Can. J. Math., Vol. XXXIX, No. 3, 1987, pp. 568-582

MATRIX SUMMABILITY OF GEOMETRICALLY DOMINATED SERIES

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1. Background and notation. One of the common uses of summability theory is found in its applications to power series. A partial listing such as [**1**]-[**5**], [**13**], and [**15**]-[**17**] might serve to remind us of the many instances of summability theory applied to power series. In some studies, the summability transformation is applied to the sequence of partial sums of the power series, while in others it is applied to the general term $a_k z^k$ as a series-to-series transformation. In [**2**] and [**5**] the transformation is applied to the coefficient sequence of the Taylor series. In the present study we investigate matrix transformations that are applied to the sequence $\{a_k z^k\}$, but we are not concerned with the usual preservation of convergence or sums. At a point within its disc of convergence, a power series exhibits more than ordinary convergence; it converges very rapidly, being dominated by a convergent geometric series. Therefore our present task is to develop a theory of matrix transformations that are required to preserve (only) convergence of series that converge geometrically. In order to state these vague ideas in precise language, it is necessary to introduce some notation.

If u is a complex number sequence and $A = [a_{nk}]$ is a summability matrix, then Au is the sequence whose n-th term is given by

$$(Au)_n = \sum_{k=0}^{\infty} a_{nk} u_k.$$

Let G denote the family of complex number sequences that are dominated by a convergent geometric sequence, i.e.,

$$G = \{u\colon u_n = O(r^n) \text{ for some } r \in (0, 1)\}.$$

We shall investigate matrix mappings of G into l_1 and of G into itself. Matrices that map l_1 into itself have been studied often (see, e.g., [**12**] and [**6**]-[**9**]), but it is obvious that G is a proper subset of l_1, so it is to be expected that G to l_1 or G to G mappings might offer a more natural tool for summing power series.

In Section 2 we study matrix mappings of G into l_1; the main result characterizes such matrices, and other results concern the preservation of

Received June 8, 1984 and in revised form April 14, 1986.

https://doi.org/10.4153/CJM-1987-026-9 Published online by Cambridge University Press

the sums of sequences in G. The content of Section 3 is a similar study of mappings from G into itself. The fourth section examines G to l_1 and G to G mapping properties of the classical summability methods of Nörlund, Euler-Knopp, Abel, and Borel.

2. Matrix mappings of G in l_1. It will be convenient to have another description of the set G. The following result, which is easily proved, gives such a characterization.

PROPOSITION 1. *The sequence u is in G if and only if*

$$\limsup_k |u_k|^{1/k} < 1.$$

If the summability matrix A maps G into l_1, we shall say that A is a $G - l$ *matrix*. The next result gives an explicit characterization of such matrices.

THEOREM 1. *The matrix A is a $G - l$ matrix if and only if*

$$(1) \qquad \sum_{n=0}^{\infty} |a_{nk}| = M_k < \infty \quad \textit{for } k = 0, 1, \ldots,$$

and

$$(2) \qquad \limsup_k M_k^{1/k} \leqq 1.$$

Proof. First assume that A satisfies (1) and (2) and suppose $u \in G$, say $|u_k| \leqq Br^k$, where $r \in (0, 1)$. Then for each n,

$$\sum_{k=0}^{\infty} |a_{nk}u_k| \leqq B \sum_{k=0}^{\infty} |a_{nk}|r^k.$$

The right-hand series is convergent because (1) and (2) together imply that for each n,

$$\limsup_k |a_{nk}|^{1/k} \leqq 1.$$

Furthermore,

$$\sum_{n=0}^{\infty} |(Au)_n| \leqq \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} |a_{nk}u_k|$$

$$\leqq \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} |a_{nk}|Br^k$$

$$\leqq B \sum_{k=0}^{\infty} M_k r^k.$$

The last series converges since, by (2), r is in the disc of convergence of the

https://doi.org/10.4153/CJM-1987-026-9 Published online by Cambridge University Press

power series $\sum_k M_k z^k$. Hence, $Au \in l_1$, and we have shown that A is a $G - l_1$ matrix.

Conversely, if A maps G into l_1, then the "basis sequences" $\{\delta_n^{(k)}\}$ are mapped into l_1; since $A\delta^{(k)}$ is the k-th column sequence of A, this means that (1) holds. Before proving the necessity of (2), we observe that the following must hold if A maps G into l_1: if $R > 1$ then for each n,

$$\sum_{k=0}^{\infty} |a_{nk}|R^{-k} < \infty.$$

Conversely, if A is a $G - l$ matrix we define the matrix B by

$$b_{nk} = a_{nk}R^{-k},$$

where $R > 1$. Then B is a $G - l$ matrix, so

$$\text{(3)} \qquad \sup_k \sum_{n=0}^{\infty} |a_{nk}R^{-k}| = M(R) < \infty.$$

Hence $M_k \leqq R$. Since this holds for every $R > 1$, (2) follows.

It is easy to give examples of $G - l$ matrices simply by citing the $l - l$ matrices in previous work such as **[6]**-**[9]**. It is more instructive, however, to give some examples of $G - l$ matrices that do not map all of l_1 into itself.

Example 1. Let D be the diagonal matrix diag$\{n\}$. It is easy to see that D maps G into l_1. (Indeed, D maps G into G.) Furthermore, if

$$f(z) = \sum_{n=0}^{\infty} a_k z^k \quad \text{and} \quad u = \{a_k z^k\},$$

then

$$\sum_{n=0}^{\infty} (Du)_n = \sum_{n=0}^{\infty} na_n z^n = z \sum_{n=0}^{\infty} \frac{d}{dz}[a_n z^n] = zf'(z),$$

and the transformed series has the same radius of convergence as the original series.

Example 2. Let M be the diagonal matrix diag$\{\beta_n\}$, where the sequence β satisfies

$$\lim\sup_n |\beta_n|^{1/n} \leqq 1.$$

Then by Theorem 1 we see that M is a $G - l$ matrix.

Although the mappings in the preceding examples preserve the convergence characteristics of members of G, the transformations will not necessarily yield a series having the same sum. We say that the matrix A is *sum-preserving over* G if for every u in G, Au is in G and

$$\sum_{n=0}^{\infty} (Au)_n = \sum_{k=0}^{\infty} u_k.$$

This following result gives a simple and explicit characterization of this property.

THEOREM 2. *The matrix A is sum-preserving over G if and only if A is a $G - l$ matrix and*

$$(4) \qquad \sum_{n=0}^{\infty} a_{nk} = 1 \quad \textit{for } k = 0, 1, \ldots .$$

Proof. The stipulation that A is a $G - l$ matrix is inherent in the definition of "sum-preserving over G." The necessity of condition (4) follows from the fact that the basis sequences $\{\delta^{(k)}\}$ are in G. Sufficiency is proved by the following calculation:

$$\sum_{n=0}^{\infty} (Au)_n = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_{nk}u_k$$

$$= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} a_{nk}u_k$$

$$= \sum_{k=0}^{\infty} (1)u_k.$$

The change in the order of summation is justified by the assumption that A is a $G - l$ matrix; thus for any u in G, $\sum_k \sum_n |a_{nk}u_k|$ is (absolutely) convergent.

Example 3. Let A be the matrix given by

$$a_{nk} = \begin{cases} n + 1, & \text{if } k = n, \\ -(n - 1), & \text{if } k = n - 1, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to verify that A satisfies properties (1), (2), and (4), so A is sum-preserving over G. It is equally obvious (by Theorem 1 of [**12**]) that A is not an $l - l$ matrix; hence, A does not preserve sums over l_1.

Since a sum-preserving $G - l$ matrix need not preserve the sum of a sequence not in G, it is natural to ask if such a matrix could "change the sum" of a (conditionally) convergent series. The answer is affirmative, and it is an immediate corollary of the corresponding result for $l - l$ matrices (see Theorem 5 of [**6**]). Therefore we state the result without proof.

PROPOSITION 2. *If $\sum x_k$ is a conditionally convergent series and w is a number, then there is a sum-preserving $G - l$ matrix A such that Ax is in l_1 and*

$$\sum_{n=0}^{\infty} (Ax)_n = w.$$

In ordinary summability one asks when a regular method sums only convergent sequences. For $G - l$ matrices the corresponding situation would be that only sequences in G are mapped into l_1. The next theorem shows that this situation can not occur.

THEOREM 3. *If A is a $G - l$ matrix, then there is a sequence $z \notin G$ such that $Az \in l_1$; i.e., there is no matrix T such that $T^{-1}[l_1] = G$.*

Proof. Assume A maps G into l_1, and let M_k denote the k-th column sum as in (1). Define the sequence z by

$$z_k = \begin{cases} \dfrac{1}{M_k(k + 1)^2}, & \text{if } M_k \geqq 1, \\[2ex] \dfrac{1}{(k + 1)^2}, & \text{if } M_k < 1. \end{cases}$$

If $M_k > 1$ for infinitely many k, then it is clear from (2) that

$$\limsup_k M^{1/k} = 1 \quad \text{and} \quad \lim_{M_k \geqq 1} M_k^{1/k} = 1.$$

Therefore $\lim_k z^{1/k} = 1$, so by Proposition 1, $z \notin G$. Also,

$$\begin{aligned} \sum_{n=0}^{\infty} |(Az)_n| &\leqq \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} |a_{nk} z_k| \\ &\leqq \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} |a_{nk}| z_k \\ &\leqq \sum_{k=0}^{\infty} M_k z_k \\ &\leqq \sum_{k=0}^{\infty} \frac{1}{(k + 1)^2} \\ &< \infty. \end{aligned}$$

Hence, $Az \in l_1$.

Although we have just shown that for a $G - l$ matrix A, $A^{-1}[l_1]$ is larger than G, the next result indicates that it is possible to exercise some choice in the sequences that are not in $A^{-1}[l_1]$.

PROPOSITION 3. *If $v \notin G$, then there is a $G - l$ matrix A such that $Av \notin l_1$.*

Proof. If $v \notin l_1$ we can take A to be the identity matrix. Suppose $v \in l_1 \sim G$; then

$$\limsup_n |v_n|^{1/n} = 1,$$

and we can choose a nonvanishing subsequence $\{v_{n(i)}\}$ such that

(5) $$\lim_i |v_{n(i)}|^{1/n(i)} = 1.$$

Now define $A = \text{diag}\{d_n\}$, where $d_n = 1/v_n$ if $n = n(i)$, $i = 1, 2, \ldots,$ and $d_n = 0$ otherwise. Thus (5) ensures that A is a $G - l$ matrix, but

$$(Av)_{n(i)} = 1 \quad \text{for } i = 1, 2, \ldots.$$

3. Matrix mappings of G into G. In this section we study summability methods that preserve geometrically dominated convergence. Such a matrix is called a *$G - G$ matrix*. In order to prove a characterization of such matrices, it is helpful to establish a preliminary result.

LEMMA. *If the matrix A maps G into G, then there is a number $r \in (0, 1)$ and a positive number sequence $\{\beta_k\}$ such that for every n and k, $|a_{nk}| \leqq \beta_k r^n$.*

Proof. In order that A maps G into G it is clearly necessary that each column of A be in G. Thus we may assume that for each k there is an $r_k \in (0, 1)$ such that

(6) $$|a_{nk}| < r_k^n \text{ for } n \text{ sufficiently large.}$$

Now suppose the conclusion of the lemma is false. This may be stated as follows: there does not exist an r in $(0, 1)$ satisfying

$$\limsup_n |a_{nk}|^{1/n} \leqq r \quad \text{for every } k.$$

Then $\limsup_k r_k = 1$, and for any s in $(0, 1)$ there exists an arbitrarily large k satisfying

(7) $$\limsup_n |a_{nk}|^{1/n} > s.$$

We now choose sequences $\{s_i\}$, $\{n(i)\}$, and $\{k(i)\}$ as follows: let s_1 be any number in $(1/2, 1)$ and choose $n(1)$ and $k(1)$ so that

$$|a_{n(1),k(1)}| \geqq (1/2)^{n(1)}.$$

Assume that $n(p)$, $k(p)$, and s_p have been defined for $p < i$. We determine s_i, $k(i)$, and $n(i)$ as follows: choose s_i in $(s_{i-1}, 1)$ such that

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$$s_i > r_t \quad \text{for } t \leqq k(i-1).$$

Next choose $k(i) > k(i-1)$ such that

(8) $$\limsup_n |a_{n,k(i)}|^{1/n} > s_i$$

and

(9) $$\sum_{k \geqq k(i)} |a_{n(i-1),k}| 2^{-k} \leqq \frac{1}{4} 2^{-k(i-1)} |a_{n(i-1),k(i-1)}|.$$

Note. We see that (8) follows directly from (7); also, (9) is justified by the fact that each row of a $G - G$ matrix must be the coefficient sequence of a power series $\sum_k a_{nk} z^k$ whose radius of convergence is at least 1, and the factor $a_{n(i-1),k(i-1)}$ is chosen to be nonzero.

We now choose $n(i) > n(i-1)$ sufficiently large so that

(10) $$n(i) > [k(i)]^2$$

and

$$4p^{n(i)} \leqq s_i^{n(i)} 2^{-k(i)},$$

where

$$p = \max_{j<i} \{r_{k(j)}\} < s_i.$$

By (8) we can also choose $n(i)$ such that

$$|a_{n(i),k(i)}| \geqq s_i^{n(i)}.$$

By (6) we have

$$|a_{n(i),k(j)}| < [r_{k(j)}]^{n(i)} \leqq p^{n(i)}$$
$$\leqq \frac{1}{4} s_i^{n(i)} 2^{-k(i)} \leqq \frac{1}{4} |a_{n(i),k(i)}| 2^{-k(i)}.$$

Thus,

(11) $$\sum_{j<i} |a_{n(i),k(j)}| 2^{-k(j)} \leqq \frac{1}{4} |a_{n(i),k(i)}| 2^{-k(i)}.$$

Now consider the sequence x defined by

$$x_k = \begin{cases} 2^{-k(i)}, & \text{if } k = k(i) \text{ for } i = 1, 2, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

Then x is obviously in G, and

$$(Ax)_n = \sum_{i=1}^{\infty} a_{n,k(i)} 2^{-k(i)}.$$

If Ax were in G there would exist an $R > 1$ such that

$$\sum_{n=0}^{\infty} |(Ax)_n|R^n < \infty.$$

In particular, $\{(Ax)_n R^n\}$ would tend to zero as $n \to \infty$. We show that this is not the case, whence Ax can not be in G. Consider the following:

$$|(Ax)_{n(i)}|R^{n(i)}$$

$$= \left|\sum_{j=1}^{\infty} a_{n(i),k(j)} 2^{-k(j)}\right| R^{n(i)}$$

$$\geqq \left\{ |a_{n(i),k(i)}|2^{-k(i)} - \sum_{j<i} |a_{n(i),k(j)}|2^{-k(j)} \right.$$

(see (11) and (9))

$$\left. - \sum_{j>i} |a_{n(i),k(j)}|2^{-k(j)} \right\} R^{n(i)}$$

$$\geqq \left\{ |a_{n(i),k(i)}|2^{-k(i)} - \frac{1}{4}|a_{n(i),k(i)}|2^{-k(i)} - \frac{1}{4}|a_{n(i),k(i)}|2^{-k(i)} \right\} R^{n(i)}$$

$$= \frac{1}{2}|a_{n(i),k(i)}|2^{-k(i)} R^{n(i)}$$

$$\geqq \frac{1}{2} s_i^{n(i)}\, 2^{-k(i)} R^{n(i)}.$$

Since $\lim_i s_i = 1$ and $R > 1$, there is a number N such that

$$s_i R \geqq L > 1 \quad \text{for } i > N.$$

Thus $i > N$ implies

$$|(Ax)_{n(i)}|R^{n(i)} \geqq \frac{1}{2} L^{n(i)} 2^{-k(i)}$$

$$> \frac{1}{2} L^{[k(i)]^2} 2^{-k(i)}$$

$$= \frac{1}{2} \{L^{k(i)}/2\}^{k(i)}$$

$$> 1,$$

for i sufficiently large. Hence, Ax is not in G, and we have shown that A is not a $G - G$ matrix.

THEOREM 4. *The matrix* A *is a* $G - G$ *matrix if and only if*

(P) *for each* $\epsilon > 0$ *there exists a constant* B *and an* r *in* $(0, 1)$ *such that*

$$|a_{nk}| \leqq Br^n(1 + \epsilon)^k \quad \text{for all } n \text{ and } k.$$

Proof. First assume A satisfies property (P) and let u be a sequence in G, say $|u_k| \leqq Cs^k$, where $s \in (0, 1)$. We apply property (P) with $\epsilon < (1/s) - 1$ and get B and $r \in (0, 1)$ satisfying

$$\left|\sum_{k=0}^{\infty} a_{nk}u_k\right| \leqq \sum_{k=0}^{\infty} Br^n(1 + \epsilon)^k Cs^k = \frac{BCr^n}{1 - (1 + \epsilon)s}.$$

The last series is convergent because $\epsilon < (1/s) - 1$ implies that $(1 + \epsilon)s < 1$. Hence, $(Au)_n = O(r^n)$, so $Au \in G$.

Conversely, suppose A is a $G - G$ matrix. By the lemma there exists a number t in $(0, 1)$ and a sequence $\{\beta_k\}$ such that

$$(12) \quad |a_{nk}| \leqq \beta_k t^n \quad \text{for all } n \text{ and } k.$$

We may assume, without loss of generality, that $1 < \beta_k < \beta_{k+1}$ for each k. Also,

$$(13) \quad \limsup_k |a_{nk}|^{1/k} \leqq 1 \quad \text{for each } n.$$

Suppose that property (P) does not hold; then there is an $\epsilon > 0$ such that for every r in $(0, 1)$ and every $B > 0$ there exist $n = n(B, r)$ and $k = k(B, r)$ satisfying

$$|a_{nk}| \geqq Br^n(1 + \epsilon)^k.$$

Let

$$r_1 = \frac{1 + t}{2} \quad \text{and} \quad r_{i+1} = \frac{1 + r_i}{2}$$

for $i = 1, 2, \ldots$. Then r_i increases to 1 and for each i, $r_i > t$. Thus for each r_i there exist $n(i)$ and $k(i)$ satisfying

$$|a_{n(i),k(i)}| \geqq ir_i^{n(i)}(1 + \epsilon)^{k(i)}.$$

We assert that

$$\lim_i n(i) = \infty \quad \text{and} \quad \lim_i k(i) = \infty.$$

For, if not, there would be a subsequence $\{i_m\}$ such that either $k(i_m) = c$ or $n(i_m) = d$, which implies that either

$$|a_{n(i_m),c}| \geqq r_{i_m}^{n(i_m)}(1 + \epsilon)^c \quad \text{or}$$
$$|a_{d,k(i_m)}| \geqq r_{i_m}^d(1 + \epsilon)^{k(i_m)},$$

i.e., either

$$\{\,|a_{n,c}|\,\}_{n=0}^{\infty} \notin G \quad \text{or}$$
$$\limsup_k |a_{dk}|^{1/k} \geqq 1 + \epsilon.$$

These statements would contradict (12) or (13), respectively. Therefore we assume, without loss of generality, that the indices $n(i)$ and $k(i)$ increase to infinity.

Now let $s = (1 + \epsilon/2)^{-1}$ and select the index sequence $\{i_p\}$ as follows: let $i_1 = 1$ and for $m > 1$ choose $i_{m+1} > i_m$ so that

$$\sum_{j=k(i_{m+1})}^{\infty} a_{n(i_m),j}|s_j \leqq t^{n(i_m)}$$

and

$$\left(\frac{1 + \epsilon}{1 + \epsilon/2}\right)^{k(i_{m+1})} > 2(1 + i_{m+1})\beta_{n(i_m)}.$$

The existence of such an i_{m+1} is ensured because

$$k(i) \geqq i \quad \text{and} \quad \sum_{j=0}^{\infty} |a_{nj}|s^j < \infty.$$

Now we have

$$\left|\sum_{j=1}^{\infty} a_{n(i_m),k(j)}s^{i_j}\right| \geqq |a_{n(i_m),k(i_m)}|s^k$$

$$- \sum_{j<m} |a_{n(i_m),k(i_j)}| - \sum_{k \geqq k(i_{m+1})} |a_{n(i_m),k}|s^k$$

$$\geqq |a_{n(i_m),k(i_m)}|s^{k(i_m)} - m\beta_{k(i_{m-1})}t^{n(i_m)} - t^{n(i_m)}$$

$$\geqq r_{i_m}^{n(i_m)}(1 + \epsilon)^{k(i_m)}s^{k(i_m)} - (m + 1)\beta_{k(i_{m-1})}t^{n(i_m)}$$

$$\geqq r_{i_m}^{n(i_m)}\left\{\left(\frac{1 + \epsilon}{1 + \epsilon/2}\right)^{k(i_m)} - (m + 1)\beta_{k(i_{m-1})}\right\}$$

$$\geqq r_{i_m}^{n(i_m)}\beta_{k(i_{m-1})}.$$

Now define $u_k = s^k$, if $k = k(i_m)$ for $m = 1, 2, \dots$, and $u_k = 0$ otherwise. Then u is obviously in G, but

$$|(Au)_{n(i_m)}| \geqq r_{i_m}^{n(i_m)},$$

so $Au \notin G$. Hence, A is not a $G - G$ matrix.

It is easy to verify that Examples 1-3 above are $G - G$ matrices. For the sake of completeness we now give an example of a $G - l$ matrix that is not a $G - G$ matrix.

Example 4. Let $\{r_k\}$ be an increasing positive number sequence having limit 1, and define the matrix A by

$$a_{nk} = r_k^n(1 - r_k).$$

Since A is nonnegative and each column sum is 1, A is a sum-preserving $l - l$ matrix and, a fortiori, a $G - G$ matrix. Now suppose $\epsilon > 0$; property (P) of Theorem 4 would require that there is a constant B and an r in (0, 1) such that

$$(14)\quad \left(\frac{r_k}{r}\right)^n \frac{(1 - r_k)}{(1 + \epsilon)^k} \leqq B \quad \text{for every } n \text{ and } k.$$

But since $\lim_k r_k = 1$, we can choose k' such that $r_{k'} > r$, and then (14) will fail for n sufficiently large. Hence, A is not a $G - G$ matrix.

We close this section with an observation concerning the preservation of sums of sequences in G. Since the basis sequences $\{\delta^{(k)}\}$ are in G, it follows that property (4) is required of a $G - G$ matrix in order that it be sum-preserving over G. Therefore a $G - G$ matrix A is sum-preserving over G if and only if (4) holds.

4. Well-known matrices as mappings on G. One of the most familiar classes of summability matrices is that of the Cesàro means ([**10**], [**14**]). It is easy to verify, however, that the columns of a Cesàro matrix of any order are not in l_1. Therefore they are not $G - l$ matrices.

The Nörlund mean N_p is represented by a lower triangular matrix in which

$$N_p[n, k] = p_{n-k}/P_n \quad \text{if } k \leqq n,$$

where

$$P_n = \sum_{k=0}^{n} p_k$$

and p is a nonnegative sequence such that $p_0 > 0$.

THEOREM 5. *The Nörlund matrix N_p is a $G - l$ matrix if and only if $p \in l$; also, N_p is a $G - G$ matrix if and only if $p \in G$.*

Proof. It is known ([**7**], Theorem 2) that if $p \in l_1$ then N_p is an $l - l$ matrix, whence N_p is a $G - l$ matrix. Conversely, if $p \notin l_1$ then by the Abel-Dini Theorem ([**11**], page 290),

$$\sum_{n=0}^{\infty} p_n/P_n = \infty,$$

so the first column of N_p is not in l_1. Thus N_p can not be a $G - l$ matrix.

Now assume $p \notin G$. If $p \notin l_1$ then the preceding argument shows that N_p is not a $G - G$ matrix. If $p \in l_1 \sim G$, then for each n,

$$p_n/P_n \geqq p_n/\left(\sum_{k=0}^{\infty} p_k\right)^{-1},$$

which implies that $\{p_n/P_n\} \notin G$. Therefore the first column of N_p is not in G, so N_p is not a $G - G$ matrix.

Conversely, if $p \in G$, say $p_k \leqq B't^k$, where $t \in (0, 1)$, then

$$N_p[n, k] \leqq B't^{n-k}/P_n \leqq Bt^{n-k}.$$

Now let $u \in G$, say $|u_k| \leqq Hr^k$, where $r \in (0, 1)$. This yields

$$\begin{aligned} |(N_pu)_n| &\leqq \sum_{k=0}^{n} Bt^{n-k}Hr^k \\ &< \frac{BHt^n}{1 - (r/t)} \quad (\text{for } t > r) \\ &= O(t^n). \end{aligned}$$

Therefore $N_pu \in G$, and we conclude that N_p is a $G - G$ matrix.

The Euler-Knopp means [**14**, pp. 56-60] are given by

$$E_r[n, k] = \begin{cases} \binom{n}{k}(1 - r)^{n-k}r^k, & \text{if } k \leqq n, \\ 0, & \text{if } k > n. \end{cases}$$

In [**7**, Theorem 4] it is shown that E_r is an $l - l$ matrix whose column sums each equal $1/r$ if and only if $r \in (0, 1]$. The first assertion of the next theorem is an obvious consequence of this.

THEOREM 6. *The matrix rE_r is a sum-preserving $G - l$ matrix if and only if $r \in (0, 1]$; also, E_r is a $G - G$ matrix if and only if $r \in (0, 1]$.*

Proof. The only part that remains to be shown is that $r \in (0, 1]$ implies that E_r is a $G - G$ matrix. If $r = 1$ then E_r is the identity matrix and the conclusion is trivial. Assume $r \in (0, 1)$ and let $u \in G$, say $|u_k| \leqq Bt^k$, where $t \in (0, 1)$. An easy calculation shows that

$$|(E_ru)_n| \leqq B[1 - r(1 - t)]^n.$$

Since r and t are in (0, 1), it follows that $1 - r(1 - t)$ is also in (0, 1), whence $E_r u \in G$. Hence, E_r is a $G - G$ matrix.

The matrix analogue A_t of the well-known Abel summability method is given by $a_{nk} = t_n(1 - t_n)^k$, where t is a null sequence in (0, 1). (See [**8**].)

THEOREM 7. *The Abel matrix A_t is a $G - l$ matrix if and only if $t \in l$; also, A_t is a $G - G$ matrix if and only if $t \in G$.*

Proof. The first assertion is an obvious variation of Theorem 1 of [**8**]. To prove the $G - G$ characterization we first note that since $t_n \in (0, 1)$, we have $|a_{nk}| \leqq t_n$ for all n and k. Therefore Theorem 4 ensures that $t \in G$ implies that A_t is a $G - G$ matrix. Conversely, if $t \notin G$ then the first column of A_t is not in G because $a_{n,0} = t_n$. Hence, A_t is not a $G - G$ matrix.

The Borel matrix method B (see [**14**, page 56]) is a variation of the more familiar Borel exponential method. The matrix B is given by

$$b_{nk} = e^{-n}n^k/k!.$$

It is known [**9**, Theorem 2] that B is an $l - l$ matrix; therefore B is a $G - l$ matrix. It can be proved by direct calculation that B is a $G - G$ matrix, but that is a special case of our next result. We extend the definition of the Borel matrix by replacing n with a positive sequence $t = t(n)$ that increases to infinity. Then the Borel matrix is given by

$$B_t[n, k] = e^{-t(n)}t(n)^k/k!.$$

As we shall see in the final two theorems, the possibility of B_t being a $G - G$ matrix depends on how rapidly $t(n)$ increases. To facilitate the proofs, we first make a simple observation:

If $u \in G$, say $|u_k| \leqq Hr^k$, where $r \in (0, 1)$, then

$$(15) \quad |(B_t u)_n| \leqq He^{-t(n)} \sum_{k=0}^{\infty} [t(n)r]^k/k! = He^{(r-1)t(n)}.$$

THEOREM 8. *If $t(n) = n^\delta$, then B_t is a $G - l$ matrix if and only if $\delta > 0$,* and *B_t is a $G - G$ matrix if and only if $\delta > 1$.*

Proof. Substituting $t(n) = n^\delta$ in (15), we get

$$(16) \quad |(B_t u)_n| \leqq He^{(r-1)n^\delta}.$$

If $\delta > 0$, then on comparing $\sum e^{(r-1)n^\delta}$ with $\sum n^{-2}$, we find that

$$\lim_n e^{(r-1)n^\delta}/n^{-2} = 0.$$

Thus $B_t u \in l_1$ and B_t is a $G - l$ matrix. If $\delta < 0$ we take $u_k = r^k$ and get

$$(B_t u)_n = e^{(r-1)n^\delta},$$

which tends to 1 as $n \to \infty$. Therefore B_t is not a $G - l$ matrix.

If $\delta > 1$ then we can rewrite (16) as

$$(17) \quad |(B_t u)_n| \leqq H[e^{(r-1)n^{\delta-1}}]^n.$$

The right-hand member of (17) is clearly in G, so we conclude that B_t is a $G - G$ matrix. On the other hand, if $\delta < 1$ then

$$\lim_n e^{(r-1)n^{\delta-1}} = 1,$$

so by taking $u_k = r^k$ we get

$$B_t u = \{e^{(r-1)n^\delta}\} \notin G.$$

THEOREM 9. *If t is a positive sequence such that B_t is a $G - G$ matrix and τ is a positive sequence such that*

$$\lim\inf_n \tau(n)/t(n) > 0,$$

then B_τ is also a $G - G$ matrix.

Proof. We assume without loss of generality that

$$\tau(n) > \delta t(n) > 0 \quad \text{for all } n.$$

Then (15) yields

$$\begin{aligned} |(B_\tau u)_n| &\leqq He^{(r-1)\tau(n)} \\ &\leqq H[e^{(r-1)t(n)}]^\delta \\ &= H[(B_t u)_n]^\delta, \end{aligned}$$

where $|u_k| \leqq Hr^k$. Since B_t is a $G - G$ matrix, $B_t u \in G$; and since $\delta > 0$, it follows that $[B_t u]^\delta$ is also in G. Hence, B_τ is a $G - G$ matrix.

The combination of Theorems 8 and 9 shows that the more rapidly that $t(n)$ increases to infinity the more likely it is that B_t is a $G - G$ matrix.

The authors would like to thank Professor D. Borwein for several helpful suggestions that significantly improved the exposition of these results.

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