PRINCIPALLY ORDERED REGULAR SEMIGROUPS

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An ordered semigroup S will be called *principally ordered* if, for every $x \in S$, there exists

$$x^* = \max\{y \in S; xyx \leq x\}.$$

Here we shall be concerned with the case where S is regular. We begin by listing some basic properties that arise from the above definition. As usual, we shall denote by $V(x)$ the set of inverses of $x \in S$.

$$x' \in V(x) \Rightarrow x' \leq x^*. \tag{1}$$

This follows immediately from the fact that $xx'x = x$.

$$x = xx^*x. \tag{2}$$

By (1), if $x' \in V(x)$ then $x = xx'x \leq xx^*x$, whence we have equality.

$$x^\circ = x^*xx^* \textit{ is the greatest inverse of } x. \tag{3}$$

In fact, by (2), we have $x^*xx^* \in V(x)$. If now $x' \in V(x)$ then, by (1), $x' = x'xx' \leq x^*xx^*$. Thus $x^\circ = x^*xx^*$ is the greatest inverse of x.

$$xx^\circ = xx^* \textit{ is the greatest idempotent in } R_x. \tag{4}$$

It is clear from (3) and (2) that $xx^\circ = xx^*$. Also, by (2), x and xx^* are $\mathscr{R}$-related. If now e is an idempotent that is $\mathscr{R}$-related to x then we have $e = xy$ and $x = ez$ for some $y, z \in S$. It follows that $xyx = ex = ez = x$ and so $y \leq x^*$ whence $e = xy \leq xx^*$.

$$x^\circ x = x^*x \textit{ is the greatest idempotent in } L_x. \tag{5}$$

This is similar to (4)

$$x \leq x^{**} \quad \text{and} \quad x \leq x^{\circ\circ}. \tag{6}$$

By (3) and (1), $x^*xx^* = x^\circ \leq x^*$ and so $x \leq x^{**}$. The second inequality follows from (3) and the fact that x is an inverse of x°.

$$x^{\circ *} = x^{**} \tag{7}$$

This follows from the observation that

$$\begin{aligned} y \leq x^{\circ *} &\Leftrightarrow x^\circ y x^\circ \leq x^\circ \\ &\Leftrightarrow x^*xx^*yx^*xx^* \leq x^*xx^* \\ &\Leftrightarrow xx^*yx^*x \leq x \\ &\Leftrightarrow x^*yx^* \leq x^* \\ &\Leftrightarrow y \leq x^{**}. \end{aligned}$$

$$x^* = x^{***} \tag{8}$$

By (6) we have $x^* \leq x^{***}$. To obtain the reverse inequality, observe that

$$xyx \leq x^{**} \Rightarrow x^*xyxx^* \leq x^* \Rightarrow xyx \leq x$$

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and hence that

$$xyx \leq x^{**} \Leftrightarrow xyx \leq x.$$

Now, by (6) and (2), we have

$$xx^{***}x \leq x^{**}x^{***}x^{**} = x^{**},$$

and therefore, by the above observation,

$$xx^{***}x \leq x$$

and so $x^{***} \leq x^*$.

$$x^{*\circ} = x^{**} \tag{9}$$

We have

$$\begin{aligned} x^{*\circ} &= x^{**}x^*x^{**} \\ &= x^{**}x^{***}x^{**} && \text{by (8)} \\ &= x^{**} && \text{by (2).} \end{aligned}$$

$$x^{\circ\circ} \leq x^{**} \tag{10}$$

This follows from the observation that

$$\begin{aligned} x^{\circ\circ} &= x^{\circ*}x^{\circ}x^{\circ*} && \text{by (3)} \\ &= x^{**}x^{\circ}x^{**} && \text{by (7)} \\ &\leq x^{**}x^*x^{**} && \text{by (1)} \\ &= x^{**}x^{***}x^{**} && \text{by (8)} \\ &= x^{**} && \text{by (2).} \end{aligned}$$

$$x^{\circ} = x^{\circ\circ\circ} \tag{11}$$

In fact,

$$\begin{aligned} x^{\circ} = x^*xx^* &= x^*x^{**}x^*xx^*x^{**}x^* \\ &= x^*x^{\circ*}x^{\circ}x^{\circ*}x^* && \text{by (7)} \\ &= x^*x^{\circ\circ}x^* \\ &= x^{***}x^{\circ\circ}x^{***} && \text{by (8)} \\ &= x^{\circ\circ*}x^{\circ\circ}x^{\circ\circ*} && \text{by (7)} \\ &= x^{\circ\circ\circ}. \end{aligned}$$

EXAMPLE 1. A perfect Dubreil–Jacotin semigroup is characterised in [**2**] as an ordered regular semigroup S in which

(α) $x^* = \max\{y \in S; xyx \leq x\}$ exists for every $x \in S$;

(β) $\xi = \max\{x \in S; x^2 \leq x\}$ exists;

(γ) $\xi x^* = x^* = x^*\xi$ for every $x \in S$.

Every perfect Dubreil–Jacotin semigroup is therefore principally ordered.

EXAMPLE 2. On the cartesian ordered set

$$S = \{(x, y, z) \in \mathbb{Z}^3; 0 \le y \le x\}$$

define a multiplication by

$$(x, y, z)(a, b, c) = (\sup\{x, a\}, y, z + c).$$

Then clearly S is an ordered semigroup. It is regular since

$$(x, y, z)(x, x, -z)(x, y, z) = (x, y, 0)(x, y, z) = (x, y, z).$$

The idempotents are the elements of the form $(x, y, 0)$, and as they form a subsemigroup S is orthodox. Since

$$\begin{aligned}(x, y, z)(a, b, c)(x, y, z) = (\sup\{x, a\}, y, 2z + c) \le (x, y, z) \\ \Leftrightarrow a \le x,\ c \le -z \\ \Leftrightarrow (a, b, c) \le (x, x, -z),\end{aligned}$$

we see that $(x, y, z)^* = (x, x, -z)$ and so S is principally ordered.

EXAMPLE 3 [The boot-lace]. Let G be an ordered group and let $x \in G$ be such that $1 < x$. Let $M = M(G; I, \Lambda; P)$ be the regular Rees matrix semigroup over G with $I = \Lambda = \{1, 2\}$ and sandwich matrix

$$P = \begin{bmatrix} x^{-1} & 1 \\ 1 & 1 \end{bmatrix}.$$

With $\{1, 2\}$ ordered by $1 < 2$, we can regard P as an isotone mapping from the cartesian ordered set $\{1, 2\} \times \{1, 2\}$ to G. Recall that the multiplication in M is given by

$$(i, a, \lambda)(j, b, \mu) = (i, ap_{\lambda j}b, \mu).$$

The set of idempotents of M is

$$E = \{(1, x, 1), (1, 1, 2), (2, 1, 1), (2, 1, 2)\}.$$

Let $\bar{E} = \bigcup_{n \ge 1} E^n$ be the subsemigroup generated by the idempotents. Then, with the convention that $x^0 = 1$, we have

$$\bar{E} = \{(i, x^n, \lambda); i, \lambda \in \{1, 2\}, n \in \mathbb{Z}\}.$$

By a result of Fitz-Gerald [5], the set $V(E^n)$ of inverse of elements in E^n is E^{n+1}. It follows that $\bar{E}$ is regular.

Consider the relation $\le$ defined on $\bar{E}$ by

$$(i, x^n, \lambda) \le (i', x^m, \lambda') \Leftrightarrow \begin{cases} n = m,\ i \le i',\ \lambda \le \lambda'; \\ \text{or } n + 1 = m,\ i \le i'; \\ \text{or } n + 1 = m,\ \lambda \le \lambda'; \\ \text{or } n + 1 < m. \end{cases}$$

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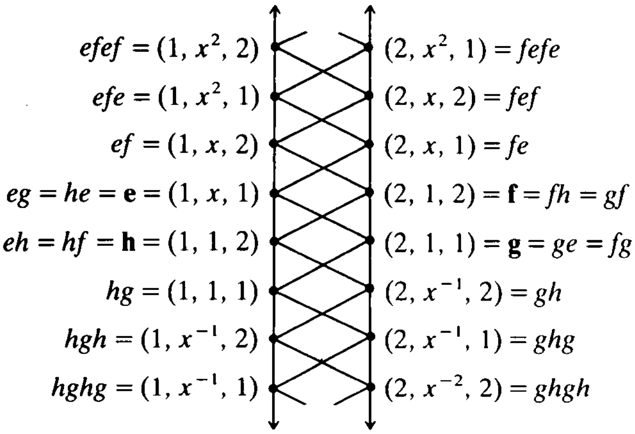
It is readily verified that this is an order on $\bar{E}$ which gives the boot-lace Hasse diagram



To see that $\bar{E}$ is thus an ordered semigroup, suppose that we have $(i, x^n, \lambda) \leq (i', x^m, \lambda')$ and compare

$$(i, x^n, \lambda)(j, x^k, \mu) = (i, x^n p_{\lambda j} x^k, \mu) = (i, \alpha, \mu),$$

$$(i', x^m, \lambda')(j, x^k, \mu) = (i', x^m p_{\lambda' j} x^k, \mu) = (i', \beta, \mu).$$

If $n = m$, $i \leq i'$ and $\lambda \leq \lambda'$ then clearly $\alpha \leq \beta$ and $(i, \alpha, \mu) \leq (i', \beta, \mu)$. If $n+1 = m$ and $i \leq i'$ then $\alpha = \beta$ when $p_{\lambda j} = 1$ and $p_{\lambda' j} = x^{-1}$, in which case $(i, \alpha, \mu) \leq (i', \beta, \mu)$; otherwise, $\alpha = x^a < x^b = \beta$ with $a + 1 \leq b$ in which case $(i, \alpha, \mu) \leq (i', \beta, \mu)$. If $n + 1 = m$ and $\lambda \leq \lambda'$, or if $n + 1 < m$, then again we have $\alpha = x^a < x^b = \beta$ with $a + 1 \leq b$ in which case $(i, \alpha, \mu) \leq (i', \beta, \mu)$. Thus we see that the multiplication in $\bar{E}$ is compatible on the right with the order; similarly, it is compatible on the left.

The ordered regular semigroup $\bar{E}$ is principally ordered. In fact, for every $n \in \mathbb{Z}$ we have

$$(1, x^n, 1)^* = (1, x^{-n+2}, 1);$$
$$(1, x^n, 2)^* = (2, x^{-n+1}, 1);$$
$$(2, x^n, 1)^* = (1, x^{-n+1}, 2);$$
$$(2, x^n, 2)^* = (2, x^{-n}, 2).$$

To see the first of these, for example, observe that

$$(1, x^n, 1)(1, x^{-n+2}, 1)(1, x^n, 1) = (1, x^n, 1)$$

whereas for the element $(2, x^{-n+1}, 2)$, which is directly opposite $(1, x^{-n+2}, 1)$ in the Hasse diagram, we have

$$(1, x^n, 1)(2, x^{-n+1}, 2)(1, x^n, 1) = (1, x^{n+1}, 1) > (1, x^n, 1).$$

It follows from this that

$$(1, x^n, 1)^* = (1, x^{-n+2}, 1).$$

Similarly, we have the other formulae. Note that in this example we have $a = a^{**}$ for every $a \in \bar{E}$.

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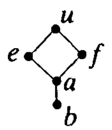
Simple calculations show that in both Examples 2 and 3 we have $x^\circ = x^*$ for every x. It is easy to construct further examples in which this identity does not hold.

EXAMPLE 4. Consider the smallest non-orthodox naturally ordered regular semigroup with a greatest idempotent (see, for example, [3]). This is the semigroup N_5 described by the following Hasse diagram and Cayley table:



	u	e	f	a	b
u	u	u	f	f	b
e	e	e	a	a	b
f	u	b	f	b	b
a	e	b	a	b	b
b	b	b	b	b	b

It is readily seen that $xux = x$ for every $x \in N_5$ and so N_5 is principally ordered with $x^* = u$ for every x. In fact, N_5 is perfect Dubreil–Jacotin.

With $\bar{E}$ as in Example 3, define a multiplication on the cartesian ordered set $N_5 \times \bar{E}$ by

$$(p, x)(q, y) = (pq, xy).$$

Then $N_5 \times \bar{E}$ is a principally ordered regular semigroup in which

$$(p, x)^* = (u, x^*).$$

In this semigroup, we have

$$(p, x)^\circ = (p, x)^*(p, x)(p, x)^* = (upu, x^\circ) = (upu, x^*).$$

It follows that, for example,

$$(b, x)^\circ = (b, x^*) < (u, x^*) = (b, x)^*.$$

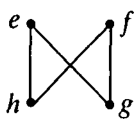
Note also that

$$(p, x)^{**} = (u, x), \qquad (p, x)^{\circ\circ} = (upu, x)$$

and so in $N_5 \times \bar{E}$ we have $x \neq x^{**}$ and $x \neq x^{\circ\circ}$ in general.

Bearing in mind Example 1 above, we recall from [2] that if S is a perfect Dubreil–Jacotin semigroup then two particular features of S are, on the one hand, that S has a greatest idempotent and, on the other, that the assignment $x \mapsto x^*$ is antitone. In a general principally ordered regular semigroup, these two properties are independent.

To see this, we refer first to Example 3. Here we have a principally ordered regular semigroup in which the idempotents form the 4-element crown



and so there is no greatest idempotent. But, as is readily verified, the assignment $x \mapsto x^*$ is antitone on $\bar{E}$.

To obtain an example of a principally ordered regular semigroup that has a greatest idempotent and in which $x \mapsto x^*$ is not antitone, we consider a subsemigroup of that of Example 2.

EXAMPLE 5. Let k be a fixed positive integer and consider the subset T_k of the semigroup S of Example 2 given by

$$T_k = \{(x, y, z) \in \mathbb{Z}^3; 0 \leq y \leq x \leq k\}.$$

Then it is readily seen that T_k is a principally ordered regular semigroup in which $(x, y, z)^* = (x, x, -z)$. Clearly, T_k has a greatest idempotent, namely the element $(k, k, 0)$, and $x \mapsto x^*$ is not antitone.

Our main objective now is to show that for a principally ordered regular semigroup S the conditions (a) S has a greatest idempotent, and (b) the mapping $x \mapsto x^*$ is antitone, are necessary and sufficient for S to be a perfect Dubreil–Jacotin semigroup. For this purpose, we first establish the following result.

THEOREM 1. *Let S be a principally ordered regular semigroup in which $x \mapsto x^*$ is antitone. Then S is naturally ordered.*

Proof. Let $\preceq$ denote the natural order on the idempotents, so that

$$e \preceq f \Leftrightarrow ef = fe = e.$$

Suppose that $e \preceq f$. Then these equalities give $fef = e$ and $efe = e$. From the latter we obtain $f \leq e^*$ whence, by (6) and the fact that $x \mapsto x^*$ is antitone, we have $e \leq e^{**} \leq f^*$. It follows that $e = fef \leq ff^*f = f$. Thus the order on S extends the natural order on the idempotents and so S is naturally ordered.

COROLLARY. *Let S be a principally ordered regular semigroup in which $x \mapsto x^*$ is antitone. Then S is locally inverse.*

Proof. This follows immediately from [**6**, Proposition 1.4].

We recall now that if E is the set of idempotents of S and if $e, f \in E$ then the *sandwich set* $S(e, f)$ is defined by

$$S(e, f) = \{g \in E; g = ge = fg, egf = ef\}.$$

A characteristic property of locally inverse semigroups is that sandwich sets are singletons. We identify them in the present situation as follows.

THEOREM 2. *Let S be a principally ordered regular semigroup in which $x \mapsto x^*$ is antitone, and let $e, f \in E$. Then $S(e, f) = \{f(ef)^\circ e\}$.*

Proof. Observe that if $g = f(ef)^\circ e$ then $g^2 = g = ge = fg$ and $egf = ef$.

Since the structure of naturally ordered regular semigroups with a greatest idempotent has been completely determined in [**3**], we shall focus our attention on principally ordered regular semigroups in which $x \mapsto x^*$ is antitone. *Unless otherwise specified, S will henceforth denote such a semigroup.* For such a semigroup, we now list properties that will lead us to our goal.

$$(\forall e \in E) \;\; e^{\infty} \in E \cap V(e) \tag{12}$$

In fact, this was shown by Saito [7, Proposition 2.2] to hold in any naturally ordered regular semigroup in which greatest inverses exist, and therefore holds in the present situation by Theorem 1 and property (3) above.

$$(\forall x, y \in S) \quad xy(xy)^\circ \leq xx^\circ, \qquad (xy)^\circ xy \leq y^\circ y. \tag{13}$$

From $y(xy)^\circ xy(xy)^\circ = y(xy)^\circ$ we deduce that

$$x \leq [y(xy)^\circ]^*,$$

and so, by (6) and the fact that $x \mapsto x^*$ is antitone,

$$y(xy)^\circ \leq [y(xy)^\circ]^{**} \leq x^*.$$

It follows that $xy(xy)^\circ \leq xx^* = xx^\circ$. Similarly, we can show that $(xy)^\circ xy \leq y^\circ y$.

$$(\forall x, y \in S) \quad (xy)^\circ = (x^\circ xy)^\circ x^\circ = y^\circ (xyy^\circ)^\circ. \tag{14}$$

Since, by (5), x and $x^\circ x$ are $\mathscr{L}$-related, and since $\mathscr{L}$ is right compatible with multiplication, we have that xy and $x^\circ xy$ are $\mathscr{L}$-related, whence

$$(a) \quad (xy)^\circ xy = (x^\circ xy)^\circ x^\circ xy.$$

It follows that

$$xy \,.\, (x^\circ xy)^\circ x^\circ \,.\, xy = xy(xy)^\circ xy = xy$$

and so

$$(x^\circ xy)^\circ x^\circ \leq (xy)^*.$$

Using this, we see that

$$\begin{aligned} xy(xy)^\circ &= xy(xy)^\circ xy(xy)^\circ \\ &= xy(x^\circ xy)^\circ x^\circ xy(xy)^\circ && \text{by } (a) \\ &\leq xy(x^\circ xy)^\circ x^\circ xx^\circ && \text{by } (13) \\ &= xy(x^\circ xy)^\circ x^\circ \\ &\leq xy(xy)^* \\ &= xy(xy)^\circ, \end{aligned}$$

whence we have

$$(b) \quad xy(xy)^\circ = xy(x^\circ xy)^\circ x^\circ.$$

It now follows that

$$\begin{aligned} (xy)^\circ &= (xy)^\circ xy(xy)^\circ \\ &= (x^\circ xy)^\circ x^\circ xy(xy)^\circ && \text{by } (a) \\ &= (x^\circ xy)^\circ x^\circ xy(x^\circ xy)^\circ x^\circ && \text{by } (b) \\ &= (x^\circ xy)^\circ x^\circ. \end{aligned}$$

Similarly, we can show that $(xy)^\circ = y^\circ (xyy^\circ)^\circ$.

$$(\forall x \in S) \quad (xx^\circ)^\circ = x^{\circ\circ} x^\circ, \qquad (x^\circ x)^\circ = x^\circ x^{\circ\circ}. \tag{15}$$

Take $y = x^\circ$ in the first equality of (14), and $x = y^\circ$ in the second.

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REMARK. Note that (13) and (14) were established by Saito [7] in the case of a naturally ordered regular semigroup with greatest inverses and in which Green's relations $\mathcal{R}$ and $\mathcal{L}$ are regular, in the sense that

$$x \le y \Rightarrow xx^\circ \le yy^\circ \quad \text{and} \quad x \le y \Rightarrow x^\circ x \le y^\circ y.$$

Later, we shall see the significance of the regularity of Green's relations $\mathcal{R}$ and $\mathcal{L}$ in the present context of a principally ordered regular semigroup in which the mapping $x \mapsto x^*$ is antitone.

$$\textit{If } e \in E \textit{ and } f^2 \le f \textit{ then } e \le f \Rightarrow e = efe. \tag{16}$$

This follows from the observation that

$$\begin{aligned} e \le f &\Rightarrow fefef \le f^5 \le f \\ &\Rightarrow efe \le f^* \le e^* \\ &\Rightarrow efe \le ee^*e = e \\ &\Rightarrow e \le f \le e^* \\ &\Rightarrow e \le efe \le ee^*e = e \\ &\Rightarrow e = efe. \end{aligned}$$

$$(\forall e \in E) \quad e^\circ \in E \Leftrightarrow e^* \in E. \tag{17}$$

For every $e \in E$ we have

$$\begin{aligned} e^\circ \in E &\Leftrightarrow e^\circ e^\circ = e^\circ \\ &\Leftrightarrow e^*ee^* \,.\, e^*ee^* = e^*ee^* \\ &\Leftrightarrow ee^*e^*e = e. \end{aligned}$$

Thus $e^\circ \in E$ implies that $e^*e^* \le e^*$. But $e^\circ \in E$ also implies, by (5), (7) and (1), that

$$e^\circ \le e^{\circ\circ}e^\circ = e^{\circ *}e^\circ = e^{**}e^\circ \le e^{**}e^*$$

whence, by (16),

$$e^{**}e^* \le e^{\circ *} = e^{**} \le e^*,$$

the last inequality following from the fact that $e \le e^*$. It follows that

$$e^* = e^*e^{**}e^* \le e^*e^*.$$

Thus $e^\circ \in E$ implies that $e^* \in E$. Conversely, if $e^* \in E$ then $ee^*e^*e = ee^*e = e$ and, from the above, we deduce that $e^\circ \in E$.

For every $x \in S$ we now consider the elements

$$\alpha_x = x^{\circ\circ}x^\circ \in E, \qquad \beta_x = x^\circ x^{\circ\circ} \in E.$$

$$(\forall x \in S) \quad \alpha_x = \alpha_x^\circ \le \alpha_x^* \in E, \qquad \beta_x = \beta_x^\circ \le \beta_x^* \in E. \tag{18}$$

Using (15) and (11), we observe that

$$\alpha_x \le \alpha_x \alpha_x^\circ = x^{\circ\circ}x^\circ(x^{\circ\circ}x^\circ)^\circ = x^{\circ\circ}x^\circ x^{\circ\circ}x^\circ = x^{\circ\circ}x^\circ = \alpha_x,$$

and so $\alpha_x = \alpha_x\alpha_x^\circ$. Similarly, we have $\alpha_x = \alpha_x^\circ\alpha_x$. It follows that

$$\alpha_x = \alpha_x\alpha_x^\circ = \alpha_x^\circ\alpha_x\alpha_x^\circ = \alpha_x^\circ,$$

and so $\alpha_x^\circ \in E$. It now follows by (17) that $\alpha_x^* \in E$.

$$\begin{aligned}&\alpha_x^* = (xx^*)^* \textit{ is the greatest idempotent above } xx^*, \textit{ and}\\ &\beta_x^* = (x^*x)^* \textit{ is the greatest idempotent above } x^*x.\end{aligned} \tag{19}$$

Clearly, $xx^* = xx^\circ \le x^{\circ\circ}x^\circ = \alpha_x \le \alpha_x^*$ where, by (18), $\alpha_x^* \in E$. Suppose now that $g \in E$ is such that $xx^* \le g$. Then we have, using (7) and (15),

$$g^{**} \ge (xx^*)^{**} = (xx^\circ)^{\circ *} = (x^{\circ\circ}x^\circ)^* = \alpha_x^* \ge \alpha_x,$$

and so $g \le g^* \le \alpha_x^*$. Thus α_x^* is the greatest idempotent above xx^*.

The maximality of the idempotent α_x^* implies that

$$\alpha_x^* = \alpha_x^*\alpha_x^{**} = \alpha_x^{**}\alpha_x^*,$$

and so

$$\alpha_x^{**} = \alpha_x^{**}\alpha_x^*\alpha_x^{**} = \alpha_x^*\alpha_x^{**} = \alpha_x^*.$$

It follows by (7), (8), and (15) that

$$\alpha_x^* = \alpha_x^{**} = (x^{\circ\circ}x^\circ)^{**} = (xx^\circ)^{\circ **} = (xx^\circ)^{***} = (xx^\circ)^* = (xx^*)^*.$$

The statement concerning β_x^* is proved similarly.

$$(\forall e, f \in E) \quad S(e,f)^* = \max\{x \in S; exf \le (ef)^{**}\}. \tag{20}$$

By Theorem 2, $S(e,f) = \{f(ef)^\circ e\} = \{g\}$, say. Then

$$\begin{aligned} x \le g^* &\Rightarrow gxg \le g\\ &\Rightarrow f(ef)^\circ e \,.\, x \,.\, f(ef)^\circ e \le f(ef)^\circ e\\ &\Rightarrow ef(ef)^* \,.\, exf \,.\, (ef)^*ef \le ef\\ &\Rightarrow (ef)^*exf(ef)^* \le (ef)^*\\ &\Rightarrow exf \le (ef)^{**}.\end{aligned}$$

Conversely, suppose that $exf \le (ef)^{**} = (ef)^{\circ *}$. Then

$$\begin{aligned} gxg &= f(ef)^\circ e \,.\, x \,.\, f(ef)^\circ e\\ &\le f(ef)^\circ(ef)^{\circ *}(ef)^\circ e\\ &= f(ef)^\circ e\\ &= g,\end{aligned}$$

and hence $x \le g^*$.

Using the above results, we can now establish the following:

THEOREM 3. *Let S be a principally ordered regular semigroup in which $x \mapsto x^*$ is antitotone. Then the following statements are equivalent:*

(*i*) $(\forall e \in E) \quad e^* = e^{**}$;
(*ii*) $(\forall e \in E) \quad e^* \in E$;
(*iii*) *S has a greatest idempotent*;
(*iv*) *S is a perfect Dubreil–Jacotin semigroup.*

Proof. $(i)\Rightarrow(ii)$: If $e^*=e^{**}$ then $e^*e^*e^*\le e^*$. But, by (16), $e\le e^*e^{**}$ gives $e^*e^{**}\le e^*$, so we have $e^*=e^*e^{**}e^*\le e^*e^*$ and so

$$e^*\le e^*e^*\le e^*e^*e^*\le e^*,$$

whence $e^*=e^*e^*$.

$(ii)\Rightarrow(iii)$: If (ii) holds then it is clear from (16) that, for every $e\in E$, e^* is the greatest idempotent above e. Suppose now that $e,f\in E$. Then, by (ii), we have $e^*,f^*\in E$. Observe now from (20) and (ii) that $e^*\le S(e^*,f^*)^*\in E$ and $f^*\le S(e^*,f^*)^*\in E$. The maximiality of e^*, f^* now gives $e^*=S(e^*,f^*)^*=f^*$. Since this holds for all $e,f\in E$, it follows that S has a greatest idempotent.

$(iii)\Rightarrow(iv)$: Suppose now that ξ is the greatest idempotent in S. Then if $x\in S$ is such that $x^2\le x$ we have

$$xxx^*x=xx\le x,$$

which gives $xx^*\le x^*$ and hence

$$x=xx^*x\le x^*x\le\xi.$$

Thus $\xi=\max\{x\in S;x^2\le x\}$.

Now since ξ is the greatest idempotent of S it follows from (19) that $\xi=\alpha_x^*$ for every $x\in S$. Thus we have

$$x^*\xi=x^*\alpha_x^*=x^*(xx^*)^*$$

and hence

$$xx^*\xi xx^*=xx^*(xx^*)^*xx^*=xx^*$$

and consequently

$$xx^*\xi x=x.$$

This gives on the one hand $x^*\xi\le x^*$. But, on the other hand, we have $x^*=x^*x^{**}x^*\le x^*\xi$. Hence we see that $x^*\xi=x^*$ for all $x\in S$, and similarly $\xi x^*=x^*$.

Thus we see that conditions (β), (γ) of [**2**, Theorem 1] hold and therefore S is a perfect Dubreil–Jacotin semigroup.

$(iv)\Rightarrow(i)$: This is immediate from the fact that in a perfect Dubreil–Jacotin semigroup we have $x^*=\xi:x$ and, for every idempotent e, $e^*=e^{**}$ (see [**2**, (8), (9)]).

THEOREM 4. *Let S be a principally ordered regular semigroup in which $x\mapsto x^*$ is antitone. Then $S^\circ=\{x^\circ;x\in S\}$ is a subsemigroup of S with the same properties.*

Proof. Let $a,b\in S^\circ$ so that, by (11), $a=a^{\infty}$ and $b=b^{\infty}$. By (13), we have $ab(ab)^\circ\le aa^\circ$ and so, by (19),

$$\alpha_a^*=(aa^\circ)^*\le[ab(ab)^\circ]^*=[ab(ab)^*]^*=\alpha_{ab}^*.$$

By the maximality of α_a^*, it follows that

$$(aa^\circ)^*=[ab(ab)^\circ]^*.$$

Consequently, we have

$$\begin{aligned}(ab)^{\circ\circ}(ab)^{\circ}ab &= [ab(ab)^{\circ}]^{\circ}ab \quad \text{by (15)}\\ &= [ab(ab)^{\circ}]^{\circ}ab(ab)^{\circ}ab\\ &= [ab(ab)^{\circ}]^{*}ab(ab)^{\circ}ab\\ &= (aa^{\circ})^{*}ab\\ &= (aa^{\circ})^{*}aa^{\circ}ab\\ &= (aa^{\circ})^{\circ}aa^{\circ}ab\\ &= (aa^{\circ})^{\circ}ab\\ &= a^{\circ\circ}a^{\circ}ab \quad \text{by (15)}\\ &= ab \quad \text{since} \quad a = a^{\circ\circ},\end{aligned}$$

and similarly $ab = ab(ab)^{\circ}(ab)^{\circ\circ}$. It follows that

$$ab = (ab)^{\circ\circ}(ab)^{\circ}ab(ab)^{\circ}(ab)^{\circ\circ} = (ab)^{\circ\circ},$$

and so $ab \in S^{\circ}$. Thus S° is a subsemigroup. It is principally ordered with $x \mapsto x^{*}$ antitone since for every $x^{\circ} \in S^{\circ}$ we have, by (7) and (9), $x^{\circ *} = x^{**} = x^{*\circ} \in S^{\circ}$.

THEOREM 5. *Let S be a principally ordered regular semigroup in which $x \mapsto x^{*}$ is antitone. Then the following statements are equivalent:*

(1) *S° is orthodox;*
(2) *S° is inverse;*
(3) *S° has a greatest idempotent;*
(4) *S is perfect Dubreil–Jacotin.*

Proof. (1)⇒(4): Suppose that S° is orthodox and let $e \in E$. Then, observing that $e^{\circ}e^{\circ\circ} \in E \cap S^{\circ}$ and $e^{\circ\circ}e^{\circ} \in E \cap S^{\circ}$, we have, using (12),

$$e^{\circ} = e^{\circ}e^{\circ\circ}e^{\circ} = e^{\circ}e^{\circ\circ} \,.\, e^{\circ\circ}e^{\circ} \in E.$$

It follows by (17) that $e^{*} \in E$ and then by Theorem 3 that S is a perfect Dubreil–Jacotin semigroup.

(4)⇒(2): In a perfect Dubreil–Jacotin semigroup we have $x^{*} = \xi : x$ and $x^{\circ} = (\xi : x)x(\xi : x)$. If $e \in E$ then $\xi : e = \xi$ and so $e^{\circ} = \xi e \xi$. Since, as shown in [**2**], we have $e = ee^{*}e = e\xi e$, it follows that $e^{\circ} \in E \cap S^{\circ}$ and that $e^{\circ\circ} = e^{\circ}$. Now if f is an idempotent in S° then clearly $f = f^{\circ\circ} = f^{\circ}$, and so every idempotent $f \in S^{\circ}$ is of the form g° for some idempotent $g(=f^{\circ}) \in S$. Now since $\xi S \xi$ is inverse we have $e^{\circ}f^{\circ} = f^{\circ}e^{\circ}$ for all $e, f \in E$. Consequently, the idempotents of S° commute and so S° is inverse.

(2)⇒(1): this is clear.

(3)⇒(4): For every $e \in E$ we have $e \leq e^{\circ\circ} \in E$, by (12). Thus, if S° has a greatest idempotent then so must S, whence S is perfect Dubreil–Jacotin by Theorem 3.

(4)⇒(3): If S is perfect Dubreil–Jacotin then, by [**2**], $\xi = \xi^{\circ} \in S^{\circ}$ and is the greatest idempotent.

EXAMPLE 6. In the semigroup $N_5 \times \bar{E}$ of Example 4 we have $(p, x)^{\circ} = (upu, x^{*})$. It follows that

$$(N_5 \times \bar{E})^{\circ} = \{(b, x^{*}), (u, x^{*}); x \in \bar{E}\} = \{b, u\} \times \bar{E},$$

which is not orthodox.

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For every idempotent $e \in S$ we know that $e \leq e^\circ \leq e^*$. We now investigate some consequences of equality occurring.

An idempotent e is maximal if and only if $e = e^*$. (21)

If e is a maximal idempotent then clearly $e = ee^\circ = e^\circ e$ and so $e^\circ = e^\circ e e^\circ = ee^\circ = e \in E$. It follows by (17) that $e^* \in E$ and hence that $e = e^*$ by the maximality of e in E. Conversely, if $e = e^*$ then $e = ee^*$ and so

$$e = e^* = (ee^*)^* = \alpha_e^*$$

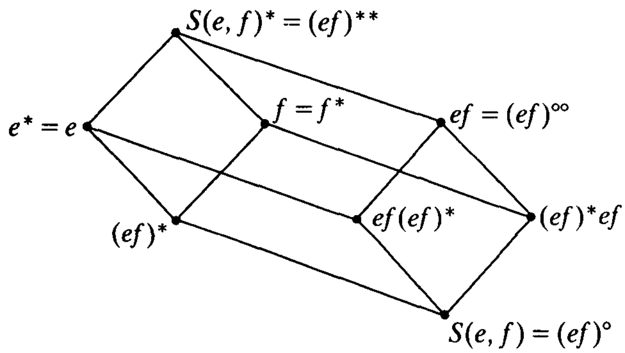
whence, by (19), e is a maximal idempotent.

If $e, f \in E$ *are such that* $e = e^\circ$ *and* $f = f^\circ$ *then*
$$(ef)^\circ = S(e, f) \in E. \tag{22}$$

Two applications of (14) give

$$(ef)^\circ = f^\circ(e^\circ eff^\circ)^\circ e^\circ = f(ef)^\circ e = S(e, f) \in E.$$

If e, f are maximal idempotents then we have the Hasse diagram (23)



By (21), we have $e = e^\circ = e^*$ and $f = f^\circ = f^*$. Now since the idempotent $S(e, f)$ absorbs e on the right we have $e \,.\, S(e, f) \in E$, and indeed $e \,.\, S(e, f) \leq e$ whence $e \,.\, S(e, f) \leq e$ by Theorem 1. Thus $e \,.\, S(e, f) \,.\, e \leq e$ and so $S(e, f) \leq e^* = e$. Similarly, we have $S(e, f) \leq f$. As $S(e, f) \in E$, we deduce that $S(e, f) \leq ef$. By Theorem 4, $ef = e^\circ f^\circ \in S^\circ$ and so $ef = (ef)^{\circ\circ}$. Now, by (22) we have $S(e, f) = (ef)^\circ \leq e$, which gives $e = e^* \leq S(e, f)^* = (ef)^{\circ *} = (ef)^{**}$ by (7), and then $(ef)^* \leq e^* = e$. Similarly, $(ef)^* \leq f$. Consider now the idempotent $(ef)^*ef$. We have

$$(a) \quad (ef)^*ef \leq e \,.\, ef = ef,$$

and, by (13),

$$(b) \quad (ef)^*ef \leq f^*f = f.$$

Similarly, $ef(ef)^* \leq ef$ and $ef(ef)^* \leq e$. Finally, since $(ef)^\circ = S(e, f) \in E$ we have

$$(ef)^\circ \leq (ef)^\circ(ef)^{\circ\circ} = (ef)^\circ ef = (ef)^*ef$$

and likewise $(ef)^\circ \leq ef(ef)^*$. The diagram now follows by (1) and (10).

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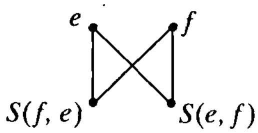
THEOREM 6. *Let S be a principally ordered regular semigroup in which $x \mapsto x^*$ is antitone. If S is not a perfect Dubreil–Jacotin semigroup then S necessarily contains a crown of idempotents of the form*



in which e, f are maximal.

Moreover, if these are the only idempotents in S then they generate the boot-lace semigroup.

Proof. If S is not perfect Dubreil–Jacotin then, by (19) and Theorem 3, S must contain at least two maximal idempotents e, f. Now for these idempotents we must have $S(e,f)$ incomparable to $S(f,e)$. For example, if we had

$$S(e,f)=(ef)^\circ \leq (fe)^\circ = S(f,e)$$

then it would follow by (13) that

$$fe(ef)^\circ \leq fe(fe)^\circ \leq ff^\circ = f$$

and consequently

$$fef = feS(e,f)f = fe(ef)^\circ f \leq f,$$

which gives the contradiction $e \leq f^* = f$. It now follows by (23) and the corresponding diagram involving the product fe that S contains a 4-element crown of idempotents as described.

Suppose now that these are the only idempotents in S. Writing $S(e,f)=(ef)^\circ = g$ and $S(f,e)=(fe)^\circ = h$, we clearly have $g = fg = ge$ and $h = hf = eh$. Now the idempotent $(ef)^\circ ef$ cannot be equal to $g=(ef)^\circ$, for if this were so we would have

$$ef = ef(ef)^\circ ef = efg = ef(ef)^\circ \in E$$

whence $ef \leq (ef)^\circ = g \leq e$ which gives $efe \leq e$ and the contradiction $f \leq e^* = e$. It follows that we must have $(ef)^\circ ef = f$ and therefore

$$gf = (ef)^\circ f = (ef)^\circ ef = f.$$

Likewise, $fe(fe)^\circ = f$ and

$$fh = f(fe)^\circ = fe(fe)^\circ = f.$$

In a similar way we can show that $eg = he = e$.

Observe now that $fe \geq he = e$, $fe \geq fh = f$ and $ef \geq eg = e$, $ef \geq gf = f$. Moreover, e, f, ef, fe are distinct since otherwise $ef \in E$ and this implies, by the maximality of e and f, the contradiction $e = f$. Next, observe that $efe \geq ef, fe$ and $fef \geq ef, fe$ with ef, fe, efe, fef distinct (since otherwise we have that $efe \in E$ whence the contradiction $e = f$). Continuing this argument, and doing so similarly with the minimal idempotents g and h, we see that the semigroup generated by the idempotents is the boot-lace semigroup as described in Example 3.

Corollary. *Let PA denote the class of principally ordered regular semigroups in which $x \mapsto x^*$ is antitone and let PDJ denote the class of perfect Dubreil–Jacotin semigroups. Then the boot-lace is the semigroup in PA\PDJ with the least number of idempotents.*

Further properties concerning the subsemigroup $\bar{E}$ generated by the set E of idempotents are the following. Note that if $x \in \bar{E}$ then $x^\circ \in \bar{E}$; this follows by [**5**].

Theorem 7. *Let S be a principally ordered regular semigroup in which $x \mapsto x^*$ is antitone. Then S is a perfect Dubreil–Jacotin semigroup if and only if $\bar{E}$ is periodic.*

Proof. $\Rightarrow$: If S is perfect Dubreil–Jacotin then for every $e \in E$ we have $\xi : e = \xi$. It follows by [**1**, Theorem 25.5] that $\xi : x = \xi$ for every $x \in \bar{E}$. Thus, for every $x \in \bar{E}$, we have

$$x = x(\xi : x)x = x\xi x,$$

and so ξ is a medial idempotent in the sense of [**4**], by Theorem 1.1 of which it follows that $\bar{E}$ is periodic.

$\Leftarrow$: Let $\bar{E}$ be periodic and let e, f be maximal idempotents of S. Observe that by (a) of (23) we have

$$ef = ef(ef)^n ef \leq (ef)^2.$$

As $\bar{E}$ is periodic, it follows that for some positive integer n we have

$$ef \leq (ef)^2 \leq \ldots \leq (ef)^n \in E.$$

Consequently, by (a), (b) of (23), by (16), and by the maximality of f, we see that

$$ef \leq (ef)^n \leq [(ef)^* ef]^* = f^* = f,$$

and similarly $ef \leq e$. By (a) again, we then have $(ef)^* ef \leq e$ whence

$$e = e^* \leq [(ef)^* ef]^* = f^* = f.$$

The maximality of e now gives $e = f$. Since this holds for all maximal idempotents e, f it follows that S has a greatest idempotent and so, by Theorem 3, is perfect Dubreil–Jacotin.

Corollary. *If $\bar{E}$ is finite then S is perfect Dubreil–Jacotin.*

Proof. If $\bar{E}$ is finite then it is necessarily periodic.

Finally, we turn our attention to the regularity of Green's relations $\mathscr{L}$ and $\mathscr{R}$. Here we have the following characterisation.

Theorem 8. *Let S be a principally ordered regular semigroup in which $x \mapsto x^*$ is antitone. Then the following statements are equivalent*:

(1) *$\mathscr{L}$ is regular on $\bar{E}$;*
(2) *$\mathscr{R}$ is regular on $\bar{E}$;*
(3) *S is a perfect Dubreil–Jacotin semigroup.*

Proof. (1)$\Rightarrow$(3): Suppose that $\mathscr{L}$ is regular on $\bar{E}$. If $e, f \in E$ are such that $e \leq f$ then $e^\circ e \leq f^\circ f$ and so

$$\beta_f^* = (f^\circ f)^* \leq (e^\circ e)^* = \beta_e^*,$$

where $f \leq f^\circ f \leq \beta_f^* \leq \beta_e^*$. Consequently, β_e^* is the greatest idempotent above e. It follows that $\alpha_e^* \leq \beta_e^*$ whence, by the maximality of α_e^*, we have

$$\alpha_e^* = \beta_e^* = \gamma_e, \text{ say.}$$

Now $e \leq \gamma_e$ gives, by (16), $e = e\gamma_e e$ and so $(e\gamma_e)^2 = e\gamma_e \mathcal{R} e$. But ee° is the greatest idempotent in the $\mathcal{R}$-class of e, and $ee^\circ \leq e\gamma_e$. Hence $ee^\circ = e\gamma_e$, whence $e^\circ = e^\circ e\gamma_e$. Similarly, we have $e^\circ e = \gamma_e e$ and so $e^\circ = \gamma_e ee^\circ$. It now follows that

$$e^\circ e^\circ = e^\circ e\gamma_e \,.\, \gamma_e ee^\circ = e^\circ \,.\, e\gamma_e e \,.\, e^\circ = e^\circ ee^\circ = e^\circ$$

and so $e^\circ \in E$. By (17) we deduce that $e^* \in E$ and then, by Theorem 3, that S is perfect Dubreil–Jacotin.

(3)$\Rightarrow$(1): If S is perfect Dubreil–Jacotin then, denoting the elements of $\bar{E}$ by $\bar{x}$, we have

$$\bar{e} \leq \bar{f} \Rightarrow \bar{e}^\circ \bar{e} = \bar{e}^* \bar{e} = \xi \bar{e} \leq \xi \bar{f} = \bar{f}^\circ \bar{f},$$

so that $\mathcal{L}$ is regular on $\bar{E}$.

The proof of (1)$\Leftrightarrow$(2) is similar.

We remark here that although, in a perfect Dubreil–Jacotin semigroup S, Green's relations $\mathcal{R}$ and $\mathcal{L}$ are regular on the subsemigroup generated by the idempotents, they are not in general regular on S. This is illustrated in the following example.

EXAMPLE 7. Consider a fixed integer $k > 1$. For every $n \in \mathbb{Z}$ let n_k denote the largest multiple of k that is less than or equal to n, so that we have

$$n_k = tk \leq n < (t+1)k.$$

It is readily seen that $\mathbb{Z}$, under the usual order and the law of composition described by $(m, n) \mapsto m + n_k$, is a principally ordered regular semigroup in which $n^* = -n_k + k - 1$. The mapping $x \mapsto x^*$ is then antitone. Since the idempotents are $0, 1, \ldots, k-1$ it follows that this semigroup is perfect Dubreil–Jacotin.

Consider now the cartesian ordered semigroup $N_5 \times \mathbb{Z}$ where N_5 is as in Example 4. This is clearly non-orthodox and perfect Dubreil–Jacotin. By Theorem 8, $\mathcal{R}$ and $\mathcal{L}$ are regular on the subsemigroup generated by the idempotents. Now for every $(x, n) \in N_5 \times \mathbb{Z}$ we have

$$(x, n)(x, n)^* = (x, n)(u, -n_k + k - 1) = (xu, n - n_k);$$
$$(x, n)^*(x, n) = (u, -n_k + k - 1)(x, n) = (ux, k - 1).$$

It is clear from these equalities that $\mathcal{L}$ is regular on $N_5 \times \mathbb{Z}$, whereas $\mathcal{R}$ is not.

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