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Units of measurement in relativistic context

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Abstract. In the Newtonian approximation of General Relativity, employed for the dynamical modelling in the solar system, the coordinates have the dimension of time and length. As these coordinates are close to their Newtonian counterpart, the adherence to the rules of the Quantity Calculus does not raise practical difficulties: the second and the metre should be used as their units, in an abstract conception of these units. However, the scaling of coordinate times, applied for practical reasons, generates controversies, because there is a lack of information about the metrics to which they pertain. Nevertheless, it is not satisfactory to introduce specific units for these scaled coordinate times.

1. Introduction

We consider in this paper the units to be used in the post-Newtonian approximations in the dynamical modeling of the solar system: the coordinates and other quantities appearing in the components of the metric.

In textbooks on general relativity, one can find a diversity of statements on the dimensions and units of coordinates, which can perplex the reader. However, their goal is often didactic: it is to explain that coordinates are not concretely measurable with clocks and rods as the Newtonian coordinates are supposed to be.

Misner, *et al.* (1973) wrote that relativistic coordinates are pure numbers (dimensionless), the "telephone numbers of events". In applications, this point of view leads to the introduction of dimensionless graduation units which bring unnecessary complications, as noted by Klioner (2008). It is much simpler to adhere to the rules of Quantity Calculus and to apply these rules in a symbolic manner, as recommended in discussions between the author and eminent metrologists (J. De Boer, T.J. Quinn).

The Quantity Calculus, which has its roots in the work of Maxwell, is not familiar outside the field of metrology. A magisterial paper 'On the History of Quantity Calculus and the International System' has been written by J. de Boer (1994/1995), designated by (JdB) in the following. Of particular interest for us, are also two recent papers by W. H. Emerson (2005) and (2008), although they refer to classical physics. As the subtleties of these papers make their reading rather difficult, I tried to retain what is essential for our purpose: the unit of coordinates and of quantities that appear in the Newtonian approximations of general relativity used in dynamical astronomy. We deal with the mechanical units only, the second, the metre and the kilogram, when we use the International System of Units (SI). Moreover, the mass M does not appear explicitly, but only through its product by the gravitational constant G, GM, which has the dimension $(\text{length})^3/(\text{time})^2$. Thus, we avoid the additional difficulties of electrostatic and electromagnetic quantities and of thermodynamics.

2. Proper time and coordinates

In General Relativity, proper time is supposed to be directly measurable. We postulate that the count of periods of an atomic transition of an unperturbed, freely falling atom provides a good measure of the theoretical proper time along its world line. In an atomic clock, the corrections (relativistic or not) are applied so that a conventional number of periods provides the unit of proper time at a specified connector of the clock. This is the basis of the so-called "definition of the second" of the SI adopted in 1967: "The second is the duration of 9 192 631 770 periods of the radiation corresponding to the transition between the two hyperfine levels of the caesium 133 atom". This definition is rather a recipe to realize a second of proper time. Thus, locally, proper time is a measurable quantity, by comparison with a standard, in a concrete manner. Note that the definition (recipe) could change within a decade in order to reach a better accuracy (presently the inaccuracy is about 10^{-15} in relative value).

The metrological problems arise in relativity with the coordinates, which are implicitly determined by the adopted form of the components of the metric, after fixing the unit of proper time and their origins (conventional for coordinate time). There is theoretically a quasi-total freedom to select the most convenient coordinates for the problem at hand and their metrological dimension is not imposed. However, in the post-Newtonian approximation, the signature of the metric tensor provides a clear distinction between time and space. This is made explicit by the coordinates which are linked to proper time by factors which are close to unity and to $1/c$ (c, velocity of light). These coordinates are measurable quantities, which, according to the *International Vocabulary of basic and general terms of metrology (VIM),* are defined in the following terms: (*measurable) quantity, attribute of a phenomenon, body or substance that may be distinguished qualitatively and determined quantitatively.* However, they cannot be measured in the concrete conception of measurement, by direct use of a physical standard (Brumberg, 1991). Following JdB, we call them "abstract quantities". The problem we have to solve is then the designation of their units.

3. Quantity Calculus and units

Klioner (2008) has already exposed the essential features of the "Quantity Calculus" for our purpose. We recall them briefly.

A quantity Q is conceptual and is independent of any system of units. It may be concrete, as the length of an object, or abstract, as a relativistic space coordinate. *Using physical quantities gives a representation* [of the laws of physics], which *is invariant with respect of the choice of units*(JdB, p. 406). For application of this representation, Q is expressed, according to ISO notations, as the product

$$Q = \{Q\} \cdot [Q], \tag{3.1}$$

where $\{Q\}$ is the numerical value of the quantity and $[Q]$ its unit in some stated system of units. Here we consider the International System SI.

If, for example, we define quantity D by a product of powers of quantities A, B, C,

$$D = A^a\, B^b\, C^c, \tag{3.2}$$

it is evident that the use of a coherent derived unit $[A]^a\,[B]^b[C]^c$ ensures that the same relation exists between quantities and between their numerical values.

This is also true in the case of sums and differences of quantities under the essential conditions that these quantities are of the same kind and that they are expressed with the same unit.

These rules are straightforward in the case of concrete measurements. For abstract quantities, their application requires an abstract (or symbolic) conception of units. This will be first illustrated by an example.

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4. Concrete and abstract points of view on units

Consider two tables of lengths L_1 and L_2 measured in meters. Their difference of lengths $L_2 - L_1$ is measured in meters, concretely, after juxtaposition of the extremities of the tables.

Now consider two time scales (relativistic or not) T_1 and T_2. In conformity with a well-established usage, the same abbreviation designates a time scale and its reading, but the reading is a quantity written in Italics, while the designation is in Roman. The numerical values of their readings at some event E are $\{T_1\}$ and $\{T_2\}$, and the difference of these values is

$$T_2 - T_1 = (\{T_2\} - \{T_1\}) \text{ seconds at event E,} \tag{4.1}$$

where $(\{T_2\} - \{T_1\})$ is a pure number.

This notation was popularized in the 1950's and was called *algebraic notation of time differences* (advocated, in particular, by W. Markowitz). It was adopted by the IAU in 1967 (Commission 31, Resolution 2) and by the International Consultative Committee of the Radiocommunications in 1970 (Recommendation 459). It is now employed let us say 'instinctively' without any difficulties although its interpretation is puzzling. If we refer to the definition of the second, the second is a duration. However, there is no duration involved in eq. (4.1); it expresses an instantaneous measurement. The reason of this paradox is that on the left of (4.1) we use "second" as an abstract unit, in a *symbolic* conception of the unit of time, both for T_1 and T_2, while on the right we refer to the concrete definition of the second of some particular quantity (supposed to be an interval of proper time). The interpretation of eq. (4.1) requires that "second" be seen as symbolic (the symbol "s") in both sides of the equal sign.

This symbolic use of SI units may be difficult to accept. It is, nevertheless, unconsciously applied, as shown in the example of data on UT1 provided by the International Earth Rotation and References System Service.

Historically, the SI was born from the need of unification of units for trade, engineers, and laboratory scientists, all of them needing concrete units that can be represented by standards (étalons). In other terms, the units were seen as quantities of the same kind as the measurands. However, it appears that quantities of a different kind may have the same expression for their units when applying the algebra as in (3.2) to the base units. For example, the moment of a force and the energy are both expressed in $\mathrm{m}^2\ \mathrm{kg}\ \mathrm{s}^{-2}$. Even in that case of concrete quantities the unit expressed by $\mathrm{m}^2\ \mathrm{kg}\ \mathrm{s}^{-2}$should be considered as symbolic and designates two different units for two different quantities. The definition of a quantity imposes the expression of its unit. However, the inverse is not true, the unit does not bring unambiguous information on the quantity that it measures. Quantities must always be defined.

Suppose that one makes a measurement of the moment of a force. He uses a dynamometer and a rule graduated in meters. These instruments bring the uncertainties of their calibration. The result of the measurement is not expressed in an ideal meter, kilogram and second that conform to their definition. Nevertheless, it is expressed in units of m^2 $\mathrm{kg}\ \mathrm{s}^{-2}$. It is, of course, possible to cover this inconsistency by the statement of an uncertainty of the numerical value. However, the uncertainty is not always mentioned and, even in the case of a concrete measurement, the expression of the unit is also symbolic.

The name "second" and the letter "s" represent symbolically (or conventionally, if one prefers) the unit for all quantities having the dimension of time, quantities which have to be precisely defined and to be referred to by their name. These quantities can be, for example (a) a duration of proper time, (b) a reading of a given time scale, (c) a difference of readings of such a time scale, (d) the difference of readings of two time scales at some

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event. In these four examples, we have quantities that are not of the same *nature:* the reading of a time scale has no extension in time, while duration has. The distinction is made in the usual language: Oxford dictionary says *Six o'clock is a point of time; six hours is a period of time.* However, in science, we use the same name of unit "second" and the same symbol "s" for these quantities. Emerson (2008) gives another interesting example, with thermodynamic temperature expressed in kelvin, K. Here the definition of thermodynamic temperature fixes exactly the zero of the scale and this is implied by the expression of a temperature in kelvin. However, in a difference of temperatures, the zero disappears. This difference has not the nature of a thermodynamic temperature. Nevertheless, the symbol K is used. This symbol is also used for the International Temperature Scale of 1990 (ITS-90), which is a different quantity.

The fact that units may be used symbolically is underlined by their writing in Roman characters, not in Italics as quantities. More convincing is to observe, as did Emerson, that they cannot always represent a concrete quantity. For example, we can conceive that m^3 represents the volume equal to that of a cube of side one meter. However, who can conceive a s^2?

5. Barycentric Celestial Reference System (BCRS)

The metric recommended for the BCRS by IAU Resolution B1.3 (2000) has the form

$$d\tau^2 = \left[-g_{00}(t, \mathbf{x}) - \frac{2}{c}\, g_{0i}(t, \mathbf{x})\, \dot{x}^i - \frac{1}{c^2}\, g_{ij}(t, \mathbf{x})\, \dot{x}^i\, \dot{x}^j \right] dt^2 \tag{5.1}$$

with the usual conventions for indices and summation. The components $g_{\mu\nu}$ of the metric are dimensionless quantities. Proper time τ is concretely measured in seconds. The velocity of light, a physical constant, is expressed in units of proper time and length, m/s. The homogeneity of the units of (5.1) requires that t (designating here Barycentric Coordinate Time, TCB) be reckoned in seconds and x^i in meters in a symbolic use of these units.

Suppose that we introduce a "graduation unit of TCB" and "a graduation unit of space coordinates". Then to keep the homogeneity of units with proper time τ, we would have to introduce factors such as second/(graduation unit of TCB). This complication serves no useful purpose. It is much easier to consider that the second and the meter, seen as symbolic, are the graduation units (Klioner, 2008).

6. Geocentric Celestial Reference System (GCRS)

Similarly, in the GCRS, the Geocentric Coordinate Time TCG and geocentric space coordinates are expressed in seconds and meters.

Expression of the difference TCB – TCG, as given by IAU Resolution B1.5(2000) would have no meaning and could not be applied to numerical values, if different units were employed for TCB and TCG.

The symbolic use of the SI units in these cases seems to be well accepted. However, the adjective "symbolic" (or equivalently "abstract") may be controversial. One must stress that they do not qualify new units, but indicate the way we use only SI units. For brevity sake we are tempted to speak of "symbolic seconds", for example, but it is more correct to say "symbolic use of the second".

7. Units of and Terrestrial Time TT and Barycentric Dynamical Time TDB

Difficulties of terminology appeared with the scaling of the coordinate times leading to the definition of TT and TDB, as linear functions of TCG and TCB. The expressions TT units and TDB units are sometimes employed.

7.1. *Terrestrial Time and International Atomic Time (TAI)*

Let us recall that IAU 2000 B1.9 recommends "that TT be a time scale differing from TCG by a constant rate: $dTT/dTCG = 1 - L_G$ where $L_G = 6.969290134 \times 10^{-10}$ is a defining constant."

The recommendation does not define explicitly TT as a new time scale, as in the case of TDB (see 7.2). However the words "differing from TCG", and the analogy with TDB, implies it.

The value of L_G was chosen so that the rate of TT is very close to the rate of proper time of a clock at any fixed point on the rotating geoid, operating in conformity with the definition of the second. The International Atomic Time TAI is a realization of TT + 32.184 s (with a constant time offset for historical reasons). TAI has, therefore, the nature of a coordinate time. However, its dissemination provides directly the unit of proper time, the second, at any fixed point on the ground (until the top of the Everest!) at better than 1×10^{-12} in relative value. Maybe this close agreement with proper time is one reason why the use of "second" as a unit of TT and TAI raised no objections. However, in orbit modeling it leads to the introduction of new quantities differing from those associated with TCG and sometimes expressed in so-called TT-units, similarly as in the case of TDB discussed below.

7.2. *Barycentric Dynamical Time*

The IAU 2006 Resolution B3 recommends

"That in situations calling for the use of a coordinate time scale that is linearly related to Barycentric Coordinate Time (TCB) and, at the geocenter, remains close to Terrestrial Time (TT) for an extended time span, TDB be defined as the following transformation of TCB

$$TDB = TCB - L_B \times (JD_{TCB} - T_0) \times 86400 + TDB_0 \tag{7.1}$$

where $T_0 = 2443144.5003725$, and $L_B = 1.550519768 \times 10^{-8}$ and $TDB_0 = -6.55 \times 10^{-5}$ s are defining constants."

Note 1 states: "JD_{TCB} is the TCB Julian date. Its value is $T_0 = 2443144.5003725$ for the event 1977 January 1 00h 00m 00s TAI at the geocenter, and it increases by one for each 86400 s of TCB".

Is equation (7.1) an equation between quantities or between numerical values?

(a) The wording "... TDB be defined..." as well as the indication "are defining constants" which follows the numerical values of these constants, indicates that TDB is to be considered as a quantity different from TCB.

(b) If nevertheless one considers that (7.1) is a relation between numerical values, that leaves the possibility to consider that it expresses a change of unit.

Let us consider the consequences of these options.

In the case (a) a new coordinate time is defined. Equation (7.1) shows that the unit of TDB is the second, seen as symbolic, as the unit of TCB. The use of TDB should imply the definition of new space coordinates expressed in metres. In order to accomplish this change of coordinates correctly, a new form of the components of the metric should be provided, introducing new quantities, to be also expressed in SI units. A difficulty (and

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may be some misunderstandings) arises from the fact that this metric has not been officially stated, although "natural" definitions are mentioned in (Klioner, 2008).

In the case (b), whichever be the adopted metric, according to the rules of the quantity calculus, the use of TDB units would require that the proper time be expressed in proper TDB seconds. In particular, if the same form of the metric as for TCB is retained (which is the main advantage of this option), the TDB second for proper time would be longer than the SI second by $1.55\ldots \times 10^{-8}$. There is a possibility to obviate this unacceptable change of unit for proper time by introduction of factors of the form (SI Unit/TDB unit) in the metric. This is like the introduction of graduation units mentioned in 1. It would not be a satisfactory solution.

8. Related problems

8.1. *Scale units*

The contact between the symbolic and the concrete point of view on units cannot be avoided. For example, it is often necessary to consider in a concrete manner the interval between two consecutive second markers of a time scale. This occurs in relativity when one needs to evaluate the rate in proper time, at some stated event, of a coordinate time. A non-relativistic example is the evaluation of the so-called "length of the day of UT1 (LOD)" in "TAI second" (note the impropriety of terms). In these cases, the astronomers often use expressions such as "the second of UT1, of TAI, of sidereal time (when considered as a time scale), the ephemeris second, etc." which appear to be different seconds. In the symbolic point of view, there is only the "second" for different quantities

Discussions on this problem took place at the 1980 meeting of the Comité Consultatif pour la Définition de la Seconde (CCDS). In the "Déclaration" of 1980 on the definition of TAI, appeared the expression "durée de l'intervalle unitaire de TAI" ("duration of the unitary interval of TAI"). However, later intervalle unitaire and unitary interval were replaced by "unité d'échelle" and "scale unit". IAU Resolution A4, 1991 uses also this terminology. As "scale unit" is widely used and well understood, I believe that we could retain its use and extend it to space coordinates.

8.2. *Some practical limitations to the symbolic point of view on units*

The symbolic use of the SI unit for quantities having the same dimension, but different definitions, is in practice limited to quantities which are nearly equal, or, in the case of scales, which have scale units nearly equal. In particular, the time scales are defined so that their rates differ by a small amount. Even in the case of sidereal time, when considered as a time scale (not an angle), the unit second has always been used without objection, although its scale unit duration differs by about 1/365 in relative value from that of UT1, TAI, etc.

8.3. *Notation of duration and dates*

In many publications, appears a typographical distinction between the notation of the unit for the date and the unit for duration. For example, the Astronomer's Handbook (1966) says "Une heure doit être notée : $4^{\mathrm{h}}\ 39^{\mathrm{s}}$ de préférence à 4 h 39 min". According to ISO, this distinction should not be made, and the symbol of the units should not be written in superscript. The symbol of the unit must appear after the full numerical value, including its decimal part if any.

9. Conclusion

In numerical applications of theoretical modeling, the requirement that the same algebraic relations exist between quantities and their numerical values imposes that there is only one unit per dimension: this is the basis of the Quantity Calculus. It is essential to note that the information on a quantity is contained it its definition, not in its unit.

In the case of quantities that cannot be measured concretely by comparison with a standard, this leads to the concept of abstract quantities and units whose logical interpretation may appear difficult. It is then possible to consider that it is a mere convention, which is simple and efficient. In particular, there is no reason to make exceptions for coordinate times and space coordinates. Their units must be the second and the meter, without adjective or qualifiers, even in the case where scaling factors are applied for convenience.

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