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THE ONE-DIMENSIONAL GAS-LUBRICATED SLIDER BEARING

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Abstract

Under the appropriate physical hypotheses, the problem of determining the pressure distribution in a gas-filled slider bearing becomes a singular perturbation problem as Λ, the bearing number, tends to infinity. This paper extends the results of an earlier one by the author to consider the case where the film profile has jump discontinuities in slope at points interior to the bearing. Application of the methods of general singular perturbation theory establishes the appropriate existence–uniqueness results for this problem, and a means is devised by which uniformly valid asymptotic approximations to the pressure distribution may be obtained for large values of Λ.

1. Introduction

A standard problem in gas lubrication theory is that of determining the pressure distribution in a thin gas film flowing between two rigid nonparallel surfaces that are in constant relative motion. Of particular interest is the case where this flow occurs under compressible and isothermal conditions, while the bounding surfaces consist of the plane $z = 0$ and a surface $z = h(x, y)$ in the region $z > 0$ with motion occurring parallel to the x-axis. When this arrangement, usually termed a *slider bearing*, is such that h is independent of y and the bearing is of infinite extent in the y direction, and while leakage of gas in this direction may be neglected, we may write the problem for the pressure $p(x, \varepsilon)$ in the film as a nonlinear two-point boundary-value problem

$$\varepsilon(h(x)^3\, pp')' - (h(x)\, p)' = 0, \quad x \in (0, 1), \tag{1.1}$$

$$p(0, \varepsilon) = p(1, \varepsilon) = 1. \tag{1.2}$$

In the above, x, $p(x, \varepsilon)$ and $h(x)$ have all been appropriately scaled, so that ε is a positive dimensionless parameter, that is, in fact, the reciprocal of the *bearing number* for the flow, Λ, a parameter depending on the geometry of the bearing, the physical properties of the gas, and the relative motion of the surfaces.

Many writers have noted that, for small ε (and hence large Λ), the problem (1.1), (1.2) is a singular perturbation problem, in the sense that the reduced problem,

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obtained by formally setting $\varepsilon = 0$ in (1.1), has no solution that will satisfy both boundary conditions (1.2) for arbitrarily assigned film profiles $h(x)$. This feature has prompted the investigation of this problem for small values of ε by the so-called method of matched asymptotic expansions, to obtain an asymptotic representation for $p(x, \varepsilon)$ valid as $\varepsilon \to 0$. Under the assumption that $h(x)$ is at least continuously differentiable on $[0, 1]$, Di Prima [1] has applied this technique to obtain a first approximation $w(x, \varepsilon)$ that is such that, formally,

$$p(x, \varepsilon) - w(x, \varepsilon) = O(\varepsilon), \tag{1.3}$$

uniformly with respect to x on $[0, 1]$. Moreover, by making further assumptions about the differentiability of $h(x)$ on $[0, 1]$, he was able, in [2], to improve the (formal) degree of this approximation.

In a more recent article, Schmitt and Di Prima [6] have applied this type of analysis to the problem where $h(x)$ is only piecewise differentiable on $[0, 1]$, but is sufficiently differentiable on subintervals of $[0, 1]$. By applying appropriate juncture conditions at the points of discontinuity of $h'(x)$, they are able to establish the general form of a function that has analogous properties to that of $w(x, \varepsilon)$ described above.

While these results are quite useful, and have been applied to a number of situations, they have their foundation on heuristic arguments, so that questions relating to the existence and uniqueness of solutions of (1.1), (1.2), and to the validity of the relation (1.3) become pertinent. The first answers to such questions were provided by Steinmetz [9] who, in a penetrating study of (1.1), (1.2), applied a shooting technique to establish the existence of a solution $p(x, \varepsilon)$ for all $\varepsilon > 0$, that satisfies

$$0 < h_{\min}/h_{\max} \leqslant p(x, \varepsilon) \leqslant h_{\max}/h_{\min}, \tag{1.4}$$

with an auxiliary proof establishing uniqueness in the above class. Further, he was able to validate (1.3), as well as to provide a corresponding nonuniform estimate for $p'(x, \varepsilon) - w'(x, \varepsilon)$.

In a later analysis of this problem [8], and under similar hypotheses to those assumed by Steinmetz, the present author examined these questions afresh, with a view to developing a rigorously based construction for the solution $p(x, \varepsilon)$. By applying the Contraction Mapping Theorem, he was able to demonstrate, for sufficiently small ε, the existence of a unique solution $p(x, \varepsilon)$ that satisfied (1.3), together with a corresponding uniform estimate for $p'(x, \varepsilon) - w'(x, \varepsilon)$. Exploitation of Steinmetz's global uniqueness result extended this local uniqueness to the class (1.4). Moreover, it was demonstrated that $w(x, \varepsilon)$ was only the initial term in a uniformly valid asymptotic development for $p(x, \varepsilon)$, and an iterative process was constructed to generate this expansion.

Fundamental to both these investigations was the assumption that $h(x)$ was (at least) twice piecewise continuously differentiable on $[0, 1]$. While Steinmetz's existence–uniqueness proof did in fact require only that $h'(x)$ be piecewise continuous, his validation of (1.3) required that $h''(x)$ be piecewise continuous. Similarly, the methods of [8] required that $h''(x)$ be continuous on $[0, 1]$, although this condition was excessive, and could be relaxed to one comparable with that of Steinmetz.

Finally, a recent analysis of the problem by Habets [3] has established the existence of a solution $p(x, \varepsilon)$ by topological degree methods that impose quite weak restrictions on $h(x)$, while separate proofs establish the uniqueness of this solution, and its asymptotic structure on subintervals of $[0, 1]$ on which $h(x)$ has appropriately smooth behaviour. His existence proof and the construction of the solution are, however, quite separate processes.

Thus, there appear to be no rigorously based methods comparable to those of [8] that provide, in a single process, the desired existence, uniqueness and asymptotic structure results for the problem (1.1), (1.2) in the case where $h'(x)$ experiences a discontinuity at some point interior to $[0, 1]$. It is our aim in this paper to adapt the methods of [8] to provide just such results under these conditions. In doing this, we will make repeated use of the results of the above reference, so that some results will be presented without proof, or with a proof outline only. In such cases, the application of the appropriate result of [8] to the proof at hand should be obvious.

It will turn out to suit our purpose best to consider the slightly more general problem

$$\varepsilon(h(x)^3 pp')' - (h(x)p)' = 0, \quad x \in (x_0, x_1), \tag{1.5}$$

$$p(x_0, \varepsilon) = \alpha(\varepsilon), \tag{1.6}$$

$$p(x_1, \varepsilon) = \beta(\varepsilon), \tag{1.7}$$

where x_0, x_1, α and β are all positive, with x_0 and x_1 independent of ε, while $\alpha(\varepsilon)$ and $\beta(\varepsilon)$ are continuous in $\varepsilon \in (0, \varepsilon_0]$. We will denote the problem defined above by the symbol $P_\varepsilon([x_0, x_1], \alpha, \beta)$ to display its dependence on these quantities. Thus, the problem defined by (1.1), (1.2) becomes $P_\varepsilon([0, 1], 1, 1)$.

In terms of problems such as this, the procedure to be adopted when $h'(x)$ has a point of discontinuity at $a \in (0, 1)$ is reasonably straightforward. We set up the problems $P_\varepsilon([0, a], 1, \lambda)$ and $P_\varepsilon([a, 1], \lambda, 1)$, where λ is a value to be constructed and, by using the techniques of [8], construct λ and solutions to both problems that have continuously joining derivatives at $x = a$. This then furnishes our solution to the original problem $P_\varepsilon([0, 1], 1, 1)$. Because our method is an iterative one, we obtain a construction that provides us with successively closer approximations to this solution, as well as giving us insight into the structure of these approximations. These results may be compared with those obtained by Schmitt and Di Prima [6].

In the following sections we will consider solutions $p(x, \varepsilon)$ of $P_\varepsilon([x_0, x_1], \alpha, \beta)$ that are positive on $[x_0, x_1]$, and so will find it convenient to expand (1.5) in the form

$$N_\varepsilon p \equiv \varepsilon p'' + \varepsilon p'^2 p^{-1} + 3\varepsilon h' h^{-1} p' - h' h^{-3} - h^{-2} p' p^{-1} = 0, \quad x \in (x_0, x_1), \tag{1.8}$$

$$p(x_0, \varepsilon) = \alpha(\varepsilon), \tag{1.9}$$

$$p(x_1, \varepsilon) = \beta(\varepsilon), \tag{1.10}$$

and refer to this as $P_\varepsilon([x_0, x_1], \alpha, \beta)$ also.

2. Results for smooth profiles

When the film profile $h(x)$ is twice continuously differentiable on the interval $[x_0, x_1]$, and the ratio

$$\sigma = h(x_0)/h(x_1) \tag{2.1}$$

satisfies a condition $\sigma \geqslant \sigma^*$, for an appropriately defined $\sigma^* \in (0, 1)$, we may prove, by an elementary adaptation of the methods of [8] that, for small enough values of ε, the problem $P_\varepsilon([x_0, x_1], \alpha, \beta)$ has a unique solution $p(x, \varepsilon)$ that satisfies the estimate

$$\| p(x, \varepsilon) - w(x, \varepsilon) \|_\kappa = O(\varepsilon), \tag{2.2}$$

for a suitably defined norm $\| . \|_\kappa$ and choices of κ and the function $w(x, \varepsilon)$.

The norm $\| . \|_\kappa$ is defined, for each $\kappa > 0$, by

$$\| \phi \|_\kappa = \sup_{[x_0, x_1]} | \phi(x, \varepsilon) | + \sup_{[x_0, x_1]} | \phi'(x, \varepsilon)/\rho(x, \varepsilon) |, \tag{2.3}$$

where ϕ is continuously differentiable with respect to x on $[x_0, x_1]$, while

$$\rho(x, \varepsilon) = 1 + \varepsilon^{-1} \exp\{-\kappa(x_1 - x)/\varepsilon\}. \tag{2.4}$$

A suitable function $w(x, \varepsilon)$ may be constructed by the methods of [1], and is given explicitly by

$$w(x, \varepsilon) = \alpha(\varepsilon)\, h(x_0)/h(x) + P((x_1 - x)/\varepsilon, \beta) - \alpha(\varepsilon)\, \sigma, \tag{2.5}$$

where $P(\tau, \beta)$ is a function defined and twice continuously differentiable with respect to $\tau \in [0, \infty)$ for each $\beta > 0$, that satisfies $P(0, \beta) = \beta$, and is defined by

$$P(\tau, \beta) - \beta + \alpha\sigma \log |(P - \alpha\sigma)/(\beta - \alpha\sigma)| = -h(x_1)^{-2}\tau. \tag{2.6}$$

Various properties of P may be established (see [9]), but the two below will suffice for our present purpose:

$$\min\{\alpha\sigma, \beta\} \leqslant P(\tau, \beta) \leqslant \max\{\alpha\sigma, \beta\} \tag{2.7}$$

and

$$\alpha\sigma - P = (\alpha\sigma - \beta) \exp\{(\beta - P)/\alpha\sigma\} \exp\{-\tau/\alpha\sigma h(x_1)^2\}. \tag{2.8}$$

These together show that the two last terms in (2.5) comprise a boundary layer correction that is exponentially small as $\varepsilon \to 0$ throughout $[x_0, x_1]$ except for the region $x - x_1 = O(\varepsilon)$, where it is significant. Moreover, the derivatives of this correction become unbounded in this region as $\varepsilon \to 0$. The first term of (2.5) is obviously the solution of the reduced problem $P_0([x_0, x_1], \alpha, \beta)$ that satisfies the boundary condition at $x = 0$.

The solution $p(x, \varepsilon)$ referred to above may be expressed in the form

$$p(x,\varepsilon) = w(x,\varepsilon) + u(x,\varepsilon), \tag{2.9}$$

where $u(x, \varepsilon)$ is the solution of the simultaneous integral equations

$$u(x,\varepsilon) = g(x,\varepsilon) + (Hf)(x,\varepsilon) + (HR[u,u'])(x,\varepsilon) \tag{2.10}$$

and

$$u'(x,\varepsilon) = g'(x,\varepsilon) + (Hf)'(x,\varepsilon) + (HR[u,u'])'(x,\varepsilon), \tag{2.11}$$

with primes denoting derivatives taken with respect to x. In the above, H is an integral operator defined by

$$(H\phi)(x,\varepsilon) = \int_{x_0}^{x_1} h(x,s,\varepsilon)\,\phi(s,\varepsilon)\,ds, \tag{2.12}$$

with $h(x, s, \varepsilon)$ the Green's function for the linear differential operator

$$L_\varepsilon u \equiv \varepsilon u'' - a(x,\varepsilon)\,u' + b(x,\varepsilon)\,u, \tag{2.13}$$

where $a(x, \varepsilon)$ and $b(x, \varepsilon)$ are given by

$$a(x,\varepsilon) = h^{-2} w^{-1} - 3\varepsilon h' h^{-1} - 2\varepsilon w' w^{-1} \tag{2.14}$$

and

$$b(x,\varepsilon) = -\varepsilon w'^2 w^{-2} + w' w^{-2} h^{-2}, \tag{2.15}$$

respectively. The function $g(x, \varepsilon)$ is known, while $f(x, \varepsilon)$ and $R[u, u']$ are given by

$$f(x,\varepsilon) = -N_\varepsilon w \tag{2.16}$$

and

$$\begin{aligned} R[u,u'] = {} & -w^{-2} h^{-2} uu' - \varepsilon(w^{-1} u'^2 - w^{-2} uu'^2 - 2w' w^{-2} uu') \\ & + w^{-2}(u+w)^{-1}(h^{-2}(u'+w') - \varepsilon(u'+w')^2)\,u^2, \end{aligned} \tag{2.17}$$

respectively.

There are a number of properties of the quantities defined above that we will find to be of considerable use in subsequent sections. These have already been exploited in [8] to establish results for the problem $P_\varepsilon([0, 1], 1, 1)$ but we briefly review them and some extensions of them here. They may be stated by means of two fundamental lemmas.

LEMMA 1. *There exist positive constants κ_0, σ^* and ε_0, all independent of ε, but dependent upon α and β, with $\sigma^* \in (0,1)$, such that, for all $\kappa \in (0,\kappa_0)$, $\sigma \geqslant \sigma^*$ and $\varepsilon \in (0,\varepsilon_0]$, the following estimates apply:*

$$\|g\|_\kappa = O(\exp(-(x_1-x_0)/\varepsilon\alpha h(x_0)\,h(x_1))), \tag{2.18}$$

$$\|Hf\|_\kappa = O(\varepsilon), \tag{2.19}$$

$$\|HR[u,u']\|_\kappa = O(1)\,\|u\|_\kappa^2, \tag{2.20}$$

and

$$\|HR[u_1,u_1']-HR[u_2,u_2']\|_\kappa = O(1)\max\{\|u_1\|_\kappa, \|u_2\|_\kappa\}\,\|u_1-u_2\|_\kappa, \tag{2.21}$$

for arbitrary continuously differentiable functions $u(x,\varepsilon)$. These estimates hold uniformly with respect to α and β for values of these parameters lying in any closed, bounded interval of positive values.

PROOF. This follows the same lines as that for the corresponding results obtained by the author in [8]. Basically, it involves the specific properties of the kernel $h(x,s,\varepsilon)$, and the estimation of integrals involving h, $h\rho$, h_x and $h_x\rho$. The details may be obtained from an examination of Section 5 of [8], where the case $\alpha=\beta=1$ is considered, and they are omitted from the present discussion.

REMARK. The constants κ_0 and σ^* may be written down explicitly, and are defined by

$$\kappa_0 = \min\{\min_x w^{-1}h^{-2}, (\alpha h(x_0)\,h(x_1))^{-1}\} \tag{2.22}$$

and

$$\sigma^* > 2\beta\alpha^{-1}[1+\surd(1+4\alpha^{-1}h(x_1)/h_{\max})]^{-1}, \tag{2.23}$$

where $h_{\max}$ is the maximum value of $h(x)$ on $[x_0,x_1]$. In most bearings of physical interest, $\sigma>1$, so that (2.23) is satisfied automatically while, when $\sigma<1$, this prescribes strict limits on the bearing geometry for the above results to hold. It should be noted that while κ_0 and σ^* *do* depend on α and β, they do not do so critically when these quantities are bounded above and below by positive limits, so that κ_0 and σ^* *may* be chosen to be independent of these parameters under such conditions.

It is clear that the estimates of Lemma 1 are sufficient for the Contraction Mapping Theorem to be applied to (2.10), (2.11) to deduce the existence of a solution $p(x,\varepsilon)$ satisfying (2.2). This is the procedure adopted in [8]. For our present purposes we require a little more, and thus seek estimates on the variation of g and Hf with respect to the parameters α and β. While the existence result above would imply continuity in these, we are able to go further, and deduce Lipschitz continuity in terms of the norm $\|\cdot\|_\kappa$. This is the result of Lemma 2 below.

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LEMMA 2. *There exist positive constants κ_0, σ^* and ε_0, independent of ε, but dependent upon α and β, with $\sigma^* \in (0,1)$* such that, for all $\kappa \in (0, \kappa_0)$, $\sigma \geqslant \sigma^*$ *and* $\varepsilon \in (0, \varepsilon_0]$,

$$\begin{aligned} &\| g(x,\varepsilon,\alpha_1,\beta_1) - g(x,\varepsilon,\alpha_2,\beta_2) \|_\kappa \\ &\qquad = O(\exp(-(x_1 - x_0)/\varepsilon\bar{\alpha}h(x_0)\,h(x_1)\{|\alpha_1 - \alpha_2| + |\beta_1 - \beta_2|\} \end{aligned} \tag{2.24}$$

and

$$\|(Hf)(x,\varepsilon,\alpha_1,\beta_1) - (Hf)(x,\varepsilon,\alpha_2,\beta_2)\|_\kappa = O(\varepsilon)\{|\alpha_1 - \alpha_2| + |\beta_1 - \beta_2|\}, \tag{2.25}$$

uniformly with respect to $\alpha_1, \beta_1, \alpha_2, \beta_2$ lying in a closed bounded interval of positive values, and with

$$\bar{\alpha} = \max\{\alpha_1, \alpha_2\}. \tag{2.26}$$

PROOF. We note from [8] that $g(x,\varepsilon,\alpha,\beta)$ takes the form

$$g(x,\varepsilon) = [\alpha\sigma - P((x_1 - x_0)/\varepsilon, \beta)]\,\xi_0(x,\varepsilon), \tag{2.27}$$

where ξ_0 is a function such that

$$\|\xi_0\|_\kappa = O(1) \quad \text{as } \varepsilon \to 0, \tag{2.28}$$

uniformly with respect to α and β for bounded positive values of these quantities, and for a suitable (positive) choice of κ. The result (2.24) now follows on application of the result (2.8) to (2.27).

The second result (2.25) follows from an adaptation of the proof of Lemma 3.2 of [8] (essentially a proof of (2.19)) to make due allowance for the dependence on α and β of quantities occurring in $N_\varepsilon w$, and application of (2.8).

REMARKS. 1. We have, for convenience, suppressed the dependence on α and β of the H's in (2.25), although this is obviously true. However, for α and β restricted as in the hypotheses, the properties of these operators remain unchanged and so we see no objection to deleting this dependence.

2. It is also evident from the lemma above that κ_0 and σ^* depend on both α_1, β_1 and α_2, β_2, and thus we would select κ_0 and σ^* from two sets of criteria analogous to (2.22) and (2.23). As it will turn out, we will make most use of this under circumstances where $\alpha_1 - \alpha_2 = O(\varepsilon)$ and $\beta_1 - \beta_2 = O(\varepsilon)$, so that these two criteria become one in the limit as $\varepsilon \to 0$.

3. One situation that will be of considerable significance occurs when $\alpha\sigma = \beta + O(\varepsilon)$, which corresponds to the case $\sigma = 1 + O(\varepsilon)$ in the case where $\alpha = \beta = 1$. When this is so, it is clear that

$$w(x,\varepsilon) = \alpha(\varepsilon)\,h(x_0)/h(x) + O(\varepsilon), \tag{2.29}$$

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that is, $w(x, \varepsilon)$ is essentially the "outer solution" $\alpha(\varepsilon) h(x_0)/h(x)$. Further, $w'(x, \varepsilon)$ is not unbounded as $\varepsilon \to 0$ anywhere on $[x_0, x_1]$, this phenomenon only occurring in the derivative $w''(x, \varepsilon)$. Even then, $\varepsilon w''(x, \varepsilon)$ is bounded as $\varepsilon \to 0$. The most significant result of this is that

$$(Hf)'(x_1, \varepsilon) = O(1), \tag{2.30}$$

which may be established by the methods used in Lemma 1, on noting that

$$\begin{aligned} |N_\varepsilon w| \leqslant k_1\{\varepsilon + |\alpha\sigma - \beta| \exp(-(x_1 - x)/\varepsilon\alpha h(x_0) h(x_1))\} \\ + k_2(x_1 - x)\varepsilon^{-1} |\alpha\sigma - \beta| \exp(-(x_1 - x)/\varepsilon\alpha h(x_0) h(x_1)) \end{aligned} \tag{2.31}$$

for positive constants k_1 and k_2 independent of ε. The estimate (2.30) is also readily shown to be uniform with respect to positive bounded α and β, a fact that is significant in the next section.

4. While our uniqueness result for the solution $p(x, \varepsilon)$ extends only to the class (2.2), we may apply Steinmetz' global uniqueness result, adapted to the problem at hand, to show that $p(x, \varepsilon)$ is, in fact, unique in the class

$$\frac{h_{\min}}{h_{\max}} \min\{\alpha, \beta\} \leqslant p(x, \varepsilon) \leqslant \frac{h_{\max}}{h_{\min}} \max\{\alpha, \beta\}, \tag{2.32}$$

where $h_{\min}$, $h_{\max}$ are the minimum and maximum values of $h(x)$ on $[x_0, x_1]$.

5. We also note that, although we have assumed $h''(x)$ to be continuous, the proof of existence of the solution $p(x,\varepsilon)$ as demonstrated in [8] requires only that $h''(x)$ be bounded and integrable, so that we could relax our hypotheses regarding this function to its being piecewise continuous on $[x_0, x_1]$.

6. It is important to observe that the derivatives p' and w' may not be close as $\varepsilon \to 0$ uniformly on $[x_0, x_1]$, but may differ at $x = x_1$ by a quantity that is $O(1)$ as $\varepsilon \to 0$. For the particular case $\alpha\sigma = \beta$, we may apply the arguments of [9], Corollary 3.1, to show that

$$u'(x_1, \varepsilon) = \alpha h(x_0)^{-1} \sigma^3 \{h'(x_1) - h'(x_0)\} + O(\varepsilon), \tag{2.33}$$

which does not vanish as $\varepsilon \to 0$ unless $h'(x_1) = h'(x_0)$. This would seem to indicate that, in general circumstances, (2.2) is the best estimate obtainable.

3. Piecewise smooth profiles

We now apply the results of the preceding section to the problem of establishing existence, uniqueness and asymptoticity properties for the problem $P_\varepsilon([0, 1], \alpha, \beta)$ when $h(x)$, the film profile, exhibits a single finite jump discontinuity in slope at $x = a$, for some $0 < a < 1$. We will find it convenient to define this jump in slope at

$x = a$ as $\Delta_a h'(x)$, where

$$\Delta_a h'(x) = h'(a-0) - h'(a+0), \tag{3.1}$$

so that $\Delta_a h'(x) \neq \infty$ for all $\varepsilon \in (0, \varepsilon_0]$. We will further assume that $h''(x)$ is sufficiently well behaved on the intervals $[0, a]$ and $[a, 1]$ for the theory of the previous section to apply. In general, this will amount to a requirement that $h''(x)$ be piecewise continuous on these subintervals, although we might expect that $h''(x)$ be continuous as a more general rule.

We will denote quantities relevant to the subintervals $[0, a]$ and $[a, 1]$ by the subscripts 1 and 2, respectively. With this notation, the results of Section 2 tell us that, for any appropriate $\lambda(\varepsilon)$, the problems $P_\varepsilon([0, \mathrm{a}], \alpha, \lambda)$ and $P_\varepsilon([a, 1], \lambda, \beta)$ have unique solutions $p_1(x, \varepsilon)$ and $p_2(x, \varepsilon)$, respectively. Moreover, if we can demonstrate that there exists a choice of λ such that $p_1'(a, \varepsilon) = p_2'(a, \varepsilon)$, where the derivatives are to be interpreted in the one-sided sense, we will have demonstrated the existence of a function $p(x, \varepsilon)$ that is continuously differentiable on $[0, 1]$, with piecewise continuous second derivative there : in fact, a solution of the problem $P_\varepsilon([0, 1], \alpha, \beta)$.

The considerations above provide us with the motivation for the methods to be applied in this section, and we obtain the desired result by constructing $\lambda(\varepsilon)$ explicitly.

We begin by noting that, corresponding to p_1 and p_2 as defined above, there correspond functions $w_1(x, \varepsilon)$ and $w_2(x, \varepsilon)$, defined by

$$w_1(x, \varepsilon) = \alpha(\varepsilon) h(0)/h(x) + P_1((a-x)/\varepsilon, \lambda) - \alpha(\varepsilon)\sigma_1, \quad x \in [0, a], \tag{3.2}$$

and

$$w_2(x, \varepsilon) = \lambda(\varepsilon) h(a)/h(x) + P_2((1-x)/\varepsilon, \beta) - \lambda(\varepsilon)\sigma_2, \quad x \in [a, 1], \tag{3.3}$$

respectively. Here, P_1 and P_2 are functions defined by a relation of the form of (2.6), with $\alpha = \alpha$, $\sigma = \sigma_1$ and $h(x_1) = h(a)$ in the case of P_1, and $\alpha = \lambda$, $\sigma = \sigma_2$ and $h(x_1) = h(1)$ in the case of P_2. The parameters σ_1 and σ_2 are defined by the ratios $h(0)/h(a)$ and $h(a)/h(1)$ respectively.

The results of Section 2 prompt us to seek functions $u_1(x, \varepsilon)$ and $u_2(x, \varepsilon)$ defined by

$$p_1(x, \varepsilon) = w_1(x, \varepsilon) + u_1(x, \varepsilon) \tag{3.4}$$

and

$$p_2(x, \varepsilon) = w_2(x, \varepsilon) + u_2(x, \varepsilon), \tag{3.5}$$

that are small in some sense. We define norms for these functions by

$$\| u_1 \|_{\kappa_1} = \sup_{[0, a]} | u_1(x, \varepsilon) | + \sup_{[0, a]} | u_1'(x, \varepsilon)/\rho_1(x, \varepsilon) | \tag{3.6}$$

and

$$\| u_2 \|_{\kappa_2} = \sup_{[a, 1]} | u_2(x, \varepsilon) | + \sup_{[a, 1]} | u_2'(x, \varepsilon)/\rho_2(x, \varepsilon) |, \tag{3.7}$$

where $\kappa_1, \kappa_2 > 0$, and the functions ρ_1 and ρ_2 are defined by

$$\rho_1(x, \varepsilon) = 1 + \varepsilon^{-1} \exp\{-\kappa_1(a-x)/\varepsilon\} \tag{3.8}$$

and

$$\rho_2(x, \varepsilon) = 1 + \varepsilon^{-1} \exp\{-\kappa_2(1-x)/\varepsilon\}, \tag{3.9}$$

respectively; we thus consider functions u_1, u_2 that are small in terms of these norms.

For given positive $\lambda(\varepsilon)$, we may apply the theory of Section 2 to this problem, and write down integral equations for u_1 and u_2 as follows:

$$u_1(x, \varepsilon) = g_1(x, \varepsilon) + (H_1 f_1)(x, \varepsilon) + (H_1 R_1[u_1, u_1'])(x, \varepsilon), \tag{3.10}$$

$$u_1'(x, \varepsilon) = g_1'(x, \varepsilon) + (H_1 f_1)'(x, \varepsilon) + (H_1 R_1[u_1, u_1'])'(x, \varepsilon) \tag{3.11}$$

and

$$u_2(x, \varepsilon) = g_2(x, \varepsilon) + (H_2 f_2)(x, \varepsilon) + (H_2 R_2[u_2, u_2'])(x, \varepsilon), \tag{3.12}$$

$$u_2'(x, \varepsilon) = g_2'(x, \varepsilon) + (H_2 f_2)'(x, \varepsilon) + (H_2 R_2[u_2, u_2'])'(x, \varepsilon), \tag{3.13}$$

respectively. In the above, g_1 and g_2 are functions analogous to the g of Section 2, while H_1 and H_2 are integral operators defined over $[0, a]$ and $[a, 1]$ in an analogous fashion to (2.12); R_1 and R_2 are defined by (2.17), with w_1 and w_2 replacing w in their respective expressions. Finally, we have

$$f_1 = -N_\varepsilon w_1 \quad \text{and} \quad f_2 = -N_\varepsilon w_2, \tag{3.14}$$

where the operator N_ε is understood to be acting on functions defined on $[0, a]$ and $[a, 1]$ in the appropriate case.

It is clear that the expressions g_i, $H_i f_i$ and $H_i R_i$, for $i = 1, 2$, satisfy the estimates given in Lemmas 1 and 2, for positive bounded values of α, λ and β. However, it must be noted that σ_1 and σ_2, as well as κ_1 and κ_2, are restricted in a like manner to that of σ and κ of Section 2. Thus the above equations hold (together with the relevant estimates on the integral operators involved) when $\sigma_1 \geqslant \sigma_1^*$, $\sigma_2 \geqslant \sigma_2^*$ and $\kappa_1 \in (0, \kappa_1^*)$, $\kappa_2 \in (0, \kappa_2^*)$, where σ_1^*, σ_2^*, κ_1^*, and κ_2^* are appropriately defined. We will return to this point later.

By our choice of the value $p_1(a, \varepsilon) = p_2(a, \varepsilon) = \lambda(\varepsilon)$, we have ensured that p_1 and p_2 join continuously at $x = a$. (Note that this does not necessarily mean that u_1 and u_2 do so.) However, to go further and ensure that the derivatives p_1' and p_2' are joined continuously at $x = a$, we must impose the condition

$$u_2'(a, \varepsilon) - u_1'(a, \varepsilon) = w_1'(a, \varepsilon) - w_2'(a, \varepsilon), \tag{3.15}$$

where derivatives at $x = a$ are to be interpreted in the one-sided sense.

Noting (3.2) and (3.3), we obtain, by an elementary calculation,

$$\begin{aligned} w_2'(a, \varepsilon) - w_1'(a, \varepsilon) = & -\lambda h(a)\, h'(a+0)/h(a)^2 + a h(0)\, h'(a-0)/h(a)^2 \\ & + \varepsilon^{-1} P_1'(0, \lambda) - \varepsilon^{-1} P_2'((1-a)/\varepsilon, \beta), \end{aligned} \tag{3.16}$$

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and thus (3.15) becomes, on noting the form for the derivative $P_1'(0,\lambda)$,

$$\begin{aligned}\{\lambda h(a)\,h'(a+0)-\alpha h(0)\,h'(a-0)\}/h(a)^2+\varepsilon^{-1}P_2'((1-a)/\varepsilon,\beta)\\ -\varepsilon^{-1}(\alpha\sigma_1-\lambda)/h(a)^2\,\lambda\\ =g_2'(a,\varepsilon)-g_1'(a,\varepsilon)+(H_2\,f_2)'(a,\varepsilon)-(H_1\,f_1)'(a,\varepsilon)\\ +(H_2\,R_2[u_2,u_2'])'(a,\varepsilon)-(H_1\,R_1[u_1,u_1'])'(a,\varepsilon).\end{aligned}$$

Rearranging this, we obtain

$$\begin{aligned}\lambda-\alpha\sigma_1 &= -\varepsilon\lambda[\lambda h(a)\,h'(a+0)-\alpha h(0)\,h'(a-0)]-\lambda h(a)^2\,P_2'((1-a)/\varepsilon,\beta)\\ &\quad+\varepsilon h(a)^2\,\lambda[g_2'(a,\varepsilon)-g_1'(a,\varepsilon)+(H_2\,f_2)'(a,\varepsilon)-(H_1\,f_1)'(a,\varepsilon)\\ &\quad+(H_2\,R_2[u_2,u_2'])'(a,\varepsilon)-(H_1\,R_1[u_1,u_1'])'(a,\varepsilon)].\end{aligned} \tag{3.17}$$

We are now in a position to set up the framework within which to apply the Contraction Mapping Theorem to the equations (3.10)–(3.13) and (3.17). For the triplets (u_1,u_2,λ) of continuously differentiable functions u_1 and u_2 and scalar λ, we may define a norm given, for each $\varepsilon>0$, by

$$\|(u_1,u_2,\lambda\|=\max\{\|u_1\|_{\kappa_1},\|u_2\|_{\kappa_2},|\lambda|\}. \tag{3.18}$$

Then it is clear that the right-hand sides of (3.10)–(3.13) and (3.17) define a map

$$\mathscr{M}_\varepsilon:(u_1,u_2,\lambda)\to(\tilde{u}_1,\tilde{u}_2,\tilde{\lambda})$$

on the metric space of such triplets normed in this way. We now consider the properties of this map.

LEMMA 3. *For $\sigma_1\geqslant\sigma_1^*$, $\sigma_2\geqslant\sigma_2^*$ and $\kappa_1\in(0,\kappa_1^*)$, $\kappa_2\in(0,\kappa_2^*)$ where σ_1^*, σ_2^*, κ_1^* and κ_2^* may be appropriately chosen, the map $\mathscr{M}_\varepsilon$ maps the ball*

$$\|(u_1,u_2,\lambda-\alpha\sigma_1)\|\leqslant m\varepsilon \tag{3.19}$$

into itself for some positive $m=O(1)$ as $\varepsilon\to0$ for all $\varepsilon\in(0,\varepsilon_0]$, with ε_0 sufficiently small and positive bounded α and β.

PROOF. By applying the results of Lemma 2 to the interval $[0,a]$, we find from equations (3.10) and (3.11) for *any* λ in the ball (3.19) and for $\sigma_1\geqslant\sigma_1^*$, $\kappa_1\in(0,\kappa_1^*)$ that

$$\|\tilde{u}_1\|_{\kappa_1}\leqslant O(\exp(-a/\varepsilon\alpha h(0)\,h(a)))+O(\varepsilon)+O(1)\|u_1\|_{\kappa_1}^2, \tag{3.20}$$

while a similar inequality may be obtained from (3.12) and (3.13) for $\|\tilde{u}_2\|_{\kappa_2}$. From (3.17) we obtain, on noting the remark at the end of Section 2,

$$|\tilde{\lambda}-\alpha\sigma_1|\leqslant O(\varepsilon)+O(1)\|u_1\|_{\kappa_1}^2+O(1)\|u_2\|_{\kappa_2}^2. \tag{3.21}$$

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Combining these results, we see that there exists an m that is $O(1)$ as $\varepsilon \to 0$, and is such that

$$\|(\tilde{u}_1, \tilde{u}_2, \tilde{\lambda} - \alpha\sigma_1)\| \leqslant m\varepsilon, \tag{3.22}$$

which establishes the result.

LEMMA 4. *Under the conditions of Lemma* 3, *the map* $\mathscr{M}_\varepsilon$ *is a contraction on the ball of functions* (*3.19*), *with contraction parameter*

$$L(\varepsilon) = O(\varepsilon), \quad \text{as } \varepsilon \to 0. \tag{3.23}$$

PROOF. The results of Lemmas 1 and 2 show that the maps defined by the right-hand sides of (3.10)–(3.13) are contractions on this ball, with contraction parameter satisfying (3.23). If we examine the right-hand side of (3.17), it follows from the results of Lemma 1 and the comment at the end of the previous section that the terms in the last set of square braces are all $O(1)$ as $\varepsilon \to 0$, uniformly with respect to λ in (3.19). Thus the last term is bounded by a term of the form $O(\varepsilon)\,\lambda$, and thus may be proved to be contractive. In the second term, the λ is multiplied by an exponentially small factor, so this term, too, is a contraction. Finally, the first term is clearly a contraction on (3.19). This establishes the result, together with the estimate (3.23) for the contraction parameter. We are now able to establish the following lemma.

LEMMA 5. *Under the conditions of Lemma* 3, *and for all* $\varepsilon \in (0, \varepsilon_0]$, *with* ε_0 *sufficiently small, the equations* (3.10)–(3.13) *and* (3.19) *have a unique solution* (u_1, u_2, λ) *that satisfies*

$$\|u_1\|_{\kappa_1} = O(\varepsilon), \quad \|u_2\|_{\kappa_2} = O(\varepsilon) \quad \text{and} \quad \lambda = \alpha\sigma_1 + O(\varepsilon) \tag{3.24}$$

for appropriate choices of κ_1 *and* κ_2.

PROOF. By applying the Contraction Mapping Theorem [4], page 27, to the map $\mathscr{M}_\varepsilon$ defined above, we see that, as a consequence of Lemmas 3 and 4, this map has a unique fixed point in the ball (3.19).

By combining the results of this section, we arrive at our basic existence result for the problem $P_\varepsilon([0,1], \alpha, \beta)$ when $h'(x)$ has a finite jump discontinuity at $x = a$.

THEOREM 1. *Let* $h(x)$ *be continuous on* $[0,1]$ *and let* $h'(x)$ *suffer a single finite jump discontinuity at* $x = a$, *for* $a \in (0,1)$. *Let* $h''(x)$ *be continuous on* $[0,a]$ *and* $[a,1]$. *Then there exist positive constants* σ_1^*, σ_2^*, κ_1^* *and* κ_2^*, *independent of* ε, *such that, for* $\sigma_1 \geqslant \sigma_1^*$, $\sigma_2 \geqslant \sigma_2^*$, $\kappa_1 \in (0, \kappa_1^*)$ *and* $\kappa_2 \in (0, \kappa_2^*)$, *and for all* $\varepsilon \in (0, \varepsilon_0]$ *with* ε_0 *sufficiently small, there exists a unique function* $p(x, \varepsilon)$, *continuously differentiable with respect to* x

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on $[0,1]$, *with continuous second derivatives on* $[0,a]$ *and* $[a,1]$, *which satisfies*

$$N_\varepsilon p = 0 \text{ on } (0,a), \quad p(0,\varepsilon) = \alpha \tag{3.25}$$

and

$$N_\varepsilon p = 0 \text{ on } (a,1), \quad p(1,\varepsilon) = \beta, \tag{3.26}$$

for given positive bounded α *and* β. *Further, with* $w_1(x,\varepsilon)$ *and* $w_2(x,\varepsilon)$ *given by (3.2) and (3.3), respectively, and* $\lambda = \alpha\sigma_1^* + O(\varepsilon)$, p *satisfies the estimates*

$$p(x,\varepsilon) - w_1(x,\varepsilon) = O(\varepsilon) \tag{3.27}$$

and

$$p'(x,\varepsilon) - w_1'(x,\varepsilon) = O(\varepsilon)\,\rho_1(x,\varepsilon), \tag{3.28}$$

uniformly on $[0,a]$, *while*

$$p(x,\varepsilon) - w_2(x,\varepsilon) = O(\varepsilon) \tag{3.29}$$

and

$$p'(x,\varepsilon) - w_2'(x,\varepsilon) = O(\varepsilon)\,\rho_2(x,\varepsilon), \tag{3.30}$$

uniformly on $[a,1]$, *where* ρ_1 *and* ρ_2 *are defined by* (3.8) *and* (3.9), *respectively.*

REMARKS. 1. Note that $p''(x,\varepsilon)$ is *not* continuous at $x = a$. In fact, this may be deduced from the equation $N_\varepsilon p = 0$, with the result that

$$\varepsilon\Delta_a p''(x,\varepsilon) = [3\varepsilon p'(a,\varepsilon)\,h(a)^{-1} - h(a)^{-3}]\,\Delta_a h'(x), \tag{3.31}$$

which is non-zero for $\Delta_a h'(x) \neq 0$.

2. Up to this point, it has been tacitly asserted that the appropriate constants σ_1^*, σ_2^*, κ_1^* and κ_2^* *do* exist and have the properties ascribed to them. The theory of Section 2 ensures that this is so, but we will write these constants (or bounds for them) down here explicitly for completeness.

The results (2.22) shows us that

$$\kappa_1^* = \min\{\min_x w_1^{-1} h^{-2}, (\alpha h(0)\,h(a))^{-1}\} \tag{3.32}$$

and

$$\kappa_2^* = \min\{\min_x w_2^{-1} h^{-2}, (\lambda h(a)\,h(1))^{-1}\}. \tag{3.33}$$

Clearly, these depend on the image point λ of the map $\mathscr{M}_\varepsilon$. However, for λ lying in the ball (3.19), it is clear that these may be chosen to be positive and independent of ε.

3. When $\lambda = \alpha\sigma_1 + O(\varepsilon)$, the inequality for σ_1 corresponding to (2.23) reduces to an identity for $\varepsilon \to 0$, and thus may be removed from further consideration. We thus have only the restriction on σ_2, which becomes

$$\sigma_2^* > 2\beta\alpha^{-1}\sigma_1^{-1}[1 + \sqrt{(1 + 4\alpha^{-1}\sigma_1^{-1} h(1)/h_{\max})}]^{-1}, \tag{3.34}$$

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where h_{max} is the maximum value of $h(x)$ on the second subinterval $[a, 1]$. This inequality sets strict limits on the geometry of the bearing considered, for our theory to apply. For example, in the case of a taper-flat slider bearing, we have $\sigma_2^* = 1$, and $h(1) = h_{max}$, so that, for $\alpha = \beta = 1$, we obtain the condition that

$$\sigma_1 > 2/(1+\sqrt{(1+4\sigma_1^{-1})}). \tag{3.35}$$

In practice, this is unlikely to be violated, since $\sigma_1 > 1$ for such bearings, but it demonstrates the implications of (3.34).

4. Retaining α and β in (3.35), we obtain

$$\sigma_1 > 2\beta\alpha^{-1}[1+\sqrt{(1+4\alpha^{-1}\sigma_1^{-1})}]^{-1}, \tag{3.36}$$

so that, for large values of α/β, we may decrease our lower bound on σ_1. Of course, it should be borne in mind that these conditions are sufficient only, and their violation does not rule out the possibility of the existence of solutions having the properties we have discussed here.

5. Finally, it is worth noting that our theory readily encompasses the case when $h''(x)$ is piecewise continuous on $(0, a)$ and $(a, 1)$.

4. Asymptotic results

A direct consequence of our application of the Contraction Mapping Theorem to the map $\mathscr{M}_\varepsilon$ is the existence of a sequence $\{\xi_n\}_{n=0}^{\infty}$ (where we use the shorthand notation $\xi_n = (u_1^{(n)}, u_2^{(n)}, \lambda^{(n)})$) which is defined by

$$\xi_n = \mathscr{M}_\varepsilon \xi_{n-1}, \quad n = 1, 2, \ldots, \tag{4.1}$$

and which converges in the norm $\|\,.\,\|$ to the (unique) fixed point of $\mathscr{M}_\varepsilon$ lying in the ball (3.19), for any choice of initial iterate ξ_0 in this ball. What is more, standard theory [4], page 28, gives us the error estimate for the nth iterate in terms of the difference $\|\xi_0 - \xi_1\|$ in the form

$$\|\xi_n - \xi\| \leqslant \frac{L(\varepsilon)^n}{1-L(\varepsilon)}\|\xi_1 - \xi_0\|, \quad n = 1, 2, \ldots, \tag{4.2}$$

where $L(\varepsilon)$ is the contraction parameter of the map $\mathscr{M}_\varepsilon$, which we have shown to be $O(\varepsilon)$.

If we choose our initial iterate $\xi_0 = (0, 0, \alpha\sigma_1)$, the scheme (4.1) applied to the equations (3.10) and (3.11) yields

$$u_1^{(1)}(x, \varepsilon) = g_1(x, \varepsilon) + (H_1 f_1)(x, \varepsilon) \tag{4.3}$$

and

$$u_1^{(1)\prime}(x, \varepsilon) = g_1'(x, \varepsilon) + (H_1 f_1)'(x, \varepsilon). \tag{4.4}$$

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Noting (2.18) applied to the case in hand, we see that both g_1 and g'_1 are exponentially small as $\varepsilon \to 0$, uniformly with respect to x on $[0, a]$, while the results of (2.19) give

$$(H_1 f_1)(x, \varepsilon) = O(\varepsilon) \tag{4.5}$$

and

$$(H_1 f_1)'(x, \varepsilon) = O(\varepsilon)\, \rho_1(x, \varepsilon), \tag{4.6}$$

uniformly on $[0, a]$. Thus

$$\| u_1^{(1)} - u_1^{(0)} \|_{\kappa_1} = O(\varepsilon) \tag{4.7}$$

for $\kappa_1 \in (0, \kappa_1^*)$ with κ_1^* suitably defined. Similarly, we obtain

$$\| u_2^{(1)} - u_2^{(0)} \|_{\kappa_2} = \mathrm{O}(\varepsilon) \tag{4.8}$$

for $\kappa_2 \in (0, \kappa_2^*)$.

Combining these results with the estimate (4.2), we obtain from (3.4), for example,

$$p_1(x, \varepsilon) = w_1(x, \varepsilon) + u_1^{(n)}(x, \varepsilon) + O(\varepsilon^{n+1}), \quad n = 1, 2, \ldots. \tag{4.9}$$

Although this gives an approximate result, it is not of great use as it stands, since the approximation w_1 is that obtained from (3.2) by evaluation at the fixed point λ of $\mathscr{M}_\varepsilon$, a quantity known only to exist at present. However, it is a simple matter to prove, from (3.2) and the results cited in Section 2, that, for iterates $\lambda^{(n)}$ in the ball (3.19),

$$w_1(x, \varepsilon) - w_1^{(n)}(x, \varepsilon) = O(|\lambda - \lambda^{(n)}|), \tag{4.10}$$

where $w_1^{(n)}(x, \varepsilon)$ denotes the function $w_1(x, \varepsilon)$ evaluated at $\lambda^{(n)}$.

Under similar circumstances, we obtain

$$w'_1(x, \varepsilon) - w_1^{(n)\prime}(x, \varepsilon) = O(|\lambda - \lambda^{(n)}|)\, \rho_1(x, \varepsilon) \tag{4.11}$$

for $\kappa_1 \in (0, \kappa_1^*)$ and suitable κ_1^*. Thus we obtain the results that, for appropriate κ_1 and κ_2,

$$p_1(x, \varepsilon) = w_1^{(n)}(x, \varepsilon) + u_1^{(n)}(x, \varepsilon) + O(\varepsilon^{n+1}), \tag{4.12}$$

$$p'_1(x, \varepsilon) = w_1^{(n)\prime}(x, \varepsilon) + u_1^{(n)\prime}(x, \varepsilon) + O(\varepsilon^{n+1})\, \rho_1(x, \varepsilon) \tag{4.13}$$

and

$$p_2(x, \varepsilon) = w_2^{(n)}(x, \varepsilon) + u_2^{(n)}(x, \varepsilon) + O(\varepsilon^{n+1}), \tag{4.14}$$

$$p'_2(x, \varepsilon) = w_2^{(n)\prime}(x, \varepsilon) + u_2^{(n)\prime}(x, \varepsilon) + O(\varepsilon^{n+1})\, \rho_2(x, \varepsilon). \tag{4.15}$$

In the above, the definitions of κ_1^* and κ_2^* (as well as σ_2^*) depend on w_1 and w_2 but, for all points λ in the ball (3.19), the defining relations (3.32), (3.33) and (3.34) may be replaced by ones involving $w_1^{(n)}$, $w_2^{(n)}$ and $\lambda^{(n)}$, in view of (4.10) and equivalent results for w_2. In fact, for small enough ε, these relations would be suitable in a form involving $w_1^{(0)}$, $w_2^{(0)}$ and $\lambda^{(0)} = \alpha\sigma_1$, with no substantial effect on the results obtained.

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Noting that $\xi_0 = (0, 0, \alpha\sigma_1)$, we obtain from (3.2), (3.3) and (4.12)–(4.14), with $p(x, \varepsilon)$ the solution of Theorem 1,

$$p(x,\varepsilon) = \begin{cases} \alpha h(0)/h(x) + O(\varepsilon), & x \in [0, a], \\ \alpha h(0)/h(x) + P_2((1-x)/\varepsilon, \beta) - \alpha h(0)/h(1) + O(\varepsilon), & x \in [a, 1], \end{cases} \tag{4.16}$$

and

$$p'(x,\varepsilon) = \begin{cases} -\alpha h(0)\, h'(x)/h(x)^2 + O(e^{-\kappa_1(a-x)/\varepsilon}), & x \in [0, a], \\ -\alpha h(0)\, h'(x)/h(x)^2 - \varepsilon^{-1} P_2'((1-x)/\varepsilon, \beta) + O(e^{-\kappa_2(1-x)/\varepsilon}), & x \in [a, 1], \end{cases} \tag{4.17}$$

for $\kappa_1 \in (0, \kappa_1^*)$, $\kappa_2 \in (0, \kappa_2^*)$, and

$$\kappa_1^* = \min\{\min_x (\alpha h(0)\, h(x))^{-1}, (\alpha h(0)\, h(a))^{-1}\}, \tag{4.18}$$

$$\kappa_2^* = \min\{\min_x ([w_2^{(0)}]^{-1} h^{-2}), (\alpha h(0)\, h(1))^{-1}\}. \tag{4.19}$$

The results above are as we might have expected, namely the "outer solution" $\alpha h(0)/h(x)$ on the first interval is a good approximation throughout that interval, while the approximation on the second exhibits the characteristic boundary-layer structure at $x = 1$. The derivatives reflect these phenomena, with errors that are no longer uniformly small as $\varepsilon \to 0$.

To improve the above situation, we calculate the first iterates. From (3.17) we obtain

$$\lambda^{(1)} = \alpha\sigma_1 + \varepsilon\alpha^2 h(0)^2 h(a)^{-1} \Delta_a h'(x) + \varepsilon h(a)^2 \alpha\sigma_1 \{(H_2 f_2)'(a,\varepsilon) - (H_1 f_1)'(a,\varepsilon)\} + \eta, \tag{4.20}$$

where η is a term that is exponentially small as $\varepsilon \to 0$. Similarly, noting the properties of g_1 and g_2, we obtain, correct to all algebraic orders, from (3.10) to (3.13),

$$u_1^{(1)}(x,\varepsilon) = (H_1 f_1)(x,\varepsilon), \quad u_1^{(1)\prime}(x,\varepsilon) = (H_1 f_1)'(x,\varepsilon) \tag{4.21}$$

and

$$u_2^{(1)}(x,\varepsilon) = (H_2 f_2)(x,\varepsilon), \quad u_2^{(1)\prime}(x,\varepsilon) = (H_2 f_2)'(x,\varepsilon), \tag{4.22}$$

while, from (4.2),

$$\| \xi_1 - \xi \| = O(\varepsilon^2). \tag{4.23}$$

By adapting (2.8) we obtain

$$\alpha\sigma_1 - P_1 = (\alpha\sigma_1 - \lambda) \exp\{(\lambda - P_1)/\alpha\sigma_1\} \exp\{-(a-x)/\varepsilon\alpha h(0)\, h(a)\}, \tag{4.24}$$

and, setting $\lambda = \lambda^{(1)}$, we obtain from this

$$\alpha\sigma_1 - P_1 = -\varepsilon\psi \exp\{-(a-x)/\varepsilon\alpha h(0)\, h(a)\} + O(\varepsilon^2), \tag{4.25}$$

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correct to leading orders in ε, where ψ is the $O(\varepsilon)$ part of (4.20). Thus

$$w_1^{(1)}(x,\varepsilon) = \alpha h(0)/h(x) + \varepsilon\psi \exp\{-(a-x)/\varepsilon\alpha h(0)\,h(a)\} + O(\varepsilon^2). \tag{4.26}$$

Differentiation and application of (4.20) gives

$$\begin{aligned} w_1^{(1)\prime}(x,\varepsilon) = &-h(0)\,h'(x)/h(x)^2 + \{\alpha h(0)\,h(a)^{-2}\,\Delta_a\,h'(x) + (H_2\,f_2)'(a,\varepsilon) \\ &-(H_1\,f_1)'(a,\varepsilon) + O(\varepsilon)\}\exp\{-(a-x)/\varepsilon\alpha h(0)\,h(a)\}. \end{aligned} \tag{4.27}$$

The properties of $H_2\,f_2$ may be applied to give

$$(H_2\,f_2)'(a,\varepsilon) = O(\varepsilon). \tag{4.28}$$

On the second interval $[a,1]$ we obtain, by a similar process,

$$w_2^{(1)}(x,\varepsilon) = \lambda^*\{h(a)/h(x) - h(a)/h(1)\} + P_2((1-x)/\varepsilon,\beta) + O(\varepsilon^2) \tag{4.29}$$

and

$$w_2^{(1)\prime}(x,\varepsilon) = -\lambda^* h(a)\,h'(x)/h(x)^2 - \varepsilon^{-1}\,P_2'((1-x)/\varepsilon,\beta) + O(\varepsilon^2), \tag{4.30}$$

where

$$\lambda^* = \alpha\sigma_1 + \varepsilon\alpha^2\,h(0)^2\,h(a)^{-1}\,\Delta_a\,h'(x) - \varepsilon h(a)^2\,\alpha\sigma_1(H_1\,f_1)'(a,\varepsilon). \tag{4.31}$$

By applying (4.26)–(4.31) to the relations (4.12)–(4.15), we arrive at the following asymptotic result.

THEOREM 2. *The solution $p(x,\varepsilon)$ of the problem $P_\varepsilon([0,1],\alpha,\beta)$, for given positive α and β that are $O(1)$ as $\varepsilon \to 0$, admits of the generalized asymptotic expansion below, where the order symbols are assumed to hold uniformly with respect to x in the relevant interval.*

$$p(x,\varepsilon) = \begin{cases} \alpha h(0)/h(x) \\ \quad + \varepsilon\{\alpha^2\,h(0)^2\,h(a)^{-1}\,\Delta_a\,h'(x) - h(a)^2\,\alpha\sigma_1(H_1\,f_1)'(a,\varepsilon)\} \\ \quad\quad \times \exp\{-(a-x)/\varepsilon\alpha h(0)\,h(a)\} \\ \quad + (H_1\,f_1)(x,\varepsilon) + O(\varepsilon^2), \quad x \in [0,a], \\ \alpha h(0)/h(x) \\ \quad + P_2((1-x)/\varepsilon,\beta) - \alpha h(0)/h(1) \\ \quad + \varepsilon\{h(a)/h(x) - h(a)/h(1)\} \\ \quad\quad \times \{\alpha^2\,h(0)^2\,h(a)^{-1}\,\Delta_a\,h'(x) - h(a)^2\,\alpha\sigma_1(H_1\,f_1)'(a,\varepsilon)\} \\ \quad + (H_2\,f_2)(x,\varepsilon) + O(\varepsilon^2), \quad x \in [a,1], \end{cases} \tag{4.32}$$

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and

$$p'(x,\varepsilon) = \begin{cases} -\alpha h(0)\, h'(x)/h(x)^2 \\ +\{\alpha h(0)\, h(a)^{-2}\, \Delta_a h'(x) - (H_1 f_1)'(a,\varepsilon)\} \exp\{-(a-x/\varepsilon\alpha h(0)\, h(a)\} \\ +(H_1 f_1)'(x,\varepsilon) + O(\varepsilon^2)\, \rho_1(x,\varepsilon), \quad x \in [0,a], \\ -\alpha h(0)\, h'(x)/h(x)^2 \\ -\varepsilon\{\alpha^2 h(0)^2\, h(a)^{-\frac{1}{2}}\, \Delta_a h'(x) - h(a)^2\, \alpha\sigma_1 (H_1 f_1)'(a,\varepsilon)\}\, h(a)\, h'(x)/h(x)^2 \\ -\varepsilon^{-1} P_2'((1-x)/\varepsilon, \beta) + (H_2 f_2)'(x,\varepsilon) + O(\varepsilon^2)\, \rho_2(x,\varepsilon), \quad x \in [a,1], \end{cases} \tag{4.33}$$

where ρ_1 and ρ_2 are defined by (3.8) *and* (3.9) *for $\kappa_1 \in (0, \kappa_2^*)$ with κ_1^* and κ_2^* defined by* (4.18) *and* (4.19), *while $\sigma_2 \geqslant \sigma_2^*$ where σ_2^* is defined by a relation of the form of* (3.34).

It is of some interest to compare the results of Theorem 2 with those obtained by Schmitt and Di Prima [6] for the case $\alpha = \beta = 1$, using the method of matched asymptotic expansions. In both sets of results, the expansions on $[0,a]$ contain a term

$$O(\varepsilon)\, \Delta_a h'(x) \exp\{-(a-x)/\varepsilon h(0)\, h(a)\}, \tag{4.34}$$

which represents a higher-order boundary layer at $x = a$. In particular, this vanishes when $\Delta_a h'(x) = 0$, which is as we would expect. The result of [6] contains no other boundary layer term at the $O(\varepsilon)$ level, the remaining term being the uniform correction

$$\varepsilon\{h'(0) - h'(x)\}\, h(0)^2/h(x), \tag{4.35}$$

while all further boundary layer effects are relegated to higher orders in ε. The remaining $O(\varepsilon)$ term in (4.32) is

$$(H_1 f_1)(x,\varepsilon) - h(a)\, h(0)\, (H_1 f_1)'(a,\varepsilon) \exp\{-(a-x)/\varepsilon h(0)\, h(a)\}, \tag{4.36}$$

where we have set $\alpha = 1$ and $\sigma_1 = h(0)/h(a)$. If we note that the function $H_1 f_1$ satisfies the boundary-value problem

$$L_{1\varepsilon}(H_1 f_1) = f_1, \quad (H_1 f_1)(0,\varepsilon) = 0 \quad \text{and} \quad (H_1 f_1)(a,\varepsilon) = 0, \tag{4.37}$$

where $L_{1\varepsilon}$ is the linear operator analogous to (2.13) with $w_1^{(0)}$ replacing w, we may apply the self-same matching techniques to this problem to show that, formally,

$$(H_1 f_1)(x,\varepsilon) = \varepsilon\{h'(0) - h'(x)\}\, h(0)^2/h(x) \\ + \varepsilon h(0)\, h(a)\, (H_1 f_1)'(a,\varepsilon) \exp\{-(a-x)/\varepsilon h(0)\, h(a)\} + O(\varepsilon^2), \tag{4.38}$$

uniformly on $[0,a]$, while, again formally,

$$(H_1 f_1)'(a,\varepsilon) = -\{h'(0) - h'(a-0)\}\, h(0)\, h(a)^{-2} + O(\varepsilon). \tag{4.39}$$

An analogous result holds for the derivative $(H_1 f_1)'(x,\varepsilon)$.

We may combine these to obtain the following (formal) results for $p(x,\varepsilon)$ and $p'(x,\varepsilon)$ on $[0,a]$, namely,

$$p(x,\varepsilon) = h(0)/h(x) + \varepsilon\{h'(0)-h'(x)\}\,h(0)^2/h(x) + h(0)^2\,h(a)^{-1}\,\Delta_a h'(x)\exp\{-(a-x)/\varepsilon h(0)\,h(a)\} + O(\varepsilon^2) \quad (4.40)$$

and

$$p'(x,\varepsilon) = -h(0)\,h'(x)/h(x)^2 + h(0)\,h(a)^{-2}\,\Delta_a h'(x)\exp\{-(a-x)/\varepsilon h(0)\,h(a)\} + O(\varepsilon), \quad (4.41)$$

which are in complete agreement with those obtained in [6].

Comparison of results on the second interval $[a,1]$ is not quite as straightforward, but still may be carried out as below. It is an elementary matter to show that, formally,

$$(H_2\,f_2)(x,\varepsilon) = \varepsilon h(0)^2\{h'(a+0)-h'(x)\}/h(x) + O(\varepsilon^2) + \theta(x,\varepsilon), \quad (4.42)$$

where θ is a boundary-layer term relevant to the region $1-x = O(\varepsilon)$. Noting (4.39) and the definition of $\Delta_a h'(x)$, we may write

$$h(0)^2\,h(a)^{-1}\,\Delta_a h'(x) - h(a)^2\,\sigma_1(H_1\,f_1)'(a,\varepsilon) = \varepsilon h(0)^2\,h(a)^{-1}\{h'(0)-h'(a+0)\} + O(\varepsilon^2). \quad (4.43)$$

Applying these results to the second part of (4.32), we obtain, on $[a,1]$, formally,

$$p(x,\varepsilon) = h(0)/h(x) + \varepsilon h(0)^2\{h'(0)-h'(x)\}/h(x) + P_2((1-x)/\varepsilon,1) - \lambda^*\sigma_2 + \theta(x,\varepsilon) + O(\varepsilon^2), \quad (4.44)$$

where λ^* is given by (4.31) with $\alpha = 1$. A similar result may be obtained for the derivatives of the above.

Since $\lambda = \lambda^* + O(\varepsilon^2)$, the last terms in (4.44) represent a boundary layer situated at $x = 1$. If we define the function $\bar{P}_2((1-x)/\varepsilon, 1)$ in an analogous way to the P_2 above, by a relation of the form of (2.6), with the exception that

$$\bar{P}_2((1-\delta)/\varepsilon, 1) \to h(0)/h(1) \quad (4.45)$$

as $\varepsilon \to 0$ for any $\delta > 0$ and independent of ε, we may write this boundary layer term in (4.44) as

$$\{\bar{P}_2((1-x)/\varepsilon,1) - h(0)/h(1)\} + G((1-x)/\varepsilon,1) - (\lambda^*-\sigma_1)\,\sigma_2 + O(\varepsilon^2), \quad (4.46)$$

with

$$G = P_2 - \bar{P}_2 + \theta. \quad (4.47)$$

The first term in (4.46) appears in the expansion on $[a,1]$ obtained in [6] while, as we can see, the second term satisfies

$$G(0,1) = 0 + O(\varepsilon^2) \quad (4.48)$$

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and represents a boundary layer at $x = 1$. This term is given explicitly in [6], and has identical properties to those we have described here. As we would expect, analogous results hold for the derivative $p'(x, \varepsilon)$ on $[a, 1]$.

Thus the rigorously derived expansions of Theorem 2 reduce to those obtained by the method of matched expansions, under appropriate conditions. Moreover, although we have been at pains to point out that our procedure was formal only, the expansions like (4.38) derived to make this reduction may be validated rigorously, since the theory of Section 2 has given us the required detailed knowledge about the behaviour of linear operators like $L_{1\varepsilon}$ occurring in (4.37). Thus a procedure somewhat like that adopted by Rosenblat [5] could be applied to verify these approximations.

Evidently the process could be continued indefinitely, with validation of the procedure at each step. However, we will content ourselves here with the results obtained in Theorem 2.

5. Conclusion

The thrust of the discussion of this paper is significant in a number of ways. It presents a method by which existence–uniqueness results may be obtained for the problem $P_\varepsilon([0, 1], \alpha, \beta)$, while at the same time it develops a rigorously based iterative procedure by which an asymptotic representation for this solution may be constructed, valid for small ε uniformly with respect to x on $[0, 1]$. While the technical details have, at times, been rather complex, the basic ideas have been kept quite simple, and although the calculations would be somewhat involved, the above scheme *could* be applied, in principle, to any order in ε. We have already shown this in Section 4, to a limited degree.

The question naturally arises as to whether the technique used in Section 3 is to be favoured over the matching techniques applied to this problem by Schmitt and DiPrima [6]. Clearly, although Theorem 2 provides us with a uniformly valid expansion for the pressure, we must resort to heuristic methods to appreciate the significance of the terms generated. Thus it would seem that, within the context of useful and easily applied expansions, the answer to the question above should be in the negative. However, the very techniques by which the existence result of Theorem 1 has been obtained provide us with a powerful tool for checking the validity of these (formal) results, so that the expansions of Theorem 2 may with confidence be expressed in terms of expansions obtained by formal arguments. This then is the main role played by the constructions of Section 4.

The iterative scheme (4.1) may be used to provide other results about the general structure of the solution $p(x, \varepsilon)$. Application of the argument applied in Section 7 of

[8] to this reveals that the solution on $[0,a]$ has a structure of the form

$$p(x,\varepsilon) = \sum_{r=0}^{n} \varepsilon^r a_{n,r}(x,\varepsilon) + e^{-\kappa_1(a-x)/\varepsilon} \sum_{r=0}^{n} \varepsilon^r b_{n,r}(x,\varepsilon) + O(\varepsilon^{n+1}), \tag{5.1}$$

for $n = 1, 2, \ldots$, and κ_1 satisfying (3.32), while $a_{n,r}$ and $b_{n,r}$ are functions that are $O(1)$ as $\varepsilon \to 0$, uniformly on $[0,a]$. A corresponding result holds for the derivative $p'(x,\varepsilon)$, as well as analogous expansions on $[a,1]$. It is apparent that (5.1) consists of an "outer" expansion and a "boundary layer correction". When x is bounded away from a as $\varepsilon \to 0$, the second term and all its derivatives vanish exponentially as $\varepsilon \to 0$ and, if we assume that each of the $a_{n,r}$ has a Poincaré type expansion in powers of ε in this region, we arrive at a statement of the existence of the "outer expansion" constructed in the matching process. When $a - x = O(\varepsilon)$, we set $a - x = \varepsilon\zeta$, and then (5.1) becomes

$$p(a-\varepsilon\zeta,\varepsilon) = \sum_{r=0}^{n} \varepsilon^r a_{n,r}(a-\varepsilon\zeta,\varepsilon) + e^{-\kappa_1\zeta} \sum_{r=0}^{n} \varepsilon^r b_{n,r}(a-\varepsilon\zeta,\varepsilon) + O(\varepsilon^{n+1}), \tag{5.2}$$

which, with further assumptions about the expansibility of $a_{n,r}$ and $b_{n,r}$ as Poincaré series in ζ and ε, gives us our "inner" or "boundary layer" expansion.

Throughout the paper there have been a number of assumptions made that are, in general, sufficient for the theory to hold. Thus, while κ_1 and κ_2 have been restricted to the open intervals $(0, \kappa_1^*)$ and $(0, \kappa_2^*)$ respectively, it should be noted that these have been sufficient to make the error estimates of Theorem 2 upper bounds only. As we have seen later in Section 4, explicit evaluation of terms could (and probably would) lead to an enlargement of these intervals. We have also had to restrict σ_2 (and σ_1 as it turns out) by the inequality (3.34), but we note that this is sufficient to cover all bearing geometries that are of physical interest. Finally, we recall that we may relax our assumptions about $h''(x)$ to allow it to be piecewise continuous on $[0,a]$ and $[a,1]$.

The extension of the results obtained to the case where $h'(x)$ has a finite number of jump discontinuities in $(0,1)$ is now obvious. If we assume that these occur at $0 < a_1 < \ldots < a_n < 1$, we merely set up the sequence of problems

$$N_\varepsilon p_i = 0, \; x \in (a_i, a_{i+1}), \quad i = 0, 1, \ldots, n, \tag{5.3}$$

where $a_0 = 0$ and $a_{n+1} = 1$,

$$p_0(0,\varepsilon) = \alpha, \quad p_0(a_1,\varepsilon) = \lambda_1, \tag{5.4}$$

$$p_n(1,\varepsilon) = \beta, \quad p_n(a_n,\varepsilon) = \lambda_n, \tag{5.5}$$

$$p_i(a_i,\varepsilon) = \lambda_i, \quad p_i(a_{i+1},\varepsilon) = \lambda_{i+1}, \quad i = 1, 2, \ldots, \mathrm{n}-1 \tag{5.6}$$

and

$$p_i'(a,\varepsilon) = p_{i+1}'(a_i,\varepsilon), \quad i = 0, 1, \ldots, n-1. \tag{5.7}$$

By linearizing about the appropriate function $w_i(x,\varepsilon)$, $i = 0, \ldots, n$, constructed as in Section 2, converting to a system of integral equations as in Section 3, and by applying the Contraction Mapping Theorem to the ball defined by

$$\|(p_0 - w_0, p_1 - w_1, \ldots, p_n - w_n, \lambda_1 - \alpha\sigma_1, \lambda_2 - \lambda_1 \sigma_2, \ldots)\| \leqslant m\varepsilon \tag{5.8}$$

for some m, where the σ_i are defined in an analogous way to σ_1 and σ_2, we may develop our existence–uniqueness theory, as well as construct expansions for the solution.

The problem may be extended in other directions. There has been some interest lately in pressure calculations at high bearing numbers when the gas flowing in the bearing is assumed to have limited slip at the bearing walls. In this case the governing equation is

$$\varepsilon(h^3 pp'(1+6k/ph))' = (ph)', \quad x \in (0,1), \tag{5.9}$$

with

$$p(0,\varepsilon) = p(1,\varepsilon) = 1, \tag{5.10}$$

where k is a dimensionless parameter termed the Knudsen number, and the other symbols have their usual meanings. This problem has been analyzed, for small ε and moderate k, by Sereny and Castelli [7], who used matching techniques to obtain results that are very similar to those of [1]. This is as we would expect. We would also envisage no difficulty in extending the existence–uniqueness theory of the present paper to the problem (5.9) and (5.10) in cases where $h'(x)$ exhibits jump discontinuities, as well as for the case of continuous $h'(x)$.

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