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Abstract

Thurston introduced shear deformations (cataclysms) on geodesic laminations–deformations including left and right displacements along geodesics. For hyperbolic surfaces with cusps, we consider shear deformations on disjoint unions of ideal geodesics. The length of a balanced weighted sum of ideal geodesics is defined and the Weil–Petersson (WP) duality of shears and the defined length is established. The Poisson bracket of a pair of balanced weight systems on a set of disjoint ideal geodesics is given in terms of an elementary 2-form. The symplectic geometry of balanced weight systems on ideal geodesics is developed. Equality of the Fock shear coordinate algebra and the WP Poisson algebra is established. The formula for the WP Riemannian pairing of shears is also presented.

1. Introduction

As a generalization of the Fenchel–Nielsen twist deformation for a simple closed curve, Thurston introduced earthquake deformations for measured geodesic laminations. Later in his study of minimal stretch maps, Thurston generalized earthquakes to shears (cataclysms), deformations incorporating left and right displacements [Thu98]. Bonahon subsequently developed the fundamental theory of shear deformations in a sequence of papers [Bon96, Bon97a, Bon97b]. At the same time, Penner developed a deformation theory of Riemann surfaces with cusps by considering shear deformations on disjoint ideal geodesics triangulating a surface [Pen87, Pen92, Pen12]. More recently shear deformations play a basic role in the Fock and Goncharov work on the quantization of Teichmüller space [FG07, FC99, FG06] and in the Kahn and Markovic work on the Weil–Petersson Ehrenpreis conjecture [KM08].

The Weil–Petersson (WP) geometry of Teichmüller space is recognized as corresponding to the hyperbolic geometry of Riemann surfaces. For example, twice the dual in the WP Kähler form of a Fenchel–Nielsen twist deformation is the differential of the associated geodesic-length function. Also for example, the WP Riemannian pairing of twist deformations is given by a sum of lengths of orthogonal connecting geodesics, see Theorem 3 and [Rie05]. An infinitesimal shear on a disjoint union of ideal geodesics is specified by weights on the geodesics with vanishing sum of weights for the edges entering each cusp. We define the length of a balanced sum of ideal geodesics and find that twice the dual in the WP Kähler form of a shear is the differential of the defined length. We then present the basic WP symplectic and Hamiltonian geometry in § 7

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with Theorem 21 and Corollaries 22–24. The results include new formulas for the Kähler form. We show that the Poisson bracket of a pair of weight systems on a common set of triangulating ideal geodesics is given in terms of an elementary 2-form computed from the weights alone. In § 8, we use the elementary 2-form to show in Theorem 29 that the Fock shear coordinate algebra introduced in the quantization of Teichmüller space is the WP Poisson algebra. The basic WP Riemannian geometry of shears is developed in § 9 with Theorem 32. We generalize Riera's WP inner product formula and show that the Riemannian pairing of two weight systems on ideal geodesics is given by the combination of an invariant of the geometry of ideal geodesics entering a cusp and a sum of lengths of orthogonal connecting geodesics.

There are challenges in calculating shear deformations. In contrast to earthquake deformations, shear deformations are in general not limits of Fenchel–Nielsen twists and a shear on a single geodesic deforms a complete hyperbolic structure to an incomplete structure. For the deformation theory larger function spaces are involved; for earthquakes geodesic laminations carry transverse Borel measures and for shears geodesic laminations carry transverse Hölder distributions. A general approach would require a deformation theory of incomplete hyperbolic structures. Rather, we follow the approach of [Wol09] and double a surface with cusps across cusps, and open cusps to collars to obtain approximating compact surfaces with reflection symmetries. Shears are then described as limits of opposing twists. Given the above expectations, the approximating formulas include individual terms that diverge with the approximation. The object is to show that diverging terms cancel and to calculate the remaining contributions. We use the Chatauby topology for representations to show that the hyperbolic structures converge and an analysis of holomorphic quadratic differentials to show that infinitesimal deformations converge.

We begin considerations in § 2 with the variation of cross ratio and geodesic length. A unified treatment is given for Gardiner's geodesic-length formula [Gar75], Riera's twist Riemannian product formula [Rie05] and the original twist–length cosine formula [Wol83]. In § 3, we review Bonahon's results on shears on compactly supported geodesic laminations and Penner's results on shears on ideal geodesics triangulating a surface with cusps. The review includes the Thurston–Bonahon theorem that shears on a maximal geodesic lamination are transitive on Teichmüller space and Penner's theorem on λ and h length global coordinate. We include the Bonahon–Sözen and Papadopoulos–Penner results that in appropriate settings the WP Kähler form is a multiple of the Thurston symplectic form. In §§ 4 and 5, beginning with hyperbolic collars and cusps, we give the geometric description of shear deformations and describe the convergence of opposing twists to shears. In § 6, we treat the convergence of infinitesimal opposing twists to infinitesimal shears. The analysis includes the convergence of holomorphic quadratic differentials. In § 7, we define the length of a balanced sum of ideal geodesics and establish the basic symplectic geometry results in Theorem 21 and the following corollaries. In Corollary 22, we show that the Poisson bracket of length functions and the shear derivative of a length function are given by evaluation of the elementary 2-form. We consider the Fock shear coordinate algebra in § 8. We use Penner's topological description of the shear coordinate bracket and compute with the elementary 2-form to show that the algebra is the WP Poisson algebra. In § 9 we begin with expansions for gradient pairings for geodesics crossing short geodesics. Then in Theorem 32, we provide the formula for the WP Riemannian pairing of balanced sums of ideal geodesics. In Example 34 we calculate the pairing for the Dedekind $\mathrm{PSL}(2;\mathbb{Z})$ tessellation to find an exact distance relation. Finally in § 10 we give the length parameter expansion for the sum of lengths of circuits about a closed geodesic.

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2. Gradients of geodesic lengths

We begin with the basics of deformation theory of Riemann surfaces [Ahl06, Hub06, IT92]. A conformal structure is described by its uniformization. An infinitesimal variation of a conformal structure is described by a variation of the identity map for the universal cover. The interesting case for the present considerations is for a Riemann surface of finite type, a compact surface with a finite number of points removed, covered by the upper half plane $\mathbb{H}$. For a vector field v on the universal cover and parameter ϵ, there is a variation of the identity map $w_\epsilon(z) = z + \epsilon v + o(\epsilon)$, for z, respectively w, conformal coordinates for the domain and range universal covers. Provided the vector field is deck transformation group invariant, the map is equivariant with respect to deck transformation groups. The range conformal structure is described by the angle measure $\arg(dw_\epsilon)$ for the differential $dw_\epsilon = w_{\epsilon,z}\,dz + w_{\epsilon,\bar{z}}\,\overline{dz}$. The expansion for the variation provides that $dw_\epsilon = w_{\epsilon,z}(dz + \epsilon v_{\bar{z}}\overline{dz}) + o(\epsilon)$, and thus $\arg(dw_\epsilon) = \arg(w_{\epsilon,z}) + \arg(dz + \epsilon v_{\bar{z}}\,\overline{dz})$. The derivative of the vector field $v_{\bar{z}}$ describes the infinitesimal variation of the conformal structure. The quantity $v_{\bar{z}}$ is an example of a Beltrami differential, a tensor of type $\partial/\partial z \otimes \overline{dz}$.

For a Riemann surface R of finite type and vector field v defined on the surface (equivalently on the universal cover and invariant by deck transformations), then $w_\epsilon(z)$ is a variation of the identity map of the surface and in effect describes a relabeling of the points of the surface: the deformation is trivial. Nontrivial deformations are given by vector fields on the universal cover; vector fields with nontrivial group cocycles relative to the deck transformation group.

We consider $B(\mathbb{H})$, the space of Beltrami differentials on $\mathbb{H}$, bounded in L^∞. By potential theory considerations, for $\mu \in B(\mathbb{H})$ there is a vector field v on $\mathbb{H}$ with $v_{\bar{z}} = \mu$, that is actually continuous on $\overline{\mathbb{H}}$ and is bounded as $O(|z|\log|z|)$ at infinity [AB60]. In particular, elements of $B(\mathbb{H})$ also describe variations of the points of $\mathbb{R}$. We are interested in the corresponding variational formula.

The cross ratio of points of $\mathbb{P}^1$ is given as

$$(p, q, r, s) = \frac{(p-r)(q-s)}{(p-s)(q-r)}$$

and for $q = s + \Delta s$ and rearranging variables, we obtain a holomorphic 1-form

$$\Omega_{pq}(z) = \frac{(p-q)\,dz}{(z-p)(z-q)} = \frac{dz}{(z-p)} - \frac{dz}{(z-q)}.$$

The cross ratio and 1-form are invariant by the diagonal action of $\mathrm{PSL}(2;\mathbb{C})$ on all variables.

There is a natural pairing of Beltrami differentials with $Q(\mathbb{H})$, the space of integrable holomorphic quadratic differentials on $\mathbb{H}$,

$$(\mu, \psi) \to \int_{\mathbb{H}} \mu\psi \quad \text{for } \mu \in B(\mathbb{H}) \quad \text{and} \quad \psi \in Q(\mathbb{H}).$$

Rational functions, holomorphic on $\mathbb{H}$, with at least three simple poles on $\mathbb{R}$ are example elements of $Q(\mathbb{H})$. The holomorphic quadratic differentials $Q(\mathbb{H})$ describe cotangents of the deformation space of conformal structures. The variational formula for points of $\mathbb{R}$ is fundamental.

THEOREM 1 (Variation of the cross ratio [Ahl61, Ahl06]). *For $p, q, r, s \in \mathbb{R}$ the variational differential of the cross ratio is*

$$d\log(p, q, r, s) = -\frac{2}{\pi}\Omega_{pq}\Omega_{rs} \in Q(\mathbb{H}).$$

The quadratic differentials $Q(\mathbb{H})$ form a pre-inner product space with a densely defined Hermitian pairing

$$\langle \phi, \psi \rangle = \int_{\mathbb{H}} \phi\bar{\psi}\,(ds^2)^{-1} \quad \text{for } \phi, \psi \in Q(\mathbb{H}) \cap L^2$$

and ds^2 the hyperbolic metric. The pairing is the WP pre-inner product [Ahl61, Wol10]. The pairing provides formal dual tangent vectors for the differentials of cross ratios

$$\operatorname{grad}\log(p,q,r,s) = \overline{(d\log(p,q,r,s))}(ds^2)^{-1}.$$

We are interested for distinct quadruples $\mathcal{P} = (p_1, p_2, r_1, r_2)$, $\mathcal{F} = (f_1, f_2, g_1, g_2)$ in the pairing

$$\langle \operatorname{grad}\log\mathcal{P}, \operatorname{grad}\log\mathcal{F} \rangle.$$

The pairing is continuous in the quadruples for all points distinct and also is continuous for (r_1, r_2) tending to (p_1, p_2) and (g_1, g_2) tending to (f_1, f_2). We will evaluate particular configurations for the pairing.

Let $\mathcal{T}$ be the Teichmüller space of homotopy marked genus g, n punctured Riemann surfaces R of negative Euler characteristic. We are interested in pairings corresponding to geometric constructions of deformations. A point of $\mathcal{T}$ is the equivalence class of a pair (R, f) with f a homeomorphism from a reference topological surface F to R. From the uniformization theorem a conformal structure determines a unique complete compatible hyperbolic metric ds^2 for R and a deck transformation group $\Gamma \subset \mathrm{PSL}(2;\mathbb{R})$ with $R = \mathbb{H}/\Gamma$. The Teichmüller space is a complex manifold with cotangent space at R represented by $Q(R)$, the space of holomorphic quadratic differentials on R with at most simple poles at punctures.

The pairing

$$(\mu, \psi) \to \int_R \mu\psi \quad \text{for } \mu \in B(R) \quad \text{and} \quad \psi \in Q(R)$$

is the ingredient for Serre duality and consequently the tangent space of $\mathcal{T}$ at R is $B(R)/Q(R)^{\perp}$ (see [Ahl61, Ahl06, Har77, Hub06, IT92]). The L^2 Hermitian pairing

$$\langle \phi, \psi \rangle = \int_R \phi\bar{\psi}\,(ds^2)^{-1}$$

is the WP cometric for $Q(R)$. The metric dual mapping

$$\phi \to \bar{\phi}(ds^2)^{-1} \in Q(R)$$

is a complex anti-linear isomorphism, since Beltrami differentials of the given form (harmonic differentials) give a direct summand of $Q(R)^{\perp}$ in $B(R)$. The metric dual mapping associates a tangent vector to a cotangent vector and so defines the WP Kähler metric on the tangent spaces of $\mathcal{T}$; the mapping is the Hermitian metric gradient.

Geodesic lengths and Fenchel–Nielsen twist deformations are geometric quantities for pairings. Associated to a nontrivial, nonperipheral free homotopy class α on the reference surface F is the length $\ell_\alpha(R)$ of the unique geodesic in the free homotopy class for R. Geodesic length is given as $2\cosh \ell_\alpha/2 = \operatorname{tr} A$ for α corresponding to the conjugacy class of $A \in \Gamma$ in the deck transformation group. Geodesic lengths are functions on Teichmüller space with a direct relationship to WP geometry. A Fenchel–Nielsen twist deformation is also associated to a closed simple geodesic. The deformation is given by cutting the surface along the geodesic α to form two metric circle boundaries, which then are identified by a relative rotation to form a new

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hyperbolic surface. A flow on $\mathcal{T}$ is defined by considering the family of surfaces $\{R_t\}$ for which at time t reference points from sides of the original geodesic are relatively displaced by t units to the right on the deformed surface. The infinitesimal generator, the Fenchel–Nielsen vector field t_α, the differential of the geodesic length and the gradient of geodesic length satisfy duality relations

$$2\,\omega_{WP}\,(\ ,t_\alpha) = d\ell_\alpha \quad \text{and equivalently} \quad 2t_\alpha = J\,\mathrm{grad}\,\ell_\alpha, \tag{1}$$

for ω_{WP} the WP Kähler form and J the complex structure of $\mathcal{T}$ (multiplication by i on $B(R)/Q(Q)^\perp$) [Wol82, Wol10]. The factor of two adjustment to our formulas as detailed in [Wol07, § 5] is included.

We are interested in the WP metric and Lie pairings of the infinitesimal deformations $\mathrm{grad}\,\ell_\alpha$ and t_α with geodesic-length functions ℓ_β. The formulas begin with Gardiner's calculation of the differential of geodesic length. We now use a single simplified approach that provides Gardiner's $d\ell_\alpha$ formula [Gar75], the cosine formula for $t_\alpha\ell_\beta$ (see [Wol83, Wol10]), the sine-length formula for $t_\alpha t_\beta \ell_\gamma$ (see [Wol83, Wol10]), as well as Riera's length–length formula for $\langle \mathrm{grad}\,\ell_\alpha, \mathrm{grad}\,\ell_\beta\rangle$ (see [Rie05, Wol10]). The approach combines Theorem 1, coset decompositions for the uniformization group and calculus calculations. An important step is identifying a telescoping sum corresponding to a cyclic group action. We present the approach.

THEOREM 2 (Gardiner's variational formula [Gar75]). *For a closed geodesic α,*

$$d\ell_\alpha = \frac{2}{\pi} \sum_{C\in\langle A\rangle\backslash\Gamma} \Omega^2_{r_A a_A}(Cz) \in Q(R)$$

with α corresponding to the conjugacy class of $A \in \Gamma$ with repelling fixed point r_A and attracting fixed point a_A.

Proof. We begin with the geodesic length. For a hyperbolic transformation A, the geodesic length is $\log(As, s, r_A, a_A)$ for s a point of $\mathbb{R}$ distinct from the fixed points. We begin with the variational formula for the cross ratio from Theorem 1. The resulting integrand is in $L^1(\mathbb{H})$ and $\mathbb{H}$ is the disjoint union

$$\bigcup_{n\in\mathbb{Z}} \bigcup_{C\in\langle A\rangle\backslash\Gamma} A^n C(\mathcal{F})$$

for $\mathcal{F}$ a Γ fundamental domain. By a change of variables the union over domains is replaced by a sum of integrands

$$d\ell_\alpha[\mu] = -\frac{2}{\pi}\Re \int_{\mathcal{F}} \mu \sum_n \sum_{C\in\langle A\rangle\backslash\Gamma} \Omega_{As\,s}(A^nCz)\Omega_{r_A a_A}(A^nCz). \tag{2}$$

The invariance of Ω by the diagonal $\mathrm{PSL}(2;\mathbb{R})$ action gives $\Omega_{pq}(A^n w) = \Omega_{A^{-n}pA^{-n}q}(w)$ and the given product of forms is

$$\Omega_{A^{-n+1}sA^{-n}s}(Cz)\,\Omega_{r_A a_A}(Cz).$$

Using the Ω partial fraction expansion, the first factor is

$$\Omega_{A^{-n+1}sA^{-n}s} = \frac{dw}{(w - A^{-n+1}s)} - \frac{dw}{(w - A^{-n}s)}$$

and the integer sum telescopes

$$\sum_{n=-N}^{N} \Omega_{A^{-n+1}sA^{-n}s} = \Omega_{A^{N+1}sA^{-N}s}$$

and as N tends to infinity, $A^{N+1}s$ tends to a_A and $A^{-N}s$ tends to r_A. (Various forms of the telescoping appear in the calculations for the cosine formula [Wol83, pp. 220–221], the sine–length formula [Wol83, pp. 223–224] and the length–length formula [Rie05, pp. 113–114].) The sum in (2) now becomes the desired sum

$$-\sum_{C\in\langle A\rangle\backslash\Gamma}\Omega^2_{r_Aa_A}(Cz). \qquad \square$$

We consider the WP Hermitian pairing of gradients $\langle \operatorname{grad}\ell_\alpha, \operatorname{grad}\ell_\beta\rangle$. By (1) the imaginary part of the pairing is

$$\Re\langle J\operatorname{grad}\ell_\alpha, \operatorname{grad}\ell_\beta\rangle = 2t_\alpha\ell_\beta = 2\sum_{p\in\alpha\cap\beta}\cos\theta_p.$$

The real part of the pairing $\langle \operatorname{grad}\ell_\alpha, \operatorname{grad}\ell_\beta\rangle$ was first evaluated by Riera [Rie05]. We now apply the above approach and with a single simpler treatment derive the real and imaginary part formulas. Riera's formula involves the logarithmic function

$$R(u) = u\log\left|\frac{u+1}{u-1}\right| - 2.$$

The function is even with a logarithmic singularity at ±1 and with the expansion

$$R(u) = 2\left(\frac{1}{3u^2}+\frac{1}{5u^4}+\frac{1}{7u^6}+\cdots\right) \quad \text{for } |u|>1.$$

In particular, for $u>1$, the function and its even derivatives are positive and the function is $O(u^{-2})$ for $u>1$. The function $R(u)$ is also given as

$$\frac{u}{2}\tanh^{-1}\frac{1}{u}-2 \quad \text{for } |u|>1 \quad \text{and} \quad \frac{u}{2}\tanh^{-1}u-2 \quad \text{for } |u|<1.$$

We present the pairing formula for the general case of a cofinite group possibly with parabolic and elliptic elements.

Theorem 3 (The complex gradient pairing [Wol83, Rie05]). *For closed primitive geodesics α,β corresponding to elements $A,B\in\Gamma$, we have for the WP pairing*

$$\langle \operatorname{grad}\ell_\alpha, \operatorname{grad}\ell_\beta\rangle = \frac{2}{\pi}\delta_{\alpha\beta}e(A)\ell_\alpha + \sum_{D\in\langle A\rangle\backslash\Gamma/\langle B\rangle}\mathcal{R}_D,$$

where $\delta_{\alpha\beta}$ is the Kronecker delta for the geodesic pair, where $e(A)$ is 2 in the special case of the axis of A having order-two elliptic fixed points and is 1 otherwise, where for the axes $\operatorname{axis}(A), \operatorname{axis}(DBD^{-1})$ disjoint in $\mathbb{H}$, then

$$\mathcal{R}_D = \frac{2}{\pi}R(\cosh d(\operatorname{axis}(A), \operatorname{axis}(DBD^{-1})))$$

and for the axes intersecting with angle θ_D, then

$$\mathcal{R}_D = \frac{2}{\pi}R(\cos\theta_D) - 2i\cos\theta_D.$$

Twist–length duality and J an isometry provide that $4\langle t_\alpha, t_\beta\rangle = \langle \operatorname{grad}\ell_\alpha, \operatorname{grad}\ell_\beta\rangle$.

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Proof. For A a hyperbolic element we write

$$\Theta_A = \sum_{C\in\langle A\rangle\backslash\Gamma} \Omega^2_{r_A a_A}$$

and from Gardiner's formula $d\ell_\alpha = (2/\pi)\Theta_A$ with

$$\langle\Theta_A, \Theta_B\rangle = -\int_{\mathbb{H}} \sum_{C\in\langle A\rangle\backslash\Gamma} \Omega^2_{r_A a_A}(Cz)\,\overline{\Omega_{Bs\,s}(z)}\,\overline{\Omega_{r_B a_B}(z)}\,(ds^2)^{-1}.$$

We first decompose each left coset $\langle A\rangle\backslash\Gamma$ by considering right $\langle B\rangle$ cosets and then move the $\langle B\rangle$ action to the two conjugate forms. The resulting sum over $\langle B\rangle$ is telescoping. In particular, we enumerate the cosets of the sum by writing for $C \in \langle A\rangle\backslash\Gamma$ the decomposition $C = DB^n$, $D \in \langle A\rangle\backslash\Gamma/\langle B\rangle$ for $n \in \mathbb{Z}$. For A, B primitive hyperbolic elements, we consider uniqueness of the presentation of an element of $\langle A\rangle D$ in the form $A^m DB^n$. A nonunique presentation is equivalent to a solution of $A^a = DB^bD^{-1}$ for a nontrivial integer pair (a, b). Since A, B each generate Γ maximal cyclic subgroups, a nontrivial solution of $A^a = DB^bD^{-1}$ provides that A is conjugate to $B^{\pm1}$ by the element D. In particular, the presentation A^mDB^n is unique except for the case $\alpha = \beta$ with $A = DB^{\pm1}D^{-1}$. In the case $\alpha = \beta$ we select the element A to represent the geodesic and the presentation is unique except for the case of D either the identity or the special case of Γ containing an order-two elliptic E with $A = EA^{-1}E$. For the special cases there is no distinction between left and right $\langle A\rangle$ cosets; we only use left cosets. The special left cosets are for the identity element and the element E.

Now for each resulting integral of the sum, change variable by writing $w = B^nz$; the effect is to move a B^{-n} action to the variable of $\Omega_{Bs\,s}\Omega_{r_Ba_B}$. Using the diagonal $\mathrm{PSL}(2;\mathbb{R})$ invariance of Ω, the B^{-n} action is moved to the quadruple of points, resulting in the telescoping sum

$$\sum_{n\in\mathbb{Z}} \Omega_{B^{n+1}sB^ns}\Omega_{r_Ba_B} = -\Omega^2_{r_Ba_B}.$$

The result is the general formula

$$\begin{aligned}\langle\Theta_A, \Theta_B\rangle = {}&-\delta_{AB^{\pm1}}e(A)\int_{\mathbb{H}} \Omega^2_{r_Ba_B}\,\overline{\Omega_{Bs\,s}}\,\overline{\Omega_{r_Ba_B}}\,(ds^2)^{-1}\\ &+\sum_{D\in\langle A\rangle\backslash\Gamma/\langle B\rangle}\int_{\mathbb{H}} \Omega^2_{r_{D^{-1}AD}a_{D^{-1}AD}}\,\overline{\Omega^2_{r_Ba_B}}\,(ds^2)^{-1},\end{aligned} \tag{3}$$

where the Kronecker delta indicates that the first integral is only present for the case that $A = B^{\pm1}$, $e(A)$ is 2 in the case of order-two elliptic fixed points on the axis of A and is otherwise 1, and for the second integral the diagonal invariance was used to move the D action to the pair of points. For each integral, a change of variable by an element of $\mathrm{PSL}(2;\mathbb{R})$ results in the inverse element applied to the tuple of points. It follows that the first integral depends only on the $\mathrm{PSL}(2;\mathbb{R})$ conjugacy class of B and the second integral depends only on the $\mathrm{PSL}(2;\mathbb{R})$ class of the pair $(D^{-1}AD, B)$. It follows that the first integral is a function of the geodesic length for B and the second integral depends only on the distance between/intersection angle of the axes.

We evaluate the integrals. The differential Ω_{pq} is continuous in p, q, including at infinity; for q tending to infinity the form limits to $dz/(z-p)$. For the first integral of (3), we take the pair of points to be 0 and ∞, to obtain for $z = re^{i\theta}$ the integral

$$-\int_{\mathbb{H}} \frac{1}{z^2\bar{z}}\,\overline{\frac{(Bs-s)}{(z-Bs)(z-s)}}\,r^2\sin^2\theta r\,dr\,d\theta,$$

which for $P = e^{i\theta}Bs, Q = e^{i\theta}s$ becomes

$$-\int_0^{\pi}\int_0^{\infty}\frac{(P-Q)}{(r-P)(r-Q)}\sin^2\theta\,dr\,d\theta$$
$$=\log\frac{(r-Q)}{(r-P)}\bigg|_0^{\infty}\int_0^{\pi}\sin^2\theta\,d\theta=\frac{\pi}{2}\log\frac{Bs}{s},$$

as expected, since $\operatorname{grad}\ell_* = 2/\pi\,\Theta_*$. For the second integral of (3), we take the first pair of points to be 0 and ∞, to obtain the integral

$$\int_{\mathbb{H}}\frac{1}{z^2}\overline{\left(\frac{(p-q)}{(z-p)(z-q)}\right)^2}r^2\sin^2\theta\,rdr\,d\theta,$$

which for $P = e^{i\theta}p, Q = e^{i\theta}q$, becomes

$$\int_0^{\pi}\int_0^{\infty}\frac{(P-Q)^2}{(r-P)^2(r-Q)^2}\sin^2\theta r\,dr\,d\theta. \tag{4}$$

The r integral has antiderivative

$$\frac{(P+Q)}{(P-Q)}\log\frac{(r-Q)}{(r-P)}-\frac{P}{(r-P)}-\frac{Q}{(r-Q)}.$$

We are evaluating an area integral and θ varies in the interval $(0,\pi)$; for $p,q\in\mathbb{R}$, θ as described, and r real positive, the quotient $(r-Q)/(r-P)$ is valued in the complex open lower half plane. The antiderivative is invariant under interchanging p,q; we now normalize p to be positive real. We use the principal branch of the logarithm; for r close to zero the argument is close to $-\pi$. Evaluating r at $0,\infty$ and integrating in θ gives

$$\frac{\pi}{2}\left(\frac{\kappa+1}{\kappa-1}\log\kappa-2\right)\quad\text{for }\kappa\text{ the ratio }q/p=(q,p,0,\infty).$$

To interpret geometrically, compare with [Rie05, p. 114], set $u=(\kappa+1)/(\kappa-1)=2(\infty,q,p,0)-1$, to obtain the complex-valued expression

$$\frac{\pi}{2}\left(u\log\frac{u+1}{u-1}-2\right).$$

For the lines $\widehat{0\infty}$ and $\widehat{pq}$ disjoint, the ratio $\kappa=p/q$ is positive and the logarithm is real, with $u=\cosh\delta_*$, for δ_* the distance between the lines. For the lines intersecting, the ratio $\kappa=q/p$ is negative and the argument of the logarithm is $-\pi$ and evaluation gives

$$\frac{\pi}{2}R(\cos\theta_*)-\frac{\pi}{2}i\pi\cos\theta_*,$$

as desired. □

The double coset enumeration admits a topological/geometric description. We consider that α and β are primitive and Γ is torsion-free. On the surface R, consider the homotopy classes rel the closed sets α,β of arcs connecting α to β. For the universal cover, fix a lifting of α to a line $\tilde{\alpha}_0$ in $\mathbb{H}$; then a connecting homotopy class on R lifts to a homotopy class of arcs connecting $\tilde{\alpha}_0$ to $\tilde{\beta}$ (a line lifting of β). The relation rel α corresponds to the relation of the $\langle A\rangle$ action

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on homotopy lifts. In particular, the nontrivial classes on R rel α, β *biject* to the classes in $\mathbb{H}$ rel $\tilde{\alpha}_0, \tilde{\beta}$, for $\tilde{\beta}$ (disjoint from $\tilde{\alpha}_0$) ranging over the line liftings of β modulo the action of $\langle A\rangle$; the nontrivial classes on R correspond to lines $\tilde{\beta}$ disjoint from $\tilde{\alpha}_0$. Let A generate the stabilizer of the line $\tilde{\alpha}_0$. We enumerate the pairs $(\tilde{\alpha}_0, \tilde{\beta})$, for $\tilde{\beta}$ distinct modulo the $\langle A\rangle$ action. For B generating the stabilizer of a line lifting of β, then line pairs $(\tilde{\alpha}_0, \tilde{\beta})$ distinct modulo the $\langle A\rangle$ action *correspond* bijectively to double cosets by the rule

$$(\tilde{\alpha}_0, \tilde{\beta}) = (\operatorname{axis}(A), \operatorname{axis}(DBD^{-1})) \quad \text{corresponds to } D \in \langle A\rangle\backslash\Gamma/\langle B\rangle.$$

The relation $\operatorname{axis}(DBD^{-1}) = D(\operatorname{axis}(B))$ is part of the correspondence. For a finite number of double cosets the corresponding axes intersect. Overall, the axes enumeration by double cosets, enumerates pairs of line liftings of α and β modulo the diagonal action of the group Γ. The geometric description comes from the description of a pair of lines. A pair of lines either intersects or has a unique perpendicular geodesic, minimizing the connecting distance. The cosine and hyperbolic cosine describe the geometry of the configurations.

The present approach to evaluating the pairing is a combination and simplification of earlier works. The role of the cyclic group in Gardiner's formula was first noted by Hejhal [Hej78, Theorem 4]. The telescoping of the cyclic group sums appears in the proofs of Theorems 3.3 and 3.4 of [Wol83] and in Theorem 2 of [Rie05], although in each case the telescoping is presented as a special feature. The basic integral (4) is simpler than found in the earlier formulations. The present approach can be applied to evaluate the second twist Lie derivatives $t_\alpha t_\beta \ell_\gamma$. The first derivative $t_\alpha \ell_\beta$ is a sum of cosines of intersection angles. A cosine is given by a cross ratio, the starting point for the above considerations.

3. Thurston shears

We are interested in Thurston shears (cataclysms) on ideal geodesics for a Riemann surface with cusps. Thurston studied the shear deformation for compact geodesic laminations [Thu98]. Bonahon developed the fundamental results in a sequence of papers [Bon96, Bon97a, Bon97b]. We present a brief summary of Bonahon's basic results following [Bon96]. In a series of works [Pen87, Pen92, Pen12], Penner developed a deformation theory of Riemann surfaces with cusps by considering shear deformations on ideal geodesics triangulating a surface. Our interests include Penner's λ-length formulas and formulas for the WP Kähler/symplectic form [PP93]. We present a brief summary of Penner's results following the exposition of the book [Pen12].

A *geodesic lamination* λ is a closed union of disjoint simple geodesics. A geodesic lamination for a compact surface R is maximal provided R–λ is a union of ideal triangles. A *transverse measure* for a geodesic lamination λ is the assignment for each transverse arc k with endpoints in λ^c of a positive Borel measure μ on the transverse arc with $\operatorname{supp}(\mu) = \lambda \cap k$. If transverse arcs k, k' are homotopic through arcs with endpoints in λ^c then the assigned measures correspond by the homotopy. The assignment $k \mapsto \mu(k)$ is additive under countable subdivision of transverse arcs. A measured geodesic lamination defines an earthquake deformation by interpreting $\mu(k)$ as the relative left shift of the λ complementary regions containing the k endpoints. By allowing left and right shifts on complementary regions, Thurston defined the shear deformation. The relative left shift of λ complementary regions again defines a functional on transverse arcs. The functional, called a *transverse cocycle*, is only finitely additive under subdivision of transverse arcs. A transverse cocycle is not given by integrating a measure, rather is given by elements of the dual of Hölder continuous functions on transverse arcs. The space of transverse cocycles $\mathcal{H}(\lambda)$ on a geodesic lamination is a finite-dimensional vector space.

Teichmüller space is the space of isotopy classes of hyperbolic metrics. A geodesic lamination is represented on each isotopy class of a hyperbolic metric. Shear deformations on a given maximal geodesic lamination parameterize Teichmüller space. A projection between leaves is defined for the lift of a lamination to the universal covering of the surface. The construction begins with the observation that the unit area horoballs in an ideal triangle are foliated by horocycles. The tangent field of the partial foliation of ideal triangles extends to a Lipschitz vector field on the universal covering; the vector field is not defined on the small trilateral regions in each ideal triangle. The Lipschitz vector field defines a projection between leaves of the lift of the lamination. The projection defines a relative displacement between lamination complementary regions. The relative displacement is finitely additive. The relative left displacement is called the *shearing cocycle* σ_R of the surface R. The transverse cocycle for the shear deformation from a surface R_1 to a surface R_2 is the difference $\sigma_{R_1} - \sigma_{R_2}$ of shearing cocycles. For a train track carrying a geodesic lamination, transverse measures are specified in terms of nonnegative weights on the track and transverse cocycles are specified in terms of real weights. We also refer to the Thurston symplectic intersection form τ for a train track. The shearing cocycles for a maximal geodesic lamination provide an embedding of Teichmüller space.

Theorem 4 [Bon96, Theorems A and B]. *The map $R \mapsto \sigma_R$ defines a real analytic homeomorphism from $\mathcal{T}$ to an open convex cone $\mathcal{C}(\lambda)$ bounded by finitely many faces in $\mathcal{H}(\lambda)$. A transverse cocycle μ is in the cone $\mathcal{C}(\lambda)$ if and only if $\tau(\mu,\nu) > 0$ for every transverse measure ν for λ.*

The R-length $\ell_\mu(R)$ of the transverse cocycle μ for λ is a generalization of the total length of a transverse measure. The R-length is defined as

$$\ell_\mu(R) = \iint_\lambda d\ell \, d\mu,$$

computed locally by first integrating hyperbolic length measure along the leaves of λ and then integrating the local function on the local space of λ leaves with respect to the Hölder distribution μ. The R-length generalizes the weighted length for weighted simple closed geodesics; R-length is given by the Thurston intersection form and the shearing cocycle as follows.

Theorem 5 [Bon96, Theorem E]. *If μ is a transverse cocycle for the maximal geodesic lamination λ and $\sigma_R \in \mathcal{H}(\lambda)$ is the shearing cocycle of the hyperbolic surface R, then $\ell_\mu(R) = \tau(\mu, \sigma_R)$.*

The Theorem 4 embedding of $\mathcal{T}$ into the vector space $\mathcal{H}(\lambda)$ provides identifications of tangent spaces $\mathbf{T}\mathcal{T}$ with $\mathcal{H}(\lambda)$. The identification enables a comparison of symplectic forms.

Theorem 6 [SB01]. *Let R be a compact hyperbolic surface with a maximal geodesic lamination λ. Then for the tangent space identifications $\mathbf{T}\mathcal{T} \simeq \mathcal{H}(\lambda)$, the WP Kähler form is a constant multiple of the Thurston intersection form.*

A *decoration* for a hyperbolic metric with cusps is the designation of a horocycle at each cusp. Decorated Teichmüller space $\mathcal{DT}$ is the space of isotopy classes of hyperbolic metrics with cusps and decorations [Pen12]. The decorated Teichmüller space is naturally fibered over Teichmüller space with fibers given by varying the horocycle lengths in a decoration. A section of the fibration is given by prescribing horocycle lengths. A decoration enables a notion of relative length for ideal geodesics. The λ-*length* of an ideal geodesic α is $\lambda(\alpha) = e^{\delta(\alpha)/2}$, where $\delta(\alpha)$ is the signed distance along α between the decoration horocycles; the distance is positive in the case that the associated horodiscs are disjoint. We are interested in the λ-lengths for the isotopy

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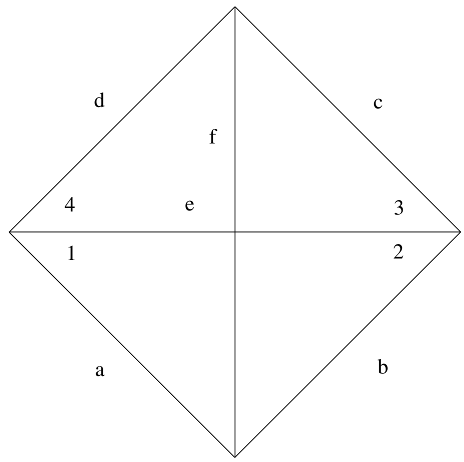

Figure 1. Adjacent ideal triangles with a second diagonal.

class of a given ideal triangulation Δ of hyperbolic metrics. An ideal triangulation for a genus g surface with n cusps has $6g-6+3n$ ideal geodesics and $4g-4+2n$ triangles.

Additional parameters are associated to an ideal triangulation. The ideal geodesics divide the decoration horocycles into segments. The *h-lengths* are the lengths of the horocycle segments. For an ideal triangle, the lengths are related by $h_{\hat{a}} = \lambda_a/\lambda_b\lambda_c$, where for the horocycle segment $\hat{a}$ the triangle opposite side is a and the triangle adjacent sides are b, c. We are particularly interested in the shear coordinates. An ideal triangle has a median. For a pair of triangles adjacent along an ideal geodesic α, drop perpendiculars from the medians to α. The *shear coordinate* for α is the signed distance between the median projections; the distance is positive if the projections lie to the right of one another along α. The shear coordinate is given simply in terms of λ-lengths and h-lengths. In Figure 1, the shear coordinate for the diagonal e is given as

$$\sigma_e = \log\frac{\lambda_b\lambda_d}{\lambda_a\lambda_c} = \log\frac{h_1}{h_4} = \log\frac{h_3}{h_2}, \quad \text{[Pen12, ch. 1, Corollary 4.16].} \tag{5}$$

The fibers of the Teichmüller fibration $\mathcal{DT} \to \mathcal{T}$ are characterized simply by constant shear coordinates.

By the classical result of Whitehead, triangulations with common vertices can be related by a sequence of replacing diagonals in quadrilaterals [Pen12, ch. 2, Lemma 1.4]. The effect on λ-lengths of replacing diagonals is given by Penner's basic Ptolemy equation $\lambda_{13}\lambda_{24} = \lambda_{12}\lambda_{34} + \lambda_{14}\lambda_{23}$ for the configuration of Figure 1 [Pen12, ch. 1, Corollary 4.6]. We also note the *coupling equation* $h_1h_2 = h_3h_4$ for the configuration of Figure 1; the equation follows from the definition of h-lengths. The λ- and h-lengths provide global coordinates for $\mathcal{DT}$.

Theorem 7 [Pen12, ch. 2, Theorems 2.5, 2.10; ch. 4, Theorems 2.6, 4.2]. *For the ideal triangulation Δ, the λ-length mapping $\mathcal{DT} \to \mathbb{R}^{\Delta}_{>0}$ is a real-analytic homeomorphism. For V the vertex sectors of the ideal triangulation, the h-length mapping $\mathcal{DT} \to \mathbb{R}^{V}_{>0}$ is a real-analytic embedding into a real-algebraic quadric variety given by coupling equations. For the ideal triangulation, the shear coordinate mapping $\mathcal{T} \to \mathbb{R}^{\Delta}$ is a real-analytic homeomorphism onto the linear subspace given by vanishing of the sum of shears around each cusp. The action of the mapping class group* MCG *is described by permutations followed by finite compositions of Ptolemy transformations.*

The WP Kähler form pulls back to the decorated Teichmüller space and has a universal expression in terms of λ- and h-lengths. We present new formulas for the pullback in § 6.

Theorem 8 [Pen12, ch. 2, Theorem 3.1]. *For an ideal triangulation Δ, the pullback WP Kähler form on $\mathcal{DT}$ is*

$$\widetilde{\omega_{WP}} = \sum_{\Delta} \widetilde{\lambda_a} \wedge \widetilde{\lambda_b} + \widetilde{\lambda_b} \wedge \widetilde{\lambda_c} + \widetilde{\lambda_c} \wedge \widetilde{\lambda_a},$$

where the sum is over ideal triangles, $\widetilde{\lambda_*} = d\log\lambda_*$ and the individual triangles have sides a, b and c in clockwise order.

The formula is given without Penner's initial 2 factor following the adjustment to our own formulas as detailed in [Wol07, § 5].

Papadopoulos–Penner establish a formula for the pullback $\widetilde{\omega_{WP}}$ in terms of h-lengths and describe identifications of spaces to establish that $2\,\widetilde{\omega_{WP}}$ coincides with Thurston's intersection form [PP93]. Specifically the authors show that their change of variable (††) transforms their formula (†) to the formula (†††); the calculation applies to the present setting by taking $\mu(greek\ index) = -\log h_{\widehat{index}}$ and $\mu(index) = \log \lambda_{index}$ and noting the factor of 2.

Corollary 9 [PP93]. *For an ideal triangulation Δ, the pullback WP Kähler form is*

$$\widetilde{\omega_{WP}} = \sum_{\Delta} \widetilde{h_\alpha} \wedge \widetilde{h_\beta} + \widetilde{h_\beta} \wedge \widetilde{h_\gamma} + \widetilde{h_\gamma} \wedge \widetilde{h_\alpha},$$

where the sum is over ideal triangles, $\widetilde{h_*} = d\log h_*$ and the individual triangles have vertex sectors α, β and γ in clockwise order.

In particular the λ to h change of coordinates is pre-symplectic.

Papadopoulos and Penner introduce the formal Poincaré dual of an ideal triangulation. The formal dual is a trivalent graph with an orientation for the edges at a vertex. A modification of the trivalent graph is a punctured null-gon train track. A set of logarithms of λ-lengths corresponds to a measure on the train track. A modification of the construction of a measured foliation from a measured train track parameterizes the space $\mathcal{DMF}$ of decorated measured foliations.

Theorem 10 [PP93, Proposition 4.1]. *The train track parameterization provides a homeomorphism of $\mathcal{DT}$ to $\mathcal{DMF}$. The homeomorphism identifies twice the pullback WP Kähler form and the Thurston intersection form $2\,\widetilde{\omega_{WP}} = \tau$.*

4. Thurston shears as limits of opposing twists

We show that weighted Fenchel–Nielsen twists with twist lines orthogonal to short geodesics converge to a Thurston shear deformation on ideal geodesics, as the short lengths tend to zero. We begin with the collars and cusp description [Bus92]. For a closed geodesic α on the surface R of length ℓ_α, normalize the universal covering for the corresponding deck transformation to be $z \to e^{\ell_\alpha} z$. The collar $\mathcal{C}(\alpha) = \{\ell_\alpha/2 \leqslant \arg z \leqslant \pi - \ell_\alpha/2\}/\langle z \to e^{\ell_\alpha} z\rangle$ embeds into R with α the core geodesic. For a cusp, normalize the universal covering for the corresponding deck transformation to be $z \to z+1$. The cusp region $\mathcal{C}_\infty = \{\Im z \geqslant \frac{1}{2}\}/\langle z \to z+1\rangle$ embeds into R. The collars about short geodesics and cusp regions are mutually disjoint in R.

In the universal cover a Fenchel–Nielsen twist deformation for a single geodesic line β is the piecewise isometry self-map of $\mathbb{H}$ with jump discontinuity across β given by a hyperbolic transformation stabilizing β. A twist deformation of magnitude t offsets the β half planes by

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a relative t units to the right, as measured when crossing β. The relative displacement of a combination of twists on disjoint lines is found as follows. For the displacement of q relative to p, consider the twist lines separating p and q (for neither point on a twist line). There is a partial ordering of lines based on containment of half planes containing p. By definition the $(n+1)$th line contains the preceding n lines in a common half plane with p. The individual twist deformations are normalized to fix p. The combined deformation map of $\mathbb{H}$ is given by left (post) composition of the individual deformations formed in the order of the lines. A basic property is that the Fenchel–Nielsen twists on a set of disjoint lines is a commutative group.

A finite collection of disjoint closed geodesics on a surface R lifts to a locally finite collection in $\mathbb{H}$ and an equivariant twist mapping is determined on relatively compact sets. For our purposes it suffices to analyze finite combinations of twists in $\mathbb{H}$.

We begin with hyperbolic cylinders and cusp regions.

Definition 11. For a hyperbolic cylinder with core geodesic γ, an opposing twist is a finite combination of weighted Fenchel–Nielsen twists with twist lines orthogonal to γ and vanishing magnitude sum. For a hyperbolic cusp region, a Thurston shear is a finite combination of weighted Fenchel–Nielsen twists with twist lines asymptotic at the cusp and with vanishing magnitude sum.

A positive shear corresponds to a right earthquake. For a Thurston shear an initial piecewise horocycle orthogonal to the twist lines with successive displacements given by the negative weights is deformed to a closed horocycle. The deformed region is complete hyperbolic with a closed horocycle, consequently is a cusp region. The vanishing magnitude sum condition is required for completeness of the deformed structure. The condition is noted in [Bon96, § 12.3] and considered in detail in [Pen12, ch. 2, § 4].

Lemma 12. *The opposing twist deformation of a hyperbolic cylinder is a hyperbolic cylinder. The core length of the deformed cylinder is bounded uniformly in terms of the initial core length and the twist weights. For a bounded number of bounded weights, the deformed core length is small uniformly as the initial core length is small.*

Proof. Opposing twist lines decompose a cylinder into bands, each isometric to a region between ultra parallel lines in $\mathbb{H}$. The twist deformation is given by translations across lines. The vanishing magnitude sum provides that a deformed cylinder is complete hyperbolic containing ultra parallel bands, consequently is a hyperbolic cylinder.

We observe that for disjoint weighted twist lines converging, Fenchel–Nielsen twists (normalized with a common fixed region) converge. For a core length ℓ, collar twist lines are represented in the band $\{1 \leqslant |z| < e^{\ell}\}$ in $\mathbb{H}$. For ℓ small, the individual twists are close to the twist line $|z| = 1$. The magnitude sum vanishing provides that for ℓ small the combined twist transformation is close to the identity. In particular, for twist weights bounded on a compact set the opposing twist is close to the identity uniformly in ℓ. The deformed core length is the translation length of $z \to ze^{\ell}$ conjugated by the opposing twists. The deformed core length is uniformly small in ℓ, as desired. □

Next we make precise the notion of opposing twists converging to a Thurston shear and also note a consequence.

Definition 13. Opposing twists for a sequence of cylinders with core lengths tending to zero geometrically converge to a Thurston shear provided the following. First, the universal coverings are normalized with the hyperbolic deck transformations for the cylinders converging in the

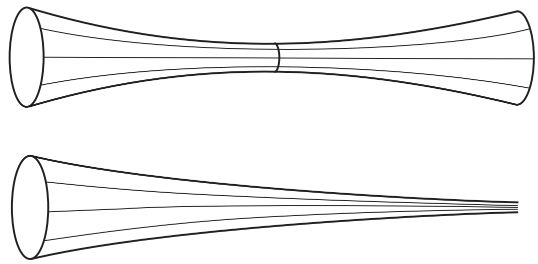

Figure 2. A hyperbolic cylinder with geodesics orthogonal to the core geodesic and a cusp region with geodesics asymptotic at the cusp.

compact open topology for $\mathbb{H}$ to the parabolic deck transformation for the cusp region. Second, for a relatively compact open set K in $\mathbb{H}$ whose projection to the cusp region contains a loop encircling the cusp, the intersection with K of the weighted twist lines for the cylinders converges to the intersection with the weighted Thurston shear lines.

Lemma 14. *Consider hyperbolic cylinders converging to a cusp region with opposing twists geometrically converging to a Thurston shear. A normalization by $\mathbb{H}$ isometries of the twist deformation maps of $\mathbb{H}$ converges to the Thurston shear in the compact open topology for $\mathbb{H}$.*

Proof. Convergence of lines intersecting a given relatively compact set in $\mathbb{H}$ provides convergence on any compact set. As noted, convergence of weighted lines in $\mathbb{H}$ provides that suitably normalized deformation maps converge in the compact open topology. □

5. Chabauty convergence and opening cusps

The points of Teichmüller space $\mathcal{T}$ are equivalence classes $\{(R, f)\}$ of Riemann surfaces with reference homeomorphisms $f : F \to R$ from a reference surface. The *complex of curves* $C(F)$ is defined as follows. The vertices of $C(F)$ are the free homotopy classes of homotopically nontrivial, nonperipheral, simple closed curves on F. A k-simplex consists of $k+1$ homotopy classes of mutually disjoint simple closed curves. For surfaces of genus g and n punctures, a maximal set of mutually disjoint simple closed curves, a *partition*, has $3g-3+n$ elements. The mapping class group acts on the complex $C(F)$.

The Fenchel–Nielsen coordinates for $\mathcal{T}$ are given in terms of geodesic lengths and lengths of auxiliary geodesic segments [Abi80, Bus92, IT92]. A partition $\mathcal{P} = \{\alpha_1, \ldots, \alpha_{3g-3+n}\}$ decomposes the reference surface F into $2g-2+n$ components, each homeomorphic to a sphere with a combination of three discs or points removed. A homotopy marked Riemann surface (R, f) is likewise decomposed into pants by the geodesics representing the elements of $\mathcal{P}$. Each component pants, relative to its hyperbolic metric, has a combination of three geodesic boundaries and cusps. For each component pants, the shortest geodesic segments connecting boundaries determine designated points on each boundary. For each geodesic α in the pants decomposition, a twist parameter τ_α is defined as the displacement along the geodesic between designated points, one for each side of the geodesic. For marked Riemann surfaces close to an initial reference marked Riemann surface, the displacement τ_α is the distance between the designated points; in general, the displacement is the analytic continuation (the lifting) of the distance measurement. For α in $\mathcal{P}$ define the *Fenchel–Nielsen angle* by $\vartheta_\alpha = 2\pi\tau_\alpha/\ell_\alpha$. The Fenchel–Nielsen coordinates for Teichmüller space for the decomposition $\mathcal{P}$ are $(\ell_{\alpha_1}, \vartheta_{\alpha_1}, \ldots, \ell_{\alpha_{3g-3+n}}, \vartheta_{\alpha_{3g-3+n}})$. The coordinates provide a real analytic equivalence of $\mathcal{T}$ to $(\mathbb{R}_+ \times \mathbb{R})^{3g-3+n}$ (see [Abi80, Bus92, IT92]).

A partial compactification, the *augmented Teichmüller space* $\overline{\mathcal{T}}$, is introduced by extending the range of the Fenchel–Nielsen parameters. The added points correspond to unions of hyperbolic surfaces with formal pairings of cusps. The interpretation of *length vanishing* is the key ingredient. For an ℓ_α equal to zero, the angle ϑ_α is not defined and in place of the geodesic for α there appears a pair of cusps; the reference map f is now a homeomorphism of $F - \alpha$ to a union of hyperbolic surfaces (curves parallel to α map to loops encircling the cusps). The parameter space for a pair $(\ell_\alpha, \vartheta_\alpha)$ will be the identification space $\mathbb{R}_{\geqslant 0} \times \mathbb{R}/\{(0, y) \sim (0, y')\}$. More generally for the partition $\mathcal{P}$, a frontier set $\mathcal{T}(\mathcal{P})$ is added to the Teichmüller space by extending the Fenchel–Nielsen parameter ranges: for each $\alpha \in \mathcal{P}$, extend the range of ℓ_α to include the value 0, with ϑ_α not defined for $\ell_\alpha = 0$. The points of $\mathcal{T}(\mathcal{P})$ in general parameterize unions of Riemann surfaces with each $\ell_\alpha = 0, \alpha \in \mathcal{P}$, specifying a pair of cusps.

We present an alternate description of the frontier points in terms of representations of groups and the Chabauty topology. A Riemann surface with punctures and hyperbolic metric is uniformized by a cofinite subgroup $\Gamma \subset \mathrm{PSL}(2;\mathbb{R})$. A puncture corresponds to the Γ-conjugacy class of a maximal parabolic subgroup. In general, a Riemann surface with punctures corresponds to the $\mathrm{PSL}(2;\mathbb{R})$ conjugacy class of a tuple $(\Gamma, \langle\Gamma_{01}\rangle, \ldots, \langle\Gamma_{0n}\rangle)$ where $\langle\Gamma_{0j}\rangle$ are the maximal parabolic classes and a labeling for punctures is a labeling for conjugacy classes. A *Riemann surface with nodes* R' is a finite collection of $\mathrm{PSL}(2;\mathbb{R})$ conjugacy classes of tuples $(\Gamma^*, \langle\Gamma^*_{01}\rangle, \ldots, \langle\Gamma^*_{0n^*}\rangle)$ with a formal pairing of certain maximal parabolic classes. The conjugacy class of a tuple is called a *part* of R'. The unpaired maximal parabolic classes are the punctures of R' and the genus of R' is defined by the relation Total area $= 2\pi(2g - 2 + n)$. A cofinite $\mathrm{PSL}(2;\mathbb{R})$ injective representation of the fundamental group of a surface is topologically allowable provided peripheral elements correspond to peripheral elements. A point of the Teichmüller space $\mathcal{T}$ is given by the $\mathrm{PSL}(2;\mathbb{R})$ conjugacy class of a topologically allowable injective cofinite representation of the fundamental group $\pi_1(F) \to \Gamma \subset \mathrm{PSL}(2;\mathbb{R})$. For a simplex σ, a point of the corresponding frontier space $\mathcal{T}(\sigma) \subset \overline{\mathcal{T}}$ is given by a collection $\{(\Gamma^*, \langle\Gamma^*_{01}\rangle, \ldots, \langle\Gamma^*_{0n^*}\rangle)\}$ of tuples with: a bijection between σ and the paired maximal parabolic classes; a bijection between the components $\{F_j\}$ of $F - \sigma$ and the conjugacy classes of parts $(\Gamma^j, \langle\Gamma^j_{01}\rangle, \ldots, \langle\Gamma^j_{0n^j}\rangle)$ and the $\mathrm{PSL}(2;\mathbb{R})$ conjugacy classes of topologically allowable isomorphisms $\pi_1(F_j) \to \Gamma^j$ (see [Abi77, Ber74]). We are interested in geodesic lengths for a sequence of points of $\mathcal{T}$ converging to a point of $\mathcal{T}(\sigma)$. The convergence of hyperbolic metrics provides that for closed curves of F disjoint from σ, geodesic lengths converge, while closed curves with essential σ intersections have geodesic lengths tending to infinity [Ber74, Wol90].

We refer to the Chabauty topology to describe the convergence for the $\mathrm{PSL}(2;\mathbb{R})$ representations. Chabauty introduced a topology of geometric convergence for the space of discrete subgroups of a locally compact group [Cha50]. A neighborhood of $\Gamma \subset \mathrm{PSL}(2;\mathbb{R})$ is specified by a neighborhood U of the identity in $\mathrm{PSL}(2;\mathbb{R})$ and a compact subset $K \subset \mathrm{PSL}(2;\mathbb{R})$. A discrete group Γ' is in the neighborhood $\mathcal{N}(\Gamma, U, K)$ provided $\Gamma' \cap K \subseteq \Gamma U$ and $\Gamma \cap K \subseteq \Gamma' U$. The sets $\mathcal{N}(\Gamma, U, K)$ provide a neighborhood basis for the topology. The $\mathrm{PSL}(2;\mathbb{R})$ topology coincides with the induced compact open topology for transformations of $\mathbb{H}$. Important for the present considerations is the following convergence characterization. A sequence of points of $\mathcal{T}$ converges to a point of $\mathcal{T}(\sigma)$, provided for each component F_j of $F - \sigma$, there exist $\mathrm{PSL}(2;\mathbb{R})$ conjugations such that restricted to $\pi_1(F_j)$ the corresponding representations converge element-wise to $\pi_1(F_j) \to \Gamma^j$ (see [Har74, Theorem 2]).

We now consider a Riemann surface R with cusps and data $\sum \mathfrak{b}_j \widehat{\beta}_j$ for a Thurston shear. The data is a weighted sum of disjoint simple ideal geodesics, geodesics with endpoints at infinity in the cusps. The weighted sum of segments entering each cusp vanishes. Double the surface across its cusps; consider the union of R and its conjugate surface $\bar{R}$ with the reflection symmetry ρ for the pair. For the geodesic $\widehat{\beta}_j$, we write β_j for the union $\widehat{\beta}_j \cup \rho(\widehat{\beta}_j)$. To open cusps, given ϵ positive, remove the area ϵ horoball at each cusp and glue the remaining surfaces by the map ρ to obtain a compact surface R_ϵ. The surface R_ϵ has a reflection symmetry (also denoted ρ) and smooth simple closed curves obtained from gluing the β_j (also denoted β_j). The construction provides a homeomorphism from a reference surface F to R_ϵ for ϵ positive and the simplex σ of short curves for F is given by the ϵ horocycles. Standard comparison estimates for metrics provide that for the uniformization hyperbolic metric, the simplex is realized by short geodesics with lengths tending to zero with ϵ. The comparison estimates also provide that on the complement of prescribed area collars about the short geodesics, the R_ϵ hyperbolic metrics converge C^∞ to the hyperbolic metric of $R \cup \bar{R}$ (see [Wol90]). The uniformization groups $\Gamma(R_\epsilon)$ for the R_ϵ, Chatauby converge to the uniformization pair $\Gamma(R), \Gamma(\bar{R})$, relative to F and the horocycle simplex σ. The uniqueness of geodesics and convergence of hyperbolic metrics provide that the geodesics $\tilde{\beta}_j$ in the free homotopy classes β_j converge uniformly on σ collar complements to $\widehat{\beta}_j \cup \rho(\widehat{\beta}_j)$ on $R \cup \bar{R}$.

We are ready to compare the effect of the Thurston shear $\sum \mathfrak{b}_j(\widehat{\beta}_j \cup -\rho(\widehat{\beta}_j))$ on $R \cup \bar{R}$ to the effect of the opposing twist $\sum \mathfrak{b}_j \tilde{\beta}_j$ on the hyperbolic metric of R_ϵ. The reflection ρ reverses orientation and notions of left/right; even though $\bar{R}$ is the mirror image, we require regions to move in the same direction by a twist; the minus sign provides the desired effect. Opposing twist deformations do not preserve the reflection symmetry. As a preliminary matter, we note from Lemma 12 for weights bounded, the opposing twist of R_ϵ has small geodesic lengths bounded in terms of ϵ. Twisting R_ϵ defines a family close to the frontier $\mathcal{T}(\sigma)$. We observe the following.

Lemma 15. *For ϵ small and weights bounded, the opposing twist $\sum \mathfrak{b}_j \tilde{\beta}_j$ of R_ϵ is Chatauby close to the Thurston shear $\sum \mathfrak{b}_j(\widehat{\beta}_j \cup -\rho(\widehat{\beta}_j))$ of $R \cup \bar{R}$. Furthermore, the infinitesimal opposing twist is close to the infinitesimal Thurston shear in the sense of infinitesimal variations of* $\mathrm{PSL}(2;\mathbb{R})$ *representations.*

Proof. In overview, the convergence of metrics provides for the compact open convergence of the twist/shear lines on $\mathbb{H}$, which in turn provides for the element-wise convergence of representations. By construction of R_ϵ, for the components F_j of $F - \sigma$, the representations $\pi_1(F_j)$ into $\mathrm{PSL}(2;\mathbb{R})$ converge element-wise and the twist lines compact open converge to shear lines. Choose generators for the limiting representations and a relatively compact open set $U \subset \mathbb{H}$, such that $CU \cap U \neq \emptyset$ for each generator C. For ϵ small, the same elements generate the representations of $\pi_1(F_j)$ and satisfy the nonempty translate intersection condition. The representations are completely determined by their action on U. A twist/shear map τ of $\mathbb{H}$ induces a variation of a representation by varying a transformation B by the conjugation $\tau B \tau^{-1}$. Only a finite number of twist/shear lines intersect U. The $\mathrm{PSL}(2;\mathbb{R})$ normalized combined twist is given by finite ordered compositions as described above. By metric convergence, as ϵ tends to zero, on U the twist lines converge uniformly and the twists converge uniformly to shears and thus the representations of the finite number of generators converge. The representations are element-wise uniformly close in ϵ. To consider the infinitesimal variations, we introduce a parameter t for $t\sum \mathfrak{b}_j \tilde{\beta}_j$ and $t\sum \mathfrak{b}_j(\widehat{\beta}_j \cup -\rho(\widehat{\beta}_j))$. The considerations provide that the initial infinitesimal variations of the generators are also close in ϵ. The infinitesimal variations of the representations are determined on generators. □

6. Infinitesimal Thurston shears and opposing twists

We are interested in geodesic length gradients. A thick–thin decomposition of hyperbolic surfaces is determined by a positive constant. The thin subset consists of those points with injectivity radius at most the positive constant; for a constant at most unity the thin subset is a disjoint union of collars and horoballs [Bus92]. Surface representations into $\mathrm{PSL}(2;\mathbb{R})$ are Chatauby close precisely when their thick subsets are Gromov–Hausdorff close [Har74]. For a sequence of hyperbolic surfaces with certain geodesic lengths tending to zero, we are interested in the magnitude and convergence of geodesic length gradients $\operatorname{grad}\ell_\alpha$ for certain geodesics α crossing the short geodesic-length collars.

Applications of convergence of surfaces and gradients include generalizing the Gardiner formula, Theorem 2, to balanced sums of ideal geodesics and generalizing twist–length duality (1) to Thurston shears and balanced sums of ideal geodesics. The basic matter is to understand the effect of Chatauby convergence for sums of the basic differential Ω^2 from § 2. We begin with convergence of hyperbolic transformations of $\mathbb{H}$.

A hyperbolic transformation with translation length ℓ, fixed points symmetric with respect to the origin and i on its collar boundary is given as

$$A = \begin{pmatrix} \cosh \ell/2 & 1/\ell \sinh \ell/2 \\ \ell \sinh \ell/2 & \cosh \ell/2 \end{pmatrix}$$

(i is distance $\log 1/\ell$ to the A axis with endpoints $\pm 1/\ell$). As ℓ tends to zero, A converges to the parabolic transformation

$$\begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}.$$

We consider a Chatauby converging sequence of surfaces with short length core geodesics and a crossing geodesic intersecting the core geodesics orthogonally. A crossing geodesic intersects collars and core geodesics. Given a segment of a crossing geodesic α in a thick region, normalize the universal coverings so that the segment lifts to a segment along the imaginary axis with highest point at i. Extend the segment by including the arcs that connect to core geodesics (the added arcs cross half collars). A core geodesic intersecting α lifts to a geodesic orthogonal to the imaginary axis. The figures for the universal covers of the surface, Figure 4, and the Chatauby limit, Figure 5, are as follows. In the figures the collar lift and its limit are shaded. In Figure 4, the left and right circular arcs orthogonal to the baseline bound a fundamental domain for a core geodesic transformation. Chatauby convergence provides that the original segments on the crossing geodesic α have length bounded and it is standard that collar boundaries converge to horocycles. Figure 5 is the limit of a sequence of Figure 4 diagrams with upper, respectively lower, shaded regions converging to upper, respectively lower, shaded regions. The crossing geodesic limits to an ideal geodesic connecting cusps.

Definition 16. For an ideal geodesic α, we write

$$d\ell_\alpha = \frac{2}{\pi} \sum_{C \in \Gamma} \Omega^2_{pq}(Cz)$$

for the infinite series, where p, q are endpoints of a lift of α to $\mathbb{H}$.

Lemma 17. *For a surface R with cusps and an ideal geodesic α, the infinite series $d\ell_\alpha$ converges. As above, consider surfaces R_ϵ with reflection symmetries obtained by doubling R across its cusps and opening cusps to obtain short length core geodesics. Consider that an ideal geodesic α*

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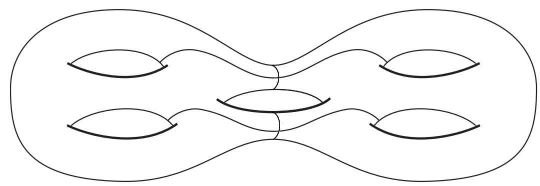

Figure 3. A symmetric compact surface with crossing and core geodesics.

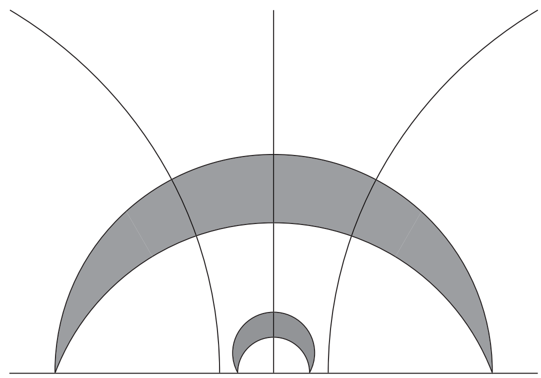

Figure 4. Crossing and core geodesics. The vertical line is the lift of the crossing geodesic. The two semicircles orthogonal to the baseline are consecutive lifts of the core geodesic. The left and right circular arcs bound a fundamental domain for the hyperbolic transformation stabilizing the larger semicircle. The shaded sectors are lifts of half collars for the core geodesic. The region bounded by the shaded sectors and the circular arcs covers a region containing a component of the thick subset of the surface.

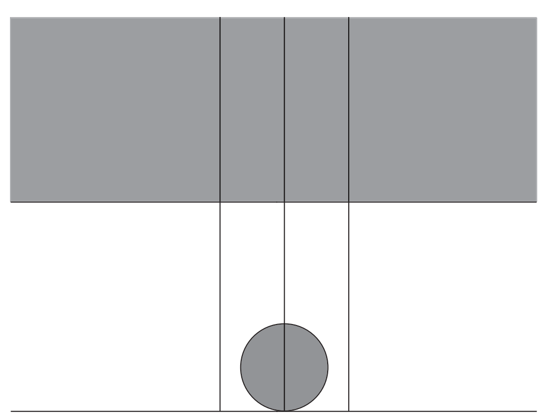

Figure 5. An ideal geodesic and horoballs. The central vertical line is the lift of the ideal geodesic connecting cusps. The left and right vertical lines bound a fundamental domain for the parabolic transformation stabilizing infinity. The shaded sectors are horoballs about the cusps. The region bounded by the shaded sectors and the vertical lines covers a region containing the thick subset of the surface.

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on R is approximated on thick subsets by closed core orthogonal geodesics α_ϵ on R_ϵ. There is a Chatauby neighborhood $\mathcal{U}$ of $R \cup \bar{R}$ such that for $R_\epsilon \in \mathcal{U}$, on thick subsets the harmonic Beltrami differentials $d\ell_{\alpha_\epsilon}(ds^2)^{-1}$ and $d\ell_\alpha(ds^2)^{-1}$ are uniformly bounded and are uniformly close.

Proof. The $d\ell_\alpha$ series are bounded by area integrals as follows. We first consider regions. In Figure 4, the unshaded region in $\mathbb{H}$ between the shaded crescents, by normalization, lies below the line $\Im z = 1$ and outside a circle tangent to $\mathbb{R}$ at 0. The integral of $|\Omega_{0\infty}|^2 = dr/r\, d\theta$ for $z = re^{i\theta}$ over the unshaded region is bounded by the integral over the region between the shaded sectors in Figure 5

$$\int_0^\pi \int_{a\sin\theta}^{\csc\theta} \frac{dr}{r}\, d\theta = \int_0^\pi \log \frac{\csc^2\theta}{a}\, d\theta = 2\pi \log 2 - \pi \log a.$$

On a thick region of a surface a holomorphic quadratic differential satisfies a mean value estimate in terms of the integral over a hyperbolic metric ball of a radius r_0 at most the injectivity radius. The thick regions of R_ϵ and R are contained in the projection of the indicated unshaded regions in Figures 4 and 5. By the standard unfolding, the absolute values of $d\ell_{\alpha_\epsilon}(ds^2)^{-1}$ and $d\ell_\alpha(ds^2)^{-1}$ at a thick point are bounded by the integral of $|\Omega_{0\infty}|^2$ over the disjoint union of r_0 balls about the orbit of the lifted point in the unshaded region, see [Wol10, ch. 8]. By the above considerations, the integrals are uniformly bounded, establishing the first result.

For the second conclusion, given δ positive, choose a relatively compact set K in the Figure 4 region between shaded crescents, such that the integral of $|\Omega_{0\infty}|^2$ over the complement between the shaded crescents is bounded by δ. The sum of evaluations of $\Omega_{0\infty}^2$ at points not in K is bounded by δ by a mean value estimate. Chatauby convergence provides convergence for the sum of evaluations of $\Omega_{0\infty}^2$ for the orbit points in K. Boundedness and convergence are established. □

Example 18 (The ideal geodesic series $d\ell_\alpha$ for a hyperbolic cusp). For a cusp uniformized at infinity with integer translation group, then the sum over the group is

$$\sum_{C\in\Gamma_\infty} \Omega_{0\infty}^2(Cz) = \sum_{n\in\mathbb{Z}} \frac{dz^2}{(z-n)^2}.$$

The formula for the integer sum gives $d\ell_\alpha = 2\pi \csc^2 \pi z\, dz^2$. From the above lemma, for a hyperbolic cylinder the series $d\ell_\alpha$ approximates the cosecant squared in the compact open topology of $\mathbb{H}$.

We now combine considerations to obtain a uniform majorant for an opposing sum of twists and gradients of geodesic-length functions. The majorant is the necessary ingredient for general limiting arguments. We codify the situation as follows.

Definition 19. A crossing configuration is a compact surface with reflection symmetry with fixed locus a finite union of small length core geodesics γ and no other geodesics having small length. A crossing geodesic α is symmetric with respect to the reflection with two intersections with the core geodesics. For a crossing configuration, a sum $\sum \mathfrak{a}_j \ell_{\alpha_j}$ of crossing geodesics length functions is balanced provided for each core geodesic γ, the weighted intersection number $\sum \mathfrak{a}_j \#(\alpha_j \cap \gamma)$ vanishes. For a surface with cusps, a formal sum $\sum \mathfrak{a}_j \ell_{\alpha_j}$ of ideal geodesics length functions is balanced provided at each cusp the weighted intersection number $\sum \mathfrak{a}_j \#(\alpha_j \cap h)$ with each small closed horocycle h vanishes.

Balanced is the precedent to the condition of the weight sum vanishing for each cusp for a Thurston shear. To prepare for a convergence argument, we first consider the distribution of mass of a harmonic Beltrami differential.

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Lemma 20. *A balanced sum $\sigma = \sum \mathfrak{a}_j \operatorname{grad} \ell_{\alpha_j}$ of gradients for a crossing configuration is bounded as follows. On the thick subset the absolute value $|\sigma|$ is uniformly bounded. On a core geodesic γ collar, uniformized as $1 \leqslant |z| \leqslant e^{\ell_\gamma}$, $\ell_\gamma \leqslant \theta \leqslant \pi - \ell_\gamma$ for $z = re^{i\theta} \in \mathbb{H}$, the balanced sum σ is bounded as*

$$O((\ell_\gamma^3 + e^{-2\pi\theta/\ell_\gamma} + e^{2\pi(\theta-\pi)/\ell_\gamma})\ell_\gamma^{-2}\sin^2\theta).$$

The bounding constants depend only on the number of crossing geodesics, the norm of the weights and a choice of Chatauby neighborhood for the limiting cusped surface.

Proof. A general bound for a harmonic Beltrami differential on a γ collar is

$$|\mu| \quad \text{is } O((|(\mu, \operatorname{grad}\log\ell_\gamma)| + (e^{-2\pi\theta/\ell_\gamma} + e^{2\pi(\theta-\pi)/\ell_\gamma})\ell_\gamma^{-2})\sin^2\theta\, M) \tag{6}$$

for M the maximum of μ on the collar boundary [Wol12, Proposition 6]. We use Theorem 3 to bound the pairings $\langle \operatorname{grad}\ell_{\alpha_j}, \operatorname{grad}\ell_\gamma\rangle$. By setup the crossing and core geodesics are orthogonal. Each core geodesic intersection contributes -2 to the pairing evaluation. From the balanced hypothesis, the weighted sum of intersection contributions vanishes. Each remaining term of the evaluation involves a connecting geodesic segment that crosses the γ half collar; the width of the half collar is $-\log\ell_\gamma$. For large distance, the formula summand $\mathcal{R}$ is approximately $e^{-2d(\gamma,\alpha)}$. In [Wol10, ch. 8] we showed that the sum of distances from α to the γ collar boundary is uniformly bounded. It follows that the contribution ℓ_γ^2 of the half collar width can be factored out of each summand. The sum evaluation is $O(\ell_\gamma^2)$, the desired bound. Lemma 17 provides the desired bound for σ on the thick subset. □

7. The symplectic geometry of lengths

There is a length interpretation for a balanced sum $\mathcal{A} = \sum \mathfrak{a}_j \ell_{\alpha_j}$ of ideal geodesics length functions as follows. Let $\mathcal{H}$ be a neighborhood of the cusps given as a union of small horoballs, one at each cusp. The length $L(\mathcal{A})$ of the balanced sum is the sum with weights $\mathfrak{a}_j$ of lengths of segments $\alpha_j \cap (R - \mathcal{H})$. The balanced condition provides that the length does not depend on the choice of horoball neighborhood $\mathcal{H}$. For a crossing configuration the length of a balanced sum $L(\mathcal{A})$ is defined in the corresponding manner. In the crossing case, the value L coincides with the sum of geodesic lengths.

The length $L(\mathcal{A})$ of a balanced sum is a generalization of the R-length of a transverse cocycle. The balanced condition at cusps is discussed in [Bon96, § 12.3], where it is noted that the condition provides a well-defined notion of length. The definition in terms of horoballs shows that the length $L(\mathcal{A})$ is given as $\sum 2\mathfrak{a}_j \log\lambda_{\alpha_j}$ for the λ-lengths of the ideal geodesics and a decoration. An example of a balanced sum is a shear coordinate σ_*, see formula (5); the sum is balanced at each vertex of the quadrilateral of Figure 1. A second example comes directly from the shear coordinates of Riemann surfaces. By Theorem 7, the sum $\sum \sigma_j \ell_{\alpha_j}$ is balanced since the sum of shear coordinates around each cusp vanishes. The adjustment of a factor of 2 to our formulas as detailed in [Wol07, § 5] is included in the following.

Theorem 21. *For a surface R with cusps and a balanced sum $\mathcal{A} = \sum \mathfrak{a}_j \ell_{\alpha_j}$ of ideal geodesics length functions, the length $L(\mathcal{A})$ is a differentiable function on the Teichmüller space of R with*

$$dL(\mathcal{A}) = \sum \mathfrak{a}_j \, d\ell_{\alpha_j} \in Q(R).$$

The formal sum $\sum \mathfrak{a}_j\alpha_j$ is data for an infinitesimal Thurston shear $\sigma_{\mathcal{A}}$ with

$$\sigma_{\mathcal{A}} = \frac{i}{2}\sum \mathfrak{a}_j \operatorname{grad} \ell_{\alpha_j}.$$

The WP twist–length duality

$$2\,\omega_{WP}\,(\ ,\sigma_{\mathcal{A}}) = dL(\mathcal{A})$$

is satisfied. In particular, the Thurston infinitesimal shear $\sigma_{\mathcal{A}}$ is a WP symplectic vector field with Hamiltonian potential function $L(\mathcal{A})/2$.

Proof. We first observe that L is a differentiable function on the $\mathrm{PSL}(2;\mathbb{R})$ representation space. For the reference surface F, a simple loop $\delta \in \pi_1(F)$ about the cusp has representation into $\mathrm{PSL}(2;\mathbb{R})$ a parabolic element that generates a maximal parabolic subgroup. Prescribing an area value (at most unity) for the quotient of a horoball by the maximal parabolic subgroup determines a horoball and horocycle. (The prescription is equivalent to a choice of decoration in the Penner approach [Pen04, Pen12].) For a pair of elements of $\pi_1(F)$ defining distinct maximal parabolic subgroups, the distance between the prescribed horocycles is a smooth function of the $\mathrm{PSL}(2;\mathbb{R})$ representation. The length L is a sum of distances between horocycles and, hence, a smooth function. The differential dL is an element of $Q(R)$. In particular, the integral of the element over small neighborhoods of the cusps is small. The construction of the function and its differential is also valid for the distance between collar boundaries.

Consider a sequence of compact surfaces R_ϵ with reflection symmetries obtained by doubling and opening the cusps of R. From Lemma 17, on thick subsets, the differentials of geodesic lengths converge uniformly to differentials for ideal geodesics. From Lemma 20, for a balanced sum, the sum of differentials is uniformly bounded in each core collar; the integral of the sum is uniformly small over small area collars. As R_ϵ limits to R, the distance between collar boundaries limits to the distance between horocycles. And for closed geodesics β contained in the thick subsets, the Fenchel–Nielsen twists on $\beta \cup \rho(\beta)$ of R_ϵ converge to the twist of $R \cup \bar{R}$ and the twist derivatives of distance converge. The considerations of Chatauby convergence and Lemmas 17 and 20 can be applied for the Fenchel–Nielsen twists on $\beta \cup \rho(\beta)$. The conclusion is again that the gradient pairing integrals over small area collars and small area horoballs are uniformly small. It follows that the pairing for a balanced sum length differential and twist converges to the limiting pairing as ϵ tends to zero. The derivative of length converges to the derivative of length. Reflection-even twists span the reflection-even tangent space. The dL formula is established.

The considerations for infinitesimal Thurston shears are analogous. The deformation is smooth and by Lemma 15 the infinitesimal deformation is a limit of opposing twists. The opposing twists satisfy $\sum \mathfrak{a}_j t_{\alpha_j} = i/2 \sum \mathfrak{a}_j \operatorname{grad} \ell_{\alpha_j}$ on the side of R_ϵ that limits to R. We find the ϵ tending to zero limit by Lemmas 17 and 20. The conclusions follow. □

We remark that symmetry is basic to considering the R_ϵ to R limit of the tangent–cotangent pairing. The reflection ρ acts on the Teichmüller spaces. With respect to the reflection ρ, the differential of the length $L(\mathcal{A})$ is even, while an opposing twist and its limit are odd. Furthermore, the Kähler form is odd with respect to the ρ action on Teichmüller space, since the reflection reverses orientation for surface integration. The above duality relation $2\,\omega_{WP}\,(\ ,\sigma_{\mathcal{A}}) = dL(\mathcal{A})$ is established for reflection even deformations of $R \cup \bar{R}$. Since a Thurston shear is reflection odd, the shear pairing $\omega_{WP}\,(\sigma_{\mathcal{B}},\sigma_{\mathcal{A}})$ evaluation requires a separate analysis.

To evaluate the pairing of Thurston shears, we introduce an elementary alternating 2-form for coefficients summing to zero. For a balanced sequence $\{a_j\}_{j=1}^p$, we consider the partial sums

$A_0 = 0, A_k = \sum_{j=1}^{k} a_j,\ 1 \leqslant k \leqslant p$, where by hypothesis $A_p = 0$. We introduce a pairing for balanced sequences

$$\omega(\{a_j\}, \{b_j\}) = \frac{1}{2}\sum_{j=1}^{p}(A_j + A_{j-1})b_j. \tag{7}$$

We explain that the pairing depends only on the joint cyclic ordering of the sequences and that the pairing is alternating. A cyclic shift in the index $j,\ 1 \leqslant j \leqslant p$, has the effect of adding a constant to the partial sums $A_j,\ 0 \leqslant j \leqslant p$. The balanced condition for the sequence $\{b_j\}$ provides that the pairing is unchanged. For the alternating property, we have summation by parts for balanced sequences $\{f_j\}$ and $\{g_j\}$ with partial sums F_k and G_k

$$\sum_{k=m}^{n-1} F_k g_{k+1} = F_n G_n - \sum_{k=m}^{n} G_k f_k.$$

In particular, we have that

$$\sum_{j=1}^{p} A_j b_j = A_p B_p - \sum_{j=1}^{p-1} B_j a_{j+1} = -\sum_{j=1}^{p} B_{j-1} a_j$$

and

$$\sum_{j=1}^{p} A_{j-1} b_j = A_p B_p - \sum_{j=1}^{p} B_j a_j = -\sum_{j=1}^{p} B_j a_j$$

using that A_0, A_p, B_0 and B_p vanish. The pairing can be written in the alternating form

$$\omega(\{a_j\}, \{b_j\}) = \frac{1}{2}\sum_{j=1}^{p} A_j b_j - B_j a_j. \tag{8}$$

We note that balanced sequences have an interpretation as tangents to the regular $(p-1)$-simplex and ω an interpretation as a closed 2-form on the regular simplex.

The form ω can be evaluated for a pair of balanced sums for a common set of disjoint ideal geodesics limiting to a cusp. For balanced sums $\mathcal{A} = \sum a_j \ell_{\alpha_j}, \mathcal{B} = \sum b_j \ell_{\alpha_j}$ and a given cusp, consider the geodesic segments limiting to the cusp; some geodesics α_j may not limit to the given cusp and some may have both ends limiting to the cusp. Choose and label a limiting geodesic as the first and enumerate limiting geodesics in the counterclockwise order about the cusp. Evaluate the form ω on the enumerated sequences of weights $\{a_j\}$ and $\{b_j\}$.

Corollary 22. *For the balanced sums $\mathcal{A} = \sum a_j \ell_{\alpha_j}$ and $\mathcal{B} = \sum b_j \ell_{\alpha_j}$ for a common set of disjoint ideal geodesics, the shear pairing is*

$$\omega_{WP}(\sigma_{\mathcal{A}}, \sigma_{\mathcal{B}}) = \frac{1}{2}\sigma_{\mathcal{A}} L(\mathcal{B}) = \frac{1}{2}\sum_{\text{cusps}} \omega(\{a_j\}, \{b_j\}).$$

The Poisson bracket for the length functions $L(\mathcal{A})$ and $L(\mathcal{B})$ is

$$\{L(\mathcal{A}), L(\mathcal{B})\} = 2\sum_{\text{cusps}} \omega(\{a_j\}, \{b_j\}).$$

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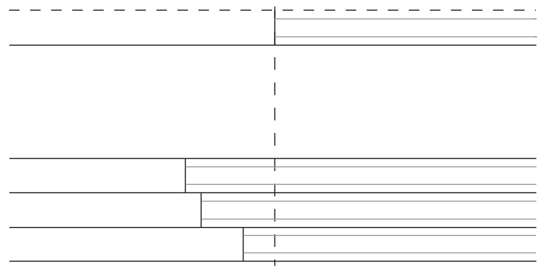

Figure 6. Shear lines at a cusp. The longer horizontal lines represent ideal geodesics ending at a cusp on the far left; the uppermost and lowermost horizontal lines are identified. The dotted vertical represents a closed horocycle in the undeformed hyperbolic structure and the shorter solid verticals form a closed horocycle after applying a shear $\sigma_{\mathcal{A}}$ for the horizontal lines. The shorter verticals are successively displaced by horizontal increments $-a_1, -a_2, \ldots, -a_p$. The shaded horizontals indicate segments along the upper and lower edges of each ideal geodesic, segments connecting the horocycles of the deformed structure.

Proof. The shear–length duality comes from Theorem 21. The first line of equations is established by finding the contribution to the change in the length $L(\mathcal{B})$ from the change in the determination of a closed horocycle at a cusp. We refer to the schematic Figure 6 for the basic geometry. To evaluate the change in length and ω, geodesic segments are labeled as described above. In the $\sigma_{\mathcal{A}}$ deformed hyperbolic structure, the distance between closed horocycles measured on the upper edge of an ideal geodesic agrees with the distance measured on the lower edge. We can compute the change in distance by averaging the change for the upper and lower edges. In Figure 6, the change in the first distance is $A_1/2$, while the change in the jth distance is $(A_j + A_{j-1})/2$. For the weighted length $L(\mathcal{B})$, the weight for the jth distance is b_j. The change in weighted distance for the given cusp is $\sum(A_j + A_{j-1})b_j/2$, as desired.

We next consider the Poisson bracket. The nondegenerate Kähler form ω_{WP} defines an isomorphism from tangent to cotangent spaces and a dual form $\widehat{\omega_{WP}}$. For the Hamiltonian length functions the Poisson bracket is defined as $\widehat{\omega_{WP}}(dL(\mathcal{A}), dL(\mathcal{B}))$. By duality the pairing is $4\,\omega_{WP}(\sigma_{\mathcal{A}}, \sigma_{\mathcal{B}})$. The final formula follows. □

There is a counterpart to Theorem 5 for the setting of shear coordinates.[1] First given an ideal triangulation Δ, Theorem 7 provides a bijection between balanced sum shears $\sum a_j \mathfrak{s}_j$ and $\mathcal{T}$ as follows, for $\mathfrak{s}_j$ denoting the shear deformations on the Δ edges. A *basepoint* $R_\Delta \in \mathcal{T}$ in Teichmüller space is determined by all shear coordinates vanishing. The surface R_Δ is constructed by gluing ideal triangles with medians on sides always matching. Each marked Riemann surface $R \in \mathcal{T}$ is given uniquely as a balanced sum shear $\sigma_R = \sum a_j(R)\mathfrak{s}_j$ of the surface R_Δ. We show the balanced sum length functions are linear in the shear coordinates as follows.

Corollary 23. *For a balanced sum $\mathcal{B} = \sum b_j \ell_{\alpha_j}$ of lengths of ideal geodesics of the triangulation Δ and a marked Riemann surface $R \in \mathcal{T}$, then*

$$L(\mathcal{B})(R) = \sum_{\text{cusps}} \omega(\{a_j(R)\}, \{b_j\}).$$

[1] Theorem 5 is formulated for left twists/shears while the present results are formulated for right twists/shears. The orientation difference explains the interchange of entries when comparing 2-forms.

Proof. First we observe that all balanced sum length functions vanish at R_Δ. Given a balanced sum $\mathcal{B} = \sum b_j \ell_{\alpha_j}$, consider the double sum of weights

$$\sum_{\text{cusps}} \sum_{\text{edges at cusp}} b_{(m,n)},$$

where the index m enumerates cusps and the index n enumerates half edges entering a cusp. The balanced sum condition is the vanishing of the inner sums. Each triangulation edge enters two cusps; the enumeration includes each triangulation edge twice. Thus, the sum of weights of a balanced sum vanishes. Since the shear coordinates of R_Δ vanish, we can introduce a decoration $\mathcal{H}$ for R_Δ such that all h-lengths have a common value. It follows that all λ-lengths have a common value λ_0. The length $L(\mathcal{B}) = \sum b_j 2\log\lambda_0$ of the balanced sum vanishes at R_Δ.

Given a surface R, the path of shears $\sigma_t = t\sum a_j(R)\mathfrak{s}_j$ connects the surfaces R_Δ and R. Corollary 22 provides that the t-derivative of $L(\mathcal{B})$ along the path has the constant value $\sum_{\text{cusps}} \omega(\{a_j(R)\}, \{b_j\})$. Integration in t provides the desired formula. □

By Theorem 7, the shear coordinates for the edges of an ideal triangulation provide a continuous immersion into Euclidean space. In particular, the shear coordinates for appropriate subsets of edges provide continuous coordinates for Teichmüller space. A procedure determining appropriate subsets of edges is given in the proof of Lemma 26 below. From Theorem 7, for a subset of shear coordinates without linear relations, the differentials of the coordinates are generically linearly independent. Furthermore from Corollary 22, for a subset of shear coordinates without linear relations there are sets of balanced sum length functions with constant full rank Poisson bracket pairing. It follows from the pointwise full rank pairing that the differentials of the shear coordinates in the subset are pointwise linearly independent on Teichmüller space. It also follows that the shear coordinates from the subset give a basis for the vector space of balanced sums of length functions.

In [Wol83, § 4], we found for surface fundamental group representations into $\mathrm{PSL}(2;\mathbb{R})$ that the Poisson bracket of trace functions is a sum of trace functions. The present result describes a simpler structure. By construction Thurston shears on a common set of ideal geodesics commute and accordingly the Poisson bracket of Hamiltonian potential length functions is constant.

We now express the 2-form ω in terms of h-lengths and use the formula to give the relation to Corollary 9.

Corollary 24. *For an ideal triangulation Δ, the pullback WP Kähler form is*

$$\widetilde{\omega_{WP}} = \sum_{\text{cusps}} \sum_{j=1}^{p} \widetilde{h}_j \wedge \widetilde{h}_{j+1},$$

where the first sum is over cusps, the second sum is over h-lengths at a cusp enumerated in counterclockwise cyclic order and $\widetilde{h}_* = d\log h_*$. For an ideal triangulation Δ, the pullback WP Kähler form is also given as

$$\widetilde{\omega_{WP}} = \frac{1}{2}\sum_{e\in\Delta} d\log\lambda_e \wedge d\sigma_e.$$

Proof. We begin with shear coordinates for $\mathcal{T}$ and the shear pairing $\omega_{WP}(\sigma_{\mathcal{A}}, \sigma_{\mathcal{B}})$ of Corollary 22 above. The coefficients $\{a_j\}, \{b_j\}$ are the evaluations of the differentials $\{d\sigma_e\}$ of the shear coordinates on the Thurston shears $\sigma_{\mathcal{A}}, \sigma_{\mathcal{B}}$. From (5) and Figure 1, the differential of a shear coordinate is $d\log h''/h'$ where h'' is the h-length clockwise from the edge and h' is the h-length

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counterclockwise from the edge. We now write the sum (7) at a cusp in terms of increments of h-lengths. We use the notation of formula (7). Introduce a decoration for the surface, write the shear coordinate increments in terms of h-length increments as $a_j = \widetilde{h}_{j-1} - \widetilde{h}_j$ and $b_j = \widetilde{g}_{j-1} - \widetilde{g}_j$, where $\widetilde{h}_*, \widetilde{g}_*$ are now the evaluations of the differential $d\log h_*$. The partial sums are $A_0 = 0$ and $A_k = \sum_{j=1}^k a_j = \widetilde{h}_p - \widetilde{h}_j$, where with the cyclic ordering $\widetilde{h}_0 = \widetilde{h}_p$ and by hypothesis $\sum_{j=1}^p \widetilde{h}_j = 0$. We find the contribution to ω from an individual increment $\widetilde{g}_k$ by considering

$$\begin{aligned}(A_k + A_{k-1})b_k &+ (A_{k+1} + A_k)b_{k+1} \\ &= (2\widetilde{h}_p - \widetilde{h}_k - \widetilde{h}_{k-1})(\widetilde{g}_{k-1} - \widetilde{g}_k) + (2\widetilde{h}_p - \widetilde{h}_{k+1} - \widetilde{h}_k)(\widetilde{g}_k - \widetilde{g}_{k+1}).\end{aligned}$$

The overall contribution is $(\widetilde{h}_{k-1} - \widetilde{h}_{k+1})\widetilde{g}_k$. We now have that

$$\begin{aligned}\omega &= \frac{1}{2}\sum_{k=1}^{p}(A_k + A_{k-1})b_k \\ &= \frac{1}{2}\sum_{k=1}^{p}\det\begin{pmatrix}\widetilde{h}_{k-1} & \widetilde{h}_k \\ \widetilde{g}_{k-1} & \widetilde{g}_k\end{pmatrix} = \sum_{k=1}^{p} d\log h_{k-1} \wedge d\log h_k\,(\sigma_{\mathcal{A}}, \sigma_{\mathcal{B}})\end{aligned}$$

and the first formula is established.

The second formula follows from Theorem 8 and formal considerations. From formula (5) we have that

$$d\log\lambda_e \wedge d\sigma_e = \widetilde{\lambda}_a \wedge \widetilde{\lambda}_e + \widetilde{\lambda}_e \wedge \widetilde{\lambda}_b + \widetilde{\lambda}_c \wedge \widetilde{\lambda}_e + \widetilde{\lambda}_e \wedge \widetilde{\lambda}_d,$$

where the ordered side pairs $(a, e), (e, b), (c, e)$ and (e, d) are in counterclockwise order relative to their containing triangles. The pairs are the side pairs of Figure 1 with one side a diagonal. Now given a pair of adjacent sides of the triangulation Δ, the pair occurs in two quadrilaterals with one of the sides being a diagonal. It follows that the sum of $d\log\lambda_e \wedge d\sigma_e$ over edges is twice the sum of Theorem 8. The second formula follows. ◻

An observation of Joergen Andersen provides a direct relation of the above to Corollary 9. The coupling equation $h_1h_2 = h_3h_4$ gives the 2-form equation $\widetilde{h}_1 \wedge \widetilde{h}_2 + \widetilde{h}_2 \wedge \widetilde{h}_3 + \widetilde{h}_3 \wedge \widetilde{h}_4 + \widetilde{h}_4 \wedge \widetilde{h}_1 = 0$ for $\widetilde{h}_* = d\log h_*$. The relation $\widetilde{h}_1 \wedge \widetilde{h}_2 + \widetilde{h}_3 \wedge \widetilde{h}_4 = \widetilde{h}_3 \wedge \widetilde{h}_2 + \widetilde{h}_1 \wedge \widetilde{h}_4$ follows. Beginning with Corollary 9 and referring to Figure 1, we observe the following. For an edge e of the triangulation, the wedge of h-lengths adjacent to e of the triangles adjacent to e can be replaced with the wedge of h-lengths for consecutive vertex sectors at the cusps at the ends of e. The replacement agrees with the orientations of the formulas. The replacement for each edge of the triangulation transforms the first adjacent by side formula to the second adjacent by vertex formula.

Example 25. The form ω for a once punctured torus.

A choice of three disjoint ideal geodesics decomposes a once punctured torus into two ideal triangles. The torus is described by edge identifying two ideal triangles to form a topological rectangle with diagonal γ, and then separately identifying the horizontal edges α and vertical edges β. The pattern of geodesics at the cusp is twofold α, γ, β. Consider the triples of balanced weights $\{a, b, -a-b\}$ and $\{c, d, -c-d\}$ for the sequence α, β and γ. For the geodesics enumerated according to the pattern at the cusp, the sequence of partial sums for the second set of weights is $A_0 = 0, A_1 = c, A_2 = -d$ and $A_3 = 0$. The sum (7) evaluates to $(ca + (c-d)(-a-b) + -db) = (ad - bc)$.

We now follow the discussion of Bonahon [Bon97b, Theorem 15] and Harer–Penner [PH92, § 2.1] for the dimension of the space of balanced sum coefficients.

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Lemma 26. *For a surface with cusps and a maximal configuration of disjoint ideal geodesics, the space of balanced sum coefficients has the same dimension as the Teichmüller space.*

Proof. Consider a configuration of ideal geodesics with weights as a graph with weighted edges. The graph is connected since ideal triangles fill in the configuration to form a connected surface. We will sequentially coalesce and remove edges, each time decreasing the number of vertices, to finally obtain a single vertex graph. For a surface with a single cusp no coalescing of edges is necessary. Otherwise, by connectedness, there is an ideal triangle with not all vertices at the same cusp. Begin with such a designated triangle. If only two vertices are at distinct cusps, then we begin by coalescing an edge connecting the distinct vertices. If all vertices are at distinct cusps then we begin by sequentially coalescing two edges of the triangle and the third edge will not be subsequently coalesced. We label the ends of edges as *incoming* or *outgoing* at coalesced vertices as follows. Label the ends of edges adjoining the first vertex as *incoming*. Coalesce the first designated edge, remove the weight and label the remaining ends of edges at the second vertex as *outgoing* for the coalesced vertex. At the coalesced vertex the weight condition is that the sum of incoming weights equals the sum of outgoing weights. To continue, take a path of edges to an uncoalesced vertex and coalesce the first edge to an uncoalesced vertex along the path. Label the new ends of edges at the coalesced vertex as the opposite type as for the initial segment of the coalesced edge. At the coalesced vertex the weight condition continues to be that the sum of incoming weights equals the sum of outgoing weights. Continue coalescing edges until only a single vertex remains. For a surface of genus g with n cusps, there are $6g-6+3n$ edges in a maximal configuration. A total of $n-1$ edges are coalesced and then $6g-6+2n+1$ edges remain. At least one edge of the initial designated triangle gives rise to an incoming–incoming edge of the final coalesced vertex. The single weight sum relation is a nontrivial condition for the weight on the incoming–incoming edge. The space of weights on the final graph has the expected dimension. □

8. The Fock shear coordinate algebra

Fock and Goncharov in their quantization of Teichmüller space introduced and worked with a Poisson algebra for the shear coordinate functions [FG07, FC99, FG06]. The quantization considerations begin with the Fock–Thurston theorem that for any ideal triangulation, the corresponding shear coordinates (without the vanishing sums about cusps condition) provide a real-analytic homeomorphism of the holed Teichmüller space to Euclidean space [Pen12, ch. 4, Theorem 4.4]. Fock proposed a Poisson structure by introducing a natural bivector, an exterior contravariant 2-tensor η and defining $\{f,g\}=\langle (df,dg),\eta\rangle$ for f,g smooth functions. A relationship to the WP Kähler form was also proposed. A bivector defines a Poisson structure with Jacobi identity provided its Schouten–Nijenhuis tensor vanishes.

Theorem 27 [FG07, FC99]. *For an ideal triangulation Δ and corresponding shear coordinates, the bivector*

$$\eta_\Delta=\sum_{\Delta}\frac{\partial}{\partial\sigma_a}\wedge\frac{\partial}{\partial\sigma_b}+\frac{\partial}{\partial\sigma_b}\wedge\frac{\partial}{\partial\sigma_c}+\frac{\partial}{\partial\sigma_c}\wedge\frac{\partial}{\partial\sigma_a}$$

is natural for the holed Teichmüller space, where the individual triangles have sides a,b and c in counterclockwise order.

Penner gave a topological description of the bracket of shear coordinates [Pen12, p. 81], a proof that the bivector is independent of triangulation and also determined the center of the algebra [Pen12, ch. 2]. For the topological description of the bracket, recall the definition of the *fat graph* dual to an ideal triangulation. To construct the fat graph G embedded in the surface, choose a vertex interior to each triangle and connect vertices by an edge when triangles are adjacent. The result is a trivalent graph with a cyclic ordering of edges at each vertex. The trivalent graph is a deformation retract of the surface.

Penner's topological description of the bracket is the following [Pen12, p. 81]. Consider an ideal triangulation Δ with dual fat graph spine G. If $a, b \in \Delta$ are distinct edges, then let ϵ_{ab} be the number of components of the complement of $\Delta \cup G$ whose frontier contains points of a and b, counted with a positive sign if a and b are consecutive in the counterclockwise order in the corresponding region, and with a negative sign if a and b are consecutive in the clockwise order.[2] Setting $\epsilon_{aa} = 0$ for each $a \in \Delta$, ϵ_{ab} takes the possible values $0, \pm 1, \pm 2$ and comprises a skew-symmetric matrix indexed by Δ. The quantity ϵ_{ab} is the count of oriented vertex sectors jointly bounded by a and b.

DEFINITION 28. The Fock shear coordinate algebra is defined by the bracket $\{\sigma_a, \sigma_b\} = \epsilon_{ab}$ for $a, b \in \Delta$.

From formula (5) and Figure 1, a shear coordinate is a balanced sum of length functions. For Riemann surfaces with cusps the WP Poisson bracket of sums of length functions is given in Corollary 22 in terms of weights and the form ω. We evaluate ω for quadrilaterals and find that the evaluation agrees with Penner's topological description of the count ϵ_{ab}.

THEOREM 29. *The Fock shear coordinate algebra is the WP Poisson algebra. The Fock shear coordinate bracket is given by the form ω.*

Proof. We begin with Corollary 22 providing that the Poisson bracket of the shear coordinates for edges e, f is $\{\sigma_e, \sigma_f\} = 2\sum_{\text{cusps}} \omega(\{a_j\}, \{b_j\})$, for $\{a_j\}, \{b_j\}$ the weights for the shears as sums of lengths of ideal geodesics. The matter is to evaluate the sum (7) for ω for the possible configurations. We first consider the case of the quadrilateral for the side e embedded in the surface and then describe necessary modifications for sides of the quadrilateral coinciding. The quadrilateral with weights for the edge e is given in Figure 7.

Referring to formula (7), the first calculation is for the partial sums A_j of edge weights. At a vertex, edges are enumerated for summation in the counterclockwise order with the *first* edge being the clockwise-most edge. Normalize the partial sums to be zero for the not listed edges preceding the first edge. The partial sums by vertex and in counterclockwise order are given in Table 1. The second calculation is for the sums $A_j + A_{j-1}$ of partial sums about vertices. The sums are given in Figure 7 by the numbers in square brackets; again sums vanish for edges not listed. Now we are ready to consider the configuration of the quadrilateral for the edge f and the sum of weights $\frac{1}{2}(A_j + A_{j-1})b_j$. The weights for f are again $0, \pm 1$ as in Figure 7. The edges e and f are necessarily distinct. First consider that f coincides with a boundary edge of the e quadrilateral. In this case the diagonal edge weight 0 for f is multiplied by the $[\pm 1]$ boundary edge weights for e and added to the ± 1 boundary edge f weight times $\frac{1}{2}$ the sum of the [2] and [2] diagonal weights for e. The result is ± 2 with the positive sign if f is counterclockwise from e. Now consider the case that the e and f quadrilaterals are either disjoint or intersect along a boundary edge. In the case of intersection along a boundary, the vanishing sum $[1] + [-1]$ of e

[2] We have reversed Penner's original sign convention given that his bivector has sides enumerated in a clockwise order, while Fock's bivector has sides enumerated in a counterclockwise order.

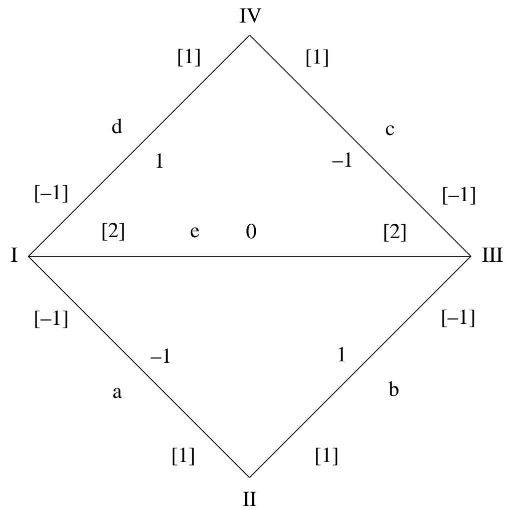

Figure 7. The quadrilateral for a triangulation edge e following formula (5). The quadrilateral sides are labeled by lowercase letters and vertices are labeled by Roman numerals. The edge weights $0, \pm 1$ refer to expressing the e shear coordinate as a balanced sum of edge lengths. The numbers in square brackets are the sums $A_j + A_{j-1}$.

Table 1. Partial weight sums in counterclockwise order about vertices.

Vertex	A_1	A_2	A_3
I	-1	-1	0
II	1	0	
III	-1	-1	0
IV	1	0	

boundary weights gives a vanishing overall contribution. This completes the calculation if the quadrilateral of e is embedded.

In general a pair of sides of the quadrilateral of e could coincide; we do not consider the special cases $(g, n) = (0, 3)$ or $(1, 1)$ where two side pairs coincide. A pair of adjacent sides could coincide by a $\frac{3}{4}$ rotation about the common vertex or opposite sides could coincide by a translation. When sides coincide the contribution to ω is found by adding the contributions from each of the relative configurations for the quadrilateral of f. The result will be $0, \pm 4$ according to adjacent or opposite sides coinciding and the e, f orientation. As already noted, we are using the adjustment [Wol07, § 5] to our formulas $2\,\omega_{WP}(\ , t_*) = d\ell_*$ in place of $\omega_{WP}(\ , t_*) = d\ell_*$ systematically used by Penner and Fock. The consequence is that our shear pairing is fourfold the Fock and Penner calculations. With this information, the shear pairing evaluations correspond and the proof is complete. □

9. The norm of a length gradient for a collar crossing geodesic

We continue to consider compact surfaces with crossing geodesics α and a reflection symmetry, see Figure 3. We consider surfaces R_ϵ obtained by doubling a surface with cusps with ideal geodesics α, and opening cusps to obtain short length core geodesics γ. We are interested in the

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products of the gradients $\operatorname{grad}\ell_\alpha$ and $\operatorname{grad}\ell_\gamma$. Theorem 3 and Lemma 20 can be combined to provide expansions for the pairings

$$\langle \operatorname{grad}\ell_\gamma, \operatorname{grad}\ell_\gamma\rangle = \frac{2}{\pi}\ell_\gamma + O(\ell_\gamma^4)$$

and

$$\langle \operatorname{grad}\ell_\alpha, \operatorname{grad}\ell_\gamma\rangle = \frac{-4}{\pi}(\#\alpha\cap\gamma) + O(\ell_\gamma^2).$$

Considerations of Chatauby convergence and sums of the differential Ω^2 from § 2 suggest the heuristic expansion $\operatorname{grad}\ell_\alpha = c_\alpha(\ell_\gamma)\operatorname{grad}\ell_\gamma + \overline{\psi(\ell_\gamma)}(ds^2)^{-1}$ with $\psi(\ell_\gamma)\in Q(R_\epsilon)$ converging to $\psi(0)\in Q(R\cup\bar{R})$. A simple argument provides that $\psi(0)$ is orthogonal to the limit of $\operatorname{grad}\ell_\gamma$. The above pairing formulas and heuristic then suggest an expansion

$$\langle \operatorname{grad}\ell_\alpha, \operatorname{grad}\ell_\alpha\rangle = \frac{8}{\pi\ell_\gamma}(\#\alpha\cap\gamma)^2 + O(1).$$

The divergence of the pairing corresponds to the geometry. The limit of $d\ell_\alpha$ is formally the differential of length of an ideal geodesic and is a holomorphic quadratic differential with double poles at cusps. The limit is not an element of $Q(R)$. Also the limiting infinitesimal deformation $\operatorname{grad}\ell_\alpha$ corresponds to opening cusps and has infinite WP norm.

We would like to now use the gradient pairing formula, Theorem 3, to find the WP pairing for balanced sums of gradients of lengths of ideal geodesics. The above considerations show that a pairing formula involves canceling divergences in ℓ_γ. The divergences appear directly in evaluating the formula. The crossing geodesic α is orthogonal to the collar core γ. Arcs along γ connect the intersection points with α. Each connecting arc provides a summand for the Theorem 3 evaluation. The connecting arcs along γ occur in families; a family consists of a simple arc and the additional arcs obtained by adjoining complete circuits of γ. With ℓ_γ tending to zero and the summand $R(\cosh\operatorname{dist})\approx 2\log 2/\operatorname{dist}$ for small distance, there is an immediate divergence. We consider the sequence of lengths as a partition for a Riemann sum and find the ℓ_γ-asymptotics of the sum.

The resulting formulas involve an elementary function, a reduced length for an ideal geodesic and a reduced connecting arcs sum formula.

Definition 30. For $0\leqslant a\leqslant 1$, define the function $\lambda(a) = a(1-a)/(2\sin\pi a)$ with value given by continuity at the interval endpoints. For a crossing geodesic α on a compact surface R with reflection symmetry, the reduced length $\operatorname{red}(\ell_\alpha)$ is the signed length of the segment connecting length 1 boundaries of the complement of collars about core geodesics. For an ideal geodesic α on a surface with cusps, the reduced length $\operatorname{red}(\ell_\alpha)$ is the signed length of the segment of the geodesic connecting the length 1 horocycles about the limiting cusps.

The function $\lambda(a)$ is symmetric about $a=\frac{1}{2}$ and satisfies $\frac{1}{8}\leqslant\lambda\leqslant 1/2\pi$. For a pair of points p,q on a circle, we write $\lambda(p,q)$ for the evaluation using the fractional part of the segment from p to q. For a hyperbolic surface without cone points the length 1 horocycles are embedded circles bounding disjoint cusp regions and $\operatorname{red}(\ell_\alpha)$ is nonnegative. For surfaces with cone points, the reduced length can be negative.

For crossing geodesics α,β on a surface with reflection symmetry or ideal geodesics α,β on a surface with cusps, we will write

$$\sum_{\alpha\,\mathrm{to}\,\beta}^{\mathrm{red}}\mathcal{R}$$

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for the reduced sum over homotopy classes rel the closed sets α, β of arcs connecting α to β, that are not homotopic to arcs along a core γ or along a horocycle and $\mathcal{R}$ the analytic quantity of Theorem 3. For the double of a surface with cusps, the symmetric homotopy classes are even with respect to the reflection; for this situation the sum is only over arcs with representatives on a chosen side of the surface. Each geodesic representative for the reduced sum intersects the thick subset of the surface and the reduced sum includes any intersection points of the ideal geodesics α and β. We assume the main result Theorem 32 and illustrate the approach with the example of a single core geodesic. The general formula depends on the pattern of crossing geodesics.

Example 31. Expansion of the WP gradient pairing for crossing geodesics α, β and a single core geodesic γ. For the core intersections $\alpha \cap \gamma = \{a_1, a_2\}, \beta \cap \gamma = \{b_1, b_2\}$ and a given positive constant c, then

$$\langle \operatorname{grad} \ell_\alpha, \operatorname{grad} \ell_\alpha \rangle = \frac{2}{\pi}\left(\frac{16}{\ell_\gamma} + \operatorname{red}(\ell_\alpha) + 4 + 2\sum_{(a_i,a_j)} \log \lambda(a_i, a_j)\right) + 2\sum_{\alpha \,\mathrm{to}\, \alpha}^{\mathrm{red}} \mathcal{R} + O(\ell_\gamma^{1-c})$$

and for $\alpha \neq \beta$

$$\langle \operatorname{grad} \ell_\alpha, \operatorname{grad} \ell_\beta \rangle = \frac{2}{\pi}\left(\frac{16}{\ell_\gamma} + 2\sum_{(a_i,b_j)} \log \lambda(a_i, b_j)\right) + 2\sum_{\alpha \,\mathrm{to}\, \beta}^{\mathrm{red}} \mathcal{R} + O(\ell_\gamma^{1-c}).$$

We are ready to consider that pairings of balanced sums on a surface with cusps are the limits of pairings of balanced sums on approximating symmetric compact surfaces. The balanced sum condition will serve to cancel the universal $16/\ell_\gamma$ leading divergence terms. To compare formulas note that a surface with cusps represents half of a compact surface. It is also important that remainder terms as in the example tend to zero with ℓ_γ.

We state the main result. For a surface with cusps, the sum over core geodesic intersections is replaced with a double sum. First, a sum over cusps and second, a sum over ordered pairs of ideal geodesic segments limiting to a cusp. Ideal geodesics are orthogonal to horocycles. The fractional part of a horocycle defined by a pair of ideal geodesics is independent of the choice of horocycle. The geometric invariant λ is evaluated by considering the intersections with any horocycle for the cusp. We present the formula for the case of a torsion-free cofinite group.

Theorem 32 (The ideal geodesic complex gradient pairing). *For a surface R with cusps and balanced sums $\mathcal{A} = \sum \mathfrak{a}_j \ell_{\alpha_j}, \mathcal{B} = \sum \mathfrak{b}_k \ell_{\beta_k}$ of ideal geodesic length functions, the WP pairing of gradients is*

$$\begin{aligned}&\langle \operatorname{grad} L(\mathcal{A}), \operatorname{grad} L(\mathcal{B}) \rangle \\ &\quad = \sum_{j,k} \mathfrak{a}_j \mathfrak{b}_k \left(\delta_{\alpha_j \beta_k} \frac{2}{\pi}(\operatorname{red}(\ell_{\alpha_j}) + 2) + \frac{2}{\pi} \sum_{\mathrm{cusps}} \sum_{\substack{\text{segments } \tilde{\alpha}_j, \tilde{\beta}_k \\ \text{limiting to the cusp}}} \log \lambda(\tilde{\alpha}_j, \tilde{\beta}_k) + \sum_{\alpha_j \,\mathrm{to}\, \beta_k}^{\mathrm{red}} \mathcal{R} \right).\end{aligned}$$

The first sum is over weights; the double sum is over ordered pairs of geodesic segments limiting to cusps. The final sum is over homotopy classes rel the closed sets α_j, β_k of arcs connecting α_j to β_k, arcs that are not homotopic into a cusp. For the homotopy class of an intersection $\alpha_j \cap \beta_k$, the function $\mathcal{R}$ is evaluated on $\cos\theta$, where θ is the intersection angle. Otherwise, the function $\mathcal{R}$ is evaluated on the hyperbolic cosine of the length of the unique minimal connecting geodesic segment. Twist–length duality and J an isometry provide that $4\langle \sigma_{\mathcal{A}}, \sigma_{\mathcal{B}} \rangle = \langle \operatorname{grad} \mathcal{A}, \operatorname{grad} \mathcal{B} \rangle$.

Proof. Begin the consideration with compact surfaces with reflection symmetries and balanced sums of geodesic-length functions converging to a surface with cusps formally doubled across the cusps. The approach is to show that the connecting arcs sums of Theorem 3 converge to the sum for the limiting surface. The individual summands are considered in terms of the geometry of the biorthogonal connecting geodesic segments.

Begin by normalizing the uniformizations to ensure Chabauty convergence of the deck transformation groups Γ. For the geodesic α, let $\tilde{\alpha}$ be a chosen geodesic line lift and $\langle A\rangle$ the cyclic group stabilizer. A fundamental interval on $\tilde{\alpha}$ is chosen; each left $\langle A\rangle$ orbit in $\Gamma\tilde{\alpha}$ and $\Gamma\tilde{\beta}$, $\tilde{\beta}$ a lift of β, has a unique biorthogonal geodesic connecting segment with one endpoint in the $\tilde{\alpha}$ fundamental interval. The considerations proceed in terms of the geometry of the second endpoint of the connecting segment. The finite number of terms corresponding to endpoints in a given compact set converge. The sums for families of connecting segments along the core geodesics provide universal divergences; the analysis is described in the next section. The remaining connecting segments have second endpoint outside a given compact set and the segments do not lie along core geodesics. The remaining segments necessarily intersect the lift of the thick subset. The remaining segments are treated according to whether the second endpoint lies in the lift of the thick or the thin subset. In the first case, the injectivity radius is bounded away from zero and the sum of such terms is uniformly bounded by applying the distant-sum method of [Wol10, ch. 8]. In the second case, the endpoint lies in the lift of a standard collar or cusp region. Hyperbolic geometry is used to show that the full sum over the stabilizing cyclic hyperbolic or parabolic group is bounded simply by the distance of the fundamental interval on $\tilde{\alpha}$ to the boundary of the region. The distant-sum and cyclic group bounds provide that the contributions from the complement of a large compact set is sufficiently small. The estimates for the various cases are combined to establish convergence of formulas.

We consider the connecting segments along a given core geodesic. We outline the approach and give a detailed treatment in the next section. The sum for a family of connecting arcs in a given direction along a core geodesic has the form

$$\sum_{n=0}^{\infty} S((a+n)\ell) \quad \text{for } S(t) = \cosh t \left(\log \frac{\cosh t+1}{\cosh t-1}\right) - 2$$

for ℓ the core length and $a\ell,\ a>0$, the distance between core intersection points. The function $S(t)$ has the initial expansion $S(t) \approx 2\log 2/t$ and for N approximately $\ell^{-1-\epsilon}, \epsilon>0$, we break up the sum

$$\begin{aligned}\sum_{n=0}^{\infty} & S((a+n)\ell) \\ &= \sum_{n=0}^{N} 2\log\frac{2}{(a+n)\ell} + \frac{1}{\ell}\sum_{n=0}^{N}\ell\left(S((a+n)\ell) - 2\log\frac{2}{(a+n)\ell}\right) + \sum_{n=N+1}^{\infty} S((a+n)\ell).\end{aligned}$$

For the first sum, we use additivity of the logarithm to obtain an expression in terms of $\log 2/\ell$ and $\log\Gamma(a+1)$ for the gamma function. Stirling's formula is then applied. For the second sum, half of the first and last sum terms are separated, then the trapezoid rule is applied to approximate the sum by an integral and an error term. The trapezoid rule provides an improved approximation in ℓ. The integral is calculated by an antiderivative. Finally the bound that $S(t)$ is $O(e^{-2t})$ for $t \geqslant t_0 > 0$, provides that the third sum is exponentially small; the consequence is

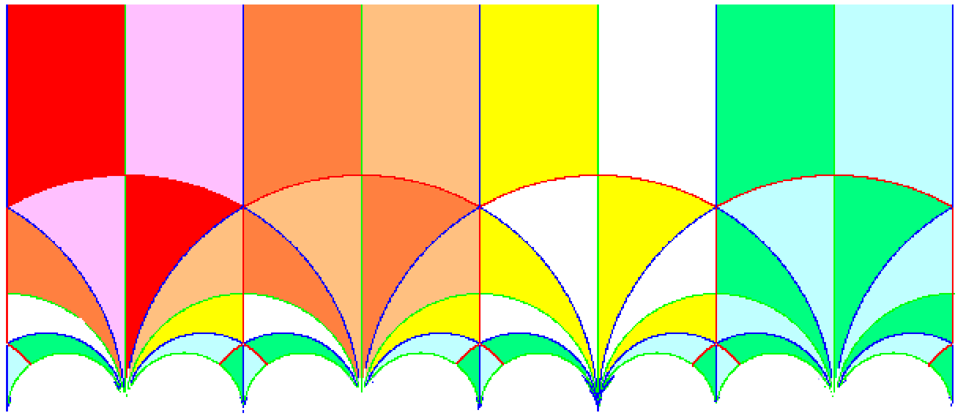

Figure 8. (Colour online) The Dedekind tessellation. Graphic created by and used with permission from Gerard Westendorp.

that for $a > 0$ the original full sum has the expansion

$$\frac{2}{\ell} + \log \frac{\Gamma(a+1)^2 \ell^{2a-1}}{2^{2a}\pi} + 2a - 1 + O(\ell^{1-\epsilon}).$$

The overall expansion for connecting arcs in the forward and reverse directions is obtained by combining the expansions for the values a and $1-a$. Identities for the gamma function are used to simplify the resulting formula and to obtain the function λ. As already noted, the ℓ-divergence is in the leading term. The balanced sum condition provides for the overall canceling of divergences in evaluating the gradient product. The proof is complete. □

Corollary 33. *For a balanced sum $\mathcal{A} = \sum \mathfrak{a}_j \ell_{\alpha_j}$ of ideal geodesic length functions and β a closed geodesic, the shear and twist derivative pairing is*

$$\sigma_{\mathcal{A}} \ell_\beta = -t_\beta L(\mathcal{A}) = \sum_j \mathfrak{a}_j \sum_{p \in \alpha_j \cap \beta} \cos\theta_p$$

for the intersection angles measured from α_j to β.

Example 34. A distance relation for the elliptic modular tessellation.

The Dedekind tessellation is the tiling of the upper half plane for the action of $\mathrm{PSL}(2;\mathbb{Z})$. The light, respectively dark, triangle tiles form a single $\mathrm{PSL}(2;\mathbb{Z})$ orbit. The reflection in the imaginary axis normalizes the group and interchanges the light and dark triangles. The tessellation vertices are fixed points of elements of the group action. There are two orbits for vertices. There are also two orbits for ideal lines. The first consists of the lines containing a single order-2 fixed point. The second consists of the lines sequentially containing an order-3, an order-2 and an order-3 fixed point. We refer to the types as 2-lines and 323-lines. We consider the lines with weights: $w = +1$ for 323-lines and $w = -1$ for 2-lines. The system of weighted lines is $\mathrm{PSL}(2;\mathbb{Z})$ invariant.

The formula of Theorem 32 provides a relation for the distances between lines for the Dedekind tessellation. For any choice $\tilde{a}$ of a 323-line and $\tilde{\alpha}$ of a 2-line, we have

$$\sum_{\text{ultraparallels to } \tilde{a}} w(\eta) R(d(\tilde{a},\eta)) - \sum_{\text{ultraparallels to } \tilde{\alpha}} w(\eta) R(d(\tilde{\alpha},\eta)) = \log \frac{3^6 \pi^4}{2^{26}}$$

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for $R(d) = u\log((u+1)/(u-1)) - 2$ and $u = \cosh d$. Ultraparallels are the tessellation lines at positive distance. Lines at zero distance are asymptotic.

We find the relation as an exercise in evaluating the formula of Theorem 32. We begin with the geometry of the tiling quotient. We work with the thrice-punctured sphere uniformized by the projectivized index 6 subgroup $P\Gamma(2) \subset \mathrm{PSL}(2;\mathbb{Z})$ of matrices congruent to the identity modulo 2. A fundamental domain for the torsion-free group $P\Gamma(2)$ is given by the 12 light and dark triangles adjacent to a given largest height nonvertical 323-line. The $P\Gamma(2)$ quotient is a tri-corner pillow with three 323-lines, labeled a, b, c and three 2-lines, labeled α, β, γ. The 2-lines separate the quotient into two ideal triangles. A 323-line enters a single cusp of the quotient, while a 2-line connects two distinct cusps. We evaluate the pairing product for the weighted balanced sum $\sigma = a + b + c - \alpha - \beta - \gamma$. The sum is $P\Gamma(2)$ invariant, thus $\operatorname{grad}\sigma \in Q(P\Gamma(2))$ by Theorem 21. The space of $P\Gamma(2)$ quadratic differentials is zero dimensional. The self-pairing of $\operatorname{grad}\sigma$ is zero.

We determine the contributions for terms on the right-hand side of the Theorem 32 formula. The evaluation corresponds to the formal expansion of the product $(a+b+c-\alpha-\beta-\gamma)^2$. The pairing is real and the initial factor $\pi/2$ can be moved to the left-hand side. We begin with the reduced length contribution. The $P\Gamma(2)$ cusps have width 2; the length 1 horocycle at infinity has height 2. For a vertical 2-line, half of the reduced length segment connects the height-2 horocycle to the order-2 fixed point at height 1. A 2-line has reduced length $2\log 2$. For a vertical 323-line, half of the reduced length segment connects the height-2 horocycle to the order-2 fixed point at height $\frac{1}{2}$. A 323-line has reduced length $4\log 2$. The reduced length contributing terms of the product are $a^2 + b^2 + c^2 + \alpha^2 + \beta^2 + \gamma^2$. The total first term reduced length contribution is $18\log 2 + 12$. We next consider the $\log\lambda$ contributions, which measure the geometry of the ideal geodesics limiting to cusps. There are two reflections stabilizing each cusp. The reflections stabilize the geodesics and provide that the intersections of the ideal geodesics with a horocycle are equally spaced and alternate by weights. The $\log\lambda$ contributing terms of the product are

$$\begin{aligned}a^2 + b^2 + c^2 + \alpha^2 + \beta^2 + \gamma^2 - 2a\beta - 2a\gamma\\ - 2b\alpha - 2b\gamma - 2c\alpha - 2c\beta + 2\alpha\beta + 2\alpha\gamma + 2\beta\gamma.\end{aligned}$$

By $\mathrm{PSL}(2;\mathbb{Z})$ symmetry, the evaluation is the same as for $3a^2 + 3\alpha^2 - 12a\beta + 6\alpha\beta$. The a^2 contribution is $2\log(\lambda(0)\lambda(\frac{1}{2}))$ given the two segments at a cusp; the α^2 contribution is $2\log\lambda(0)$ given the two limiting cusps; the $a\beta$ contribution is $2\log\lambda(\frac{1}{4})$ given the symmetry of λ and the $\alpha\beta$ contribution is $\log\lambda(\frac{1}{2})$. The evaluations are $\lambda(0) = 1/(2\pi)$, $\lambda(\frac{1}{4}) = 3\sqrt{2}/32$ and $\lambda(\frac{1}{2}) = \frac{1}{8}$. The total $\log\lambda$ contribution is

$$6\,\log\frac{1}{16\pi} + 6\,\log\frac{1}{2\pi} + -24\,\log\frac{3\sqrt{2}}{32} + 6\,\log\frac{1}{8}.$$

We next consider the contribution from ideal geodesics intersecting. The intersection product contributing terms are $2ab + 2ac + 2bc - 2a\alpha - 2b\beta - 2c\gamma$. The geodesic intersections ab, ac and bc are twofold. From the formula the total intersection contribution is

$$2\cdot 3\cdot R\left(\cos\frac{\pi}{3}\right) + 2\cdot 3\cdot R\left(\cos\frac{2\pi}{3}\right) - 2\cdot 3\cdot R\left(\cos\frac{\pi}{2}\right) = 6\log 3 - 12$$

as follows. The leading 2-factors are from the formal expansion of σ^2. The 3-factors are from the symmetry of the triples a, b, c and α, β, γ. The first and second terms correspond to the fact that distinct 323-lines intersect twice. The R-evaluations $R(\cos\pi/3) = (\log 3)/2 - 2$ and $R(\cos\pi/2) = -2$ are elementary. The final contribution of the right-hand side of the overall formula is the sum for the nontrivial connecting geodesics. We start with the formal expansion $\sigma^2 = a\sigma + b\sigma + c\sigma - \alpha\sigma - \beta\sigma - \gamma\sigma$. By $\mathrm{PSL}(2;\mathbb{Z})$ symmetry the evaluation is the same as for $3a\sigma - 3\alpha\sigma$. Connecting geodesics are enumerated by lifting to the universal cover. Given lifts $\tilde{a}$ and $\tilde{\alpha}$, the desired sums are obtained. The overall relation now follows. We note that the lines asymptotic to $\tilde{a}$ and $\tilde{\alpha}$ correspond to the limits of lines with connecting segments along core geodesics; the $\log\lambda$ terms account for the combined contribution of the asymptotic lines.

10. The geodesic circuit sum

We consider the contribution to the Theorem 3 sum corresponding to connecting geodesics given by circuits about a fixed closed geodesic. Such a circuit sum enters when the geodesics α and β are orthogonal to a common closed geodesic. The summands are evaluations of the function

$$S(t) = \cosh t\left(\log\frac{\cosh t + 1}{\cosh t - 1}\right) - 2.$$

The consideration is for the length parameter ℓ expansion of the infinite sum of circuits. The application to Theorem 32 requires an expansion with remainder term tending to zero for small ℓ. Simple analysis gives that the expansion begins with terms divergent in ℓ. We provide the expansion.

Theorem 35. *For a and ϵ positive, the circuit sum has the expansion*

$$\sum_{n=0}^{\infty} S((a+n)\ell) = \frac{2}{\ell} + \log\frac{\Gamma(a+1)^2\ell^{2a-1}}{2^{2a}\pi} + 2a - 1 + O(\ell^{1-\epsilon})$$

for the gamma function $\Gamma(z)$.

Corollary 36. *For ϵ positive, the circuit sum for $0 < a < 1$ has the expansion*

$$\sum_{n=-\infty}^{\infty} S((a+n)\ell) = \frac{4}{\ell} + 2\log\frac{\Gamma(a+1)\Gamma(2-a)}{2\pi} + O(\ell^{1-\epsilon})$$
$$= \frac{4}{\ell} + 2\log\frac{a(1-a)}{2\sin\pi a} + O(\ell^{1-\epsilon}),$$

and for $a = 1$ has the expansion

$$\sum_{n=1}^{\infty} S(n\ell) = \frac{2}{\ell} + \log\frac{\ell}{4\pi} + 1 + O(\ell^{1-\epsilon}).$$

Proof of Corollary. Since $S(t)$ is an even function the first sum can be rewritten as $\sum_{n=0}^{\infty} S((a+n)\ell) + S((1-a+n)\ell)$ and the theorem is applied. The gamma function identities $\Gamma(z+1) = z\Gamma(z)$ and $\Gamma(1-z)\Gamma(z)\sin\pi z = \pi$ are applied to obtain the desired expression. Finally, the case $a = 1$ is a direct application of the theorem. □

Proof of Theorem. We begin with properties of the summand $S(t)$. The summand has the small-t expansion $S(t) = 2\log 2/t - 2 + O(t^2\log t)$ and the large-t expansion $S(t) = O(e^{-2t})$. We also consider the function

$$F(t) = S(t) - 2\log\frac{2}{t}$$

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and write

$$F(t) = (\cosh t - 1)\left(\log \frac{\cosh t + 1}{\cosh t - 1}\right) + \log \frac{t^2(\cosh t + 1)}{4(\cosh t - 1)} - 2.$$

We note that for small t, since $\cosh t - 1$ is $O(t^2)$ and $t^2/(\cosh t - 1)$ is analytic it follows that $F(t)$ has second derivative bounded by $-\log t$ for small t. □

We are ready to begin the overall considerations and write the sum in the form of Riemann sums, adding in and subtracting out a $2\log 2/t$ contribution

$$\begin{aligned}\sum_{n=0}^{\infty} S((a+n)\ell) &= \sum_{n=0}^{N} 2\log \frac{2}{(a+n)\ell} \\ &\quad + \frac{1}{\ell}\sum_{n=0}^{N} \ell\left(S((a+n)\ell) - 2\log\frac{2}{(a+n)\ell}\right) \\ &\quad + \frac{1}{\ell}\sum_{n=N+1}^{\infty} \ell S((a+n)\ell) \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned} \tag{9}$$

We consider the right-hand sums in order. For the first sum we have

$$2\sum_{n=0}^{N} \log \frac{2}{(a+n)\ell} = 2(N+1)\log\frac{2}{\ell} + 2\sum_{n=0}^{N}\log\frac{1}{(a+n)}.$$

The right-hand sum is $-2\log\prod_{n=0}^{N}(a+n) = -2\log\Gamma(a+N+1)/\Gamma(a+1)$. We apply Stirling's formula $\log\Gamma(z) = \frac{1}{2}\log 2\pi/z + z(\log z - 1) + O(1/z)$ to find that

$$\begin{aligned}\mathrm{I} = {}& 2(N+1)\log\frac{2}{\ell} + 2\log\Gamma(a+1) - 2\left(a + N + \frac{1}{2}\right)\log(a+N+1) \\ & + 2(a+N+1) - \log 2\pi + O(N^{-1})\end{aligned}$$

and noting that $\log(a+N+1) = \log(a+N) + 1/(a+N) + O(N^{-2})$ gives the desired final expansion

$$\begin{aligned}\mathrm{I} = {}& 2(N+1)\log\frac{2}{\ell} + 2\log\Gamma(a+1) - 2(a+N)\log(a+N) \\ & - \log(a+N+1) + 2(a+N) - \log 2\pi + O(N^{-1}).\end{aligned} \tag{10}$$

For the second sum of (9) we refer to the trapezoid rule approximation for an integral. The approximation uses weights $\frac{1}{2}$ for the first and last sum terms and unit weights for the remaining terms. The error bound is in terms of the second derivative of $F(t)$ on the interval $[a\ell, (a+N)\ell]$ and the square of the partition size. The approximation gives the expansion

$$\begin{aligned}\mathrm{II} = {}& \frac{1}{\ell}\int_{a\ell}^{(a+N)\ell} F(t)\,dt + \frac{1}{2}(F(a\ell) + F((a+N)\ell)) \\ & + O(\ell\,|[a\ell,(a+N)\ell]|\max|F''|).\end{aligned}$$

We set $(a+N) = \ell^{-\epsilon}$ and consider terms in order from right to left. Given the small-t logarithmic bound for F'' the remainder is bounded as $O(\ell^{1-2\epsilon})$. Given the large-t exponential decay $S(t)$ and the small-t expansion of $S(t)$, then

$$F((a+N)\ell) = -2\log\frac{2}{(a+N)\ell} + O(e^{-\ell^{-\epsilon}}) \quad \text{and} \quad F(a\ell) = -2 + O(\ell^{2-\epsilon}).$$

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The next step is to include the contribution of sum III. The sum is replaced with the corresponding integral. Since the integrand is exponentially decreasing on the interval, the replacement remainder is exponentially small. The considerations combine to give the expansion

$$\mathrm{II} + \mathrm{III} = -\frac{2}{\ell}\int_{a\ell}^{(a+N)\ell} \log\frac{2}{t}\,dt + \frac{1}{\ell}\int_{a\ell}^{\infty} S(t)\,dt - 1 - \log\frac{2}{(a+N)\ell} + O(\ell^{1-2\epsilon}).$$

The first integrand has antiderivative $t\log 2/t + t$. The second integrand $S(t)$ has antiderivative

$$\sinh t\left(\log\frac{\cosh t + 1}{\cosh t - 1}\right),$$

which has the large-t expansion $2 + O(e^{-2t})$. We evaluate the integrals to find the contribution

$$\begin{aligned}\mathrm{II} + \mathrm{III} = &-2N\log 2 + 2(a+N)\log((a+N)\ell) - 2(a+N)\\ &- 2a\log a\ell + 2a + \frac{2}{\ell} - 2a\log\frac{2}{a\ell} - 1 - \log\frac{2}{(a+N)\ell} + O(\ell^{1-2\epsilon}).\end{aligned}$$

The next step is to combine with expansion (10) and note again that $(a+N)\ell = \ell^{-\epsilon}$ to find the desired final expansion

$$\mathrm{I} + \mathrm{II} + \mathrm{III} = \frac{2}{\ell} + \log\frac{\Gamma(a+1)^2\ell^{2a-1}}{2^{2a}\pi} + 2a - 1 + O(\ell^{1-\epsilon}).$$

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