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FINITELY GENERATED SUBGROUPS OF HNN GROUPS

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1. Introduction. In this paper we give sufficient conditions for an HNN group to have the following two properties:

(1) any two finitely generated subgroups intersect in a finitely generated subgroup;

(2) every finitely generated subgroup containing a non-trivial subnormal subgroup has finite index.

The following result is a particular case of the main theorem.

1.1 THEOREM. *The group* $G = \langle A, x | xux^{-1} = v\rangle$ *where* A *is free and* $u, v \in A$, *has property* (2) *if* u *is not a proper power and at least one of* u, v *does not generate* A, *and has property* (1) *if neither* u *nor* v *is a proper power.*

This does not include the result of Moldavanskiĭ [**4**] that $\langle y, x | xyx^{-1} = y^k\rangle$ has property (1) for all k. However we show briefly (at the end of § 3) how our proof modifies to include this case. The question of whether in general property (1) is preserved if the restriction on v is removed, remains unanswered.

To permit a concise statement of the main theorem we make the following definition. A subgroup U of a group A is called $AMFI$ (for "almost malnormal-finitely involved" (see [**1**])) if there exists a left transversal T for U in A containing the identity e, such that

$$U(T\backslash\{e\}) = (T\backslash\{e\})V_1 \tag{3}$$

for some finite subset $V_1 \subseteq U$; and for every coset Ha of every finitely generated subgroup $H \leqq A$,

$$Ha \subseteq TV_2(a^{-1}Ha \cap U) \tag{4}$$

for some finite subset $V_2 \subseteq U$, depending on Ha (cf. [**1**]).

We now state the theorem. For undefined terms see [**3**].

1.2 THEOREM. *Let* G *denote the* HNN *group* $\langle A, x | xU_{-1}x^{-1} = U_1\rangle$. *The following conclusions hold:*

(i) *if both* U_1 *and* U_{-1} *are* $AMFI$ *in* A; *for all finitely generated subgroups* H *of* G, $H \cap U_1$ *is finitely generated; and* A *has property* (1), *then* G *has property* (1);

(ii) *if* U_1 *is* $AMFI$ *in* A, $U_1 \neq A$, *and* H *is a finitely generated subgroup of* G *containing a subnormal subgroup* N *of* G, $N \nleq U_1$, *then* H *has finite index in* G.

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It is unknown whether in (i) the condition that U_{-1} be AMFI can be removed.

Theorem 1.1 follows immediately from this and the result that in a free group a cyclic subgroup whose generator is not a proper power, is AMFI [**1**, Theorem 6.1], except for one case easily dealt with separately. Since a finite subgroup is always AMFI, the theorem also applies when U_1 is finite. This was originally proved by Cohen [**2**] and P. C. Oxley [**5**].

The proof generalizes easily to give the corresponding result for the HNN group $\langle A, x_i | x_i U_{-i} x_i^{-1} = U_i \rangle$ where i ranges over any index set, and where in part (i) we demand that for all i both U_i and U_{-i} are AMFI in A, and in part (ii) that for at least one i, $U_i \neq A$ and U_i is AMFI in A.

The proof of 1.2 uses the same basic ideas as in [**1**], in conjunction with Cohen's subgroup theorem for HNN groups [**2**]. It is hoped that the present paper and [**1**] will together lead to some general criterion for a one-relator group to possess properties (1) and (2).

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2. Preliminary definitions and lemmas. For the detailed definition of the HNN group $\langle A, x | x U_{-1} x^{-1} = U_1 \rangle$ (denoted throughout by G) the reader is referred to [**3**]. We recall only that U_1 and U_{-1} are isomorphic subgroups of A and as implied by the notation, for all $u_{-1} \in U_{-1}$, $x u_{-1} x^{-1} = u_{-1}\varphi$ for some given isomorphism $\varphi : U_{-1} \to U_1$. These relations, one for each generator of U_{-1}, together with those of some presentation of A, give a presentation defining G. Let T_1, T_{-1} be left transversals for U_1, U_{-1} respectively, in A. Cohen [**2**] observes that every element g of G has a unique *normal form*

$$g = t_1 x^{\epsilon_1} \dots t_n x^{\epsilon_n} a \tag{5}$$

where $\epsilon_i = \pm 1$, $t_i \in T_{\epsilon_i}$, $a \in A$, and if $\epsilon_{i-1} = -\epsilon_i$ then $t_i \notin U_{\epsilon_i}$. Write $a = tu$ where $t \in T_1$, $u \in U_1$. Then t and those t_i such that $\epsilon_i = 1$ will be termed the *T_1-syllables* of g; if $g \notin A$, $x^{\epsilon_n} a$ will be called the *ending* of g and u the *U_1-ending* of g; $t_1 x^{\epsilon_1}$ is the *beginning* of g. The elements $t_1 x^{\epsilon_1} \dots t_i x^{\epsilon_i}$ and $t_1 x^{\epsilon_1} \dots t_i x^{\epsilon_i} t_{i+1}$ $(i = 0, 1, \dots, n)$ where $t_{n+1} = t$, will be called *initial segments* of g; in particular we shall call ga^{-1} the *principal initial segment* and all other initial segments except gu^{-1} will be said to be *proper*. The elements $t_i x^{\epsilon_i} \dots t_n x^{\epsilon_n} a$, $x^{\epsilon_i} \dots t_n x^{\epsilon_n} a$ will be called *terminal segments*. Finally the *length*, $l(g)$, of g is defined to be n.

The key lemma for the proof of Theorem 1.1 is the following.

2.1 LEMMA. *Suppose $G = \langle A, x | x U_{-1} x^{-1} = U_1 \rangle$ where U_1 is AMFI in A. Let T_1 be a left transversal for U_1 in A satisfying* (3) *and* (4) *and let T_{-1} be any left transversal for U_{-1} in A, containing e. Let H be any finitely generated subgroup of G such that $g U_1 g^{-1} \cap H$ is finitely generated for all $g \in G$. Then if S denotes the set of all elements of H with endings of the form $x^{-1} tu$, $t \in T_1 \backslash \{e\}$, $u \in U_1$,* *and $\hat{S}$ denotes the set obtained from S by deleting the U_1-endings u from the elements of S, there exists a finite subset $V \subseteq U_1$ such that*

$$S \subseteq \hat{S}V(H \cap U_1) \tag{6}$$

(*Clearly if the subscript* 1 *is replaced throughout by* -1, *the lemma will remain true.*)

For the proof of this, among other things, we shall need some of the results of Cohen [**2**] on subgroups of HNN groups. To formulate these the concept of a cress is used: our definition is slightly less general than Cohen's. A *cress* for a subgroup $H \leqq G$, relative to T_1, T_{-1}, both containing e, is a pair (C_1, C_{-1}) of right transversals for H in G, satisfying the following conditions (7)-(9):

(7) for all $g \in G$ with normal form (5) where $a = tu$ $(t \in T_1, u \in U_1)$

(i) if $g \in C_1$ then $gu^{-1} \in C_1$;

(ii) if $g \in C_1 \cup C_{-1}$, then $ga^{-1} \in C_1 \cap C_{-1}$;

(iii) if $g \in C_1 \cup C_{-1}$ with $a = e$ then $gx^{-\epsilon_n} \in C_{\epsilon_n}$;

(8) if R_1 is the set of all $g \in C_1$ with $u = e$, then R_1 is a complete double coset representative system for G modulo (H, U_1) (and similarly for R_{-1});

(9) if D is the set of all $g \in C_1 \cup C_{-1}$ with $a = e$, then D is a complete double coset representative system modulo (H, A).

The method of construction of a cress for any subgroup of G is given by Cohen [**2**]. In fact his construction yields a minimal cress, i.e. a cress such that for every $d \in D$, d has smallest length in HdA.

We now elaborate the part of Cohen's subgroup theorem that we shall need. He chooses a particular generating set for $H \leqq G$ in terms of a given cress (C_1, C_{-1}) for H. Let $\varphi_1 : G \to C_1$ and $\varphi_{-1} : G \to C_{-1}$, denote the coset representative functions for cosets Hg. For each $k \in C_1$, $a \in A$, write $\sigma(k, a) = ka((ka)\varphi_{-1})^{-1}$. If $k = da_1$ where $d \in D$, $a_1 \in A$, then it is easy to check that $\sigma(k, a) \in dAd^{-1} \cap H$. For each $k \in C_1$ define $\tau(k) = kx((kx)\varphi_{-1})^{-1}$. The *Kuroš rewriting process* for the elements of H in terms of the elements $\sigma(k, a)$ and $\tau(k)$ is as follows. Suppose $h \in H$ has canonical form (5), and write

$$\gamma_i = (t_1x^{\epsilon_1} \ldots t_i)\varphi_{-1},\ \delta_i = (t_1x^{\epsilon_1} \ldots t_ix_i^{\epsilon_i})\varphi_1 \quad (i = 1, \ldots, n). \tag{10}$$

The rewritten expression for h is obtained by replacing t_i by

$$\delta_{i-1}t_i\gamma_i^{-1} = \sigma(\delta_{i-1}, t_i) \quad (i = 1, \ldots, n), \tag{11}$$

and x^{ϵ_i} by

$$\gamma_ix^{\epsilon_i}\delta_i^{-1} \quad (i = 1, \ldots, n), \tag{12}$$

and a by δ_na. These replacements leave h unchanged. We now show how (5) can be written as a product of certain of the $\sigma(k, a)$ and $\tau(k)$. First suppose $\epsilon_i = 1$ and write $\gamma_i\varphi_1 = d\hat{t}\,\hat{u}$ where $d \in D$, $\hat{t} \in T_1$, $\hat{u} \in U_1$. Then

$$\gamma_ix\delta_i^{-1} = \sigma(\gamma_i\varphi_1, e)^{-1} \cdot \gamma_i\varphi_1 \cdot x \cdot \delta_i^{-1} = \sigma(\gamma_i\varphi_1, e)^{-1}\, d\hat{t}\,\hat{u}x\delta_i^{-1} =$$
$$\sigma(\gamma_i\varphi_1, e)^{-1} \cdot \tau(d\hat{t})\,(d\hat{t}x)\varphi_{-1}x^{-1}\hat{u}x\delta_i^{-1},$$

whence

$$\gamma_i x \delta_i^{-1} = \sigma(\gamma_i \varphi_1, e)^{-1} \cdot \tau(d\hat{t}_1) \cdot \sigma(\delta_i, v^{-1})^{-1}, \tag{13}$$

where $v = x^{-1}\hat{u}x \in U_{-1}$. The case $\epsilon_i = -1$ is dealt with similarly: the details are omitted.

It follows that H is generated by the $\sigma(k, a)$ and the $\tau(k)$ $(k \in C_1)$. It is easy to see that in fact it suffices to include only those $\tau(k)$ for which $k \in R_1$. Write Q_1 for the set of all elements $\tau(k)$ where $k \in R_1$, and write Q_2 for the set of all $d \in D\backslash\{e\}$ such that, if d ends in x^{-1} then $dAd^{-1} \cap H \nleqq dU_1d^{-1}$, and if d ends in x then $dAd^{-1} \cap H \nleqq dU_{-1}d^{-1}$. The following is immediate from [**2**, Lemmas 1, 2].

2.2 Lemma. *Let $H \leqq G = \langle A, x | xU_{-1}x^{-1} = U_1\rangle$. If H is finitely generated then Q_1 and Q_2 are finite. If in addition $gU_1g^{-1} \cap H$ is finitely generated for all $g \in G$, then $gAg^{-1} \cap H$ is finitely generated for all $g \in G$. Conversely if Q_1 and Q_2 are finite and for all $g \in G$, $gAg^{-1} \cap H$ is finitely generated, then H is finitely generated.*

Write $R = \{kx | k \in R_1, \tau(k) \neq e\}$ and denote by P the set of all T_1-syllables of elements of the set $R \cup Q_2 \cup (R\varphi_{-1})$.

2.3 Corollary. *If $H \leqq G$ is finitely generated then P is finite.*

The next two lemmas are rather technical.

2.4 Lemma. *Suppose $H \leqq G$, $H \nleqq A$, and that we have a cress relative to T_1, T_{-1} for H in G, as defined above. Let $h \in H\backslash A$ have normal form* (5) *and let γ_i, δ_i be defined as in* (10). *Then for all i, $1 \leqq i \leqq n$, such that $\delta_i \notin A$, the principal initial segment of δ_i is an initial segment of some element of $R \cup Q_2 \cup (R\varphi_{-1})$. It follows that if an initial segment d of h, ending in $x^{\pm 1}$, is its own representative* (*i.e. if $d \in D$*) *then d is an initial segment of some element of $R \cup Q_2 \cup (R\varphi_{-1})$.*

Proof. For $1 \leqq i \leqq n$ write $\delta_i = d_i\tau_i u_i$ where $d_i \in D$, $\tau_i \in T_1$, $u_i \in U_1$, and suppose that for some particular i, $1 \leqq i \leqq n$, d_i is not an initial segment of any element of $R \cup Q_2 \cup (R\varphi_{-1})$. Suppose first that $\epsilon_i = 1$. As in (13), write $\gamma_i\varphi_1 = d\hat{t}\hat{u}$. By (7), (9), $d = d_{i-1}$ (where d_0 is defined as e). Now if $\tau(d_{i-1}\hat{t}) \neq e$, then $d_{i-1}\hat{t}x \in R$, and therefore $(d_{i-1}\hat{t}x)\varphi_{-1} \in R\varphi_{-1}$, and, by (7), (9), and since $\epsilon_i = 1$, $(d_{i-1}\hat{t}x)\varphi_{-1} = d_i a_1$ for some $a_1 \in A$, which contradicts the assumption that d_i is not an initial segment of any element of $R \cup Q_2 \cup (R\varphi_{-1})$. Hence $\tau(d_{i-1}\hat{t}) = e$, whence $d_i = d_{i-1}\hat{t}x$. If $\epsilon_i = -1$ one obtains in a very similar way that $d_{i-1} = d_i tx$, where $(\gamma_i x^{-1})\varphi_1 = d_i tu$, $t \in T_1$, $u \in U_1$. We have thus shown that for those d_i, $1 < i < n$, which are not initial segments of any elements of $R \cup Q_2 \cup (R\varphi_{-1})$, either $d_i = d_{i-1}\hat{t}x$ (in the case $\epsilon_i = 1$) or $d_{i-1} = d_i tx$ (in the case $\epsilon_i = -1$). Now since $d_n = e = d_0$, it is easy to see that this can happen for no i. This completes the proof.

2.5 LEMMA. *Let $H \leqq G$ and choose a cress for H relative to T_1, T_{-1}. Suppose that $h \in H\backslash A$ has normal form $t_1x^{\epsilon_1} \ldots t_nx^{-1}a$. Then $a = u_1p^{-1}a_1$, where $u_1 \in U_1$, $p \in P$, and $a_1 \in A \cap H$.*

Proof. We have $\delta_n = (t_1x^{\epsilon_1} \ldots t_nx^{-1})\varphi_1 = (ha^{-1})\varphi_1$. Thus since $ha^{-1} \in HA$, whose representative in the cress is e, it follows that $\delta_n = pu$, say, where $p \in T_1$ and $u \in U_1$. Also $\delta_n a \in A \cap H$: call this element a_1. Then $a = \delta_n^{-1}a_1 = u^{-1}p^{-1}a_1$. The proof will be complete once it is proved that $p \in P$.

Suppose that $p \notin P$. Then $\tau(p) = e$ (note that $p \in R_1$); that is $(px)\varphi_{-1} = px$, and, by (7), (9), $\gamma_n = pxa_2$ where $a_2 \in A$. Thus px is the principal initial segment of γ_n (and δ_{n-1}) and therefore, by Lemma 2.4, an initial segment of some element of $R \cup Q_2 \cup R\varphi_{-1}$. Thus $p \in P$, a contradiction. Therefore $p \in P$.

Proof of Lemma 2.1. Suppose that $h \in H\backslash A$ has the normal form $t_1x^{\epsilon_1} \ldots t_nx^{-1}a$ where $a \notin U_1$, and that we have a cress for H in G relative to T_1, T_{-1}. By Lemma 2.5, $a = u_1p^{-1}a_1$ where $u_1 \in U_1$, $p \in P$, $a_1 \in A \cap H$. The element $p^{-1}a_1 = (p^{-1}a_1p)p^{-1}$ lies in the coset $(A \cap p^{-1}Hp)p^{-1}$. Since H and, for all $g \in G$, $gU_1g^{-1} \cap H$ are finitely generated, by Lemma 2.2 so is $A \cap p^{-1}Hp$. Thus since T_1 satisfies (4), there exists a finite subset $V_p \subseteq U_1$, depending on p and H only, such that

$$(A \cap p^{-1}Hp)p^{-1} \subseteq T_1V_p(p(p^{-1}Hp)p^{-1} \cap U_1) = T_1V_p(H \cap U_1).$$

Thus for each $a \in A\backslash U_1$ such that $x^{-1}a$ is the ending of some element of H, there is an element $p \in P$ and $u_1 \in U_1$ such that $a \in u_1^{-1}(T_1\backslash\{e\})V_p(H \cap U_1)$. Now for a given H, V_p depends only on p. Since P is finite (by Corollary 2.3), $\bigcup_{p \in P} V_p = V_1$ say, is also finite. Hence the set X of all $a \in A\backslash U_1$ as above, is contained in $U_1(T_1\backslash\{e\})V_1(H \cap U_1)$. By condition (3), $U_1(T_1\backslash\{e\}) \subseteq (T_1\backslash\{e\})V_2$ for some finite subset $V_2 \subseteq U_1$. Write $V = V_2V_1$. Then $X \subseteq T_1V(H \cap U_1)$, from which the desired result (6) follows.

We need one final lemma. Suppose $H \leqq G$ and that we have constructed a cress for H in G relative to transversals T_1, T_{-1} both containing e. We wish to relate the property that a subgroup $H \leqq G$ is finitely generated, to the finiteness of the number of certain double cosets that are "double ended" in the following special senses. A *double ended* (H, U_ϵ) double coset $(\epsilon = \pm 1)$ is one that contains at least one element ending in $x^{-\epsilon}a$ where $a \in A\backslash U_\epsilon$ and another element ending in $x^{-\epsilon}$. Also, an (H, A) double coset is called *double ended* if it contains at least one element ending in each of x and x^{-1}.

2.6 LEMMA (cf. [**2**, Lemma 1]). *Let $H \leqq G = \langle A, x | xU_{-1}x^{-1} = U_1\rangle$ and suppose R and Q_2 are defined as above in terms of a cress for H in G. Then $R \cup Q_2$ is finite if and only if there are only finitely many double ended (H, U_1) cosets, and the same is true of either the double ended (H, U_{-1}) cosets or the double ended (H, A) cosets.*

Remark. It is plausible that this remains true when the second condition is

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removed. A proof that this is so would correspondingly strengthen Theorems 1.1 and 1.2.

Proof of 2.6. Suppose there are only finitely many double ended (H, U_1) double cosets. First we show that this implies that there are only finitely many elements of $R \cup Q_2$ ending in x^{-1}. In fact it suffices to prove this for Q_2 since every element of R ends in x, by definition. Thus suppose $d \in Q_2$ ends in x^{-1}. Then by definition of Q_2, $H \cap dAd^{-1} \nleq dU_1d^{-1}$; let $a \in A \backslash U_1$ be such that $dad^{-1} \in H$. Then HdU_1 is double ended since it contains da with ending $x^{-1}a$, and d with ending x^{-1}. Hence there are only finitely many elements of $R \cup Q_2$ ending in x^{-1}. Next suppose that $g \in R \cup Q_2$ has an initial segment of the form dtx, in normal form, where $d \in D$, $t \in T_1 \backslash \{e\}$. In the first place, if there is an initial segment dt_1x, $t_1 \in T_1$, of some other element of $R \cup Q_2$, such that $t_1 \neq t$, then the double coset $HdtU_1$ is double ended since it contains an element ending in x^{-1}, and also an element ending in $x^{-1}u_1t_1^{-1}t$, for some $u_1 \in U_1$, which has the form $x^{-1}a$, where $a \in A \backslash U_1$. In the second place, if d ends in x^{-1} then $HdtU_1$ contains dt which ends in $x^{-1}t$, $t \in T_1 \backslash \{e\}$, and an element ending in $x^{-1}u$ for some $u \in U_1$, whence $HdtU_1$ is double ended. These two observations and the fact that only finitely many elements of $R \cup Q_2$ end in x^{-1}, imply that there exists a finite subset Y of initial segments of elements of $R \cup Q_2$ and a finite number of (possibly infinite) chains of the form

$$e,\ t_1x,\ t_1xt_2x,\ t_1xt_2xt_3x,\ \ldots, \tag{14}$$

where $t_i \in T_1$ and every element in (14) is an initial segment of its successor, such that every element of $R \cup Q_2$ has the reduced form yz where $y \in Y$ and z belongs to one of the chains. (This is easiest seen by considering the graph whose vertices are the elements of $R \cup Q_2 \cup \{e\}$, where two vertices v_1, v_2 with $l(v_1) < l(v_2)$ are joined by an edge if $v_2 = v_1tx^{\epsilon}$ (where $t \in T_{\epsilon}$) for $\epsilon = +1$ or -1.) Now if infinitely many of the elements of the chain (14) lie in $R \cup Q_2$, then none of them is in R, since none of the elements of R is in C_1 while all their initial segments are in C_1. If there are only finitely many double ended (H, U_{-1}) cosets then analogously to the first part of the proof it can be shown that only finitely many elements of Q_2 end in x. Alternatively, if there are only finitely many double ended (H, A) cosets then only finitely many of the elements of (14) are in Q_2, since otherwise HgA would be double ended for all g in (14). This completes the proof of sufficiency.

We now turn to the converse. Thus suppose $R \cup Q_2$ is finite. Let $g \in R_1$ be such that HgU_1 is double ended and write $g = dt$ where $t \in T_1$ and $d \in D$. There exist elements $h_1, h_2 \in H$ such that h_1g ends in $x^{-1}u$, where $u \in U_1$, and h_2g ends in $x^{-1}a$ where $a \in A \backslash U_1$. It is easy to verify that then either h_1^{-1} or h_2^{-1} has as an initial segment the element dtx. Then either $dtx \in R$ or $\tau(dt) = e$ in which case $dtx \in D$. Thus in either case $dt \in C_1$ (by (7)) whence by Lemma 2.4, dt is an initial segment of some element of $R \cup Q_2 \cup (R\varphi_{-1})$. Since the latter is finite it follows that the number of double ended (H, U_1) cosets is finite.

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In the same way, if $dt \in R_{-1}(d \in D, t \in T_{-1})$ is such that $HdtU_{-1}$ is double ended, then dtx^{-1} is an initial segment of some element h of H. If $dtx^{-1} \in D$ then dtx^{-1} is an initial segment of some element of $R \cup Q_2 \cup (R\varphi_{-1})$ by Lemma 2.4. If $dtx^{-1} \notin D$, write $(dtx^{-1})\varphi_1 = d_1t_1u_1$, where $d_1 \in D$, $t_1 \in T_1$, $u_1 \in U_1$. If $\tau(d_1t_1) \neq e$, then $d_1t_1x \in R$, and since $Hd_1t_1xU_{-1} = HdtU_{-1}$ and R is finite there can be only finitely many such elements $dt \in R_{-1}$. On the other hand if $\tau(d_1t_1) = e$ then $(d_1t_1x)\varphi_{-1} = d_1t_1x \in D$; now

$$(d_1t_1x)\varphi_{-1} = ((d_1t_1u_1)u_1^{-1}x)\varphi_{-1} = (dtu_2^{-1})\varphi_{-1} = dtu_3 \text{ (by (8))}$$

where $u_3 \in U_{-1}$, and therefore $t = u_3 = e$. Thus $dt = d$, which by Lemma 2.4 is an initial segment of some element of $R \cup Q_2 \cup (R\varphi_{-1}$. We have thus proved that there are only finitely many double ended (H, U_{-1}) cosets.

Finally if $HdA\,(d \in D)$ is double ended then it is again not difficult to show that d is an initial segment of some element of H. By Lemma 2.4 d is then an initial segment of some element of $R \cup Q_2 \cup (R\varphi_{-1})$, and since this set is finite the number of double ended (H, A) cosets must also be finite.

3. The finitely generated intersection property. Theorem 1.2(i) follows immediately from Lemmas 2.2 and 2.6, and the following result.

3.1 LEMMA. *Let $G = \langle A, x | xU_{-1}x^{-1} = U_1\rangle$ satisfy the hypotheses of Theorem* 1.2(i) *and let T_1, T_{-1} be left transversals for U_1, U_{-1} respectively, in A satisfying* (3) *and* (4). *If H and K are finitely generated subgroups of G then for both $\epsilon = +1$ and -1 each intersection of an (H, U_ϵ) coset with a (K, U_ϵ) coset contains only finitely many $(H \cap K, U_\epsilon)$ cosets containing elements with endings of the form $x^{-\epsilon}a$, $a \in A \backslash U_\epsilon$.*

Proof. It suffices to prove the assertion for $\epsilon = +1$ (since the proof for $\epsilon = -1$ is entirely analogous). If $HgU_1 \cap KgU_1$ contains infinitely many $(H \cap K, U_1)$ cosets containing elements with endings of the form $x^{-1}a$, $a \in A\backslash U_1$, then it is not difficult to show that this implies that $g^{-1}(HgU_1 \cap KgU_1)(= g^{-1}HgU_1 \cap g^{-1}KgU_1)$ has the same property, provided $H \cap K \nleqq A$. (The case $H \cap K \leqq A$ is easily dealt with.) Thus it suffices to prove that $HU_1 \cap KU_1$ cannot contain infinitely many $(H \cap K, U_1)$ cosets with the specified form of ending (by renaming $g^{-1}Hg$ and $g^{-1}Kg$ if necessary). (This observation is due to Cohen [**2**].) Thus suppose on the contrary that $HU_1 \cap KU_1$ does contain infinitely many such cosets. The proof now continues much as that of [**1**, Lemma 4.1]. Let $\{s_1, s_2, \ldots\}$ be a set of representatives of a countably infinite set of distinct $(H \cap K, U_1)$ cosets such that for all i, s_i ends in $x^{-1}\tau_i$ say, where $\tau_i \in T_1\backslash\{e\}$. Write $s_i = h_iu_i = k_iv_i(i = 1, 2, \ldots)$ where $h_i \in H, k_i \in K, u_i, v_i \in U_1$. Let T_H and T_K be left transversals in U_1 for $H \cap U_1$ and $K \cap U_1$ respectively. Then as in [**1**, Lemma 4.1] we may assume $u_i^{-1} \in T_H$ and $v_i^{-1} \in T_K$, for all i. Then $h_i = s_iu_i^{-1}$, $k_i = s_iv_i^{-1}$ where $u_i^{-1} \in T_H$ and $v_i^{-1} \in T_K$ and u_i^{-1} and v_i^{-1} are the respective U_1-endings of h_i and k_i. By Lemma 2.1 all but finitely many of the u_i are equal and the same is true of the v_i. Hence there exists a pair j, l, $j \neq l$, such that $u_j = u_l$ and $v_j = v_l$. But then $s_j s_l^{-1} = h_j h_l^{-1} = k_j k_l^{-1} \in H \cap K$, which contradicts the choice of s_j, s_l as representatives of distinct $(H \cap K, U_1)$ cosets.

Remark. It is perhaps instructive to see how the above theory can be extended to include the groups $\langle x, y | xy^k x^{-1} = y \rangle$ which Moldavanskiĭ [**4**] proved (in a simple, direct way) to have property (1). It follows from Lemma 2.6 that the assumption in Theorem 1.2 of the existence of a transversal T_{-1} for U_{-1} in A satisfying (3) and (4), can be replaced by the hypothesis that, for all finitely generated groups H, K of G, each intersection of an (H, A) coset with a (K, A) coset contain only finitely many double ended $(H \cap K, A)$ cosets. With this altered hypothesis it is not difficult to show that if A is cyclic and $A = U_1$, then G has property (1), i.e. that Moldavanskiĭ's groups have property (1).

4. Subnormal subgroups. We now prove part (ii) of Theorem 1.2. The proof is split up into lemmas much as in [**1**, §5].

4.1 Lemma (cf. [**1**, Lemma 5.1]). *Let $G = \langle A, x | xU_{-1}x^{-1} = U_1 \rangle$ where $U_1 \neq A$, $U_{-1} \neq A$, and let H be a finitely generated subgroup containing a subnormal subgroup N of G such that for some integer i, $N \nleq x^i U_1 x^{-i}$. Then H has finite index in G if and only if, for all $g \in G$, $g^{-1}Hg \cap U_1$ has finite index in U_1.*

(Note that the assumption on N implies that $N \nleq A$.) We first prove the following lemma.

4.2 Lemma (cf. [**1**, Lemma 5.2]). *Let G be as above (with U_1, $U_{-1} \neq A$) and choose left transversals T_1, T_{-1}, containing e, for U_1, U_{-1} in A. If N is a subnormal subgroup as above, then N contains: an element with beginning tx, $t \in T_1 \backslash \{e\}$, and ending $x^{-1}a$, $a \in A \backslash U_1$; another with beginning x and ending $x^{-1}u$, $u \in U_1$; and a third beginning in τx^{-1}, $\tau \in T_{-1}$, and ending in xa_1, $a_1 \in A$.*

Proof. It clearly suffices to show that whenever $N \trianglelefteq K < G$ where K contains such a triple of elements and $N \nleq A$, then N also contains such a triple. To prove this suppose that g_1, g_2, $g_3 \in K$ are, in order, elements of the kind described in the lemma. Let $y \in N \backslash A$. First suppose y begins in $\hat{t}x^{\epsilon}$ and ends in $x^{-\epsilon}a_2$, where $\epsilon = \pm 1$. If $\epsilon = -1$ then $g_1 y g_1^{-1}$, $g_2 y g_2^{-1}$ and y lie in N and have the requisite properties, while if $\epsilon = +1$ then the same is true of $g_1 g_3 y g_3^{-1} g_1^{-1}$, $g_2 g_3 y g_3^{-1} g_2^{-1}$ and $g_3 y g_3^{-1}$. Suppose next that y begins in $\hat{t}x$ and ends in xa_3. One of g_1, g_2 has beginning different from $\hat{t}x$; without loss of generality suppose g_1 begins in tx where $t \neq \hat{t}$. Then $g_1^{-1} y g_1 \in N$ and has beginning of the form τx and ending of the form $x^{-1}a$, which was one of the cases first considered. Finally if y has beginning of the form τx^{-1} and has ending of the form $x^{-1}a$ then consider instead y^{-1} which falls into the preceding case.

Proof of 4.1. We first show that HgU_1 is double ended if and only if $g^{-1}HgU_1$ is double ended. Thus suppose y, $z \in HgU_1$ have respective endings $x^{-1}a$, $a \in A \backslash U_1$, and $x^{-1}u$, $u \in U_1$. If $g \in A$, then $g^{-1}y$ and $g^{-1}z$ lie in $g^{-1}HgU_1$ and have the required endings. Suppose that $g \notin A$ and first that g begins in tx^{ϵ}

where $\epsilon = \pm 1$, $t \in T_\epsilon$. If y begins in $\tau x^{-\epsilon}$, then $g^{-1}y$ ends in $x^{-1}u_1a$, for some $u_1 \in U_1$, and similarly, if z begins in $\tau x^{-\epsilon}$ then $g^{-1}z$ has ending $x^{-1}u_2$, $u_2 \in U_1$. If y begins in τx^ϵ, then by Lemma 4.2 there is an element $w \in H$ beginning in $\hat{t}x^{-\epsilon}$ and ending in $x^\epsilon a_2$. Then $g^{-1}wy \in g^{-1}HgU_1$ and has ending of the form $x^{-1}a$, $a \in A\backslash U_1$, and similarly if z begins in $\hat{t}x^{-\epsilon}$, then $g^{-1}wz$ has ending of the same form as that of z. The proof of the converse is similar and is omitted.

By Lemma 4.2 for every $H \leqq G$ such that H contains a subnormal subgroup not contained in A, HU_1 is double ended. By what we have just proved this implies that for such $H \leqq G$, all (H, U_1) cosets are double ended. Thus if in addition H is finitely generated, then by Lemmas 2.2 and 2.6 there are only finitely many (H, U_1) cosets in G. It follows that H has finite index in G if and only if $(U_1 : g^{-1}Hg \cap U_1)$ is finite for all $g \in G$.

Theorem 1.2 (ii) is immediate from Lemma 4.1 (with the observation that the case $U_{-1} = A$ is easily dealt with on its own merits) and the following.

4.3 Lemma. *Let $G = \langle A, x | xU_{-1}x^{-1} = U_1\rangle$ where U_1 and A satisfy the hypotheses of part* (ii) *of Theorem* 1.2, *let T_1 be a left transversal for U_1 in A satisfying* (3) *and* (4) *and let T_{-1} be any left transversal for U_{-1} in A, containing e. Let H be a finitely generated subgroup containing a subnormal subgroup N of G, $N \nleqq A$. Then for all $g \in G$, $g^{-1}Hg \cap U_1$ has finite index in U_1.*

Proof. Suppose that for some $g \in G$, $g^{-1}Hg \cap U_1$ has infinite index in U_1. We may suppose $g = e$ by replacing H by $g^{-1}Hg$ and N by $g^{-1}Ng$. Let W be a left transversal for $H \cap U_1$ in U_1. We shall show that every subnormal subgroup N, $N \nleqq A$, has the property that for some finite subset $S \subseteq U_1$, infinitely many elements from the set SW lie in left cosets of $H \cap U_1$ in U_1 containing U_1-endings of elements of N which begin in elements of the form tx and have endings of the form $x^{-1}a$, $a \notin U_1$. This is trivially true for G itself. Suppose that

$$N = N_0 \lhd N_1 \lhd \ldots \lhd N_l = G \quad (l > 0)$$

is a shortest subnormal chain beginning with N and, as inductive hypothesis, that the above property is possessed by subnormal subgroups with shorter subnormal chains. Thus N_1 is assumed to have the property. Suppose $g_1 \in N_1\backslash A$ has the appropriate beginning and ending and has U_1-ending u. By Lemma 4.2 there is an element $g_2 \in N\backslash A$ beginning with τx^{-1} and ending in xa_1. Then $g_1^{-1}g_2g_1 \in N$, has U_1-ending in V_1u, where V_1 is a fixed finite subset of U_1, since T_1 satisfies condition (3), and has the requisite beginning and ending. Thus N (and hence H) has the property that infinitely many elements from the set V_1SW lie in left cosets of $H \cap U_1$ in U_1, determined by U_1-endings of elements of H. This completes the inductive step since V_1S is finite. However what we have proved contradicts (6) of Lemma 2.1.

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