A NEW APPROACH TO THE KOSZUL PROPERTY IN REPRESENTATION THEORY USING GRADED SUBALGEBRAS

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We dedicate this paper to the memory of Ed Cline.

Abstract   Given a quasi-hereditary algebra B, we present conditions which guarantee that the algebra $\operatorname{gr} B$ obtained by grading B by its radical filtration is Koszul and at the same time inherits the quasi-hereditary property and other good Lie-theoretic properties that B might possess. The method involves working with a pair $(A, \mathfrak{a})$ consisting of a quasi-hereditary algebra A and a (positively) graded subalgebra $\mathfrak{a}$. The algebra B arises as a quotient $B = A/J$ of A by a defining ideal J of A. Along the way, we also show that the standard (Weyl) modules for B have a structure as graded modules for $\mathfrak{a}$. These results are applied to obtain new information about the finite dimensional algebras (e.g., the q-Schur algebras) which arise as quotients of quantum enveloping algebras. Further applications, perhaps the most penetrating, yield results for the finite dimensional algebras associated with semisimple algebraic groups in positive characteristic p. These results require, at least at present, considerable restrictions on the size of p.

1. Introduction

In the study of rational representations of a reductive group G over an algebraically closed field K of characteristic $p > 0$, the finite dimensional modules with composition factors having highest weights in some fixed finite saturated set (poset ideal) Γ of dominant weights identify (by means of a category equivalence) with the finite dimensional modules for a finite dimensional algebra B. The homological algebra of B closely parallels that of G; for example, given $M, N \in B\text{-mod}$, $\operatorname{Ext}_B^\bullet(M, N) \cong \operatorname{Ext}_G^\bullet(M, N)$, where, on the right-hand side, M, N are identified with rational G-modules. By varying Γ, the representation theory of G can be largely recaptured from that of the algebras B.

The algebras B are examples of quasi-hereditary algebras, which have been extensively studied in their own right, but often with an eye toward applications to

representation theory. When $G = GL_n$, the famous Schur algebras arise this way [27]. The representation/cohomology theory of a (Lusztig) quantum enveloping algebra U_ζ at a root of unity also can be studied by means of similarly defined finite dimensional algebras. The q-Schur algebras of Dipper and James [20] (see also [19, 22] for exhaustive treatments) are the most well-known examples.

This paper presents new results on the structure of the graded algebras

$$\operatorname{gr} B := \bigoplus_{n \geqslant 0} \operatorname{rad}^n B / \operatorname{rad}^{n+1} B$$

attached to some of the important algebras B alluded to in the previous two paragraphs. One dominant theme of our own work, often with Ed Cline, has been a search for Koszul structures in the quasi-hereditary algebras of interest in modular representation theory of algebraic groups. Koszul algebras (see [9, 46], and also [41]) have very strong homological properties, intertwined with very strong grading properties. In particular, a Koszul algebra B is positively graded, and $B \cong \operatorname{gr} B$. However, it is very difficult, in general, to obtain a good positive grading on an algebra B, or establish an isomorphism $B \cong \operatorname{gr} B$. Indeed, it is possible for a finite dimensional algebra to enjoy many standard Lie-theoretic properties, including all the known homological consequences of Koszulity, yet fail to be Koszul or even to have a positive grading consistent with its radical filtration; see [17, 3.2] for a specific example. Fortunately, in the example given there, the algebra $\operatorname{gr} B$ is Koszul. With this and other insights from [17] as a starting point, this paper considers the general question of what good properties present in a finite dimensional algebra B imply that the graded algebra $\operatorname{gr} B$ is Koszul, requiring at the same time that $\operatorname{gr} B$ keep these good properties (e.g., quasi-heredity, parity conditions, . . .). Of course, in many circumstances, results about $\operatorname{gr} B$ can then be translated back to B. Although our methods are entirely algebraic (in the classical sense), the reader may be reminded of Mumford's notion of 'projective methods' in [39, p. 137], where he viewed the process of forming graded algebras from filtrations (and extracting useful information from the process) as one of the cornerstones of his view of algebraic geometry. The idea has, of course, occurred to classical algebraists, too, but it has been very difficult to say much about $\operatorname{gr} B$. This is still true of the results in this paper, which often require difficult proofs, though we do provide a general framework for them.

One new ingredient in this general framework is the use (when they exist) of Koszul subalgebras. This approach is developed in detail in Part I, where it forms the major theme. The reader is referred to the introduction of Part I for a more detailed discussion. Although B itself does often have a Koszul subalgebra, it turns out to be more flexible to require that such a subalgebra lies inside an algebra A which has B has a homomorphic image. The basic ingredients in this setup include the following:

(1) B is a quasi-hereditary algebra and B-mod has an abstract Kazhdan–Lusztig theory in the sense of [13].

(2) $B = A/J$ is the quotient of a quasi-hereditary algebra A by a certain defining ideal J. Thus, B has weight poset Γ, a poset ideal in the weight poset Λ of A.

(3) A has a Koszul subalgebra $\mathfrak{a}$ such that $(\mathrm{rad}\,\mathfrak{a})A = \mathrm{rad}\,A$.

The goal, achieved in §§ 6 and 7, under additional hypotheses, is to show that $\mathrm{gr}\,B$ is Koszul, and that a large list of properties of B transfer to $\mathrm{gr}\,B$. These include the properties of being quasi-hereditary, and of having a module category with an abstract Kazhdan–Lusztig theory. Furthermore, B and $\mathrm{gr}\,B$ share the same Kazhdan–Lusztig polynomials and have the same homological dual $B^! = \mathrm{Ext}_B^\bullet(B/\mathrm{rad}\,B, B/\mathrm{rad}\,B)$. The multiplicities $[\mathrm{rad}^n \Delta(\lambda)/\mathrm{rad}^{n+1}\Delta(\lambda) : L(\nu)]$ of irreducible modules $L(\nu)$ in the radical layers of standard (Weyl) modules for B agree with the corresponding multiplicities for $\mathrm{gr}\,B$, and the latter can be calculated as coefficients of inverse Kazhdan–Lusztig polynomials. In fact, the graded module category of $\mathrm{gr}\,B$ has the very strong property of having a graded Kazhdan–Lusztig theory in the sense of [13], and the calculation of graded multiplicities is one of its many consequences.

Another result is the elementary Proposition 7.7, which takes $B = A$ (with no quasi-heredity or Kazhdan–Lusztig theory assumption) and proves Koszulity directly for $\mathrm{gr}\,B$ under simple hypotheses on the subalgebra $\mathfrak{a}$. While these hypotheses do not hold for our applications in Part II, the result provides an interesting model, and should at least be of interest to mathematicians working with finite dimensional algebras.

In our applications, starting with the finite dimensional algebra B associated with an quantum enveloping algebra or a simple simply connected group, quite a bit of effort is required to properly choose the pair $(A, \mathfrak{a})$ with all the required properties (which include (1)–(3) above as well as further homological conditions). This project is carried out in Part II. For example, the Koszul subalgebra comes about from work by Andersen–Jantzen–Soergel [2] (and more recently Riche [47]). See the introduction to Part II for more discussion.

The proofs of the results in this paper have independent interest, apart from the results themselves, because of the algebraic methods involved.[1] In addition, this paper lays the groundwork for several sequels. For example, it plays an essential role in [43], giving new results on filtrations of q-Specht modules for Hecke algebras. Similarly, some of the theorems and methods of this paper are used in obtaining new graded quasi-hereditary algebras over DVRs in [44], which in turn are essential in applications to p-filtrations for Weyl modules of algebraic groups planned in [45].

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[1] Using geometric methods, quoting results asserted in [7], it seems likely that the finite dimensional algebras B studied in the quantum case (see § 8 in Part II), as well as the algebras $\mathrm{gr}\,B$, can be shown to be Koszul in the regular weight case. However, it seems first necessary to complete some of the arguments in [7]. We will return to this in [43, § 7]. These geometric methods are insufficient, however, for obtaining the positive characteristic results of § 10. Even in the quantum case, simply knowing Koszulity of B does not imply any connection with the Koszul algebra structure of $\mathfrak{a}$. In our approach, this is built in, and provides an important theme for this paper and subsequent work [45]. *Added Oct. 7, 2011:* Recently, Shan, Vasserot and Varagnolo (arXiv:1107.0146) have proved a parabolic–singular duality theorem in the affine Lie algebra case which shows, at least, that the q-Schur algebras are Koszul. It also likely implies (using [43, Appendix, § 9]) that the algebras B above (of § 8, Part II) are Koszul, even in the singular case. The proofs are a mixture of algebraic and geometric methods.

Recently, there has been much activity regarding graded algebras. See [9, 11, 49], together with their references. All these papers deal with either Koszul or at least positively graded algebras. For the $\mathbb{Z}$-graded case, see, for example, [12] and its references. Our studies here are independent of these works, as is, to some extent, the main thread of our interests. The authors have had a long involvement in the study of positively graded quasi-hereditary algebras and corresponding homological questions; see, e.g., [14–17, 41, 42]. These papers were part of a general program aimed at achieving an understanding in an algebraic way of some of the geometric ingredients which led to a proof of the Kazhdan–Lusztig conjecture [8, 12], and the positive characteristic Lusztig conjecture for large primes [2, 32, 33]. Some of our work was inspired by a preprint of Beilinson and Ginzburg on Koszul algebras, an early version of [9], which had algebraic consequences beyond the Kazhdan–Lusztig conjecture. Our efforts hit an obstacle, however, in paper [17], when we discovered the example [17, 3.2] mentioned above. The approach of this paper in some sense pushes beyond that obstacle, and raises the prospect of further progress in this program.

We heartily thank the referee for suggestions on improvement of the manuscript. In particular, the referee suggested changing the original title ('New graded methods in representation theory') to a more descriptive one. We have tried to do this. As discussed above, either title refers to the process of transferring good properties of B to the graded algebra $\operatorname{gr} B$ (e.g., quasi-heredity), and then working with this positively graded algebra to go further. The existence of the subalgebra $\mathfrak{a}$ makes it possible for this line of attack to succeed.

Part I: theory

Part I, which consists of §§ 2–7, develops some new theoretical techniques for studying the representation theory of certain finite dimensional algebras (all taken over some fixed based field K). As a guide to the reader, we now describe the approach in Part I in more detail. In this discussion, and in Part I itself, A often denotes a general finite dimensional algebra, which could be either the algebra A or B above, or even, in some cases, $\mathfrak{a}$.

As noted in the introduction, given a finite dimensional algebra A, the positively graded algebra $\operatorname{gr} A$ is formed simply by using the radical series of A; see (2.0.5) below. Given a finite dimensional A-module M, there is an analogous finite dimensional graded $\operatorname{gr} A$-module $\operatorname{gr} M$; see (2.0.6) below. Section 2 introduces this and related elementary concepts. If A is a (positively) graded algebra, and if X, Y are graded A-modules, then the extension groups $\operatorname{Ext}_A^n(X, Y)$, $n = 0, 1, 2, \ldots$, are computed in the category A-mod of *ungraded* A-modules, while the extension groups $\operatorname{ext}_A^n(X, Y)$, $n = 0, 1, 2, \ldots$, are computed in the category A-grmod of *graded* A-modules. (This notation is taken from the standard reference [9].) Connecting the graded ext-groups to the ungraded Ext-groups is easy; just use the simple formula (2.0.2) below.

Unfortunately, the functor, $\operatorname{gr} : A\text{-mod} \to \operatorname{gr} A\text{-mod}$, $M \mapsto \operatorname{gr} M$, from A-modules to $\operatorname{gr} A$-modules is not exact. Nevertheless, it is sometimes possible to extend the functor

gr so that it is defined on extensions, i.e., elements of $\mathrm{Ext}_A^1(M, N)$. This process, which we call the 'gr-construction', is studied in §3. As far as we know, this idea and the results of this section are completely new.

Some explanation of the notation in these first sections is required, since the results are often cast in terms of a finite dimensional algebra denoted as A and sometimes in terms of a finite dimensional algebra denoted as $\mathfrak{a}$. The reason for this apparent dichotomy is that, for later applications, $\mathfrak{a}$ will often be a (very special) subalgebra of A. We will need the results of §3 for both $\mathfrak{a}$ and A, and we have tried to state results for the algebra (i.e., for $\mathfrak{a}$ or A) for which the later application is relevant.

Usually, $\mathfrak{a}$ will be a normal subalgebra of A. This means simply that $\mathfrak{a}$ is an augmented algebra with augmentation ideal $\mathfrak{a}_+$ such that $A\mathfrak{a}_+ = \mathfrak{a}_+A$ (necessarily a two-sided ideal of A). In this case, the quotient algebra $A/\mathfrak{a}_+A$ is usually denoted as $A//\mathfrak{a}$.

In later results, $\mathfrak{a}$ will also be positively graded, tightly graded (i.e., $\mathfrak{a} \cong \mathrm{gr}\,\mathfrak{a}$), or (especially) Koszul. The basic conceptual underpinning of this work aims to study the representation theory of a finite dimensional algebra A(often with a focus on a quotient algebra B) which has a normal Koszul subalgebra $\mathfrak{a}$ which is large in the sense that $(\mathrm{rad}\,\mathfrak{a})A = \mathrm{rad}\,A$. In particular, this means that $A//\mathfrak{a}$ is semisimple. Various other conditions will be imposed on the pair $(A, \mathfrak{a})$, but as we see in Part II, these conditions can be verified in applications.

From §4 on, the algebra A is often taken to be quasi-hereditary in the sense of [13]. A more recent account of the basic material on quasi-hereditary algebras is contained in [19, Appendix C]. The section lays out the essential parity conditions – originally developed in [15–17] – that later play an important role. The most basic condition requires that, if A is a quasi-hereditary algebra with weight poset Λ having a length function $l : \Lambda \to \mathbb{Z}$, then, stating just one of the two dual conditions (see §4),

$$\mathrm{Ext}_A^n(\Delta(\lambda), L(\nu)) \neq 0 \Rightarrow l(\lambda) - l(\nu) \equiv n \bmod 2, \quad \forall \lambda, \nu \in \Lambda, n \in \mathbb{N}. \qquad \text{(KL)}$$

This particular (KL) condition is a special case of a much stronger condition

$$\mathrm{Ext}_A^n(\mathrm{rad}^i \Delta(\lambda), L(\nu)) \neq 0 \Rightarrow l(\lambda) - l(\nu) - i \equiv n \bmod 2, \quad \forall \lambda, \nu \in \Lambda, n, i \in \mathbb{N}. \quad \text{(SKL}')$$

In these expressions, $\Delta(\lambda)$ (resp., $L(\nu)$) is the standard (resp., irreducible) module associated with the weight λ (resp., ν) in Λ. As indicated in §4, there are dual versions of these two conditions, involving the irreducible modules $L(\nu)$ and costandard modules $\nabla(\lambda)$, which are automatic when the above versions hold and the category A-mod has a duality.[2] Of course, for an arbitrary quasi-hereditary algebra, the properties (KL) or (SKL$'$) may not hold. In fact, the validity of (KL), or of (SKL$'$), has very strong consequences. For example, in standard examples involving semisimple algebraic groups in positive characteristic, condition (KL) is equivalent to the validity of the Lusztig

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[2] A duality on an abelian category $\mathcal{C}$ is a contravariant equivalence $\mathfrak{d} : \mathcal{C} \to \mathcal{C}$. When $L \in \mathcal{C}$ is irreducible, we usually require $\mathfrak{d}L \cong L$. In categories which allow for $\mathbb{Z}$-gradings of objects, we require instead that $\mathfrak{d}L \cong L$ when L is irreducible of pure grade 0. Also assume that, if M is any graded object, then $\mathfrak{d}M(n) = (\mathfrak{d}M)(-n)$, where $M \mapsto M(n)$ is the 'shift' operator for the graded structure (given in our context by $M(n)_i = M_{n-i}$, for integers n, i).

character formula; see [15]. Also, a main result, proved in [17], shows that when (SKL′)
holds, then the algebra gr A is Koszul, quasi-hereditary, and retains the parity properties
enjoyed by A, as well as its Kazhdan–Lusztig polynomials. The algebra gr A has a graded
Kazhdan–Lusztig theory in the sense of [13].

Section 6 concerns a pair $(A, \mathfrak{a})$ in which A is quasi-hereditary with weight poset
Λ and $\mathfrak{a}$ is a tightly graded subalgebra. The conditions on this pair are laid out
at the start of §6. (The material in §5 is a technical digression, capitalizing on the
fact that, in our applications, the algebras are homomorphic images of certain Hopf
algebras.) Theorem 6.4 establishes that certain of the standard modules $\Delta(\lambda)$ for A,
as well as other A-modules, have a natural structure as graded $\mathfrak{a}$-modules, generated
by their terms in grade 0. The main tool in establishing this important result is the
difficult Lemma 6.3, which inductively approaches $\mathfrak{a}$-gradings one radical layer at a time.
Theorem 6.5 establishes that, if Γ is a poset ideal in Λ and if $B = A/J$ is the associated
quotient quasi-hereditary algebra, then gr B is quasi-hereditary (subject to a condition
on Γ).

Finally, §7 is the heart of Part I. In the main result, Theorem 7.3, we have a pair
$(A, \mathfrak{a})$ as above, but with $\mathfrak{a}$ Koszul. There, B is a quasi-hereditary quotient algebra of A
associated with a poset ideal Γ. It is assumed that B-mod satisfies the (KL) property
above and a further homological condition stated in Hypothesis 7.1(2) is needed. The
conclusion is that B satisfies the (SKL′) condition, and so gr B is Koszul and has other
strong properties as discussed two paragraphs above (as well as the introduction). See
Part II (including its introduction) for further discussion of how the Koszulity property
hypothesized for $\mathfrak{a}$ is known in the cases of interest there.

2. Preliminaries on gradings

We work with finite dimensional algebras A (or $\mathfrak{a}$) over a field K. Let A-mod be the
category of left A-modules which are finite dimensional over K. (All 'A-modules' in
this paper will be assumed to be finite dimensional, unless otherwise indicated.) There
is given a set Λ in bijection with the set $\mathrm{Irr}(A)$ of isomorphism classes of irreducible
A-modules. For $\lambda \in \Lambda$, let $L(\lambda)$ be an irreducible module whose isomorphism class
corresponds to λ.

Often A is positively graded, i.e.,

$$A = \bigoplus_{n\in\mathbb{N}} A_n, \quad A_m A_n \subseteq A_{m+n}, \quad (m, n \in \mathbb{N}).$$

(The term 'graded algebra' will always mean 'positively graded algebra'.) Let A-grmod
be the category of left A-modules M which are finite dimensional K-modules and which
are $\mathbb{Z}$-graded, i.e., $M = \bigoplus_{n\in\mathbb{Z}} M_n$ over A. Thus, $M_n = 0$ if $|n| \gg 0$, and, for $m, n \in \mathbb{Z}$,
$A_m M_n \subseteq M_{m+n}$. If $M, N \in A$-grmod, $\mathrm{Hom}_{A\text{-grmod}}(M, N) \equiv \hom_A(M, N)$ is the R-module of
A-module homomorphisms $f : M \to N$ such that $f(M_n) \subseteq N_n$, for all integers n.

Given a graded A-module M, M_n is called the nth grade of M; it is naturally an
A_0-module, and, therefore, can be regarded as an A-module through the surjective
map $A \twoheadrightarrow A/(\sum_{i>0} A_i) \cong A_0$. For $M \in A$-grmod and $r \in \mathbb{Z}$, let $M(r) \in A$-grmod be the

graded A-module obtained by shifting the grading on M by r places to the right; i.e., $M(r)_n = M_{n-r}$. Then

$$\text{Hom}_A(M, N) = \bigoplus_{r \in \mathbb{Z}} \text{hom}_A(M, N(r)). \tag{2.0.1}$$

An irreducible A-module L is regarded as an irreducible graded A-module concentrated in grade 0.

We say $M \in A\text{-grmod}$ is generated in grade i if $AM_i = M$. Let $A\text{-grmod}_i$ be the full subcategory of $A\text{-grmod}$ consisting of modules generated in grade i.

When A is graded, the category $A\text{-grmod}$ is an abelian category with enough projective objects. The bifunctors $\text{ext}_A^n(-, -) : A\text{-grmod} \times A\text{-grmod} \to K\text{-mod}$, $n = 0, 1, \ldots$, are therefore defined. As with (2.0.1), these functors are related to the bifunctors $\text{Ext}_A^\bullet(-, -)$ in $A\text{-mod}$ by an isomorphism

$$\text{Ext}_A^\bullet(M, N) \cong \bigoplus_{r \in \mathbb{Z}} \text{ext}_A^\bullet(M, N(r)), \quad M, N \in A\text{-grmod} \tag{2.0.2}$$

preserving homological degrees.

Given a graded algebra A, let $\text{Forget} = \text{Forget}_A : A\text{-grmod} \to A\text{-mod}$ be the exact 'forgetful functor' which assigns to any $M \in A\text{-grmod}$ the underlying A-module in which the grading is ignored.[3] For $P \in A\text{-grmod}$, P is projective if and only if $\text{Forget}\, P$ is projective in $A\text{-mod}$.

Let A be a finite dimensional K-algebra. We do not necessarily assume that A is graded. Let $\text{rad}\, A$ be the radical of A and, for $r \in \mathbb{N}$, write $\text{rad}^r A = (\text{rad}\, A)^r$. Similarly, if $M \in A\text{-mod}$, put $\text{rad}^r M = \text{rad}_A^r M := (\text{rad}^r A)M$. More generally, given integers r, s satisfying $0 \leqslant r \leqslant s$, put

$$M^{r,s} = \text{rad}^r M / \text{rad}^s M \in A\text{-mod}. \tag{2.0.3}$$

Thus, $M^{0,s} = M/\text{rad}^s M$. We sometimes write $M^r = M^{r,\infty}$ for $\text{rad}^r M$. Also, let $0 = \text{soc}_0 M \subseteq \text{soc}_{-1} M \subseteq \cdots$ be the socle series of M. Thus, $\text{soc}_{-1} M$ is the socle of M, while for $i > 1$, $\text{soc}_{-i} M \subseteq M$ is the unique submodule containing $\text{soc}_{-i+1} M$ such that $\text{soc}_{-i} M / \text{soc}_{-i+1} M = \text{soc}(M/\text{soc}_{-i+1} M)$.

A graded algebra $A = \bigoplus_{n \in \mathbb{N}} A_n$ is *tightly graded* if A_0 is semisimple, and, for $n \geqslant 1$, $A_n = A_1^n$. It is Koszul provided that, if L, L' are irreducible modules with grading concentrated in grade 0, then

$$\text{ext}_A^n(L, L'(r)) \neq 0 \Rightarrow n = r.$$

Equivalently, if $P_\bullet \twoheadrightarrow L$ is a minimal projective resolution in $A\text{-grmod}$, then the head of P_i, $i \geqslant 0$, is generated in grade i. Any Koszul algebra is tightly graded.

For any positively graded algebra A, $M \in A\text{-grmod}_0$ is said be linear, or to have a linear resolution, if there graded projective resolution $P_\bullet \twoheadrightarrow M$ in which each P_i generated in grade i. (Such a resolution is necessarily minimal.) See [38] for more on

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[3] While useful in this preliminary section, we will generally be more informal with the forgetful functor, using the same symbol M for a graded or an ungraded module, as determined by context.

this topic. Thus, A is Koszul provided every irreducible module L (viewed as a graded module concentrated in grade 0) is linear. Even if A is Koszul, not every $M \in A\text{-grmod}_0$ need be linear. However, linearity for M is equivalent to an ext-vanishing condition as stated for L.

Returning to the general finite dimensional algebra A, the associated graded algebra

$$\operatorname{gr} A := \bigoplus_{i \geqslant 0} \operatorname{rad}^i A / \operatorname{rad}^{i+1} A = \bigoplus_{i \geqslant 0} A^{i,i+1} \tag{2.0.4}$$

is tightly graded. If $A = \bigoplus_{n \geqslant 0} A_n$ is itself tightly graded, there is a natural isomorphism

$$\iota_A : A \xrightarrow{\sim} \operatorname{gr} A \tag{2.0.5}$$

of graded algebras which send $x \in A_i \subseteq \operatorname{rad}^i A$ to its image $[x]$ in $\operatorname{rad}^i A / \operatorname{rad}^{i+1} A$. It induces a category equivalence $\iota_{A*} : \operatorname{gr} A\text{-grmod} \xrightarrow{\sim} A\text{-grmod}$. If $M \in A\text{-mod}$, putting

$$\operatorname{gr} M = \operatorname{gr}_A M := \bigoplus_{n \geqslant 0} \operatorname{rad}^n M / \operatorname{rad}^{n+1} M \in \operatorname{gr} A\text{-grmod}_0 \tag{2.0.6}$$

defines an additive functor $\operatorname{gr} : A\text{-mod} \to \operatorname{gr} A\text{-grmod}_0$, i.e., a functor from the category of A-modules (generated in grade 0) to the category of graded modules for the tightly graded algebra $\operatorname{gr} A$. While not left or right exact, gr preserves surjections. Additionally, if $P \twoheadrightarrow M$ is a projective cover in $A\text{-mod}$, then $\operatorname{gr} P \twoheadrightarrow \operatorname{gr} M$ is a projective cover in $\operatorname{gr} A\text{-grmod}$. Since $(\operatorname{gr} A)/(\operatorname{rad} \operatorname{gr} A) \cong A/\operatorname{rad} A$, the module categories $\operatorname{gr} A\text{-mod}$ and $A\text{-mod}$ for the algebras $\operatorname{gr} A$ and A have the same irreducible modules (and completely reducible) modules.

Also, define $\operatorname{gr}^{\bullet} M = \operatorname{gr}_A^{\bullet} M := \bigoplus_{i \leqslant 0} \operatorname{soc}_{i-1} M / \operatorname{soc}_i M \in \operatorname{gr} A\text{-grmod}$. Clearly, $\operatorname{gr}^{\bullet} : A\text{-mod} \to \operatorname{gr} A\text{-grmod}$ is an additive functor, preserving injections. Both gr and $\operatorname{gr}^{\bullet}$ make sense for right A-modules M (resulting in graded right $\operatorname{gr} A$-modules). Since the linear dual $(-)^*$ on modules reverses left and right modules, $(\operatorname{gr} M)^* = \operatorname{gr}^{\bullet} M^*$ and $(\operatorname{gr}^{\bullet} M)^* = \operatorname{gr} M^*$, for any left or right A-module M.

A short exact sequence $0 \to L \to M \to N \to 0$ in $A\text{-mod}$ defines a short exact sequence $0 \to \operatorname{gr}^{\#} L \to \operatorname{gr} M \to \operatorname{gr} N \to 0$ in $\operatorname{gr} A\text{-grmod}$, in which $\operatorname{gr}^{\#} L = \operatorname{gr}_M^{\#} L$ (after regarding L as a submodule of A) is given explicitly as

$$\operatorname{gr}^{\#} L = \bigoplus_{s \geqslant 0} (L \cap \operatorname{rad}^s M)/(L \cap \operatorname{rad}^{s+1} M) \in \operatorname{gr} A\text{-grmod}. \tag{2.0.7}$$

If L is completely reducible, then $\operatorname{Forget}(\operatorname{gr}^{\#} L) \cong L$, although the isomorphism is not canonical.

Proposition 2.1. *Suppose that $\mathfrak{a}$ is a tightly graded algebra. There is a natural isomorphism*

$$\iota_{\mathfrak{a}*} \circ \operatorname{gr} \circ \operatorname{Forget}|_{\mathfrak{a}\text{-grmod}_0} \cong \operatorname{Id}_{\mathfrak{a}\text{-}grmod_0}$$

of functors. In other words (and more informally), if $M \in \mathfrak{a}\text{-grmod}_0$, and if $\operatorname{gr} M$ is given a graded $\mathfrak{a}$-structure via the isomorphism $\iota_{\mathfrak{a}} : \mathfrak{a} \xrightarrow{\sim} \operatorname{gr} \mathfrak{a}$, then $M \cong \operatorname{gr} M$.

Proof. The isomorphism $\iota_{\mathfrak{a}} : \mathfrak{a} \to \mathrm{gr}\,\mathfrak{a}$ sends $x \in \mathfrak{a}_i$ to $[x]_i \in \mathrm{rad}^i\mathfrak{a}/\mathrm{rad}^{i+1}\mathfrak{a}$. Define a map $M \to \mathrm{gr}\,M$ similarly. For $m \in M_j$, we have $xm \in M_{i+j}$, and $xm \mapsto [xm]_{i+j} = [x]_i[m]_j = x[m_j]$, $\forall i, j \in \mathbb{Z}$. So our map is a homomorphism. It is clearly bijective, and the lemma is proved. $\square$

Corollary 2.2. *Let* $\mathfrak{a}$ *be a tightly graded algebra. If* $M, M' \in \mathfrak{a}$-grmod_0 *satisfy* $Forget(M) \cong Forget(M')$ *(i.e., if* M *and* M' *are isomorphic as ungraded* $\mathfrak{a}$-*modules), then* $M \cong M'$ *in* $\mathfrak{a}$-grmod.

For another argument for this result, without the hypothesis of tight grading, see [28, Theorem 4.1].

3. The gr-functor and extensions

We work with two finite dimensional algebras $\mathfrak{a}$ and A over a field K, with $\mathfrak{a}$ usually a subalgebra of A. The functor $\mathrm{gr} = \mathrm{gr}_{\mathfrak{a}} : \mathfrak{a}$-mod $\to \mathrm{gr}\,\mathfrak{a}$-grmod defined in §2 can, in favorable circumstances, be defined on extensions, i.e., on elements in $\mathrm{Ext}^1_A(M, L)$ for certain A-modules M, L.

Let $M \in A$-mod and let $0 \to \Omega \to P \xrightarrow{\nu} M \to 0$ be a short exact sequence in which ν is a projective cover of M. We obtain a commutative diagram in $\mathrm{gr}\,A$-grmod:

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\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{gr}^{\#}\Omega & \longrightarrow & \mathrm{gr}\,P & \xrightarrow{\mathrm{gr}\,\nu} & \mathrm{gr}\,M & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathrm{gr}^{\#}(\Omega/\mathrm{rad}\,\Omega) & \longrightarrow & \mathrm{gr}(P/\mathrm{rad}\,\Omega) & \longrightarrow & \mathrm{gr}\,M & \longrightarrow & 0 \quad (3.0.1) \\
& & \wr\,\downarrow\,\omega & & & & & & \\
& & \Omega/\mathrm{rad}\,\Omega & & & & & &
\end{array}
$$

in which $\mathrm{gr}\,\nu$ is a projective cover map. Here $(\mathrm{gr}^{\#}\Omega)_i = (\Omega \cap \mathrm{rad}^i P)/(\Omega \cap \mathrm{rad}^{i+1} P)$, $i \geqslant 0$, and $\mathrm{gr}^{\#}(\Omega/\mathrm{rad}\,\Omega)$ is defined similarly. The second row is induced by the exact sequence $0 \to \Omega/\mathrm{rad}\,\Omega \to P/\mathrm{rad}\,\Omega \to M \to 0$. The surjection $\mathrm{gr}\,P \twoheadrightarrow \mathrm{gr}(P/\mathrm{rad}\,\Omega)$ induces the top left vertical surjection. The isomorphism ω is fixed, but is not canonical. We forget the gradings, and regard (3.0.1) as a diagram in $\mathrm{gr}\,A$-mod. Thus, for any completely reducible A-module L, there is an injection

$$
\begin{aligned}
\mathrm{Ext}^1_A(M, L) &\cong \mathrm{Hom}_A(\Omega, L) \\
&\cong \mathrm{Hom}_A(\Omega/\mathrm{rad}\,\Omega, L) \hookrightarrow \mathrm{Hom}_{\mathrm{gr}\,A}(\mathrm{gr}^{\#}\Omega, L) \\
&\cong \mathrm{Ext}^1_{\mathrm{gr}\,A}(\mathrm{gr}\,M, L), \quad (3.0.2)
\end{aligned}
$$

natural in L. Thus, we obtain a natural transformation

$$
\begin{aligned}
\mathrm{gr} = \mathrm{gr}_{A,M,\nu,\omega} &: \mathrm{Ext}^1_A(M, -)|_{\text{completely reducible}} \\
&\longrightarrow \mathrm{Ext}^1_{\mathrm{gr}\,A}(\mathrm{gr}\,M, -)|_{\text{completely reducible}} \quad (3.0.3)
\end{aligned}
$$

of functors on the category of completely reducible A-modules; it depends on the projective cover $\nu : P \twoheadrightarrow M$ and the isomorphism ω in (3.0.1).

Proposition 3.1. *Suppose that $\mathfrak{a}$ is a subalgebra of a finite dimensional algebra A with* $(\mathrm{rad}\,\mathfrak{a})A = \mathrm{rad}\,A$, *and M is an A-module with a projective cover $P \xrightarrow{\nu} M$ in which P is also projective as an $\mathfrak{a}$-module. Then the choices ν, ω above for A and M may also be used for $\mathfrak{a}$ and the restriction of M to $\mathfrak{a}$. For each completely reducible A-module L, we have a commutative diagram*

$$
\begin{array}{ccc}
\mathrm{Ext}^1_A(M, L) & \xrightarrow{\;\;\mathrm{gr}\;\;} & \mathrm{Ext}^1_{\mathrm{gr}\,A}(\mathrm{gr}\,M, L) \\
\downarrow & & \downarrow \\
\mathrm{Ext}^1_{\mathfrak{a}}(M, L) & \xrightarrow{\;\;\mathrm{gr}\;\;} & \mathrm{Ext}^1_{\mathrm{gr}\,\mathfrak{a}}(\mathrm{gr}\,N, L)
\end{array}
$$

where the horizontal maps arise from the gr-construction and the vertical maps are induced by 'restriction' through the algebra homomorphisms $\mathfrak{a} \to A$ and $\mathrm{gr}\,\mathfrak{a} \to \mathrm{gr}\,A$.

Proof. The condition $(\mathrm{rad}\,\mathfrak{a})A = \mathrm{rad}\,A$ implies by induction on n that $\mathrm{rad}^n A = \mathrm{rad}^n \mathfrak{a} A := (\mathrm{rad}\,\mathfrak{a})^n A$. (Observe that $\mathrm{rad}^2 A = (\mathrm{rad}\,\mathfrak{a})A(\mathrm{rad}\,A) = (\mathrm{rad}\,\mathfrak{a})(\mathrm{rad}\,A) = (\mathrm{rad}\,\mathfrak{a})(\mathrm{rad}\,\mathfrak{a})A = (\mathrm{rad}\,\mathfrak{a})^2 A$, etc.) Therefore, given an A-module M, $\mathrm{rad}^n_A M = \mathrm{rad}^n_{\mathfrak{a}} N$, so $\mathrm{gr}_A M = \mathrm{gr}_{\mathfrak{a}} M$. In particular, a completely reducible A-module remains completely reducible upon restriction to $\mathfrak{a}$. Also, if $M \in A$-mod, then the head of M as an A-module becomes upon restriction to $\mathfrak{a}$ the head of $M|_{\mathfrak{a}}$.[4]

Thus, since $\nu : P \twoheadrightarrow M$ is a projective cover in A-mod and $P|_{\mathfrak{a}}$ is projective, ν is also a projective cover in $\mathfrak{a}$-mod. It follows that if the isomorphism ω in (3.0.1) is chosen for A-mod, this isomorphism also works for $\mathfrak{a}$-mod. The commutative diagram thus follows from the naturality of the restriction maps $\mathrm{Hom}_A(-, -) \to \mathrm{Hom}_{\mathfrak{a}}(-, -)$ and $\mathrm{Hom}_{\mathrm{gr}\,A}(-, -) \to \mathrm{Hom}_{\mathrm{gr}\,\mathfrak{a}}(-, -)$. $\square$

Let $\mathfrak{a}$ be tightly graded and $M \in \mathfrak{a}\text{-grmod}_0$. Taking the projective cover $P \xrightarrow{\nu} M$ in $\mathfrak{a}$-grmod, both Ω and $\Omega/\mathrm{rad}\,\Omega$ belong to $\mathfrak{a}$-grmod, and the isomorphism ω in (3.0.1) can be assumed to be in $\mathfrak{a}$-grmod. Also, $\Omega \cong \mathrm{gr}^{\#}\Omega$ in $\mathfrak{a}$-grmod. By Proposition 2.1, there are isomorphisms $P \cong \mathrm{gr}\,P$ and $M \cong \mathrm{gr}\,M$ compatible with the isomorphism $\iota_{\mathfrak{a}}$. Putting all this together, we get the following result.

Proposition 3.2. *Suppose that the finite dimensional algebra $\mathfrak{a}$ is tightly graded and that $M \in \mathfrak{a}\text{-grmod}_0$. For each completely reducible $\mathfrak{a}$-module L with a graded $\mathfrak{a}$-module structure concentrated in grade 0, and each integer s, we have a commutative diagram*

$$
\begin{array}{ccc}
\mathrm{Ext}^1_{\mathfrak{a}}(M, L) & \xrightarrow{\;\;\sim\;\;} & \mathrm{Ext}^1_{\mathrm{gr}\,\mathfrak{a}}(\mathrm{gr}\,M, L) \\
\uparrow & & \uparrow \\
\mathrm{ext}^1_{\mathfrak{a}}(M, L(s)) & \xrightarrow{\;\;\sim\;\;} & \mathrm{ext}^1_{\mathrm{gr}\,\mathfrak{a}}(\mathrm{gr}\,M, L(s))
\end{array}
$$

---

[4] Applied to extensions of M by L (or of $\mathrm{gr}\,M$ by L), this property of heads implies that both vertical restriction maps in the display of the proposition are injective. See Proposition 3.8.

in which the vertical maps are the natural inclusions defined by (2.0.2), the top horizontal map is an isomorphism given by the gr-*construction* (*determined by* Ω *and* ν *described above*), *and the bottom horizontal map is the natural identification induced by Proposition* 2.1.

Remark 3.3. The proposition provides a (largely obvious) interpretation of the gr-construction in the special case in which $\mathfrak{a}$ is tightly graded, $M \in \mathfrak{a}$-grmod$_0$, and the extension in $\operatorname{Ext}^1_{\mathfrak{a}}(M, L)$ arises from a graded extension $0 \to L(s) \to E \xrightarrow{\mu} M \to 0$. It says that the result of applying the gr-construction to this element of $\operatorname{Ext}^1_{\mathfrak{a}}(M, L)$ is (if nonzero) just the element of $\operatorname{Ext}^1_{\operatorname{gr}\mathfrak{a}}(\operatorname{gr} M, L)$ arising from the extension $0 \to L(s) \longrightarrow \operatorname{gr} E \xrightarrow{\operatorname{gr}\mu} \operatorname{gr} M \to 0$ in $\operatorname{gr}\mathfrak{a}$-grmod. This is one of the few explicit interpretations that we have for the gr-construction. Also, no projective cover is needed, and neither are any other choices. Simply apply gr to the (ungraded version of) μ, giving a surjection; the gr-construction is obtained by passing to a natural kernel diagram using L. (This works even when the original extension is split, though E will not be generated in degree 0 when $s \neq 0$, in the split case.) One cannot hope for such a simple interpretation in general, since this procedure always results in an extension obtained by forgetting the grading on a graded extension. No such functor could be additive, if extensions in different degrees by the same irreducible were to exist in its image, for a fixed M and $\operatorname{gr} M$.

Theorem 3.4. *Suppose that* A *is a finite dimensional algebra over a field and that* $P \in A$-mod *is projective. Then, for each non-negative integer* r *and each completely reducible* A-*module* L, *the* gr-*construction induces an isomorphism*

$$\operatorname{Ext}^1_A(P/\operatorname{rad}^r P, L) \xrightarrow{\sim} \operatorname{Ext}^1_{\operatorname{gr} A}(\operatorname{gr}(P/\operatorname{rad}^r P), L).$$

Proof. The projective cover $P \xrightarrow{\nu} P/\operatorname{rad}^r P$ has kernel $\Omega = \operatorname{rad}^r P$. Clearly, in this case, the kernel $\operatorname{gr}^{\#}\Omega$ of $\operatorname{gr}\nu$ is isomorphic to $(\operatorname{gr}\Omega)(r)$. Note that $(\Omega/\operatorname{rad}\Omega)(r)$ is the head of $(\operatorname{gr}\Omega)(r)$. Consequently, the surjection $\operatorname{gr}^{\#}\Omega \twoheadrightarrow \operatorname{gr}^{\#}(\Omega/\operatorname{rad}\Omega) \xrightarrow{\omega} \Omega/\operatorname{rad}\Omega$ induces an isomorphism on heads, and so an isomorphism $\operatorname{Hom}_A(\Omega, L) \xrightarrow{\sim} \operatorname{Hom}_{\operatorname{gr} A}(\operatorname{gr}^{\#}\Omega, L)$, proving the theorem. $\square$

Corollary 3.5. *Let* A *be any finite dimensional algebra, and suppose that* D *and* L *are completely reducible* A-*modules. Then the* gr-*construction gives an isomorphism* $\operatorname{gr} : \operatorname{Ext}^1_A(D, L) \xrightarrow{\sim} \operatorname{Ext}^1_{\operatorname{gr} A}(D, L)$.

Remarks 3.6. (a) Any choice of the gr-construction may be used above. We have used a cover ν with kernel $\Omega = \operatorname{rad}^r P$, but all such covers are isomorphic as maps. Thus, the heads of Ω and $\operatorname{gr}^{\#}\Omega$ must be isomorphic for the kernel Ω of any cover ν, and any surjection $\operatorname{gr}^{\#}\Omega \to \operatorname{gr}^{\#}(\Omega/\operatorname{rad}\Omega) \xrightarrow{\omega} \Omega/\operatorname{rad}\Omega$ must induce an isomorphism on heads, leading to the isomorphism of the theorem.

(b) We also have an isomorphism $\operatorname{Ext}^1_{\operatorname{gr} A}(\operatorname{gr}(P/\operatorname{rad}^r P), L) \cong \operatorname{ext}^1_{\operatorname{gr} A}(\operatorname{gr}(P/\operatorname{rad}^r P), L(r))$, if L is given a pure graded structure in grade 0. (This follows because $\operatorname{gr}^{\#}\Omega$ is generated in grade r.)

(c) The proof shows that $\omega : \operatorname{gr}^{\#}\Omega/\operatorname{rad}\Omega \to \Omega/\operatorname{rad}\Omega$ may be taken to be the identity map on $\operatorname{rad}^r P/\operatorname{rad}^{r+1}P$. Thus, if $0 \to L \to E \xrightarrow{\mu} P/\operatorname{rad}^r P \to 0$ is an extension, then its image under the gr-construction is represented by an extension $0 \to L \to \operatorname{gr}E \xrightarrow{\operatorname{gr}\mu} \operatorname{gr}(P/\operatorname{rad}^r P) \to 0$. (Indeed, if the first extension is viewed as a push-out of $f : \Omega \to L$, the second is the push-out of $\operatorname{gr}f$, if the gradings are ignored.) A similar phenomenon occurs in Remark 3.3. In both cases, the extensions over $\operatorname{gr}A$ that are involved are graded, with L given some pure grade. (See (b) above.) Another situation similar to the present one occurs in Proposition 4.1(c); see the remark following its proof.

In the following result, part (b) shows that, while the algebras $\mathfrak{a}$ and $\operatorname{gr}A$ need not be Morita equivalent, $\mathfrak{a}$ can strongly influence the homological algebra of $\operatorname{gr}A$. This is a dominant theme in this paper.

Theorem 3.7. *Let $\mathfrak{a}$ be a tightly graded subalgebra of the finite dimensional algebra A such that $(\operatorname{rad}\mathfrak{a})A = \operatorname{rad}A$. Let $M \in A$-mod be such that $M|_{\mathfrak{a}} \in \mathfrak{a}\text{-grmod}_0$, and assume that M has a projective cover $P \xrightarrow{\nu} M$ in A-mod in which $P|_{\mathfrak{a}}$ is projective. Also, assume that A_0 is a Wedderburn complement for A containing $\mathfrak{a}_0$, and that M_0 is A_0-stable. Let L be a completely reducible A-module.*

(a) *The projective cover $P \xrightarrow{\nu} M$ can be assumed, upon restriction to $\mathfrak{a}$, to be a projective cover in $\mathfrak{a}$-grmod. Also, $P|_{\mathfrak{a}} \in \mathfrak{a}\text{-grmod}_0$.*

(b) *Any choice of the gr-construction gives an isomorphism*

$$\operatorname{gr} : \operatorname{Ext}^1_A(M, L) \xrightarrow{\sim} \operatorname{Ext}^1_{\operatorname{gr}A}(\operatorname{gr}M, L).$$

Proof. We first prove (a). Observe that

$$A \otimes_{A_0} M_0 \longrightarrow M, \quad a \otimes m \mapsto am,$$

is a projective cover of M in A-mod. Similarly, $\mathfrak{a} \otimes_{\mathfrak{a}_0} M_0 \to M$, $x \otimes m \mapsto xm$, is a $\mathfrak{a}_0$-projective cover. However, the map $\mathfrak{a} \otimes_{\mathfrak{a}_0} M_0 \to A \otimes_{A_0} M_0$ is surjective, since it covers the head of $A \otimes_{A_0} M_0$. But, since some projective cover P of M as an A-module is also $\mathfrak{a}$-projective, all such covers must be $\mathfrak{a}$-projective. Hence, the map $\mathfrak{a} \otimes_{\mathfrak{a}_0} M_0 \to A \otimes_{A_0} M_0$ is a split surjection, and thus an isomorphism of $\mathfrak{a}$-modules (since the $\mathfrak{a}$-heads of $A \otimes_{A_0} M_0$ and $\mathfrak{a} \otimes_{\mathfrak{a}_0} M_0$ are isomorphic to M_0).

Now consider (b). Since any $\operatorname{gr} : \operatorname{Ext}^1_A(M, L) \to \operatorname{Ext}^1_{\operatorname{gr}A}(\operatorname{gr}M, L)$ is an injection, it will be an isomorphism if and only if it is an isomorphism for some particular choice of ν and ω. Thus, by (a), we can assume that $P \xrightarrow{\nu} M$ restricts to a projective cover in $\mathfrak{a}$-grmod. In the commutative diagram (3.0.1), we can take gr to be either $\operatorname{gr}_{\mathfrak{a}}$ or gr_A, and rad to be $\operatorname{rad}_{\mathfrak{a}}$ or rad_A, without changing the spaces. But at the level of graded $\mathfrak{a}$-modules, it is obvious that the kernel of $\operatorname{gr}^{\#}\Omega \twoheadrightarrow \operatorname{gr}^{\#}(\Omega/\operatorname{rad}\Omega)$ is precisely $\operatorname{rad}(\operatorname{gr}^{\#}\Omega)$. Thus, the mapping $\operatorname{gr} : \operatorname{Ext}^1_A(M, L) \to \operatorname{Ext}^1_{\operatorname{gr}A}(\operatorname{gr}M, L)$ is an isomorphism. $\square$

We conclude this section by recording the elementary result mentioned in footnote §. It is interesting that the hypotheses (and thus the conclusions) remain valid when A is replaced by an algebra B which is a homomorphic image of A.

Proposition 3.8. *Let $\mathfrak{a} \to A$ be a homomorphism of algebras such that, if A is regarded as a left $\mathfrak{a}$-module, then $(\operatorname{rad}\mathfrak{a})A = \operatorname{rad} A$. Then, for A-modules M, L with L completely reducible, the natural map $\operatorname{Ext}^1_A(M, L) \to \operatorname{Ext}^1_{\mathfrak{a}}(M, L)$ is injective.*

Proof. It suffices to treat the case in which L is irreducible. If (*) $0 \to L \to E \to M \to 0$ is a non-split extension of M by L in A-mod, then $L \subseteq \operatorname{rad} E$. But $\operatorname{rad} E = (\operatorname{rad} A)E = (\operatorname{rad} a)AE = \operatorname{rad}_{\mathfrak{a}} E$, the radical of E as an $\mathfrak{a}$-module. This means that (*) is a non-split extension of M by L when the modules are restricted to $\mathfrak{a}$. This proves the required injectivity. $\qquad\square$

4. Quasi-hereditary algebras

Let A be a quasi-hereditary algebra (QHA) over a field K with finite (weight) poset Λ. For $\lambda \in \Lambda$, let $L(\lambda)$, $\Delta(\lambda)$, $\nabla(\lambda)$ be the corresponding irreducible, standard, and costandard modules, respectively, in the highest weight category (HWC) A-mod. Thus, $\Delta(\lambda)$ (resp., $\nabla(\lambda)$) has head (resp., socle) isomorphic to $L(\lambda)$, and other composition factors $L(\mu)$ satisfy $\mu < \lambda$ in the poset Λ. Assume that A is *split*, i.e., $\dim \operatorname{End}_A(L(\lambda)) \cong K$ for all $\lambda \in \Lambda$. Let $P(\lambda) \twoheadrightarrow L(\lambda)$ (resp., $\nabla(\lambda) \hookrightarrow Q(\lambda)$) be the projective cover (resp., injective envelope) of $L(\lambda)$. Let A-mod(Δ) (resp., A-mod(∇)) be the full subcategory of A-mod consisting of modules with a Δ-filtration (resp., ∇-filtration), i.e., have filtrations with sections $\Delta(\lambda)$ (resp., $\nabla(\lambda)$), $\lambda \in \Lambda$. Thus, $P(\lambda) \in A$-mod(Δ) and $Q(\lambda) \in A$-mod(∇). In any Δ-filtration (resp., ∇-filtration) of $P(\lambda)$ (resp., $Q(\lambda)$), the top (resp., bottom) section is isomorphic to $\Delta(\lambda)$ (resp., $\nabla(\lambda)$); other sections are isomorphic to $\Delta(\mu)$ (resp., $\nabla(\mu)$) for some $\mu > \lambda$.

If $\lambda, \mu \in \Lambda$ and $n \in \mathbb{N}$, then, by [15, Lemma 2.2],

$$\dim \operatorname{Ext}^n_A(\Delta(\lambda), \nabla(\mu)) = \delta_{\lambda,\mu}\delta_{n,0}. \tag{4.0.1}$$

For a non-empty poset ideal Γ in Λ, let A-mod$[\Gamma]$ be the full subcategory of the module category A-mod consisting of modules all of whose composition factors $L(\gamma)$ satisfy $\gamma \in \Gamma$. There exists an idempotent ideal (sometimes called a defining ideal) J of A such that A-mod$[\Gamma]$ is Morita equivalent to A/J-mod. In addition, A/J is a QHA. The exact inclusion functor $i_* : A$-mod$[\Gamma] \to A$-mod (which is just inflation through the map $A \to A/J$) admits a left adjoint $i^* : A$-mod $\to A$-mod$[\Gamma]$ and a right adjoint $i^! : A$-mod $\to A$-mod$[\Gamma]$. Explicitly, given $M \in A$-mod, i^*M (resp., $i^!M$) is the largest quotient module (resp., submodule) of M lying in A-mod$[\Gamma]$. It will be convenient, given $M \in A$-mod, to denote i^*M also by M_Γ. A basic property states that, given $M, N \in A$-mod$[\Gamma]$, there is a natural isomorphism (preserving grades)

$$\operatorname{Ext}^\bullet_{A/J}(M, N) \cong \operatorname{Ext}^\bullet_A(i_*M, i_*N). \tag{4.0.2}$$

Generally, we denote i_*M simply by M, so $\operatorname{Ext}^\bullet_{A/J}(M, N) \cong \operatorname{Ext}^\bullet_A(M, N)$.

The QHA algebra A/J above is often denoted simply as A_Γ. In fact, it is the largest quotient module of A (regarded as a left A-module) whose composition factors $L(\gamma)$ satisfy $\gamma \in \Gamma$.

Proposition 4.1. *Let P be a projective module for a finite dimensional algebra A. Let $L \in A$-mod be completely reducible and let $r \geqslant 0$.*

(a) *The natural map $\operatorname{Ext}_A^1(P/\operatorname{rad}^r P, L) \to \operatorname{Ext}_A^1(\operatorname{rad}^{r-1} P/\operatorname{rad}^r P, L)$ induced by the inclusion $\operatorname{rad}^{r-1} P/\operatorname{rad}^r P \hookrightarrow P/\operatorname{rad}^r P$ is injective.*

(b) *If A is a QHA, Γ is a poset ideal in Λ, and all weights of composition factors of L lie in Γ, then the natural map $\operatorname{Ext}_A^1(P_\Gamma/\operatorname{rad}^r P_\Gamma, L) \to \operatorname{Ext}_A^1(\operatorname{rad}^{r-1} P_\Gamma/\operatorname{rad}^r P_\Gamma, L)$ is an injection.*

(c) *Continuing* (b), *if $\dim \operatorname{Ext}_A^1(P_\Gamma/\operatorname{rad}^r P_\Gamma, L) = \dim \operatorname{Ext}_{\operatorname{gr} A}^1(\operatorname{gr}(P_\Gamma/\operatorname{rad}^r P_\Gamma), L)$, then the restriction map $\operatorname{Ext}_{\operatorname{gr} A}^1(\operatorname{gr}(P_\Gamma/\operatorname{rad}^r P_\Gamma), L) \to \operatorname{Ext}_{\operatorname{gr} A}^1(\operatorname{rad}^{r-1} P_\Gamma/\operatorname{rad}^r P_\Gamma, L)$ is an injection.*

Proof. We first prove (a). The commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{rad}^{r-1}P & \longrightarrow & P & \longrightarrow & P/\operatorname{rad}^{r-1}P & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{rad}^{r-1}P/\operatorname{rad}^r P & \longrightarrow & P/\operatorname{rad}^r P & \longrightarrow & P/\operatorname{rad}^{r-1}P & \longrightarrow & 0
\end{array}
$$

has exact rows, so the long exact sequence of $\operatorname{Ext}_A^\bullet$ provides a commutative diagram

$$
\begin{array}{ccc}
\operatorname{Hom}_A(\operatorname{rad}^{r-1}P, L) & \overset{\sim}{\longleftarrow} & \operatorname{Hom}_A(\operatorname{rad}^{r-1}P/\operatorname{rad}^r P, L) \\
\downarrow & & \downarrow{\alpha} \\
\operatorname{Ext}_A^1(P/\operatorname{rad}^{r-1}P, L) & = & \operatorname{Ext}_A^1(P/\operatorname{rad}^{r-1}P, L) \\
\downarrow & & \downarrow \\
\operatorname{Ext}_A^1(P, L) & \longleftarrow & \operatorname{Ext}_A^1(P/\operatorname{rad}^r P, L) \\
& & \downarrow{\beta} \\
& & \operatorname{Ext}_A^1(\operatorname{rad}^{r-1}P/\operatorname{rad}^r P, L)
\end{array}
$$

in which the columns are exact. To show that β is injective, we show that α is surjective. However, because L is completely reducible, the top horizontal map is an isomorphism. Since P is projective, $\operatorname{Ext}_A^1(P, L) = 0$. The result follows by a diagram chase.

Now (b) follows from (a), using (4.0.2) and the fact that if $P \in A$-mod is projective, then $P_\Gamma := j^* P \in A$-mod$[\Gamma]$ is projective.

To prove (c), write $P_\Gamma^{0,r}$ for $P_\Gamma/\operatorname{rad}^r P_\Gamma$ and $P_\Gamma^{r-1,r}$ for $\operatorname{rad}^{r-1} P_\Gamma/\operatorname{rad}^r P_\Gamma$. Without loss, we can assume that L is irreducible. Let Ω be the kernel of the surjection $P \twoheadrightarrow P_\Gamma^{0,r}$.

Claim. No composition factor $L(\nu)$ of $\Omega/(\operatorname{rad} \Omega + \operatorname{rad}^r P)$ satisfies $\nu \in \Gamma$. Otherwise, there is a nonzero homomorphism $\Omega \to L(\nu)$ with kernel $R \supseteq \operatorname{rad} \Omega + \operatorname{rad}^r P$ and with $\nu \in \Gamma$.

The composition factors of $M := P/R$ are just those of $P_\Gamma^{0,r} = P/\Omega$ and $L(\nu)$. Thus, all composition factors of M have highest weight in Γ. Since $\operatorname{head} M = \operatorname{head} P = \operatorname{head} P_\Gamma$, M is a homomorphic image of P_Γ. Since M is a homomorphic image of $P/\operatorname{rad}^r P$ by construction, its radical length is at most r. Hence, M is a homomorphic image of $P_\Gamma/\operatorname{rad}^r P_\Gamma = P_\Gamma^{0,r}$. However, $\dim M > \dim P_\Gamma^{0,r}$, contradicting the existence of $L(\nu)$, and proving the claim.

Now put $Y = \operatorname{rad} \Omega + \operatorname{rad}^r P$, and write $\Omega/\operatorname{rad} \Omega = X/\operatorname{rad} \Omega \oplus Y/\operatorname{rad} \Omega$, a direct sum of (completely reducible) A-submodules for some A-submodule X of Ω containing $\operatorname{rad} \Omega$. By the previous paragraph, L is not a composition factor of $X/\operatorname{rad} \Omega$. Let $a = \dim \operatorname{Ext}_A^1(P_\Gamma^{0,r}, L)$. Obviously, $\Omega/\operatorname{rad} \Omega$, and thus Ω/X has a quotient module Ω/Z isomorphic to $L^{\oplus a}$ (since $\dim \operatorname{Hom}_A(\Omega, L) = \dim \operatorname{Ext}_A^1(P_\Gamma^{0,r}, L) = a$). Put $E = P/Z$. This gives an exact sequence $0 \to L^{\oplus a} \to E \to P_\Gamma^{0,r} \to 0$. Notice that $L^{\oplus a} \cong \Omega/Z = (X + \operatorname{rad}^r P)/Z = (Z + \operatorname{rad}^r P)/Z = (Z + (\operatorname{rad}^r A)P)/Z \cong \operatorname{rad}^r A(P/Z)$ identifies with $\operatorname{rad}^r E$. Passage to $\operatorname{gr} E$ gives an exact sequence $0 \to (L^{\oplus a})(r) \to \operatorname{gr} E \to \operatorname{gr} P_\Gamma^{0,r} \to 0$ of graded $\operatorname{gr} A$-modules. Regarding $\operatorname{gr} E$ as an ungraded extension, the push-out ρ_π via any projection $\pi : L^{\oplus a} \to L$ is non-split. (It clearly arises from a graded push-out in which $L^{\oplus a}$ and L have degree r. Observe that $\operatorname{gr} E$ and, thus, its graded push-out – a homomorphic image – are generated in degree 0.) Let $\pi_1, \ldots, \pi_a$ be the standard projections $L^{\oplus a} \to L$, and put $\rho_i = \rho_{\pi_i}$. Let $c_1, \ldots, c_a \in K$ and let $\pi = \sum c_i \pi_i : L^{\oplus a} \to L$. All maps $L^{\oplus a} \to L$ are obtained in this way, and $\pi \neq 0$ if and only if some $c_i \neq 0$. The push-out ρ_π is $\sum c_i \rho_{\pi_i} = \sum c_i \rho_i$. In particular, $\sum c_i \rho = 0$ if and only if $\rho_\pi = 0$, which occurs if and only if $\pi = 0$, or, equivalently, all $c_i = 0$. That is, $\rho_1, \ldots, \rho_a$ are linearly independent in $\operatorname{Ext}_{\operatorname{gr} A}^1(\operatorname{gr} P_\Gamma^{0,r}, L)$. By dimension considerations, these elements form a basis for $\operatorname{Ext}_{\operatorname{gr} A}^1(\operatorname{gr} P_\Gamma^{0,r}, L)$. Also, if $\pi \neq 0$, then ρ_π remains non-split upon pull-back through the map $P_\Gamma^{r-1,r} \hookrightarrow (\operatorname{gr} P_\Gamma^{0,r})_{r-1} \subseteq P_\Gamma^{0,r}$. (Observe that the $\operatorname{gr} A$-module $(\operatorname{gr} E)_{r-1} \oplus (\operatorname{gr} E)_r$ is generated in degree $r-1$.) This pull-back is the image of ρ_π under the restriction map $\operatorname{Ext}_{\operatorname{gr} A}^1(P_\Gamma^{0,r}, L) \to \operatorname{Ext}_{\operatorname{gr} A}^1(P_\Gamma^{r-1,r}, L)$. This proves (c). $\qquad\square$

Remark 4.2. After establishing the claim in the proof of (c), the proof may be concluded by using an appropriate choice of the gr-construction. Indeed, the implicit correspondence of push-outs of $\operatorname{gr} E$ and E in the proof as given can be seen to arise from a gr-construction. The gr construction behaves here similarly to how it behaves in the context of Theorem 3.4 and Remarks 3.6(b, c). Note also that, by the injectivity (3.0.2) of the gr-construction, the equality of dimensions assumed in the hypothesis of (c) is equivalent to an isomorphism via the gr-construction. In practice, this is checked using Theorem 3.7(b).

Suppose that $A = \bigoplus_{n \geqslant 0} A_n$ is a graded QHA with weight poset Λ; see [14–17]. For each $\lambda \in \Lambda$, $L(\lambda) \in A$-grmod denotes the irreducible A-module $L(\lambda)$ viewed as a graded A-module concentrated in grade 0. Also, $\Delta(\lambda), \nabla(\lambda), P(\lambda), Q(\lambda) \in A$-grmod. We can assume that $\Delta(\lambda)$ and $P(\lambda)$ are generated in grade 0, and that, dually, $\operatorname{soc} \nabla(\lambda) \subseteq \nabla(\lambda)_0$ and $\operatorname{soc} Q(\lambda) \subseteq Q(\lambda)_0$. Furthermore, $P(\lambda)$ has a filtration by graded submodules with sections of the form $\Delta(\nu)(s)$, $s \geqslant 0$. (Thus, the 'top' section of $P(\lambda)$ is $\Delta(\lambda)$ in A-grmod.)

Of course, $P(\lambda) \to \Delta(\lambda)$ is the projective cover in A-grmod. Similar statements hold for $Q(\lambda)$.

Proposition 4.3. *Suppose that A is a graded QHA over a field K. Let Q be a finite dimensional graded A-module which has a filtration by graded submodules with sections $\nabla(\tau)(s)$, $\tau \in \Lambda$, $-s \in \mathbb{N}$. For $\tau \in \Lambda$, $-s \in \mathbb{Z}$, the number $[Q : \nabla(\lambda)(s)]$ of occurrences of $\nabla(\tau)(s)$ as a section in such a filtration is given by $[Q : \nabla(\tau)(s)] = \dim \hom_A(\Delta(\lambda)(s), Q(\mu))$.*

By (4.0.1), $\dim \operatorname{ext}_A^n(\Delta(\lambda)(r), \nabla(\mu)(s)) = \delta_{n,0}\delta_{\lambda,\mu}\delta_{r,s}$, which implies the stated result. If $Q = Q(\mu)$ is the injective envelope in A-grmod of $L(\mu)$, then $[Q(\mu) : \nabla(\tau)(s)] = [\Delta(\tau)_{-s} : L(\mu)]$, a graded form of Brauer–Humphreys reciprocity; see [16, Proposition 1.2.4(a)].

Let A be an arbitrary QHA. As defined in [17, § 2], the HWC A-mod (or just A) satisfies (by definition) the (SKL$'$) condition[5] with respect to a function $l : \Lambda \to \mathbb{Z}$, provided that, for $\lambda, \mu \in \Lambda$, $n, i \in \mathbb{N}$,

$$\begin{cases} (1) & \operatorname{Ext}_A^n(\Delta^i(\lambda), L(\mu)) \neq 0 \Rightarrow n \equiv l(\lambda) - l(\mu) + i; \quad \text{and} \\ (2) & \operatorname{Ext}_A^n(L(\mu), \nabla_{-i}(\lambda)) \neq 0 \Rightarrow n \equiv l(\lambda) - l(\mu) + i. \end{cases} \tag{SKL$'$}$$

Here $\Delta^i(\lambda) := \operatorname{rad}^i \Delta(\lambda)$ and $\nabla_{-i}(\lambda) := \nabla(\lambda)/\operatorname{soc}_{-i}\nabla(\lambda)$. If Γ is a poset ideal in Λ, (4.0.2) implies that if A-mod satisfies (SKL$'$), then A/J-mod also satisfies (SKL$'$) with respect to $l|_\Gamma$. Indeed, it is only necessary that condition (SKL$'$) holds for $\lambda, \mu \in \Gamma$. Similar remarks hold for the (KL) property discussed in the next paragraph.

If condition (SKL$'$) holds for $i = 0$, then A-mod is said to satisfy the Kazhdan–Lusztig property (KL) (with respect to $l : \Lambda \to \mathbb{Z}$). More precisely, the (KL) property holds for A-mod (with respect to $l : \Lambda \to \mathbb{Z}$) provided that, for $\lambda, \mu \in \Lambda$, $n \in \mathbb{N}$,

$$\begin{cases} (1) & \operatorname{Ext}_A^n(\Delta(\lambda), L(\mu)) \neq 0 \Rightarrow n \equiv l(\lambda) - l(\mu); \quad \text{and} \\ (2) & \operatorname{Ext}_A^n(L(\mu), \nabla(\lambda)) \neq 0 \Rightarrow n \equiv l(\lambda) - l(\mu). \end{cases} \tag{KL}$$

The property (KL) implies surjectivity of the natural map $\operatorname{Ext}_A^n(L(\lambda), L(\mu)) \to \operatorname{Ext}_A^n(\Delta(\lambda), L(\mu))$ $\forall n \in \mathbb{N}$, $\forall \lambda, \mu \in \Lambda$; see [16, Theorem 4.3]. This means that $\operatorname{Ext}_A^n(\Delta(\lambda), L(\mu)) \to \operatorname{Ext}_A^n(\operatorname{rad}\Delta(\lambda), L(\mu))$ is the zero map $\forall n \in \mathbb{N}$, $\forall \lambda, \mu \in \Lambda$. Thus,

$$\operatorname{Ext}_A^n(\Delta(\lambda), L(\mu)) \to \operatorname{Ext}_A^n(\Delta^r(\lambda), L(\mu)) \quad \text{is the zero map,}$$
$$\forall r > 0, \forall n \in \mathbb{N}, \forall \lambda, \mu \in \Lambda. \tag{4.3.1}$$

A graded QHA A (or its graded module category) is said to have a graded Kazhdan–Lusztig theory, or satisfy the graded Kazhdan–Lusztig property, with respect to $l : \Lambda \to \mathbb{Z}$, provided the following holds: Given $\lambda, \mu \in \Lambda$, if either

---

[5] A stronger condition, in which $L(\mu)$ is replaced by $\nabla(\mu)$ in the first line and by $\Delta(\mu)$ in the second line of (SKL$'$), is called the strong Kazhdan–Lusztig property (SKL). It can be shown that (SKL) $\Rightarrow$ (SKL$'$), and that (SKL) is equivalent to (SKL$'$) holding for all algebras eAe, with e an idempotent associated with a poset coideal in Λ; see [17, § 2.4]. Unfortunately, not all of these idempotents may be chosen from the subalgebra $\mathfrak{a}$ that we will consider later, and so we work with (SKL$'$) only, in dealing with A. The property (SKL) will hold for $\operatorname{gr} A$ once (SKL$'$) is established for A [17, Theorem 2.4.2].

$\operatorname{ext}_A^n(\Delta(\lambda), L(\mu)(m)) \neq 0$ or $\operatorname{ext}_A^n(L(\mu), \nabla(\lambda)(m)) \neq 0$, then $m = n \equiv l(\lambda) - l(\mu) \mod 2$. (Equivalently, given property (KL), the standard modules are linear and the costandard modules have a dual property.) In this case, A is a Koszul algebra; see [15–17].

Given $\lambda, \nu \in \Lambda$, the Kazhdan–Lusztig polynomial $P_{\nu,\lambda}$ is defined as[6]

$$P_{\nu,\lambda} = t^{l(\lambda)-l(\nu)} \sum_{n=0}^{\infty} \dim \operatorname{Ext}_A^n(L(\lambda), \Delta(\nu)) t^{-n} \in \mathbb{Z}[t, t^{-1}]. \qquad (4.3.2)$$

In practice (see §8), $P_{\nu,\lambda} \in \mathbb{Z}[t^2]$ identifies with the Kazhdan–Lusztig polynomial of a Coxeter group.

Theorem 4.4 ([17, Theorem 2.2.1, Remark 2.2.2(b)]). *If A is a QHA such that (SKL$'$) holds relative to $l : \Lambda \to \mathbb{Z}$, then $\operatorname{gr} A$ is a graded QHA with poset Λ. Also, the graded Kazhdan–Lusztig property holds for $\operatorname{gr} A$, so $\operatorname{gr} A$ is a Koszul algebra. In addition, A-mod and $\operatorname{gr} A$-mod have the same Kazhdan–Lusztig polynomials. Finally, the standard (resp., costandard) module in $\operatorname{gr} A$-mod indexed by $\lambda \in \Lambda$ is $\operatorname{gr} \Delta(\lambda)$ (resp., $\operatorname{gr} \nabla(\lambda)$).*

Finally, we end this section with the following important remark.

Remark 4.5. Let A be a QHA with weight poset Λ, and let $l : \Lambda \to \mathbb{Z}$ be given. It is important to know whether the highest weight category A-mod satisfies the (KL) property with respect to l stated above. In fact, considerable effort has been devoted to conditions which are equivalent to the validity of the (KL) property. Some of these are laid out in [15, § 5]. One of these equivalent conditions is the validity of the Lusztig character formula LCF (suitably formulated). Rather than repeat this material (see, in particular, [15, Theorem 5.3 & Corollary 5.4]), we mention, looking ahead to §10 below (and using the notation there), that the LCF holds for p-regular weights in the Janzten region Γ_{Jan} for a semisimple simple algebraic group G (over an algebraically closed field of positive characteristic $p \geqslant h$) provided that, given $\lambda = w \cdot \lambda^- \in \Gamma_{\text{Jan}}$, $\lambda^- \in C^-$, the character equality

$$\operatorname{ch} L(\lambda) = \sum_{y \leqslant x \in W_p} (-1)^{l(w)-l(y)} P_{y,w}(-1) \operatorname{ch} \Delta(y \cdot \lambda^-)$$

is valid. See [15, § 5] for more details. Similar comments hold for U_ζ-mod discussed in §8.

---

[6] Strictly speaking, this is the left Kazhdan–Lusztig polynomial. The analogous right Kazhdan–Lusztig polynomial $P_{\nu,\lambda}^R$ is defined by the same formula, but replacing $\dim \operatorname{Ext}_A^n(L(\lambda), \Delta(\nu))$ by $\dim \operatorname{Ext}_A^1(\Delta(\nu), L(\lambda))$. However, in all the cases that we are interested in, the category A-mod has a duality, so $P_{\nu,\lambda} = P_{\nu,\lambda}^R$.

5. Hopf algebras and further results on extensions

Let U be a Hopf algebra over a field K having a normal Hopf subalgebra u. Assume that the antipodes on U and u are surjective.[7] For U-modules M, N, $\operatorname{Hom}_K(M, N)$ has a natural U-module structure given by $(u \cdot f)(m) = \sum u_{(1)} f(S(u_{(2)})m)$ for $f \in \operatorname{Hom}_K(M, N)$, $m \in M$, $u \in U$. (Here $S : U \to U$ is the antipode and $\Delta(u) := \sum u_{(1)} \otimes u_{(2)}$, the image of u under the comultiplication $\Delta : U \to U \otimes U$.) The surjectivity of the antipode S implies that $\operatorname{Hom}_U(M, N) = \operatorname{Hom}_K(M, N)^U$; see [4, Proposition 2.9]. Let $U^\dagger := U//u$, the quotient of U by the Hopf ideal $u_+ U = U u_+$ generated by the augmentation ideal u_+ of u. We are given finite dimensional augmented algebras A and $\mathfrak{a}$ over K, with $\mathfrak{a}$ a normal (augmented) subalgebra of A. Assume that there are compatible surjective algebra homomorphisms

$$
\begin{array}{ccc}
u & \twoheadrightarrow & \mathfrak{a} \\
\downarrow & & \downarrow \\
U & \twoheadrightarrow & A
\end{array}
\qquad (5.0.1)
$$

in which the vertical arrows are inclusions. We also assume that

$$
\operatorname{rad} A = (\operatorname{rad} \mathfrak{a}) A. \qquad (5.0.2)
$$

Thus, given an A-module L, then L is completely reducible if and only if $L|_\mathfrak{a}$ is completely reducible.

Then
$$
\operatorname{Hom}_A(M, N) = \operatorname{Hom}_U(M, N) = \operatorname{Hom}(M, N)^U = \operatorname{Hom}_u(M, N)^{U^\dagger} = \operatorname{Hom}_\mathfrak{a}(M, N)^{U^\dagger},
$$
for $M, N \in A\text{-mod}$. When $N = L$ is a completely reducible, versions of this identity exist for $\operatorname{gr} A$ and $\operatorname{gr} \mathfrak{a}$. Thus, if M is a $\operatorname{gr} A$-module, then $M/\operatorname{rad} M$ is a $\operatorname{gr} A/(\operatorname{rad} \operatorname{gr} A) \cong A/\operatorname{rad} A$-module, and

$$
\begin{aligned}
\operatorname{Hom}_{\operatorname{gr} A}(M, L) &\cong \operatorname{Hom}_{\operatorname{gr} A}(M/\operatorname{rad} M, L) \\
&\cong \operatorname{Hom}_A(M/\operatorname{rad} M, L) \\
&\cong \operatorname{Hom}_\mathfrak{a}(M/\operatorname{rad} M, L)^{U^\dagger} \\
&\cong \operatorname{Hom}_{\operatorname{gr} \mathfrak{a}}(M/\operatorname{rad} M, L)^{U^\dagger} \\
&\cong \operatorname{Hom}_{\operatorname{gr} \mathfrak{a}}(M, L)^{U^\dagger}.
\end{aligned}
$$

The intermediate isomorphisms explain the isomorphism $\operatorname{Hom}_{\operatorname{gr} A}(M, L) \cong \operatorname{Hom}_{\operatorname{gr} \mathfrak{a}}(M, L)^{U^\dagger}$. Finally, if M is a graded $\operatorname{gr} A$-module and L is a graded completely reducible module,

$$
\begin{aligned}
\hom_{\operatorname{gr} A}(M, L) &\cong \hom_{\operatorname{gr} A}(M/\operatorname{rad} M, L) \\
&\cong \bigoplus_{r \in \mathbb{Z}} \operatorname{Hom}_{\operatorname{gr} A}((M/\operatorname{rad} M)_r, L_r) \\
&\cong \bigoplus_{r \in \mathbb{Z}} \operatorname{Hom}_A((M/\operatorname{rad} M)_r, L_r)
\end{aligned}
$$

---

[7] This assumption is automatic for the Hopf algebras that we consider.

$$\cong \bigoplus_{r \in \mathbb{Z}} \operatorname{Hom}_{\mathfrak{a}} \left((M/\operatorname{rad} M)_r, L_r \right)^{U^{\dagger}}$$

$$\cong \hom_{\operatorname{gr} \mathfrak{a}} (M/\operatorname{rad} M, L)^{U^{\dagger}}$$

$$\cong \hom_{\operatorname{gr} \mathfrak{a}} (M, L)^{U^{\dagger}}.$$

Proposition 5.1. *Assume the setup of (5.0.1) and (5.0.2). Let L be a completely reducible A-module. Let M be an A-module satisfying the following property $(*)$: there is a projective cover $P \xrightarrow{\nu} M$ in which P is a projective $\mathfrak{a}$-module. Then there are isomorphisms $\operatorname{Ext}^1_A(M, L) \cong \operatorname{Ext}^1_{\mathfrak{a}}(M, L)^{U^{\dagger}}$ and $\operatorname{Ext}^1_{\operatorname{gr} A}(\operatorname{gr} M, L) \cong \operatorname{Ext}^1_{\operatorname{gr} \mathfrak{a}}(\operatorname{gr} M, L)^{U^{\dagger}}$, which are natural with respect to morphisms between A-modules satisfying $(*)$. Finally, if L is given a graded structure, $\operatorname{ext}^1_{\operatorname{gr} A}(\operatorname{gr} M, L) \cong \operatorname{ext}^1_{\operatorname{gr} \mathfrak{a}}(\operatorname{gr} M, L)^{U^{\dagger}}$, which are also natural (giving graded homomorphisms) with respect to morphisms in A-mod satisfying $(*)$. (In particular, there are natural actions of $U^{\dagger}$ on all the indicated Ext-groups above.)*

Proof. There exist exact sequences

$$\begin{cases} 0 \to \Omega \to P \xrightarrow{\nu} M \to 0, \\ 0 \to \operatorname{gr}{}^{\#}\Omega \to \operatorname{gr} P \xrightarrow{\operatorname{gr} \nu} \operatorname{gr} M \to 0, \end{cases}$$

resulting in isomorphisms

$$\begin{cases} \operatorname{Ext}^1_A(M, L) \cong \operatorname{Hom}_A(\Omega, L), \\ \operatorname{Ext}^1_{\operatorname{gr} A}(\operatorname{gr} M, L) \cong \operatorname{Hom}_{\operatorname{gr} A}(\operatorname{gr}{}^{\#}\Omega, L), \\ \operatorname{Ext}^1_{\mathfrak{a}}(M, L) \cong \operatorname{Hom}_{\mathfrak{a}}(\Omega, L), \\ \operatorname{Ext}^1_{\operatorname{gr} \mathfrak{a}}(\operatorname{gr} M, L) \cong \operatorname{Hom}_{\operatorname{gr} \mathfrak{a}}(\operatorname{gr}{}^{\#}\Omega, L). \end{cases}$$

In fact, by (5.0.2), $P \twoheadrightarrow M$ (resp., $\operatorname{gr} P \twoheadrightarrow \operatorname{gr} M$) is a projective cover of M (resp., $\operatorname{gr} M$) in $\mathfrak{a}$-mod (resp., $\operatorname{gr} \mathfrak{a}$-mod). Also, if L has a graded structure, then $\operatorname{Ext}^1_{\operatorname{gr} A}(\operatorname{gr} M, L) \cong \hom_{\operatorname{gr} A}(\operatorname{gr}{}^{\#}\Omega, L)$ and $\operatorname{ext}^1_{\operatorname{gr} \mathfrak{a}}(\operatorname{gr} M, L) \cong \hom_{\operatorname{gr} \mathfrak{a}}(\operatorname{gr}{}^{\#}\Omega, L)$. We have previously shown that:

(i) $\operatorname{Hom}_{\mathfrak{a}}(\Omega, L)$ has a natural $U^{\dagger}$-structure, and $\operatorname{Hom}_A(\Omega, L) \cong \operatorname{Hom}_{\mathfrak{a}}(\Omega, L)^{U^{\dagger}}$;

(ii) $\operatorname{Hom}_{\operatorname{gr} \mathfrak{a}}(\operatorname{gr}{}^{\#}\Omega, L)$ has a natural $U^{\dagger}$-structure, and $\operatorname{Hom}_{\operatorname{gr} A}(\operatorname{gr}{}^{\#}\Omega, L) \cong \operatorname{Hom}_{\operatorname{gr} \mathfrak{a}}(\operatorname{gr}{}^{\#}\Omega, L)^{U^{\dagger}}$;

(iii) $\hom_{\operatorname{gr} \mathfrak{a}}(\operatorname{gr}{}^{\#}\Omega, L)$ has a natural $U^{\dagger}$-structure, and $\hom_{\operatorname{gr} A}(\operatorname{gr}{}^{\#}\Omega, L) \cong \hom_{\operatorname{gr} \mathfrak{a}}(\operatorname{gr}{}^{\#}\Omega, L)^{U^{\dagger}}$.

These isomorphisms are natural in the category of modules satisfying $(*)$, and can be used to define the $U^{\dagger}$-action and the isomorphisms in the statement of the proposition. $\qquad\square$

6. Quasi-hereditary structures on gr B

The algebra B of the title of this section will be a introduced above Theorem 6.5. It will be a QHA quotient of a QHA A. The algebra A will be part of a pair $(A, \mathfrak{a})$ satisfying the following conditions.

Hypothesis 6.1. (1) *A is a split QHA over the field* K *with poset* Λ *(see* § *4).*

(2) $\mathfrak{a} = \bigoplus_{n \geqslant 0} \mathfrak{a}_n$ *is a tightly graded subalgebra of* A *such that the pair* $(A, \mathfrak{a})$ *satisfies the setup given in (5.0.1) for a pair* $(U, \mathfrak{u})$ *of Hopf algebras. In particular,* $\mathfrak{a}$ *is an augmented, normal subalgebra of* A.

(3) $(\mathrm{rad}\,\mathfrak{a})A = \mathrm{rad}\,A$ *as in (5.0.2). In particular,* $\mathrm{rad}\,A \subseteq \mathfrak{a}_+ A$, *so* $A/\mathfrak{a}_+ A$ *is semisimple.*

(4) $\mathfrak{a}_0 \subseteq A_0$, *a Wedderburn complement of* A.

Condition (4) is automatic when $\mathfrak{a}_0$ and $A/\mathrm{rad}\,A$ are separable algebras over K. It will hold when $\mathfrak{a}_0$ and $A/\mathrm{rad}\,A$ are split semisimple, or if K has characteristic 0 or is algebraically closed. (These will be the cases in § 8.) In fact, the separability of $A/\mathrm{rad}\,A$ guarantees that A has a subalgebra (Wedderburn complement) A_0' mapping isomorphically onto $A/\mathrm{rad}\,A$ under the quotient map $\pi : A \twoheadrightarrow A/\mathrm{rad}\,A$. If $\mathfrak{a}_0'$ is the subalgebra of A_0' mapping isomorphically to $\pi(\mathfrak{a}_0)$ by π, then $\mathfrak{a}_0'$ and $\mathfrak{a}_0$ are both Wedderburn complements for $E := \pi^{-1}(\pi(\mathfrak{a}_0))$. By the Wedderburn–Malcev theorem, there exists $x \in \mathrm{rad}\,A = \mathrm{rad}\,E$ such that $\mathfrak{a}_0 = (1 - x)\mathfrak{a}_0'\,(1 - x)^{-1}$. Then

$$\mathfrak{a}_0 \subseteq A_0 := (1 - x)A_0'\,(1 - x)^{-1},$$

as required.

An (A, A_0)-module is defined to be a pair (M, M') with M, M' modules for A, A_0, respectively, and with $M' \subseteq M|_{A_0}$. In addition, the pair (M, M') is defined to have a (non-negative) $\mathfrak{a}$-graded structure if $M|_{\mathfrak{a}}$ is a non-negatively $\mathfrak{a}$-graded module and if $M' = M_0$ is its grade 0-term.

Lemma 6.2. *Suppose that* (M, M_0) *is a* $\mathfrak{a}$-graded (A, A_0)-module *such that* $M_i = 0$ *for* $i \geqslant r$ *for a given integer* $r > 0$. *Let* L *be a completely reducible* A-module, *viewed as a homogeneous* $\mathfrak{a}$-module concentrated in grade 0. *Let* E *be an* A-extension of M by L *which is also an* $\mathfrak{a}$-graded extension of M by $L(r)$. *Then the grade 0 term* E_0 *of the* $\mathfrak{a}$-grading *may be rechosen to be an* A_0-submodule of E, *with* $E_1, \dots, E_r$ *remaining the same, and* E_0 *having image* M_0 *in* M. *(Thus,* (E, E_0) *is an* $\mathfrak{a}$-graded (A, A_0)-module *with* (M, M_0) *as a natural homomorphic image.)*

Proof. Write $E|_{\mathfrak{a}} = E_0 \oplus E_1 \oplus \cdots \oplus E_r$, the direct sum of its grades. Since M_0 is A_0-stable, the extension E of M by L defines an A_0-extension of M_0 by L. Since A_0 is semisimple, there is an A_0-submodule E_0' of E mapping isomorphically onto M_0. Since $\mathfrak{a}_i E_r = 0$ for $i > 0$, it follows that $\mathfrak{a}_i E_0' \subseteq E_i$ for $i > 0$. Therefore, $E = E_0' \oplus E_1 \oplus \cdots \oplus E_r$ defines a

new grading of E as an $\mathfrak{a}$-module. By construction, (E, E_0') is an (A, A_0)-module which is graded as an $\mathfrak{a}$-module. $\qquad\square$

Fix a proper poset ideal Θ of Λ, and let $\lambda \in \Theta$. To simplify the notation in the remainder of this section, we define

$$\Delta = P_\Theta(\lambda), \qquad\qquad (6.2.1)$$

the largest quotient of the projective cover $P(\lambda)$ of $L(\lambda)$ in A-mod all of whose composition factors $L(\theta)$ satisfy $\theta \in \Theta$. Thus, if $\Theta = \{\nu \in \Lambda \mid \nu \leqslant \lambda\}$, then $\Delta = \Delta(\lambda)$, which motivates the notation.

In applying the lemma below to Theorem 6.4, we will assume, inductively, that $\Delta^{0,r}|_\mathfrak{a}$ has a graded $\mathfrak{a}$-module structure with the 0-grade term $(\Delta^{0,r})_0$ stable under the action of A_0. In particular, Theorem 3.7(a) allows Propositions 3.1 and 3.2, with $M = \Delta^{0,r}$, to be used simultaneously provided the A-projective cover of $\Delta^{0,r}$ (which is also the projective cover of Δ and $L(\lambda)$) is $\mathfrak{a}$-projective.

Lemma 6.3. *Let $\Delta \in A$-mod be as in (6.2.1) for the fixed $\lambda \in \Theta$. Let $r \geqslant 1$. Let $L \in A$-mod be completely reducible with summands $L(\gamma)$, $\gamma \in \Theta$. Assume, for each $\theta \in \Theta$, that the projective cover $P(\theta)$ of $L(\theta)$ in A-mod is projective for $\mathfrak{a}$. Finally, assume that $\Delta^{0,r}|_\mathfrak{a} \in \mathfrak{a}\text{-grmod}_0$ so that $\Delta_0^{0,r}$ is A_0-stable. Then there is a commutative diagram*

$$
\begin{array}{ccccc}
\mathrm{Ext}_A^1(\Delta^{0,r}, L) & \xrightarrow{\alpha} & \mathrm{Ext}_\mathfrak{a}^1(\Delta^{0,r}, L) & \xleftarrow{\beta} & \mathrm{ext}_\mathfrak{a}^1(\Delta^{0,r}, L(r)) \\
& \nwarrow{\scriptstyle f} & \uparrow{\scriptstyle \rho} & & \uparrow{\scriptstyle \sigma} \\
\mathrm{Ext}_{\mathrm{gr}A}^1(\mathrm{gr}\Delta^{0,r}, L) & \xleftarrow[\sim]{\delta} & \mathrm{Ext}_{\mathrm{gr}\mathfrak{a}}^1(\mathrm{gr}\Delta^{0,r}, L)^{U^\dagger} & \xleftarrow{\epsilon} & \mathrm{ext}_{\mathrm{gr}\mathfrak{a}}^1(\mathrm{gr}\Delta^{0,r}, L(r))^{U^\dagger} \\
\downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \tau} & (I) & \downarrow{\scriptstyle \phi} \\
\mathrm{Ext}_{\mathrm{gr}A}^1(\Delta^{r-1,r}, L) & \xleftarrow[\sim]{\kappa} & \mathrm{Ext}_{\mathrm{gr}\mathfrak{a}}^1(\Delta^{r-1,r}, L)^{U^\dagger} & \xleftarrow[\sim]{\zeta} & \mathrm{ext}_{\mathrm{gr}\mathfrak{a}}^1(\Delta^{r-1,r}, L(r))^{U^\dagger} \\
& & \downarrow{\scriptstyle \pi} & (II) & \downarrow{\scriptstyle \theta} \\
& & \mathrm{Ext}_{\mathrm{gr}\mathfrak{a}}^2(\mathrm{gr}\Delta^{0,r-1}, L) & \xleftarrow{\iota} & \mathrm{ext}_{\mathrm{gr}\mathfrak{a}}^2(\mathrm{gr}\Delta^{0,r-1}, L(r))
\end{array}
\qquad (6.3.1)
$$

in which ξ, ι are injective, the maps ζ, κ, δ, f are isomorphisms, and also $\ker \pi = \mathrm{Im}\, \tau$ and $\ker \theta = \mathrm{Im}\, \phi$.

Proof. We begin by establishing the existence of the map f and by defining the maps $\delta, \epsilon, \rho, \sigma$. In order to obtain the top two rows of the diagram, consider a smaller diagram

$$
\begin{array}{ccc}
\operatorname{Ext}^1_A(\Delta^{0,r}, L) & \xrightarrow{\ \mu\ } & \operatorname{Ext}^1_{\operatorname{gr} A}(\operatorname{gr} \Delta^{0,r}, L) \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\delta'} \\
\operatorname{Ext}^1_{\mathfrak{a}}(\Delta^{0,r}, L) & \xrightarrow{\ \nu\ } & \operatorname{Ext}^1_{\operatorname{gr}\mathfrak{a}}(\operatorname{gr} \Delta^{0,r}, L) \\
\uparrow{\scriptstyle\beta} & & \uparrow{\scriptstyle\epsilon'} \\
\operatorname{ext}^1_{\mathfrak{a}}(\Delta^{0,r}, L(r)) & \xrightarrow{\ \omega\ } & \operatorname{ext}^1_{\operatorname{gr}\mathfrak{a}}(\operatorname{gr} \Delta^{0,r}, L(r))
\end{array}
$$

The three horizontal maps are all induced by the gr-construction, and the vertical maps are the 'obvious' ones. Commutativity is a consequence of Propositions 3.1 and 3.2. A projective cover of $\Delta^{0,r}$ is also a projective cover of $L(\lambda)$ and hence $\mathfrak{a}$-projective by hypothesis. The maps μ, ν, ω are isomorphisms, by Proposition 3.2 and Theorem 3.7(b) (applied to $M = \Delta^{0,r}$). Now ρ (resp., σ, ϵ) is obtained by restricting ν^{-1} (resp., ω^{-1}, ϵ') to the submodule of $U^\dagger$-fixed points. We have defined the upper right-hand square and the top row of (6.3.1). We now consider the construction of the diagonal map f. Proposition 5.1 and its argument show that the images of α and δ' are $\operatorname{Ext}^1_{\mathfrak{a}}(\Delta^{0,r}, L)^{U^\dagger}$ and $\operatorname{Ext}^1_{\operatorname{gr}\mathfrak{a}}(\operatorname{gr} \Delta^{0,r}, L)^{U^\dagger}$, respectively. Both α and δ' are injective, giving isomorphisms to their images. Let

$$
\delta : \operatorname{Ext}^1_{\operatorname{gr}\mathfrak{a}}(\operatorname{gr} \Delta^{0,r}, L)^{U^\dagger} \to \operatorname{Ext}^1_{\operatorname{gr} A}(\operatorname{gr} \Delta^{0,r}, L)
$$

be the map inverse to δ' on its image. Define

$$
f : \operatorname{Ext}^1_{\operatorname{gr}\mathfrak{a}}(\operatorname{gr} \Delta^{0,r}, L)^{U^\dagger} \to \operatorname{Ext}^1_A(\Delta^{0,r}, L)
$$

to be the composition of δ with μ^{-1}.

Our construction shows that $\rho = \alpha \circ f$ and $\rho \circ \epsilon = \beta \circ \sigma$, and that f is an isomorphism. This establishes all the claimed interrelationships of the maps in the top two rows of the four-row diagram, and the claimed properties of these maps. The isomorphism κ is obtained using Proposition 5.1, and the commutativity of the square to which it belongs (along with the definitions of ξ and τ) is obvious.

Boxes (I) and (II) *without* $(-)^{U^\dagger}$ are commutative as evident parts of long exact sequences of Ext and ext, with the maps ϵ, ζ, ι from right to left standard injections provided by (2.0.2). Also, ϵ and ζ have factorizations fitting into a commutative diagram:

$$
\begin{array}{ccccc}
\operatorname{Ext}^1_{\operatorname{gr}\mathfrak{a}}(\operatorname{gr} \Delta^{0,r}, L)^{U^\dagger} & \xleftarrow{\ \supseteq\ } & \operatorname{ext}^1_{\operatorname{gr} A}(\operatorname{gr} \Delta^{0,r}, L(r)) & \xleftarrow{\ \sim\ } & \operatorname{ext}^1_{\operatorname{gr}\mathfrak{a}}(\operatorname{gr} \Delta^{0,r}, L(r))^{U^\dagger} \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{Ext}^1_{\operatorname{gr}\mathfrak{a}}(\Delta^{r-1,r}, L)^{U^\dagger} & \xleftarrow{\ \supseteq\ } & \operatorname{ext}^1_{\operatorname{gr} A}(\Delta^{r-1,r}, L(r)) & \xleftarrow{\ \sim\ } & \operatorname{ext}^1_{\operatorname{gr}\mathfrak{a}}(\Delta^{r-1,r}, L(r))^{U^\dagger}
\end{array}
$$

in which the right-hand box is commutative by the naturality in Proposition 5.1. The commutativity of the left-hand box follows from naturality in Proposition 5.1 together with the obvious naturality of (2.0.2). The fact that ϵ and ζ are the composites of the top row and bottom row, respectively, is easily seen from the construction of these maps in terms of syzygies. Note that box (II) without the $(-)^{U^\dagger}$ gives maps whose restrictions to $\operatorname{ext}^1_{\operatorname{gr}\mathfrak{a}}(\operatorname{gr}\Delta^{0,r}, L(r))^{U^\dagger}$ and $\operatorname{ext}^1_{\operatorname{gr}\mathfrak{a}}(\Delta^{r-1,r}, L(r))^{U^\dagger}$ are candidates for ϵ and ζ, and it is really these restrictions that we have shown factor as above with images in $\operatorname{Ext}^1_{\operatorname{gr}\mathfrak{a}}(\operatorname{gr}\Delta^{0,r}, L)^{U^\dagger}$ and $\operatorname{Ext}^1_{\operatorname{gr}\mathfrak{a}}(\operatorname{gr}\Delta^{0,r}, L)^{U^\dagger}$, respectively. Thus our argument establishes the existence of box (II), including the $(-)^{U^\dagger}$.

The A-modules Δ and $\Delta^{0,r}$ have the projective cover $P = P(\lambda)$. It is assumed that $P|_\mathfrak{a}$ is projective for $\mathfrak{a}$. Thus, the hypothesis of Theorem 3.7(b) is satisfied with $M = \Delta^{0,r}$. Hence, any choice of the gr-construction gives an isomorphism

$$\operatorname{Ext}^1_A(\Delta^{0,r}, L) \xrightarrow{\sim} \operatorname{Ext}^1_{\operatorname{gr}A}(\operatorname{gr}\Delta^{0,r}, L),$$

and so gives an equality of dimensions of these two spaces. Then Proposition 4.1(c) implies that ξ is injective as required.

The map ζ is an isomorphism, since the containment

$$\operatorname{Ext}^1_{\operatorname{gr}\mathfrak{a}}(\Delta^{r-1,r}, L)^{U^\dagger} \supseteq \operatorname{ext}^1_{\operatorname{gr}A}(\Delta^{r-1,r}, L(r))$$

factors as

$$\operatorname{Ext}^1_{\operatorname{gr}\mathfrak{a}}(\Delta^{r-1,r}, L)^{U^\dagger} \cong \operatorname{Ext}^1_{\operatorname{gr}A}(\Delta^{r-1,r}, L) \supseteq \operatorname{ext}^1_{\operatorname{gr}A}(\Delta^{r-1,r}, L(r)).$$

Proposition 5.1 has been used again for the isomorphism. The containment is an equality, since, for the tightly graded algebra $\operatorname{gr}A$, graded extensions between irreducible modules are split unless the grades differ by 1. □

Theorem 6.4. *Assume that Hypothesis 6.1 holds, and $\Delta = P_\Gamma(\lambda)$, where Γ is a poset ideal in Λ and $\lambda \in \Gamma$. Assume, for each $\gamma \in \Gamma$, that the projective cover $P(\gamma)$ of $L(\gamma)$ in A-mod is $\mathfrak{a}$-projective. Then Δ has an $\mathfrak{a}$-graded structure generated in grade 0 such that Δ_0 is A_0-stable. Consequently, these conclusions hold if $\Delta = \Delta(\lambda)$, $\lambda \in \Gamma$.*

Proof. We prove by induction on r that $\Delta^{0,r+1}$ has an $\mathfrak{a}$-graded structure generated in grade 0 such that $\Delta_0^{0,r+1}$ is A_0-stable, assuming that a similar result holds for $\Delta^{0,r}$. Let $L = \Delta^{r,r+1}$ in Lemmas 6.2 and 6.3. Let $\chi \in \operatorname{Ext}^1_A(\Delta^{0,r}, L)$ be the extension corresponding to $0 \to \Delta^{r,r+1} \to \Delta^{0,r+1} \to \Delta^{0,r} \to 0$. If $\alpha(\chi) \in \operatorname{Im}(\beta)$, we can put an $\mathfrak{a}$-graded structure on $\Delta^{0,r+1}$, compatible with that on $\Delta^{0,r}$, with $\Delta^{r,r+1}$ pure of grade r. Then Lemma 6.2 applied with $M = \Delta^{0,r}$ completes the induction. However, since f is an isomorphism in (6.3.1) there is an element $\chi' \in \operatorname{Ext}^1_{\operatorname{gr}\mathfrak{a}}(\operatorname{gr}\Delta^{0,r}, L)^{U^\dagger}$ with $f(\chi') = \chi$, and so $\rho(\chi') = \alpha(\chi)$. Write $\tau(\chi') = \zeta(\chi'')$, with $\chi'' \in \operatorname{ext}^1_{\operatorname{gr}\mathfrak{a}}(\Delta^{r-1,r}, L(r))^{U^\dagger}$. Then $\iota\theta(\chi'') = \pi\tau(\chi') = 0$. Thus, $\theta(\chi'') = 0$ since ι is injective. Hence, $\chi'' = \phi(\chi''')$ with $\chi''' \in \operatorname{ext}^1_{\operatorname{gr}\mathfrak{a}}(\operatorname{gr}\Delta^{0,r}, L(r))^{U^\dagger}$. $\epsilon(\chi''')$ and χ' have the same image under τ and hence are equal, using the injectivity of ξ and the isomorphisms δ, κ. Now we have $\alpha(\chi) = \rho(\chi') = \rho(\epsilon(\chi''')) = \beta(\sigma(\chi'''))$, as desired. □

We prove the main result of this section, establishing that, given a poset ideal Γ of Λ, if the hypotheses of Theorem 6.4 holds for Γ, then $\operatorname{gr} B$ is a QHA for $B := A_\Gamma$.

Theorem 6.5. *Assume that A satisfies Hypothesis 6.1. Let Γ be a (non-empty) poset ideal in Λ such that, given $\gamma \in \Gamma$, the projective cover $P(\gamma)$ of $L(\gamma)$ in A-mod is projective as an $\mathfrak{a}$-module. Let $B = A_\Gamma := A/J$ be such that B-mod $\cong A$-mod$[\Gamma]$ (for some ideal J of A). Then $\operatorname{gr} B$ is a QHA with weight poset Γ. Also, the $\operatorname{gr} B$-modules $\operatorname{gr} \Delta(\gamma)$, $\gamma \in \Gamma$, are the standard modules in the HWC $\operatorname{gr} B$-mod.*

Proof. Induction on $|\Gamma|$: If $|\Gamma| = 1$, then $B \cong K$, and the result is trivial. Thus, assume that the result holds for any proper ideal Θ of Γ. In particular, let $\tau \in \Gamma$ be maximal, and form the poset ideal $\Theta := \Gamma \setminus \{\tau\}$ of Λ. By induction, $C := B_\Theta$ is QHA with weight poset Θ and with standard modules $\operatorname{gr} \Delta(\theta)$, $\theta \in \Theta$.

For $\gamma \in \Theta$, the projective cover $P(\gamma)$ of $L(\gamma)$ in A-mod is, by hypothesis, a projective $\mathfrak{a}$-module. Thus, by Theorem 6.4, $P_\Theta(\gamma) \in \mathfrak{a}\text{-grmod}_0$, and $P_\Theta(\gamma)_0$ is A_0-stable. Therefore, the hypotheses of Theorem 3.7 hold with $M = P_\Theta(\gamma)$ and $L = L(\mu)$ for any $\mu \in \Lambda$, so Theorem 3.7(b) gives an isomorphism $\operatorname{Ext}_A^1(P_\Theta(\gamma), L(\mu)) \cong \operatorname{Ext}_{\operatorname{gr} A}^1(\operatorname{gr} P_\Theta(\gamma), L(\mu))$. But, if $\mu \in \Theta$, then (4.0.2) implies that $\operatorname{Ext}_A^1(P_\Theta(\gamma), L(\mu)) \cong \operatorname{Ext}_C^1(P_\Theta(\gamma), L(\nu)) = 0$, because $P_\Theta(\gamma)$ is projective in C-mod. Therefore, $\operatorname{Ext}_{\operatorname{gr} A}^1(\operatorname{gr} P_\Theta(\gamma), L(\mu)) = 0$, for all $\mu \in \Theta$. However, there is an evident surjection $\operatorname{gr} P(\gamma) \twoheadrightarrow \operatorname{gr} P_\Theta(\gamma)$ whose kernel has no quotient $L(\mu)$ with $\mu \in \Theta$, by the Ext^1-vanishing just proved. (Note that $\operatorname{gr} P(\gamma)$ is the projective cover of $L(\gamma)$ in $\operatorname{gr} A$-mod, so it has a simple head.) It follows that $(\operatorname{gr} P(\gamma))_\Theta \cong \operatorname{gr} P_\Theta(\gamma)$ as a $(\operatorname{gr} A)_\Theta$-module. Therefore, letting $\gamma \in \Theta$ vary:

$$\operatorname{gr}(A_\Theta) \cong (\operatorname{gr} A)_\Theta. \tag{6.5.1}$$

Let $J = \ker(B \to B_\Theta)$. (We sometimes denote B_Θ by C.) In the notation of §2, $\operatorname{gr} B/\operatorname{gr}^{\#} J \cong \operatorname{gr} C$. Clearly, $\operatorname{gr}^{\#} J$ is an ideal in $\operatorname{gr} B$. Since τ is maximal in Γ, $P_B(\tau) \cong \Delta(\tau)$ (i.e., $\Delta(\tau)$ is the projective cover of $L(\tau)$ in B-mod). It follows that $\operatorname{gr} \Delta(\tau)$ is the projective cover of $L(\tau)$ in $\operatorname{gr} B$-mod.

Since B is a QHA, $J \cong \Delta(\tau)^{\oplus n}$ for some $n > 0$. Next, the head of $\operatorname{gr}^{\#} J$ has only copies of $L(\tau)$ in it. (According to (6.5.1), $\operatorname{gr} C$ is the largest quotient of $\operatorname{gr} A$ with all composition factors $L(\mu)$, $\mu \in \Theta$. So it is the largest such quotient of $\operatorname{gr} B$.) Also, there are at most n copies of $L(\tau)$ in this head. Since $\dim J = \dim \operatorname{gr}^{\#} J$, it follows that $\operatorname{gr}^{\#} J \cong \operatorname{gr} \Delta(\tau)^{\oplus n}$. Thus, $\operatorname{gr}^{\#} J$ is a heredity ideal and $\operatorname{gr} B$ is a QHA with the indicated standard modules. $\square$

7. Koszul structures on $\operatorname{gr} B$

In this section, let A be a QHA with weight poset Λ and let Γ be a (non-empty) poset ideal in Λ. In the notation of §4, write A-mod$[\Gamma] \cong B$-mod where $B = A/J$ for a suitable (idempotent) ideal of A. Since B is a QHA, it has finite global dimension. We make the following assumptions.

Hypothesis 7.1. *Let A, B be as above. The following statements hold.*

(1) *There exists a subalgebra* $\mathfrak{a}$ *of* A *such that the pair* $(A, \mathfrak{a})$ *satisfies Hypothesis* 6.1.

(2) *Let* N *be the global dimension of* B. *For each* $\nu \in \Gamma$, *the projective cover* $P(\nu)$ *of* $L(\nu)$ *in* A-mod *is projective as an* $\mathfrak{a}$-module. *Moreover, if* $N > 1$, *there is an exact complex*

$$0 \to \Omega_{N-1} \to P_{N-2} \to \cdots \to P_0 \to L(\nu) \to 0$$

in A-mod *in which each* P_i *is a projective* A-module such that $P_i|_{\mathfrak{a}}$ *is also projective.*

Let $\mathfrak{b}$ be the image of $\mathfrak{a}$ under the quotient map $A \to B$.

Proposition 7.2. *Assume Hypothesis* 7.1. *For* $M, L \in B$-mod *with* L *completely reducible, we have that the natural maps*

$$\begin{cases} (1) & \mathrm{Ext}_A^n(M, L) \to \mathrm{Ext}_{\mathfrak{a}}^n(M, L) \\ (2) & \mathrm{Ext}_B^n(M, L) \to \mathrm{Ext}_{\mathfrak{b}}^n(M, L) \end{cases}$$

are injective, for all $n \geqslant 0$.

Proof. First, we prove the assertion $(*)$ below for A-modules M, L with L completely reducible.

$(*)$ Suppose that there is a non-negative integer m and an exact complex

$$0 \to \Omega_m \to P_{m-1} \to \cdots \to P_0 \to M \to 0$$

in A-mod such that each P_i is projective and such that each $P_i|_{\mathfrak{a}}$ is also projective. (We take this as a vacuous statement if $m = 0$.) Then $\mathrm{Ext}_A^n(M, L) \to \mathrm{Ext}_{\mathfrak{a}}^n(M, L)$ is injective for $n \leqslant m + 1$.

Assertion $(*)$ follows from Proposition 3.8 when $m = 0$. So, without loss of generality, assume that $m > 1$ and $n > 1$. The kernel M' of the surjection $P_i \twoheadrightarrow M$ satisfies the hypothesis of $(*)$, provided we replace m by $m - 1$. However, we may assume by induction that $\mathrm{Ext}_A^{n-1}(M', L) \to \mathrm{Ext}_{\mathfrak{a}}^{n-1}(M', L)$ is injective for $n \leqslant m$ (equivalently, $n - 1 \leqslant m - 1$). But $\mathrm{Ext}_A^{n-1}(M', L) \cong \mathrm{Ext}_A^n(M, L)$ and $\mathrm{Ext}_{\mathfrak{a}}^{n-1}(M', L') \cong \mathrm{Ext}_{\mathfrak{a}}^{n-1}(M, L)$. This proves the assertion $(*)$. Its hypothesis is satisfied for $m \leqslant N - 1$ by Hypothesis 7.1, while $\mathrm{Ext}_A^n(M, L) = \mathrm{Ext}_B^n(M, L) = 0$ for $n > N$ by (4.0.2). Thus (1) holds.

To prove assertion (2) (for B-modules), it is only necessary to apply assertion $(*)$, taking $M, L \in B$-mod, to the top horizontal map of the commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}_A^n(M, L) & \longrightarrow & \mathrm{Ext}_{\mathfrak{a}}^n(M, L) \\ \big\uparrow {\scriptstyle\sim} & & \big\uparrow \\ \mathrm{Ext}_B^n(M, L) & \longrightarrow & \mathrm{Ext}_{\mathfrak{b}}^n(M, L) \end{array}$$

The left-hand vertical map is an isomorphism by (4.0.2). The required injectivity of (2) follows. $\qquad\square$

In earlier sections, we worked with the standard modules $\Delta(\lambda)$, $\lambda \in \Lambda$, in the highest weight category A-mod. These results have evident dual formulations for the

modules $\nabla(\lambda)$. These dual formulations often do not need to be proved separately if the category A-mod has a duality $\mathfrak{d} : A\text{-mod} \to (A\text{-mod})^{\text{op}}$. (Thus, $\mathfrak{d}$ is a contravariant category equivalence such that $\mathfrak{d}L \cong L$ for all irreducible A-modules.) In this case, $\mathfrak{d}\Delta^i(\lambda) \cong \nabla_{-i}(\lambda)$, $\forall \lambda \in \Lambda, \forall i \in \mathbb{N}$, and $\text{Ext}_A^\bullet(M, N) \cong \text{Ext}_A^\bullet(\mathfrak{d}N, \mathfrak{d}M)$, $\forall M, N \in A\text{-mod}$. Thus, in the presence of a duality, the verification of condition (SKL$'$), as stated in §4, simplifies to just checking just one of the two conditions, say condition (1) involving the $\Delta^i(\lambda)$. In the applications in §8, a natural duality is present.

Theorem 7.3. *Assume that Hypothesis 7.1 above holds and that A-mod or B-mod has a duality $\mathfrak{d}$. Also, assume that $\mathfrak{a}$ is Koszul, and that B-mod has the (KL) property with respect to a given length function l. Then B satisfies the (SKL$'$) property with respect to l.*

Proof. Using (4.0.2), it suffices to show that the condition (SKL$'$) holds for the algebra A as long as $\lambda, \mu \in \Gamma$. Also, given $\lambda \in \Gamma$, $L(\lambda)$ has a projective cover in A-mod which is $\mathfrak{a}$-projective. Thus, by Theorem 6.4, $\Delta(\lambda)$ has a $\mathfrak{a}$-graded structure generated by its grade 0 term and satisfying the property that $\Delta(\lambda)|_{\mathfrak{a}}$ is A_0-stable.

In the discussion below, we take $\lambda, \mu \in \Gamma$. It suffices to show that (SKL$'$)(1) holds for each $i > 0$ since the assumed (KL) property (for weights in Γ) implies the case $i = 0$. Denote $\Delta(\lambda)$ by Δ, and let $\Delta' = \Delta|_{\mathfrak{a}} \in \mathfrak{a}\text{-grmod}_0$. Put $L = L(\mu)$. We will make use of the following commutative diagram (with the evident maps):
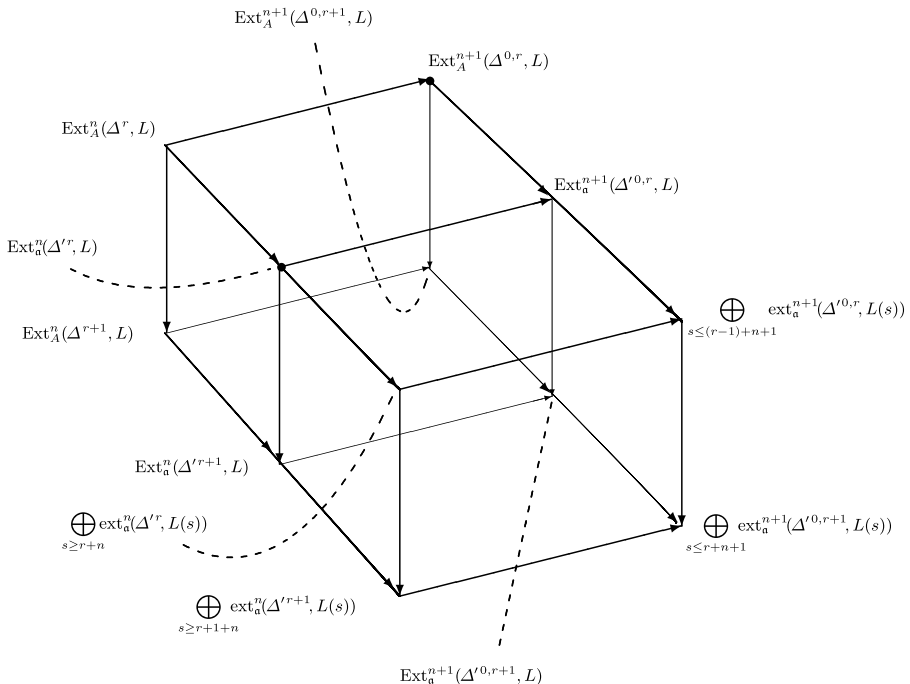
To prove (SKL′) for $r = i > 0$, it suffices to show

$$\text{Ext}_A^n(\Delta^{r,r+1}, L) \to \text{Ext}_A^n(\Delta^r, L) \quad \text{is surjective, for } n \geqslant 0. \tag{7.3.1}$$

Indeed, assuming (7.3.1), if $\text{Ext}_A^n(\Delta^r, L) \neq 0$, there is an irreducible constituent $L(\tau)$ of $\Delta^{r,r+1} = \Delta^r/\Delta^{r+1}$ such that $\text{Ext}_A^n(L(\tau), L) \neq 0$, whence $n \equiv l(\tau) - l(\mu) \mod 2$. On the other hand, any irreducible constituent $L(\tau)$ of $\Delta^{r,r+1}$ necessarily satisfies $l(\tau) \equiv r + l(\lambda) \mod 2$, using [15, Corollary 5.4(b)]. So, finally, $n \equiv l(\lambda) = l(\mu) + r \mod 2$, as required in (SKL′). But (7.3.1) is clear for $n = 0$, so assume that $n > 0$. Then, relabelling n as $n + 1$, it suffices to show

$$\text{Ext}_A^n(\Delta^r, L) \to \text{Ext}_A^n(\Delta^{r+1}, L) \quad \text{is the zero map, } \forall r > 0, \forall n \in \mathbb{N}. \tag{7.3.2}$$

The maps $\text{Ext}_A^n(\Delta^{r+1}, L) \to \text{Ext}_A^{n+1}(\Delta^{0,r+1}, L)$ and $\text{Ext}_A^{n+1}(\Delta^{0,r+1}, L) \to \text{Ext}_{\mathfrak{a}}^{n+1}(\Delta'^{0,r+1}, L)$ are injective by (4.3.1) and Proposition 7.2, respectively.

Since $\mathfrak{a}$ is Koszul, we can assume that $s \leqslant r + n + 1$ in the isomorphism $\text{Ext}_{\mathfrak{a}}^{n+1}(\Delta'^{0,r+1}, L) \cong \bigoplus_{s \in \mathbb{Z}} \text{ext}_{\mathfrak{a}}^{n+1}(\Delta'^{0,r+1}, L(s))$. The Koszul property gives similar (indicated) constraints in the commutative diagram above. Following $x \in \text{Ext}_A^n(\Delta^r, L)$ through the maps above, and considering alternative paths in the lower plane, the image y of x in $\bigoplus_{s \leqslant r+n+1} \text{ext}_{\mathfrak{a}}^{n+1}(\Delta'^{0,r+1}, L(s))$ lies in $\text{ext}_{\mathfrak{a}}^{n+1}(\Delta'^{0,r+1}, L(s))$ with $s = r + n + 1$. Also, it comes by a path in the upper plane of the diagram from an element in $\bigoplus_{s \leqslant (r-1)+n+1} \text{ext}_{\mathfrak{a}}^{n+1}(\Delta'^{0,r+1}, L(s))$ with $s = r + n$. The vertical maps on ext's preserve the terms with coefficients $L(s)$ for any given s. So $y = 0$ and the image of x in $\text{Ext}_A^n(\Delta^{r+1}, L)$ is zero. Thus, (7.3.2) and, hence, (7.3.1) hold. □

Corollary 7.4. *Assume that the hypothesis of Theorem 7.3 holds. Then* $\text{gr } B$ *is a graded QHA with the same poset* Γ *as* B, *and with standard modules* $\text{gr } \Delta(\lambda)$, $\lambda \in \Gamma$. *Also,* $\text{gr } B$ *satisfies the graded Kazhdan–Lusztig property with respect to* $l|_\Gamma$. *In particular,* $\text{gr } B$ *is a Koszul algebra. It has the same Kazhdan–Lusztig polynomials as* B. *The standard modules for* $\text{gr } B$ *are linear (and the costandard modules satisfy a dual property).*

This follows by combining Theorems 7.3 and 4.4. The linearity of standard modules (and the dual property for costandard modules) is essentially part of the definition of a graded Kazhdan–Lusztig property; see §4 and the discussion above (2.0.4).

Suppose that C is a finite dimensional algebra over a field K with $L_1, \ldots, L_m$ representatives from the isomorphism classes of distinct irreducible modules. Put $M = \bigoplus L_i$. The homological dual $C^!$ is Morita equivalent to the Yoneda Ext-algebra $\text{Ext}_C^\bullet(M, M)$.

Corollary 7.5. *Assume that the hypotheses of Theorem 7.3 hold. Then* $B^!$*-grmod is a graded HWC with a graded Kazhdan–Lusztig theory. It has weight poset* Γ^{op} *(opposite poset to* Γ*), length function* $l^{\text{op}} = -l$, *and Kazhdan–Lusztig polynomials* $F_{\nu,\lambda} := (-t)^{l(\nu)-l(\lambda)} \sum_n [\text{gr } \Delta(\nu) : L(\lambda)(n)] (-t)^{-n}$. *In particular,* $B^!$ *is a Koszul algebra. Also,* $(\text{gr } B)^! \cong B^!$.

Proof. By Theorem 7.3, B satisfies the condition (SKL$'$), so all of the assertions except the last follow from [16, Theorem 3.8]. Finally, by [17, Theorem 2.2.1(a)], $B^{!!} \cong \mathrm{gr}\,B$. Hence, $(\mathrm{gr}\,B)^! \cong B^{!!!} \cong B^!$. □

The polynomials $F_{\nu,\lambda}$, $\lambda, \mu \in \Gamma$, in Corollary 7.5 identify with the inverse Kazhdan–Lusztig polynomials $Q_{\lambda,\nu}$ associated with the Kazhdan–Lusztig polynomials $P_{\lambda,\nu}$ of $\mathrm{gr}\,B$-mod (or equivalently B-mod). These polynomials are defined by the condition that $\sum_{\lambda \leqslant \nu \leqslant \lambda'} (-1)^{l(\lambda')-l(\nu)} Q_{\lambda,\nu} P_{\nu,\lambda'} = \delta_{\lambda,\lambda'}$. See [15, § 3] – where our $Q_{\lambda,\nu}$ here is denoted as $Q_{\nu,\lambda}$.

In the formula for $F_{\nu,\lambda}$ in Corollary 7.5, $[\mathrm{gr}\,\Delta(\nu) : L(\lambda)(n)] \neq 0$ if and only if $L(\lambda)$ appears as a constituent of $\mathrm{rad}^n \Delta(\lambda)/\mathrm{rad}^{n+1} \Delta(\lambda)$. Thus, in that case, $l(\lambda) - l(\nu) \equiv n \bmod 2$ (a fact already used in the proof of Theorem 7.3). Therefore, in the expression for $F_{\nu,\lambda}$, the '$-t$' can be replaced by 't'. We have maintained the $-t$ in the definition of $F_{\nu,\lambda}$ to be consistent with [15].[8]

Remark 7.6. If Hypothesis 7.1 holds with $B = A$, it is possible to prove a 'bare-bones' version of Koszulity in an elementary way. The argument for Koszulity of $\mathrm{gr}\,B$ in this case does not require the Hopf algebra setting of § 5 or results of § 6, and is given below in a concise form. Nevertheless, while it provides an interesting model, there are reasons which prevent this simple and elegant result from being a cornerstone in our theory. First, the hypotheses do not actually occur in any of the regular weight cases of §§ 8–10; a main reason is that we must use appropriately truncated projective modules for B. Second, the conclusion does not give us everything we want. Even if the hypotheses were to apply for concluding that $\mathrm{gr}\,B$ is Koszul, and if we also assume the (much deeper) quasi-hereditary property of $\mathrm{gr}\,B$ from § 6, we would still not be able to conclude that $\mathrm{gr}\,B$-mod satisfied the (KL) property, even when B has such a theory. Also, we would not have the other conclusions of Corollaries 7.4 and 7.5. All of these conclusions are very valuable for the larger program of this paper, which is to transport all or most good properties (especially those arising in Lie theory) known for an algebra B to $\mathrm{gr}\,B$, while at the same time providing conditions which show that $\mathrm{gr}\,B$ is Koszul.

Nevertheless, an approach alternative to that of Theorem 7.3 would be welcome for establishing these other conclusions, and we raise the existence of effective alternatives as an open question. We hope to provide some variations ourselves in a future work [45] studying gradings in the setting of integral quasi-hereditary algebras and highest weight categories [16, 24].

Proposition 7.7. *Suppose that B is a finite dimensional algebra with a subalgebra $\mathfrak{a}$ satisfying the following conditions.*

(1) $\mathfrak{a}$ *is positively graded and Koszul.*

(2) $(\mathrm{rad}\,\mathfrak{a})B = \mathrm{rad}\,B$.

---

[8] We take this opportunity to record that the discussion of these polynomials in [17, (2.1.3)] contains a misprint, which is corrected by changing the two separate occurrences of t and $-t$ to either both $-t$, or both t.

(3) *Every projective B-module is projective as an $\mathfrak{a}$-module. (Equivalently, the $\mathfrak{a}$-module B is projective as a left $\mathfrak{a}$-module.)*

Then $\operatorname{gr} B$ *is Koszul.*

Proof. If P is a projective B-module, $\operatorname{gr} P$ is projective as a $\operatorname{gr} B$ module and as an $\mathfrak{a}$-module (as follows from the projectivity of $\operatorname{gr} \mathfrak{a} \cong \mathfrak{a}$), and every projective $\operatorname{gr} B$ module arises this way. If P has a simple head as a B-module, then $\operatorname{gr} P$ has a simple head as a $\operatorname{gr} B$-module, and is semisimple as an $\mathfrak{a}$-module; also $\operatorname{gr} P$ may be regarded as a graded projective module generated in degree 0 (for either $\operatorname{gr} B$ or $\mathfrak{a}$). Henceforth, we will just consider projective modules for $\operatorname{gr} B$, and just denote them by P to simplify the notation. For any irreducible $\operatorname{gr} B$-module L, regarded as graded with pure degree 0, let $\cdots \to P_n \to \cdots \to P_0 \to L \to 0$ be a minimal graded projective resolution of L. This resolution is also minimal as an ungraded projective resolution, and as a graded or ungraded projective resolution of the (completely reducible) restriction of L to $\mathfrak{a}$. Thus, by Koszulity of $\mathfrak{a}$, the head of P_n is pure of degree n as an $\mathfrak{a}$-module. This graded module P_n, and its (graded) head, are just the restriction to $\mathfrak{a}$ of the graded module P_n for $\operatorname{gr} B$ and its $\operatorname{gr} B$ head. Thus, the (graded) head of P_n, viewed as a graded $\operatorname{gr} B$-module, has grade n. Since this holds for all irreducible modules L for $\operatorname{gr} B$, the algebra $\operatorname{gr} B$ is Koszul. $\qquad\square$

Part II: applications

Part II, which consists of §§ 8–10, takes up applications of Part I to specific finite dimensional algebras arising in representation theory. These algebras include:

(1) Quasi-hereditary algebras $B = U_\Gamma$ attached to a quantum enveloping algebras $U = U_\zeta$ in characteristic 0 at an eth root of unity (e odd). For the definition of the quotient algebra U_Γ of U, see (8.1.1). Here the non-empty, finite weight poset Γ is required to consist of e-regular weights. Thus, it must hold that $e \geqslant h$ (the Coxeter number of U_ζ; in fact, it will be necessary to assume that $e > h$). See § 8.

(2) Regular blocks of q-Schur algebras $S_q(n, r)$. The section on these is largely a formal recasting of the results in (1) in the special case of type A. See § 9.

(3) Quasi-hereditary algebras $B = U_\Gamma$ attached to the distribution algebra $U = \operatorname{Dist}(G)$ of a simple, simply connected algebraic group G in (large) positive characteristic p; see § 10 for more details, including the definition of U_Γ. Here also the non-empty finite weight poset Γ is required to consist of p-regular weights.

In these cases, considerable effort is required to establish the various hypotheses needed for the results of Part I. For example, in case (3), the validity of the Lusztig character formula (LCF) for regular weights in the Jantzen region is essential. (See Remark 4.5.) With current technology, this validity is known for p very large. See [26] for a specific bound. It is expected that $p \geqslant h$ will ultimately be shown to suffice. The validity of the LCF means that, by the result of Andersen–Jantzen–Soergel [2], with $p > h$, the p-regular blocks of the restricted universal enveloping algebra u are Koszul.

The conditions given in Hypothesis 6.1 and 7.1 need to be checked. In this setup, there is a pair $(A, \mathfrak{a})$ as in Part I, and $B = U_\Gamma$, where Γ is a poset ideal in the weight poset Λ of $A = U_\Lambda$. (Thus, $B = A/J$ for a defining ideal of A and B also equals A_Γ in the notation of § 4.) In fact, the process begins with B and Γ, and then A and Λ are chosen to guarantee the hypotheses. The verification takes some effort, especially in the positive characteristic case of § 10. Meaningful results here require at least that $p \geqslant 2h - 2$ and that p is large enough that LCF holds in the Jantzen region. Additional requirements on p also appear in § 10, such as $p \geqslant 4h - 5$ in Theorems 10.8 and 10.9, and even $p \geqslant 2N(h-1) - 1$, for a certain N with $2 \leqslant N \leqslant (h-1)|\Phi|$ in Theorem 10.6. However, we expect that all additional requirements on p can be eventually removed (leaving only $p \geqslant 2h - 2$ and p large enough that LCF holds in the Jantzen region).

In each case, once the above program is carried out, the results of § 7 are available for $\mathrm{gr}\, B$, as is discussed in the introduction.

8. Quantum quasi-hereditary algebras

Let Φ be a finite root system, spanning a Euclidean space $\mathbb{E}$ having inner product (x, y). Assume that Φ is irreducible. For $\alpha \in \Phi$, let $\alpha^\vee := \frac{2}{(\alpha, \alpha)}$. Let $\mathfrak{g}$ be the complex simple Lie algebra with root system Φ. Choose a set $\Pi = \{\alpha_1, \ldots, \alpha_s\}$ of simple roots, and let Φ^+ be the corresponding set of positive roots. For $1 \leqslant i \leqslant s$, $\varpi_i \in \mathbb{E}$ is the fundamental dominant weight, defined by $(\varpi_i, \alpha_j^\vee) = \delta_{i,j}$, $1 \leqslant j \leqslant s$; let $X^+ = \mathbb{N}\varpi_1 \oplus \cdots \oplus \mathbb{N}\varpi_s$ be the cone of dominant weights in $X := \mathbb{Z}\varpi_1 \oplus \cdots \oplus \mathbb{Z}\varpi_s$. Let $\rho = \varpi_1 + \cdots + \varpi_s$ and $h = (\rho, \alpha_0^\vee) + 1$ (the Coxeter number), where $\alpha_0 \in \Phi^+$ is the maximal short root.

Fix an integer $e > 1$. Some restrictions, discussed below, will be imposed on e.

For $\lambda, \mu \in X^+$, define $\lambda \leqslant \mu \iff \mu - \lambda$ is a sum of positive roots; hence, X^+ is a poset. If $Y \subseteq X^+$, it is regarded as a poset by restricting the relation $\leqslant$ to it – thus, Y is a subposet of X^+. For example, $\Gamma_{e\text{-reg}} = \Gamma_{\mathrm{reg}}$ denotes the subposet of X^+ consisting of e-regular dominant weights, i.e., $\lambda \in \Gamma_{\mathrm{res}}$ if and only if $\lambda \in X^+$ and $(\lambda + \rho, \alpha^\vee) \not\equiv 0 \bmod e$, $\forall \alpha \in \Phi$. Then $\Gamma_{\mathrm{reg}} \neq \emptyset \iff e \geqslant h$. Put $X_1^+ = X_{1,e}^+ := \{\lambda \in X^+ \mid 0 \leqslant (\lambda, \alpha^\vee) < e, \forall \alpha \in \Pi\}$, the set of e-restricted dominant weights.

If $Z \neq \emptyset$ is a subset of a poset Y, then $\{y \in Y \mid y \leqslant z, \text{ for some } z \in Z\}$ is called the poset ideal in Y generated by Z. For example, $\Gamma_{e\text{-res}} = \Gamma_{\mathrm{res}}$ is the poset ideal in X^+ generated by X_1^+. Also, let $\Gamma_{\mathrm{res,reg}} = \Gamma_{\mathrm{reg}} \cap \Gamma_{\mathrm{res}}$, the poset ideal in Γ_{reg} generated by the e-regular, e-restricted weights. (Γ_{res} does not generally consist entirely of e-restricted weights.)

For the rest of this section, unless otherwise noted, the following assumptions on the integer e will be required. First, $e > 1$ is odd.[9] We require also that $e > h$. In addition, in types B, C and F_4, assume that e is not divisible by 4, and in type G_2, assume that e is not divisible by 3.

Let $K = \mathbb{Q}(\zeta)$ be the cyclotomic field obtained by adjoining a primitive eth root of unity $\zeta = \sqrt[e]{1}$ to $\mathbb{Q}$. Let $\mathbb{U} = \mathbb{U}(\mathfrak{g})$ be the quantum enveloping algebra over the function

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[9] It seems likely that this assumption can be removed. This assumption is only made so that we can apply [2], written before treatments of quantum groups allowing even roots of 1. The case when $e = 2e'$ with e' odd can be easily treated, using [1] and the Comparison Theorem [40], [18, Theorem 2].

field $\mathbb{Q}(v)$ associated with C.[10] It has generators $E_1, \ldots, E_s, F_1, \ldots, F_s, K_1^{\pm 1}, \cdots, K_s^{\pm 1}$ satisfying the familiar relations [31, (4.3)]. Let $U_\mathbb{Z}$ be the $\mathbb{Z}[v, v^{-1}]$-form of $\mathbb{U}$ generated by the 'quantum' divided powers $E_i^{(m)}, F_i^{(m)}$, together with the elements $K_i^{\pm 1}, 1 \leqslant i \leqslant s$, $m \in \mathbb{N}$. Let $U' = K \otimes_{\mathbb{Z}[v, v^{-1}]} U_\mathbb{Z}$, with K viewed as a $\mathbb{Z}[v, v^{-1}]$-algebra by specializing $v \mapsto \zeta$. Finally, let $U_\zeta = U'/\langle K_1^e - 1, \ldots, K_s^e - 1 \rangle$ be the quotient of U' by the ideal generated by the central elements $K_i^e - 1$. The subalgebra u_ζ of U_ζ generated by (the images of) the elements $E_i, F_i, K_i^{\pm 1}$ is a finite dimensional, normal Hopf subalgebra of U_ζ of dimension $e^{\dim \mathfrak{g}}$. Furthermore, $U_\zeta // u_\zeta \cong U(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}$.

The category U_ζ-mod of finite dimensional, integrable, type 1 U_ζ-modules is a HWC with poset X^+ having standard (resp., costandard, irreducible) modules $\Delta_\zeta(\lambda)$ (resp., $\nabla_\zeta(\lambda), L_\zeta(\lambda)$), $\lambda \in X^+$. Our restrictions on e mean that for $\lambda \in X^+$, the character $\operatorname{ch} L_\zeta(\lambda)$ is given by the LCF. See [48, § 7].[11]

The set X_1^+ of e-restricted dominant weights indexes the irreducible u_ζ-modules; in fact, given $\lambda \in X_1^+$, $L_\zeta(\lambda)|_{u_\zeta}$ is the corresponding irreducible u_ζ-module. Given $\lambda \in X_1^+$, we let $Q_\zeta(\lambda) \in U_\zeta$-mod be the projective cover for $L_\zeta(\lambda)$. As a u_ζ-module, $Q_\zeta(\lambda)$ is the projective cover for $L_\zeta(\lambda)|_{u_\zeta}$. Furthermore, $Q_\zeta(\lambda) \cong T_\zeta(2(e-1)\rho + w_0\lambda)$ is the indecomposable tilting module in U_ζ-mod of highest weight $2(e-1)\rho + w_0\lambda$, where $w_0 \in W$ has maximal length in the Weyl group W of Φ.

The following standard result follows easily from the tensor product theorem. (For the $\Leftarrow$ direction, argue on u_ζ-isotypic components as in [30, I, (2.14)(3)].)

Lemma 8.1. *Any $M \in U_\zeta$-mod is completely reducible if and only if its restriction $M|_{u_\zeta}$ to u_ζ is completely reducible.*

Suppose that Γ is a finite poset ideal in Γ_{reg}, and let J_Γ be the annihilator in U_ζ of the full subcategory U_ζ-mod$[\Gamma]$ of U_ζ-mod. Put

$$A := U_\Gamma := U_\zeta / J_\Gamma \tag{8.1.1}$$

and let $\mathfrak{a}$ be the image of u_ζ in A. The algebra U_ζ has an anti-involution ι (stabilizing u_ζ) such that, for $\lambda \in X^+$, $L_\zeta(\lambda)^{*\iota} \cong L_\zeta(\lambda^\star)$, where $\lambda^\star := -w_0\lambda$. Composing ι with an appropriate graph automorphism yields an anti-involution ι' which induces a duality $\mathfrak{d}$ on the category A-mod. There is an induced duality on $\operatorname{gr} A$-mod and for $\operatorname{gr} A$-grmod. (In the latter case, a module of pure grade n has a dual in pure grade $-n$.)

The standard and costandard modules for a QHA algebra are determined by its weight poset.[12] For details on the following result, see [24, Proposition 3.5, Corollary 2.5, Proposition 2.1.2].

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[10] For further details on the quantum enveloping algebra, especially at roots of unity used below, including the small quantum group, see [4, 5, 31, 36, 37].

[11] The length function is explained above Theorem 8.4.

[12] We have stated this result for a finite ideal of e-regular dominant weights, but, in fact, it remains valid for any finite ideal in X^+. Since most results are established for regular weights, it seems simplest to concentrate on this case.

Lemma 8.2. *For any finite poset ideal Γ in Γ_{reg}, $A = U_\Gamma$ is a QHA (and hence, finite dimensional), satisfying A-mod $\cong U_\zeta$-mod$[\Gamma]$. The standard, costandard, irreducible modules in the HWC A-mod (with poset Γ) are the $\Delta_\zeta(\lambda)$, $\nabla_\zeta(\lambda)$, $L_\zeta(\lambda)$, $\lambda \in \Gamma$.*

A block E in the finite dimensional algebra u_ζ is called *e-regular* if its irreducible modules are of the form $L_\zeta(\lambda)|_{u_\zeta}$ for $\lambda \in \Gamma_{res,reg}$. Let

$$u_\zeta' = \prod_{E \ e\text{-regular}} E, \quad \text{a direct factor of } u_\zeta. \tag{8.2.1}$$

A finite non-empty poset ideal Γ in Γ_{reg} is called *e-fat* if, given any *e*-regular, *e*-restricted dominant weight λ, $2(e-1)\rho + w_0(\lambda) \in \Gamma$. The reader can verify that, if Γ is *e*-fat, then it contains all *e*-dominant, *e*-restricted weights. Moreover, for such a λ, the U_ζ-projective cover $Q_\zeta(\lambda)$ of $L_\zeta(\lambda)$ belongs to U_ζ-mod$[\Gamma]$, the full subcategory of U_ζ-mod with modules whose composition factors $L_\zeta(\gamma)$ satisfy $\gamma \in \Gamma$. This is because $2(e-1)\rho + w_0(\lambda)$ is the highest weight of the U_ζ-module $Q_\zeta(\lambda)$.

Lemma 8.3. *Let Γ be an e-fat poset ideal in Γ_{reg} and let $A := U_\Gamma$. Then under the quotient map $\phi : U_\zeta \twoheadrightarrow A$, the algebra u_ζ' maps isomorphically onto a normal subalgebra $\mathfrak{a}$ of A.*

Furthermore, $\mathfrak{a}$ is a Koszul algebra satisfying $(\mathrm{rad}\,\mathfrak{a})A = \mathrm{rad}\,A$.

Proof. Since Γ is *e*-fat, for any $\lambda \in \Gamma_{res,reg}$, we have $Q_\zeta(\lambda) \in A$-mod. Thus, u_ζ' acts faithfully on the category A-mod, so $u_\zeta' \cap J_\Gamma = 0$. Hence, ϕ maps u_ζ' isomorphically onto its image $\mathfrak{a}$ in A, which must necessarily be a subalgebra (i.e., it contains the identity of A). Since the LCF holds for U_ζ, [2, Proposition 18.17, p. 256] shows that the algebra $\mathfrak{a} \cong u_\zeta'$ is Koszul. In addition, $\mathfrak{a}$ is the image of the normal subalgebra u_ζ under the map $U_\zeta \to A$, so $\mathfrak{a}$ is normal in A.

It remains to prove that $(\mathrm{rad}\,\mathfrak{a})A = \mathrm{rad}\,A$. For this, it is enough to check that $(\mathrm{rad}\,\mathfrak{a})P = \mathrm{rad}\,P$, for each projective indecomposable module $P \in A$-mod. However, such a PIM P can be constructed as the largest A-module quotient of a U_ζ-module PIM $Q_\zeta(\lambda) = Q_\zeta(\lambda_0) \otimes L(\lambda_1)^{(1)}$, where $\lambda = \lambda_0 + e\lambda_1 \in \Gamma$, with $\lambda_0 \in X_1^+$, $\lambda_1 \in X^+$. But $\mathrm{rad}_{U_\zeta} Q_\zeta(\lambda) = (\mathrm{rad}_{U_\zeta} Q_\zeta(\lambda_1)) \otimes L(\lambda_1)^{(1)} = \mathrm{rad}_{u_\zeta}(Q_\zeta(\lambda))$, and the desired equation follows by passing to the homomorphic image P. We use here that $Q_\zeta(\lambda_0)$ is also a PIM in the category of u_ζ-modules. $\square$

The equation $(\mathrm{rad}\,\mathfrak{a})A = \mathrm{rad}\,A$ at the end of the lemma does not require that Γ be *e*-fat, since radicals are preserved under homomorphic images. It will be useful to formulate a 'fattening' procedure. If $\xi \in X^+$, write $\xi = \xi_0 + e\xi_1$, for $\xi_0 \in X_1^+$ and $\xi_1 \in X^+$. Put

$$\mathfrak{f}_e(\xi) = 2(e-1)\rho + w_0\xi_0 + e\xi_1 \in X^+. \tag{8.3.1}$$

Thus, $\mathfrak{f}_e(\xi)$ is the highest weight of the U_ζ-module PIM $Q_\zeta(\xi) = Q_\zeta(\xi_0) \otimes L(\xi_1)^{(1)}$. For $\emptyset \neq \Psi \subseteq X^+$, let $\Psi(-1)$ be the poset ideal in X^+ generated by Ψ, and, for $n \geqslant 0$,

define $\Psi(n)$ as the poset ideal in X^+ generated by all $\mathfrak{f}_e(\xi)$, with $\xi \in \Psi(n-1)$.[13] In particular, if $\xi \in \Psi$, then the composition factors of $Q_\zeta(\xi)$ all have highest weights in $\Psi(0)$. More generally, if M is a U_ζ-module all of whose composition factors $L_\zeta(\xi)$ satisfy $\xi \in \Psi$, then there is, for any non-negative integer n, an exact sequence

$$0 \to \Omega_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0 \qquad (8.3.2)$$

of U_ζ-modules in which all the modules P_i are projective U_ζ-modules and all composition factors of P_i have highest weights in $\Psi(i) \subseteq \Psi(n-1)$.

Observe that the $P_i|_{u_\zeta}$, $i = 0, \ldots, n-1$, are projective modules in u_ζ-mod.

Let $W_e = W \ltimes e\mathbb{Z}\Phi$ be the affine Weyl group generated by the Weyl group W and the normal subgroup of translations by e-multiples of roots. Given $\lambda \in X$, there exists a unique $\lambda^- \in \overline{C^-}$ (the closure of C^-) such that $\lambda = w \cdot \lambda^- := w(\lambda^- + \rho) - \rho$ for some $w \in W_e$. Among all possible $w \in W_e$ satisfying $w \cdot \lambda^- = \lambda$, let w_λ have minimal length. Thus, if W_{e,λ^-} is the stabilizer in W_e of λ^-, w_λ is a minimal left coset representative of the parabolic subgroup W_{e,λ^-} in W_e. Also, define $l : X \to \mathbb{Z}$ by putting $l(\lambda) := l(w_\lambda)$. Given dominant weights λ, ν, put $P_{\nu,\lambda} = 0$ unless $\lambda^- = \nu^-$. If $\lambda^- = \nu^-$, set $P_{\nu,\lambda} = P_{w_\nu,w_\lambda}$, the Kazhdan–Lusztig polynomial for the elements w_ν, w_λ in the Coxeter group W_e. Also, define $Q_{\nu,\lambda} = Q_{w_\nu,w_\lambda}$ if $\lambda^- = \nu^-$, and $= 0$ if $\lambda^- \neq \nu^-$. Here, $Q_{x,y} \in \mathbb{Z}[t^2]$ is the inverse Kazhdan–Lusztig polynomial associated with the pair (x, y).

The theorem below concerns the category of U_ζ-modules which involve only e-regular dominant weights. If Γ is a finite poset ideal in Γ_{reg}, the $P_{\lambda,\nu} \in \mathbb{Z}[t^2] \subseteq \mathbb{Z}[t]$ are the Kazhdan–Lusztig polynomials of the HWC $B = U_\Gamma$. Because the LCF holds for all $\lambda \in \Gamma$, the HWC B-mod satisfies the (KL) property with respect to $l : \Gamma \to \mathbb{Z}$, $\lambda \mapsto l(w_\lambda)$; see [15, Remark 5.8]. In other words, the $i = 0$ part of condition (SKL$'$) given in §4 holds.

Theorem 8.4. *Let Γ be a finite poset ideal in Γ_{reg} and let $B = U_\Gamma$. The category of ungraded modules for the graded algebra $\operatorname{gr} B$ is a HWC with poset Γ and a duality $\mathfrak{d}$, which also induces a duality on $\operatorname{gr} B$-grmod. The category $\operatorname{gr} B$-grmod satisfies the graded Kazhdan–Lusztig property with the same Kazhdan–Lusztig polynomials as B-mod (which are, in turn, classical Kazhdan–Lusztig polynomials of W_e). In particular, $\operatorname{gr} B$ is a QHA with a Koszul grading. For $\lambda \in \Gamma$, the associated standard module is $\operatorname{gr} \Delta_\zeta(\lambda)$, which is linear as a graded $\operatorname{gr} B$-module.*

Proof. We will apply Theorem 7.3. Let N be the global dimension of B. We can assume that $N > 0$, and we set $\Lambda := \Gamma(N-1)$ and $A := U_\Lambda$. Let $\mathfrak{a}$ be the image of u'_ζ in A. We must check that Hypothesis 7.1 holds for the QHAs A and B, and the subalgebra $\mathfrak{a}$ of A.

By construction, Λ is e-fat. Then Lemma 8.3 (with Λ here playing the role of Γ there) says that $u'_\zeta \cong \mathfrak{a}$ is Koszul (and thus, tightly graded), and a normal subalgebra of A. (In fact, the map $U_\zeta \twoheadrightarrow A$ maps the Hopf subalgebra u_ζ onto $\mathfrak{a}$.) The lemma also states that $(\operatorname{rad} \mathfrak{a})A = \operatorname{rad} A$. Thus, $(A, \mathfrak{a})$ satisfies Hypothesis 6.1.

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[13] If ξ is e-regular, then so is $\mathfrak{f}_e(\xi)$. If $\Psi \subseteq \Gamma_{\text{reg}}$, then take $\Psi(n)$, $n \geqslant -1$, to be the poset ideal in Γ_{reg} generated by all $\mathfrak{f}_e(\xi)$ for $\xi \in \Psi(n-1)$.

Therefore, Hypothesis 7.1(1) holds. The discussion above for (8.3.2) with $n = N - 1$ implies that Hypothesis 7.1(2) is satisfied. Finally, as observed just before the statement of the theorem, B-mod satisfies the (KL) property with respect to the length function $l: \Gamma \to \mathbb{Z}$, $\lambda \mapsto l(\lambda) = l(w_\lambda)$.

The duality hypothesis in Theorem 7.3 is satisfied, by the discussion above Lemma 8.2. Therefore, all the hypotheses of Theorem 7.3 hold, so this implies that B-mod satisfies the (SKL$'$) property.

Therefore, the conclusion of the theorem follows from Corollary 7.4. $\qquad\square$

The theorem implies results on the homological dual $B^!$ and on the composition factors of the radical sections of the modules $\Delta_\zeta(\lambda)$. These facts are contained in parts (a) and (c) of the following result. Using translation functors, it is possible to obtain information in the case of non-e-regular weights – this is the content of part (b). In the result below, the $Q_{\lambda,\nu}$ are the inverse Kazhdan–Lusztig polynomials discussed right above Theorem 8.4.

Corollary 8.5. (a) *Let B be as in the previous theorem and let $B^! = \mathrm{Ext}^\bullet_B(B/\mathrm{rad}\,B, B/\mathrm{rad}\,B)$ be its homological dual. Then $B^!$ is a graded QHA having a graded Kazhdan–Lusztig theory with length function $l^{\mathrm{op}} = -l: \Gamma^{\mathrm{op}} \to \mathbb{Z}$ and Kazhdan–Lusztig polynomials $F_{\lambda,\nu} := Q_{\nu,\lambda}$, for $\nu < \lambda$ in Γ. In particular, $B^!$ is a Koszul algebra. Also, $B^! \cong (\mathrm{gr}\,B)^!$.*

(b) *Suppose that $\lambda \in X^+$ (not necessarily e-regular). Then $\Delta_\zeta(\lambda)$ has a U_ζ-filtration*

$$\Delta_\zeta(\lambda) = F^0(\lambda) \supseteq F^1(\lambda) \supseteq \cdots \supseteq F^m(\lambda) = 0$$

in which each section $F^i(\lambda)/F^{i+1}(\lambda)$, $0 \leqslant i < m$, is a completely reducible U_ζ-module satisfying the following property: for $n \geqslant 0$ and $\nu \in X^+$, the multiplicity of $L_\zeta(\nu)$ in the section $F^n(\lambda)/F^{n+1}(\lambda)$ equals the coefficient of $t^{l(\lambda)-l(\nu)-n}$ in the polynomial $Q_{\nu,\lambda}$.

(c) *If $\lambda \in \Gamma_{reg}$, then in (b) above, we can take $F^n(\lambda) = \mathrm{rad}^n \Delta_\zeta(\lambda)$.*

Proof. Parts (a) and (c) follow from Corollary 7.5 and the discussion following it. Now consider (b), and suppose that $\lambda^- \in \overline{C^-} \setminus C^-$ is not e-regular. Let T be the translation operator from the category of U_ζ-modules having composition factors with highest weights of the form $w \cdot (-2\rho) \in X^+$ to the category of U_ζ-modules having composition factors with highest weights of the form $w \cdot \lambda^- \in X^+$. Then $T\Delta_\zeta(w_\lambda \cdot (-2\rho)) \cong \Delta_\zeta(\lambda)$. Also, if $\lambda = w \cdot \lambda^- \in X^+$, then $TL(w \cdot (-2\rho)) = L(\lambda)$ if and only if $w = w_\lambda$, and equals 0, otherwise.

Let $F^i(\lambda) := T\,\mathrm{rad}^i \Delta_\zeta(\lambda)$. Since T is exact, $F^\bullet(\lambda)$ is a U_ζ-filtration of $\Delta_\zeta(\lambda)$ with completely reducible sections. In fact, $F^i(\lambda)/F^{i+1}(\lambda) \cong T\mathrm{rad}^i \Delta_\zeta(\lambda)/\mathrm{rad}^{i+1}\Delta_\zeta(\lambda)$. Suppose that $n \geqslant 0$ and $\nu \in X^+$. To determine the multiplicity of $L_\zeta(\nu)$ in $F^n(\lambda)/F^{n+1}(\lambda)$, we assume that $\nu^- = \lambda^-$ (otherwise, the multiplicity is zero). Then ν is in the upper e-closure of $w_{\lambda^-} \cdot C^-$. In fact, $w_{\lambda^-} \cdot C^-$ is the unique alcove containing ν in its upper closure. Since the multiplicity of $L_\zeta(w_\lambda \cdot (-2\rho))$ in $\mathrm{rad}^n \Delta_\zeta(\lambda)/\mathrm{rad}^{n+1}\Delta_\zeta(\lambda)$ is the coefficient of $t^{l(\lambda)-l(\nu)-b}$ in $Q_{w_\nu,w_\lambda} = Q_{w_\nu \cdot (-2\rho), w_\lambda \cdot (-2\rho)}$, part (b) is proved. $\qquad\square$

Remarks 8.6. (a) Translating into the language of quantum enveloping algebras yields a remarkable fact. Let $\mathbb{E}^! := \bigoplus_{\lambda,\mu \in \Gamma_{\text{reg}}} \text{Ext}^\bullet_{U_\zeta}(L_\zeta(\lambda), L_\zeta(\mu))$ be the full Ext-algebra of U_ζ on irreducible modules having regular highest weights. Then $\mathbb{E}^!$ (an algebra without identity) is 'almost Koszul' in the sense that if $e = e_\Gamma$ is the idempotent projection corresponding to a poset ideal Γ in Γ_{reg}, then $e\mathbb{E}^!e$ is Koszul.

(b) In [2], it is conjectured that the full small quantum enveloping algebra u_ζ is Koszul. This conjecture would provide a step toward establishing Theorem 8.4 (and Corollary 8.5) for the singular blocks, although our approach would still require that the singular blocks satisfy a parity condition (have a Kazhdan–Lusztig theory [15]).

(c) In [34, Conjecture 4.1], Lin conjectured (in the context of algebraic groups in positive characteristic) an equality of Ext^1-groups between irreducible modules having regular weights and irreducible modules having weights on a wall. If this is true in the quantum enveloping algebra setting, it seems likely that translation operators from a 'regular' standard module to a 'singular' standard module would preserve radical filtrations. In that case, Corollary 8.5(c) would extend to arbitrary standard modules.

Theorem 8.7. *Let $\lambda \in \Gamma_{reg}$. There is a positive grading $\Delta_\zeta(\lambda) = \bigoplus_{n \geqslant 0} \Delta_\zeta(\lambda)_n$ of $\Delta_\zeta(\lambda)$ as a module for the Koszul algebra u'_ζ. The graded module $\Delta_\zeta(\lambda)$ is generated in degree 0. As a $u'_\zeta/\text{rad}\, u'_\zeta$-module, $\Delta_\zeta(\lambda)_0 \cong L_\zeta(\lambda)$.*

Proof. Choose an e-fat poset ideal Λ in Γ_{reg} containing λ. Then Theorem 6.4 applies. □

Theorem 6.4 contains more information. In particular, a Wedderburn complement A_0 for $A = U_\Lambda$ can be chosen, containing $\mathfrak{a}_0$, such that $\Delta_\zeta(\lambda)_0$ is A_0-stable.

9. The Dipper–James q-Schur algebras $S_q(n,r)$ in characteristic 0

Here n, r are positive integers, and $q = \zeta^2$ for ζ a primitive eth root of 1 for an integer $e > n = h$. As in the previous section, we assume that e is odd, unless otherwise explicitly noted. Let $\Lambda^+(n,r)$ (resp., $\Lambda(n,r)$) be the set of partitions (resp., compositions) of r with at most n nonzero parts. The dominance order $\trianglelefteq$ on partitions makes $\Lambda^+(n,r)$ into a poset. Let $\Lambda^+_{e\text{-reg}}(n,r)$ be the set of e-regular partitions in $\Lambda^+(n,r)$, i.e., $\lambda \in \Lambda^+(n,r)$ is e-regular if and only if no nonzero part of λ is repeated e times.[14] A partition $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda^+(n,r)$ is called chamber e-regular provided that, for all i, j satisfying $1 \leqslant i < j \leqslant n$, it holds that $\lambda_i - i \not\equiv \lambda_j - j \mod e$. Clearly, the set $\Lambda^{+,\sharp}(n,r)$ of all chamber e-regular partitions is a (usually proper) subset of $\Lambda^+_{e\text{-reg}}(n,r)$.

Let $\widetilde{H}_r$ be the Hecke algebra over $\mathbb{Z}[\mathbf{q}, \mathbf{q}^{-1}]$ for the Coxeter group $W = \mathfrak{S}_r$ (symmetric group of degree r) with fundamental reflections $S = \{s_1, \ldots, s_{r-1}\}$, $s_i = (i, i+1)$. Here $\mathbf{q}$ is an indeterminate. Each composition $\lambda \in \Lambda(n,r)$ defines a (right) $\mathbf{q}$-permutation module

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[14] Also, λ is e-restricted (or just restricted) if its transpose λ' is e-regular.

$x_\lambda \widetilde{H}_r$, and the **q**-Schur algebra over $\mathbb{Z}[\mathbf{q}, \mathbf{q}^{-1}]$ is the endomorphism algebra

$$\widetilde{S}_{\mathbf{q}}(n, r) := \mathrm{End}_{\widetilde{H}_r} \left(\bigoplus_{\lambda \in \Lambda(n,r)} x_\lambda \widetilde{H}_r \right). \tag{9.0.1}$$

(See [19, 25] for more details.) The space $\bigoplus_{\lambda \in \Lambda(n,r)} x_\lambda \widetilde{H}_r$ identifies with $V^{\otimes r}$ (V a free $\mathbb{Z}[\mathbf{q}, \mathbf{q}^{-1}]$-module of rank n) which has a natural right $\widetilde{H}_r$-action. Definition (9.0.1), due to Dipper and James [20], behaves well with respect to base change to any commutative ring $\mathcal{R}$ over $\mathbb{Z}[\mathbf{q}, \mathbf{q}^{-1}]$. In particular, we can take $\mathcal{R} = K = \mathbb{Q}(\zeta)$, specializing $\mathbf{q} \mapsto \zeta^2$ and put $S_q(n, r) := \widetilde{S}_{\mathbf{q}}(n, r) \otimes_{\mathbb{Z}[\mathbf{q}, \mathbf{q}^{-1}]} K$. There is a surjective homomorphism $U_\zeta(\mathfrak{gl}_n) \twoheadrightarrow S_q(n, r)$ [19, Theorem 14.24] (see also [25, Theorem 6.3] and [23]). Here $U_\zeta(\mathfrak{gl}_n)$ is the quantum enveloping algebra over $\mathbb{Q}(\zeta)$ corresponding to the general linear Lie algebra $\mathfrak{gl}_n$; the action of $U_\zeta(\mathfrak{gl}_n)$ on 'quantum tensor space' $V^{\otimes r}$ comes about by the action of $U_\zeta(\mathfrak{gl}_n)$ on its natural module V of dimension n. (We will argue momentarily that there is a similar surjection using $U_\zeta(\mathfrak{sl}_n)$ for $\mathfrak{sl}_n$.)

The q-Schur algebra $S_q(n, r)$ is a QHA with poset $\Lambda^+(n, r)$. The arguments in §8 can be applied (with some care) to obtain similar results about $S_q(n, r)$, viewing the latter as a quotient of $U_\zeta(\mathfrak{gl}_n)$. However, it is also possible to use $\mathfrak{sl}_n$ and quote the results of §8 more directly. We briefly describe this $\mathfrak{sl}_n$ approach.

Label the simple roots $\Pi = \{\alpha_1, \ldots, \alpha_{n-1}\}$ of Φ (a root system of type A_{n-1}) in the usual way. Fix r and let $\Gamma_{n,r}$ be the subset (poset ideal) of X^+ consisting of dominant weights $\lambda = \sum_{i=1}^{n-1} a_i \varpi_i$ such that $r \geqslant \sum i a_i \equiv r \mod n$. Given $\lambda \in \Lambda^+(n, r)$, put $\bar{\lambda} = \sum_{i=1}^{n-1} (\lambda_i - \lambda_{i-1}) \varpi_i \in X^+$. Then $\lambda \mapsto \bar{\lambda}$ is a poset isomorphism $\Lambda^+(n, r) \to \Gamma_{n,r}$. Under this isomorphism, the set $\Lambda^{+,\sharp}(n, r)$ of chamber e-regular weights corresponds to the set of e-regular elements in $\Gamma_{n,r}$.

Now form the QHA algebra $A := U_{\Gamma_{n,r}}$ (as above Lemma 8.2, taking $U_\zeta = U_\zeta(\mathfrak{sl}_n)$ and $\Gamma = \Gamma_{n,r}$, allowing posets of non-regular weights). For convenience, also denote A by $A_{\Gamma_{n,r}}$. Then the dimension of $A_{\Gamma_{n,r}}$ is the sum of squares of dimensions of standard modules, and it is found to be the same as the dimension of $S_q(n, r)$. The inclusion $U_\zeta(\mathfrak{sl}_n) \subseteq U_\zeta(\mathfrak{gl}_n)$ induces an algebra homomorphism $A_{\Gamma_{n,r}} \to S_q(n, r)$, and an exact functor $T : S_q(n, r)\text{-mod} \to A_{\Gamma_{n,r}}\text{-mod}$ which is described explicitly as the restriction of $U_\zeta(\mathfrak{gl}_n)$-modules (with composition factors having highest weights in $\Lambda^+(n, r)$) to $U_\zeta(\mathfrak{sl}_n)$-modules (with composition factors having highest weights in $\Gamma_{n,r}$). The functor T takes standard, costandard, and irreducible modules to standard, costandard, and irreducible modules, respectively, and it is compatible with the poset isomorphism. Therefore, the Comparison Theorem [40], [18, Theorem 2] implies that T is a category equivalence. Since the functor T here arises as a 'restriction' through a map of algebras, the latter algebra map must be an isomorphism.

Let u'_ζ be the sum of the regular blocks in the small quantum group $u_\zeta(\mathfrak{sl}_\zeta)$.

Now Theorems 8.4 and 8.7, and Corollary 8.5 imply the following result.

Theorem 9.1. *Let n, r be positive integers, and $e > n$ an odd integer. If B is any block of $S_q(n, r)$ corresponding to a subset (poset ideal) of $\Lambda^{+,\sharp}(n, r)$ of chamber e-regular*

partitions, then $\operatorname{gr} B$ *is a QHA with a Koszul grading. Also,* $B^! \cong (\operatorname{gr} B)^!$ *is a Koszul algebra (and a QHA). In addition, for* $\lambda \in \Lambda^{+,\sharp}(n,r)$, *the standard (Weyl) module* $\Delta_q(\lambda)$ *has a positive grading, generated in grade* 0, *for the Koszul algebra* u'_ζ.

The standard modules for the algebra $\operatorname{gr} B$ *are the modules* $\operatorname{gr} \Delta_q(\lambda)$, $\lambda \in \Lambda^{+,\sharp}(n,r)$, *and the multiplicities of the irreducible modules in the various sections* $\operatorname{rad}^i \Delta_q(\lambda)/\operatorname{rad}^{i+1} \Delta_q(\lambda)$ *are coefficients of inverse Kazhdan–Lusztig polynomials.*

Again, the only need for the assumption that e is odd is to be able to quote [2] as in footnote ††.

As in Theorem 8.4, B-mod, $\operatorname{gr} B$-mod, and $\operatorname{gr} B$-grmod have compatible dualities. The modules $\operatorname{gr} \Delta_q(\lambda)$ are linear, and $\operatorname{gr} B$-mod has a graded Kazhdan–Lusztig theory (cf. §4). Both B-mod and $\operatorname{gr} B$-mod have to satisfy the property (KL) with the same (classical) Kazhdan–Lusztig polynomials.

Remark 9.2. It is interesting to compare these results on q-Schur algebras with Ariki's recent work [6]. Ariki obtains a $\mathbb{Z}$-grading on the q-Schur algebra $S_q(n,r)$, assuming that $q = \zeta^2$ is a primitive e'th root of unity with $e' \geqslant 4$. (Thus, $e' = e$ if e is odd, and $e' = e/2$ if e is even (allowed in this case).) In addition, when $n \geqslant r$, he obtains a formula for $[\Delta_q(\lambda) : L_q(\nu)(r)]$ similar to that given in Corollary 8.5(b) for the multiplicity of $L_\zeta(\nu)$ in $F^i(\lambda)/F^{i+1}(\lambda)$. Each standard module $\Delta_q(\lambda)$ is given a specific $\mathbb{Z}$-grading. Since Corollary 8.5 requires that $e > h = n$, and since the Hecke algebra H_r (obtained from $\widetilde{H}_r$ by specializing $\mathbf{q}$ to q), and hence $S_q(n,r)$, are semisimple if $e' > r$, Ariki's results and those above apply to different situations.

In the preprint [43], we eliminate the hypothesis $e > n$ from Corollary 8.5(b) in many cases, and completely so in type $\mathfrak{sl}_n$. Thus, the comparison of the two theories becomes more meaningful. In particular, one can ask whether there is an adjustment (regrading) of Ariki's graded q-Schur algebra A to give a graded isomorphism $A \cong \operatorname{gr} A$, or even ask whether there is a regrading making A Koszul.

10. Representations in positive characteristic of simple algebraic groups

Let p be a prime integer, and continue to assume that Φ is a root system in the Euclidean space $\mathbb{E}$ as in §8. Use the notation of §8 involving Φ, taking $e = p$ in this section. For example, $W_p := W \ltimes p\mathbb{Z}\Phi$, etc. The Jantzen region in X^+ is the set

$$\Gamma_{\mathrm{Jan}} := \{x \in X^+ \mid (x + \rho, \alpha_0^\vee) \leqslant p(p - h + 2)\}.$$

As in §8 (put taking $e = p$), Γ_{res} denotes the poset ideal in X^+ generated by the set $X_1^+ := X_{1,p}^+$ of p-restricted dominant weights. Put $C^- := \{x \in \mathbb{E} \mid (x + \rho, \alpha^\vee) \leqslant 0, \alpha \in \Pi, (x + \rho, \alpha_0^\vee) > -p\}$, the fundamental anti-dominant alcove. If $p \geqslant 2h - 3$, $X_1^+ \subset \Gamma_{\mathrm{Jan}}$, so Γ_{res} is a poset ideal in Γ_{Jan}. We will always assume (unless otherwise explicitly noted) that $p \geqslant h$; thus, $-2\rho \in C^-$; and, in particular, $\Gamma_{\mathrm{reg}} \neq \emptyset$. Given a p-regular dominant weight λ, write $\lambda = w \cdot \lambda^-$ for $w = w_\lambda \in W_p$, $\lambda^- \in C^- \cap X$. (The element w_λ is uniquely determined.) We define $l(w) = l(w_\lambda)$ to be the length of λ. We will use this length function on Γ_{reg} or its ideals below (and as the length function for a (KL) property).

Let G be a simple simply connected algebraic group over an algebraically closed field K of positive characteristic p. Assume G defined and split over the prime field $\mathbb{F}_p$. Then G has root system denoted Φ with respect to a maximal split torus T. For $\lambda \in X^+$, let $L(\lambda)$ be the irreducible rational G-module of highest weight λ. Let $U = \mathrm{Dist}(G)$ be the distribution algebra (hyperalgebra) of G. The category of rational G-modules is equivalent to the category of locally finite U-modules.

For a finite ideal Γ in the poset of p-regular dominant weights, define (just as in (8.1.1) for $U = U_\zeta$) the algebra U_Γ to be the quotient $U_\Gamma = U/J_\Gamma$, where J_Γ is the annihilator of U-mod$[\Gamma]$ (the full subcategory of finite dimensional rational G-modules generated by the $L(\gamma)$, $\gamma \in \Gamma$). As in the quantum case of § 8, A_Γ is a QHA with weight poset Γ. (For instance, one needs only to mimic the argument in [24, Proposition 3.5] mentioned in § 8.)

Let $F : G \to G$ be the Frobenius morphism with kernel G_1, the first infinitesimal subgroup of G. If $V \in G$-mod (the category of rational G-modules), let $V^{(1)} := F^*V$, the pull-back of V through F. If $\nu_0 \in X_1^+$, let $Q(\nu_0)$ be the projective indecomposable cover of the irreducible G_1-module $L(\nu_0)|_{G_1}$. If $p \geqslant 2h - 2$, then $Q(\nu_0)$ has a unique compatible structure as a rational G-module [30, § 11.1]. In fact, this module identifies with the indecomposable tilting module $T(2(p-1)\rho + w_0\nu_0)$ of highest weight $2(p-1)\rho + w_0\nu_0$ in G-mod.

Let $\Xi \subseteq X^+$. Define $\Xi_1 \subseteq X^+$ as follows: given $\lambda \in \Xi$, write $\lambda = \lambda_0 + p\lambda_1$, where, as usual, $\lambda_0 \in X_1^+$, $\lambda_1 \in X^+$. Now put $\Xi_1 = \{\lambda_1 \mid \lambda \in \Xi\}$. Put

$$a_1(\Xi) := \max_{\gamma \in \Xi_1}\{(\gamma, \alpha_0^\vee)\}.$$

The reader may check that, if $\lambda \in X^+$ belongs to $W_p \cdot 0$ and if $(\lambda, \alpha_0^\vee) < 2p - 2h + 2$, then $\lambda = 0$. In this case, $H^1(G, L(\lambda)) = 0$. For the definition of $\Lambda(0)$ in the lemma below, see the discussion below (8.3.1) for $\Psi(0)$.

Lemma 10.1. *Assume that $p \geqslant 2h - 2$. Let Λ be a finite poset ideal in X^+ or Γ_{reg}.*

(a) *If $a_1(\Lambda) < p - h + 1$, then*

$$\mathrm{Ext}^1_G(Q(\mu_0) \otimes L(\mu_1)^{(1)}, L(\lambda)) = 0, \quad \forall \lambda, \mu \in \Lambda.$$

(b) *If $a_1(\Lambda) + a_1(\Lambda(0)) < 2p - 2h + 2$, then, for each $\mu \in \Lambda$, the module $Q(\mu_0) \otimes L(\mu_1)^{(1)}$ is the projective cover of $L(\mu)$ in $A_{\Lambda(0)}$-mod.*

Proof. First, we prove (b). The construction of $\Lambda(0)$ guarantees that the highest weight of $Q := Q(\mu_0) \otimes L(\mu_1)^{(1)}$ belongs to $\Lambda(0)$. Thus, Q is an $A_{\Lambda(0)}$-module whose head is obviously $L(\mu)$. To prove that Q is projective as an $A_{\Lambda(0)}$-module, we need only check that $\mathrm{Ext}^1_G(Q, L(\nu)) = 0$ for any $\nu \in \Lambda(0)$. This is equivalent to showing that $\mathrm{Ext}^1_G(Q(\mu_0), L(p\mu_1)^* \otimes L(\nu)) = 0$, $\forall \nu \in \Lambda(0)$. The hypothesis guarantees that all the composition factors of $L(p\mu_1)^* \otimes L(\nu)$ have the form $L(\tau)^{(1)} \otimes L(\nu_0)$ with the only possible $\tau \in W_p \cdot 0$ being 0 itself. The required vanishing follows from a standard Hochschild–Serre spectral sequence argument, using the normal subgroup scheme G_1 of G. Thus, (b) holds. Finally, (a) follows from a similar vanishing argument, replacing the condition '$\nu \in \Lambda(0)$' with '$\nu \in \Lambda$'. $\qquad\square$

Elementary methods for checking the hypotheses of Lemma 10.1 will be given in Lemma 10.4 and Corollary 10.5, as well as (the proof of) the corollaries below.

Corollary 10.2. *Assume that $p \geqslant 2h - 2$. Let Λ be a finite poset ideal in X^+ or Γ_{reg} satisfying $a_1(\Lambda) < p - h + 1$ (as in (a) of the above lemma). If $A = A_\Lambda$ and if $\mathfrak{a}$ denotes the image of u in A, then $(\mathrm{rad}\,\mathfrak{a})A = \mathrm{rad}\,A$. In particular, the latter equality holds if Λ is any finite poset ideal in Γ_{res} or $\Gamma_{res,reg}$.*

Proof. First, $a_1(\Lambda) \leqslant \frac{p-1}{p}(h-1) < h - 1$ if Λ is contained in Γ_{res}. If $p \geqslant 2h - 2$, then $a_1(\Lambda) < p - h + 1$. Thus, the second assertion of the corollary follows from the first. To prove the first assertion, Lemma 10.1(a) shows that, for each $\mu \in \Lambda$, the largest quotient Q_Λ of $Q = Q(\mu_0) \otimes L(\mu_1)^{(1)}$ with composition factors having highest weights in Λ is the projective cover of $L(\mu)$ in A-mod. However, we clearly have $(\mathrm{rad}\,u)Q = \mathrm{rad}\,Q$, the latter taken in the category G-mod. Thus, $\mathrm{rad}\,Q_\Lambda = (\mathrm{rad}\,\mathfrak{a})Q_\Lambda$ in A-mod. The left A-module A is a direct sum of modules Q_Λ, with μ varying over elements of Λ (allowing repetitions). Thus, $\mathrm{rad}\,A = (\mathrm{rad}\,\mathfrak{a})A$, completing the proof. (We remark that much of the argument parallels that for the quantum case (the last assertion of Lemma 8.3).) □

If $\Lambda \subseteq \Gamma_{reg}$, then $\mathfrak{a}$ is the also the image of u', the sum of the p-regular blocks in u.

For a given finite non-empty poset ideal Γ of X^+ or Γ_{reg}, let $N = N(\Gamma)$ be the global dimension of $A = U_\Gamma$. Then A is a QHA, so N is a finite integer. An estimate on N, assuming that $\Gamma \subseteq \Gamma_{Jan}$ consists of p-regular weights and the LCF holds on Γ, can be obtained as follows. First,

$$N = \max\{n \mid \mathrm{Ext}_G^n(L(\mu), L(\nu)) \neq 0, \mu, \nu \in \Gamma\}.$$

By [15, Theorem 3.5], N is the maximum $n_1 + n_2$ for which $\mathrm{Ext}_G^{n_1}(L(\mu), \nabla(\tau)) \neq 0$ and $\mathrm{Ext}_G^{n_2}(\Delta(\tau), L(\nu)) \neq 0$ for some $\tau \in \Gamma$. The dimensions of $\mathrm{Ext}_G^m(\Delta(\tau), L(\nu))$ and $\mathrm{Ext}_G^m(L(\nu), \nabla(\tau))$ are zero if $\nu \notin W_p \cdot \tau$. Otherwise, this (common) dimension is the coefficient of $q^{\frac{l(w)-l(y)-m}{2}}$ in a Kazhdan–Lusztig polynomial $P_{y,w}(q)$ for $y, w \in W_p$ with $y^{-1} \cdot \tau = w^{-1} \cdot \nu \in C^-$. Obviously, the maximum value of m can be no more than the maximum value of $l(w) - l(y)$, with $y^{-1} \cdot \tau, w^{-1} \cdot \nu \in C^-$. The element y of minimal length such that $y^{-1} \cdot \tau \in C^-$, for some $\tau \in \Gamma$, is $y = w_0$. If $\nu \in \Gamma$ with $w^{-1} \cdot \nu \in C^-$, we have $w = w_0 d$ with $l(w) = l(w_0) + l(d)$, computing lengths with respect to C^- (i.e., taking the fundamental reflections for the Coxeter group W_p to be those in the walls of C^-). The number $l(d)$ can be computed explicitly in terms of ν: In the spirit of [30, II, 6.6], for any $\lambda \in \Gamma_{reg}$, let $\mathbf{d}(\lambda) := \sum_{\alpha \in \Phi^+} n_\alpha$ with $p n_\alpha < (\lambda + \rho, \alpha^\vee) < p(n_\alpha + 1)$. For $\nu \in \Gamma$, $w^{-1} \cdot \nu \in C^-$, we can now calculate the length of d above. Define $w' := w_0 d w_0$, so $(w')^{-1} \cdot \nu \in w_0 \cdot C^- =: C^+$. Of course, $0 \in C^+$. Then $\mathbf{d}(\nu)$ counts the number of hyperplanes of the form $\{x \mid (x + \rho, \alpha^\vee) = pm\}$, $m \in \mathbb{Z}$, which separate ν from 0, or equivalently, ν from $(w')^{-1} \cdot \nu \in C^+$. This number is the length $l'(w')$, computed taking the fundamental reflections to be in the walls of C^+. Thus, $l'(w') = l(w_0 w' w_0) = l(d)$. So $l(d) = \mathbf{d}(\nu)$.

Proposition 10.3. *Let Γ be a finite non-empty poset ideal of p-regular weights contained in Γ_{Jan} for which the LCF holds. Let $N = N(\Gamma)$ be the global dimension*

of $A = U_\Gamma$. Then

$$N \leqslant 2 \max_{\nu \in \Gamma} \{\mathbf{d}(\nu)\}.$$

(If $\Gamma = \Gamma_{res,reg} := \Gamma_{res} \cap \Gamma_{reg}$, $N \leqslant 2\mathbf{d}((p-2)\rho) \leqslant (h-1)|\Phi|$.)

Hypothesis 7.1 requires the following result. Recall that if $\emptyset \neq \Psi \subset \Gamma_{reg}$ and if $m \geqslant -1$, the poset ideal $\Psi(m)$ is defined after (8.3.1) in §8.

Lemma 10.4. *Let Λ be a poset ideal in X^+ or Γ_{reg}. Then for any integer $m \geqslant -1$,*

$$a_1(\Lambda(m)) \leqslant a_1(\Lambda) + 2(m+1)(h-1)$$

with strict inequality whenever $m \geqslant 0$.

Proof. Clearly, the inequality holds for $m = -1$. By induction, it suffices to treat the case $m = 0$. By definition, $\Lambda(0)$ is the poset ideal generated by the weights $\mathfrak{f}_p(\lambda) = 2(p-1)\rho + w_0\lambda_0 + p\lambda_1 = (p-2)\rho + w_0\lambda_0 + p(\lambda_1 + \rho)$, $\lambda \in \Lambda$. Let γ be a dominant weight such that $\gamma \leqslant \mathfrak{f}_p(\lambda)$. Thus,

$$p(\gamma_1, \alpha_0^\vee) \leqslant (p-2)(\rho, \alpha_0^\vee) + pa_1(\Lambda) + p(h-1).$$

Dividing by p gives $(\gamma, \alpha_0^\vee) < h - 1 + a_1(\Lambda) + h - 1 = a_1(\Lambda) + 2(h-1)$, as required. $\square$

The following corollary is an easy consequence of Lemma 10.4.

Corollary 10.5. *Let Λ be a poset ideal in Γ_{res} or in $\Gamma_{res,reg}$. If $m \geqslant -1$, then*

$$a_1(\Lambda(m)) < (2m+3)(h-1).$$

In addition, if $p \geqslant (2m+3)(h-1)$, then

$$a_1(\Lambda(m-1)) + a_1(\Lambda(m)) < 2p - 2h + 2.$$

The positive characteristic version of Theorem 8.4 requires that the LCF holds on $\Gamma_{reg} \cap \Gamma_{Jan}$. By the result of Andersen–Janzten–Soergel [2], this formula holds if $p \gg 0$ (depending on Φ). Some specific bounds are provided in [26]. If Γ is a finite non-empty poset ideal in the poset $\Gamma_{res,reg}$, there is a finite dimensional algebra $B = U_\Gamma$, which is a quotient of the distribution (Hopf) algebra $U = \text{Dist}(G)$ of G such that B-mod identifies with the full subcategory of finite dimensional rational G-modules which have composition factors $L(\gamma)$, $\gamma \in \Gamma$. This fact follows from [24, §§ 2, 3] (which was quoted in §8 for the similar quantum result), but the reader may wish to consult the earlier treatment in [21, § 3.2] by Donkin (where $\text{Dist}(G)$ is denoted as $\text{hy}(G)$ and called the hyperalgebra of G). Necessarily, the algebra B is a QHA with standard, costandard and irreducible modules the corresponding modules in G-mod. Of course, the restricted enveloping algebra u of G is a Hopf subalgebra of $\text{Dist}(G)$.

We remark that the categories B-mod, gr B-mod, and gr B-grmod all have compatible dualities for the algebras $B = U_\Gamma$ below for any poset. See the parallel discussion in §8.

Theorem 10.6. *Assume that G is a simple simply connected algebraic group over a field of characteristic $p \geqslant 2h - 2$. Also, assume that the LCF holds for all p-regular weights*

in Γ_{Jan}. Let Γ be a poset ideal in $\Gamma_{res,reg}$, and let N be the global dimension of $B := U_\Gamma$ or $N = 2$ in the case where this global dimension is 1. (By Proposition 10.3, $N \leqslant (h-1)|\Phi|$.) Assume also that $p \geqslant 2N(h-1) - 1$. Then $\operatorname{gr} B$ has a graded Kazhdan–Lusztig theory. In particular, the algebra $\operatorname{gr} B$ is a QHA with a Koszul grading. Its standard modules have the form $\operatorname{gr} \Delta(\lambda)$, $\lambda \in \Gamma$, and they are linear in $\operatorname{gr} B$-grmod.

Proof. Let u' be the sum of the p-regular blocks of the restricted enveloping algebra u of G. Since the LCF holds and $p > h$ (excluding $p = h = 2$, a trivial case), the algebra u' is a Koszul algebra by [2, Proposition 18.17, p. 256]. If B has global dimension 0, the theorem holds trivially, so we may assume that $N \geqslant 2$. Define $\Lambda = \Gamma_{res,reg}(N-2)$. Set $A := U_\Lambda$. If $\lambda_0 \in X_1^+ \cap \Gamma_{res}$, the condition that $p \geqslant 2h - 2$ means that the u'-projective cover $Q(\lambda_0)$ of $L(\lambda_0)$ is a rational G-module. In addition, $\Gamma_{res,reg}(0) \subseteq \Lambda$, so that $Q(\lambda_0) \in A$-mod.[15] So $u' \to A$ is an injection, mapping the Koszul algebra u' isomorphically onto its image $\mathfrak{a}$ in A.

The condition that $p \geqslant 2N(h-1) - 1$ is readily checked, using Lemma 10.4, to imply that $a_1(\Lambda) < p - h + 1$, so Corollary 10.2 implies that $(\operatorname{rad} \mathfrak{a})A = \operatorname{rad} A$. Thus, Hypothesis 6.1 holds for $(A, \mathfrak{a})$. Also, Lemma 10.1 and Corollary 10.5 imply that condition (2) of Hypothesis 7.1 holds. By assumption, B-mod satisfies the (KL) property. Theorem 7.3 implies that B satisfies (SKL$'$). Now the theorem follows from the discussion of § 4 on graded Kazhdan–Lusztig theories. $\qquad\square$

Remarks 10.7. (a) We expect that the condition that $p \geqslant 2N(h-1) - 1$ in Theorem 10.6 can be removed, just leaving the conditions that $p \geqslant 2h - 2$ and that p is large enough that the LCF holds for all p-regular weights in Γ_{Jan}. The improvement should be a consequence of our study [44, 45] (still in progress), of integral versions of the algebras $\operatorname{gr} B$ and p-filtrations of standard modules. The conclusion that $\operatorname{gr} B$ is a QHA can already be obtained in this paper under roughly comparable conditions. See Theorem 10.8 below.

Of course, current bounds [26] on p required for the validity of the LCF are much larger even than the current requirement that $p \geqslant 2N(h-1) - 1$ in Theorem 10.6.

(b) Under the same hypotheses as for Theorem 10.6, the analogue of Corollary 8.5 holds. In particular, the homological dual $B^! := \operatorname{Ext}_G^\bullet(B/\operatorname{rad} B, B/\operatorname{rad} B)$ is a Koszul QHA algebra, isomorphic to the homological dual of $\operatorname{gr} B$, $B = U_\Gamma$. In addition, for $\lambda \in \Gamma$, the multiplicities $[\operatorname{gr} \Delta(\lambda) : L(\mu)(n)]$ in the radical filtration of $\Delta(\lambda)$ are given explicitly by inverse Kazhdan–Lusztig polynomial coefficients.[16] (A generic version of this radical filtration multiplicity result had been proved in [3, 35].)

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[15] The condition $\Gamma_{res,reg}(0) \subseteq \Lambda$ is similar to and plays the same role here as the assumption that Λ be e-fat in § 8.

[16] A similar multiplicity result for an explicit semisimple series can be obtained by reduction mod p from the quantum radical filtration in § 8 for Weyl modules $\Delta(\lambda)$ with $\lambda \in \Gamma_{Jan}$ a regular weight, when $p > h$ and the LCF holds on Γ_{Jan}. Parity considerations imply the needed complete reducibility. (No claim is made here that this series is the characteristic p radical series.) As in Corollary 8.5(b) and its proof, this semisimple series can be translated to an explicit semisimple series for Weyl modules associated with singular weights in Γ_{Jan}.

(c) To obtain an analogue for the Schur algebra $S(n, r)$, fix n and choose p large enough that the LCF holds for all regular weights in the Jantzen region, for G of type SL_n. Let $N = h|\Phi| = n^2(n-1)/2$, and also assume that $p \geqslant 2N(n-1) - 1$. (As in (a), we expect that it will turn out that this condition can be replaced by $p \geqslant 2n - 2$.) Now let $r \geqslant p$ be any integer such that (in the notation of § 10), the regular weights in $\Gamma_{n,r}$ are contained in $\Gamma_{\mathrm{res,reg}}$. (For example, $r - p < n$ works.) Then the above result is applicable for any regular block (or product of blocks) of $B_{\Gamma_{n,r}} \cong S(n, r)$ (the latter isomorphism following as in § 10 for the q-Schur algebra).

Potentially, such a characteristic p-analogue of Theorem 9.1 could hold without any p-chamber regularity condition on the weights, or any condition on p, other than perhaps $p^2 > r$ – the hypothesis of the James conjecture [29]. However, such a generalization to singular weights is beyond our current methods. (Computer calculations by Carlson, found at http://www.math.uga.edu/~jfc/schur.html, support the speculation that gr $S(n, r)$ is Koszul under the hypothesis that $p^2 > r$.)

In the following result, Theorem 6.5 is used to improve the bound on p in Theorem 10.6 as regards the conclusion that gr B is a QHA.

Theorem 10.8. *Assume that G is a simple simply connected algebraic group over a field of characteristic $p \geqslant 4h - 5$. Also, assume that the LCF holds for all p-regular weights in Γ_{Jan}. Let Γ be a poset ideal in $\Gamma_{res,reg}$ and let $B := U_\Gamma$. Then gr B is a QHA with weight poset Γ whose standard modules are the gr $\Delta(\lambda)$, $\lambda \in \Gamma$.*

We postpone the proof of this theorem until the following result is established.

Theorem 10.9. (a) *Assume that G is a simple simply connected algebraic group over a field of characteristic $p \geqslant 4h - 5$. Assume also that the LCF holds for all $\lambda \in \Gamma_{res,reg} \subseteq \Gamma_{Jan}$. For any $\lambda \in \Gamma_{res,reg}$, there is a positive grading $\Delta(\lambda) = \bigoplus_{n \geqslant 0} \Delta(\lambda)_n$ of $\Delta(\lambda)$ as a module for the Koszul algebra u'. As a graded u'-module, $\Delta(\lambda)$ is generated in grade 0. As a $u'/\mathrm{rad}\, u'$-module, $\Delta(\lambda)_0 \cong L(\lambda) \cong L(\lambda_0) \otimes L(\lambda_1)^{(1)}$.*

(b) *Additionally, assume that the full (i.e., not just the regular part u') restricted enveloping algebra u is Koszul (which holds for $p \gg h$ by [47]). Then for any $\lambda \in \Gamma_{Jan}$, $\Delta(\lambda)$ has a positive grading as a module for u. As a graded u-module, $\Delta(\lambda)$ is generated in grade 0. Also, as a $u/\mathrm{rad}\, u$-module, $\Delta(\lambda)_0 \cong L(\lambda) \cong L(\lambda_0) \otimes L(\lambda)^{(1)}$ (the irreducible G-module of highest weight λ).*

Proof. (a) The condition $p \geqslant 4h - 5 \geqslant 2h - 2$ guarantees that the u-projective cover $Q(\lambda_0)$ of $L(\lambda_0)$ is a G-module for $\lambda_0 \in X_1^+$. (See the discussion above Lemma 10.1.) Let $\Gamma = \Gamma_{\mathrm{res,reg}}$ and $\Lambda = \Gamma(0)$. Also, Lemma 10.1(b) shows that $Q = Q(\lambda_0) \otimes L(\lambda_1)^{(1)}$ is the projective cover of $L(\lambda)$ in A_Λ-mod for any $\lambda \in \Gamma$. Observe that $a_1(\Gamma) + a_1(\Lambda) = a_1(\Gamma) + a_1(\Lambda(0)) < 2p - 2h + 2$ by Corollary 10.5 since $p \geqslant 4h - 5 \geqslant 3h - 3$. So, by Lemma 10.1(b), $Q := Q(\lambda_0) \otimes L(\lambda_1)^{(1)}$ is the projective cover in $A := A_\Lambda$-mod of $L(\lambda)$. Of course, Q is also $\mathfrak{a}$-projective, where $\mathfrak{a}$ is the isomorphic image of u' in A.

Finally, to apply Theorem 6.4 with $A = A_\Lambda$ and $\lambda \in \Gamma$, it is necessary to check that $(\operatorname{rad} a)A = \operatorname{rad} A$ (which is part (3) of Hypothesis 6.1). This follows from Corollary 10.2, since $a_1(\Lambda) < a_1(\Gamma) + 2(h-1) \leqslant h - 2 + 2(h-1) = (4h-5) - h + 1 \leqslant p - h + 1$ by Lemma 10.4. This proves (a).

(b) The proof is similar. $\qquad\square$

Proof of Theorem 10.8. Let $\Lambda = \Gamma_{\mathrm{res,reg}}(0)$ and let Γ be the given poset ideal. The proof of Theorem 10.9 shows that Hypothesis 6.1 holds for $(A, \mathfrak{a})$, where $A = U_\Lambda$, provided $p \geqslant 4h - 5$. The theorem follows from Theorem 6.5. $\qquad\square$

In both of the above two proofs, it was only needed that $\mathfrak{a}$ be tightly graded. We expect that it will turn out that Theorem 10.9(a) can be improved to include all p-regular dominant weights (at least when $p \geqslant 2h - 2$ and the LCF holds in the Jantzen region). The current version of [45] proves this for $p \gg h$.

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