



RESEARCH ARTICLE

Relative rank and regularization

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Received: 4 August 2022; Revised: 21 September 2023; Accepted: 30 January 2024

2020 Mathematics Subject Classification: Primary – 11T06, 11B30, 13F20

Abstract

We introduce a new concept of rank – *relative rank* associated to a filtered collection of polynomials. When the filtration is trivial, our relative rank coincides with *Schmidt rank* (also called *strength*). We also introduce the notion of *relative bias*. The main result of the paper is a relation between these two quantities over finite fields (as a special case, we obtain a new proof of the results in [21]). This relation allows us to get an accurate estimate for the number of points on an affine variety given by a collection of polynomials which is of high relative rank (Lemma 3.2). The key advantage of relative rank is that it allows one to perform an efficient regularization procedure which is *polynomial* in the initial number of polynomials (the regularization process with Schmidt rank is far worse than tower exponential). The main result allows us to replace Schmidt rank with relative rank in many key applications in combinatorics, algebraic geometry, and algebra. For example, we prove that any collection of polynomials $\mathcal{P} = (P_i)_{i=1}^c$ of degrees $\leq d$ in a polynomial ring over an algebraically closed field of characteristic $> d$ is contained in an ideal $\mathcal{I}(\mathcal{Q})$, generated by a collection \mathcal{Q} of polynomials of degrees $\leq d$ which form a regular sequence, and \mathcal{Q} is of size $\leq Ac^A$, where $A = A(d)$ is independent of the number of variables.

1. Introduction

In [23], Schmidt introduced a notion of complexity, the *Schmidt rank* (also called the *h*-invariant)¹ for a collection of polynomials $\mathcal{P} = (P_i)_{i=1}^c$, and used it to obtain conditions on the existence of integer solutions for a system of equations $\{P_i(x) = 0\}_{i=1}^c$ defined over the rational numbers. The most natural polynomials on which to define this quantity are homogeneous ones. If P is a polynomial of degree d , we denote by \tilde{P} the degree d homogeneous component of P .

Definition 1.1 (Schmidt rank). The *Schmidt rank* of a homogeneous polynomial $P \in \mathbb{F}[x_1, \dots, x_n]$ is the minimal length of a presentation $P = \sum_{i=1}^r Q_i R_i$, where for all $i \in [r]$, Q_i, R_i are homogeneous polynomials of degree $< \deg(P)$, and is denoted by $rk(P)$. For a general polynomial P we set $rk(P) = rk(\tilde{P})$. The rank of a collection of homogeneous polynomials $\mathcal{P} = (P_i)_{i=1}^c$ is

$$rk(\mathcal{P}) = \min\{rk(a_1 P_1 + \dots + a_c P_c) : 0 \neq a \in \mathbb{F}^c\}.$$

If \mathcal{P} is a general collection of polynomials, we set $rk(\mathcal{P}) = rk(\tilde{\mathcal{P}})$, where $\tilde{\mathcal{P}} = (\tilde{P}_i)_{i=1}^c$.

¹This notion of complexity was reintroduced later in the work of Ananyan and Hochster [2] in their proof of Stillman's conjecture where it is called *strength*.

Now assume $\mathbb{F} = \mathbb{F}_q$ is a finite field and fix a nontrivial character $\chi : \mathbb{F} \rightarrow \mathbb{C}$. We define the *bias* of a polynomial as follows.

Definition 1.2. For a function $P : \mathbb{F}^n \rightarrow \mathbb{F}$, we define

$$\text{bias}(P) = |\mathbb{E}_{x \in \mathbb{F}^n} \chi(P(x))|,$$

where $\mathbb{E}_{x \in E} := \frac{1}{|E|} \sum_{x \in E}$ for a finite set E , with the convention $\mathbb{E}_{x \in \emptyset} = 0$. This quantity may of course depend on χ , but we omit it from the notation.²

The relation between bias and rank can already be traced to the work of Schmidt; indeed, in his paper [23], Schmidt showed that over the complex field, the Schmidt rank of a polynomial is proportional to the codimension of the singular locus of the associated variety, and used this for estimating related exponential sums.

In [10], Green and Tao observed that one can make the relation independent of the number of variables:³

Theorem 1.3 (Bias implies low rank). *Let $\mathbb{F} = \mathbb{F}_q$ be a finite field, let $0 \leq d < \text{char}(\mathbb{F})$, and let $s > 0$. There exists a constant $C = C(\mathbb{F}, s, d)$, such that the following holds: For any n , $P \in \mathbb{F}[x_1, \dots, x_n]$ of degree $\leq d$ with $\text{bias}(P) \geq q^{-s}$, we have $\text{rk}(P) \leq C$.*

This result was subsequently extended to include the case $d \geq \text{char}(\mathbb{F})$ by Kauffman and Lovett in [13]. Bhowmick and Lovett [3] showed that the bound can be made independent of the field \mathbb{F} in the sense that we can take $C = C(s, d)$. Returning to the case $d < \text{char}(\mathbb{F})$, effective bounds for C were given by Milićević [21] who proved

$$C = A(d)(1+s)^{B(d)}.$$

Independently, Janzer [12] proved a similar, slightly weaker result with $s \log q$ instead of s .

Conjecture 1.4. *For all finite fields \mathbb{F} with $\text{char}(\mathbb{F}) > d$, we have $C(\mathbb{F}, s, d) = A(d)(1+s)$.*

Conjecture 1.4 was recently proved in the case $d = 3$ in [1], [5] independently.

A key feature of high rank collections of polynomials, that can be easily derived from Theorem 1.3, is that all fibers of the map $\mathcal{P} : \mathbb{F}^n \rightarrow \mathbb{F}^c$ are essentially of the same size.

Theorem 1.5 (Size of fibers). *Let $d > 0$. Let $\mathbb{F} = \mathbb{F}_q$ be a finite field of characteristic $> d$. There exists $A = A(d)$, such that the following holds: For any n , any collection of polynomials $\mathcal{P} = (P_i)_{i=1}^c$ in $\mathbb{F}[x_1, \dots, x_n]$ of degrees $\leq d$ and $\text{rank} > Ac^A$, any $a, b \in \mathbb{F}^c$, we have $||\mathcal{P}^{-1}(a)|/|\mathcal{P}^{-1}(b)| - 1| < q^{-1}$.*

Using model theoretic techniques, one can translate the data in Theorem 1.5 to an estimate of the dimension of the fibers of the map $\mathcal{P} : \mathbb{A}^n \rightarrow \mathbb{A}^c$ for any algebraically closed field of characteristic zero or $> \max \deg P_i$ (see, e.g. [15, 16]).

In many proofs using the notion of Schmidt rank, a key procedure is regularization (e.g. [23], [3], [6]): given a collection of polynomials, one applies procedure replacing the original collection of polynomials \mathcal{P} with a new collection \mathcal{Q} that is of high Schmidt rank compared to its size, (so Theorem 1.5 holds) and such that the ideal generated by \mathcal{P} is contained in the ideal generated by \mathcal{Q} . The drawback in this regularization procedure is that the size of \mathcal{Q} is far worse than even tower exponential in the size of \mathcal{P} (the bound that can be derived from Schmidt's argument was worked out by Wooley in [25]).

We introduce a new notion of rank, *relative rank*, associated with a filtered collection of polynomials, both in the algebraic and analytic contexts. We replace the ambient space \mathbb{F}^n by certain affine varieties, and we replace Schmidt rank by an appropriate relative rank. The definition of relative rank is somewhat technical; we defer it to the next section (Definition 2.1). We remark that the new notion of relative

²For multilinear polynomials, $\text{bias}(P)$ does not depend on the character χ ; in this case, $-\log_q(\text{bias}(P))$ was introduced in [11] as the *analytic rank* of P .

³They use a slightly weaker definition of rank, but the results in their paper imply the theorem as stated.

rank coincides with the Schmidt rank and the analytic rank when the filtration consists of one set, but is different otherwise.

Our main result (Theorem 2.2), described in detail in the next section, is a relation between relative rank and relative bias, similar to the relation described above between rank and bias. The key advantage of our new notion of rank is that while it retains many of the properties of the ranks described above, for example, Theorem 1.5 remains valid, the regularization procedure with respect to this notion is *polynomial* in the size of the original collection. As such, we are able to give good quantitative bounds on a variety of problems in which a regularization procedure is used. We provide an example of an algebraic application in Theorem 2.6.

In the interest of exposition, we will start by presenting a special case of our main result. This special case is actually the result we initially set out to prove. First, we give our definitions of relative rank and relative bias.

Definition 1.6 (Relative rank). The relative rank of a homogeneous polynomial P on a collection of homogeneous polynomials $\mathcal{Q} = (Q_1, \dots, Q_m)$ is

$$rk_{\mathcal{Q}}(P) := \min\{rk(P + \sum_{i=1}^m R_i Q_i) : \deg(R_i) + \deg(Q_i) \leq \deg(P) \forall i \in [m]\}.$$

Note that whenever $\deg(Q_i) > \deg(P)$, this implies $R_i = 0$. For general polynomials, with $\tilde{P}, \tilde{\mathcal{Q}}$ the corresponding homogeneous polynomial and collection, set $rk_{\mathcal{Q}}(P) := rk_{\tilde{\mathcal{Q}}}(\tilde{P})$.

Definition 1.7 (Relative bias). The relative bias of a function $P : \mathbb{F}^n \rightarrow \mathbb{F}$ on a subset $X \subset \mathbb{F}^n$ is

$$bias_X(P) = |\mathbb{E}_{x \in X} \chi(P(x))|.$$

For a collection $\mathcal{Q} = (Q_i)_{i \in [m]}$ of polynomials in $\mathbb{F}[x_1, \dots, x_n]$, denote the zero locus by

$$Z(\mathcal{Q}) = \{x \in \mathbb{F}^n : Q_i(x) = 0 \forall i \in [m]\}.$$

A special case of our main result is then:

Theorem 1.8. Let \mathbb{F} be a finite field and $0 \leq d < \text{char}(\mathbb{F})$. There exist constants $A(d), B(d)$, such that if $\mathcal{Q} = (Q_i)_{i=1}^m$ is a collection of polynomials with degrees $\leq d$ with $rk(\mathcal{Q}) > A(m + s)^B$ and P is a polynomial of degree $\leq d$ with $bias_{Z(\mathcal{Q})}(P) \geq q^{-s}$, then we have

$$rk_{\mathcal{Q}}(P) \leq A(1 + s)^B.$$

We briefly pause here and compare this to the result of Milićević (and Janzer). For $\mathcal{Q} = \emptyset$, this gives a new proof of their result. Moreover, when $m \leq s$ or when \mathcal{Q} is composed of polynomials of degree $\geq \deg(P)$, the above theorem, in fact, follows from their result. But it is new in the case that \mathcal{Q} is composed of $\gg s$ polynomials of lower degree than P .

2. Main theorem

We now give our definitions in full and state our main theorem.

Definition 2.1.

1. The relative rank of a collection of homogeneous polynomials $\mathcal{P} = (P_i)_{i \in [m]}$ on another collection of polynomials \mathcal{Q} is

$$rk_{\mathcal{Q}}(\mathcal{P}) = \min\{rk_{\mathcal{Q}}(a \cdot \mathcal{P}) : 0 \neq a \in \mathbb{F}^m\}.$$

If \mathcal{P} is a general collection of polynomials with corresponding homogeneous collection $\tilde{\mathcal{P}}$, then $rk_{\mathcal{Q}}(\mathcal{P}) = rk_{\tilde{\mathcal{Q}}}(\tilde{\mathcal{P}})$.

2. A tower

$$\mathcal{Q} = (\mathcal{Q}_i)_{i \in [h]}$$

of **height** h is composed of h collections of polynomials $\mathcal{Q}_i = (\mathcal{Q}_{i,j})_{j \in [m_i]}$ which we call **layers**, such that the polynomials in each layer all have the same degree d_i . The **degree** of the tower is $\max(d_1, \dots, d_h)$. The **dimension** of the tower is $m_1 + \dots + m_h$. We denote the truncated tower $\mathcal{Q}_{<i} = (\mathcal{Q}_j)_{j \in [i-1]}$.

3. The tower \mathcal{Q} is (A, B, s) -regular if for every $i \in [h]$, we have

$$rk_{\mathcal{Q}_{<i}}(\mathcal{Q}_i) > A(m_i + m_{i+1} + \dots + m_h + s)^B.$$

The main theorem we prove is the following:

Theorem 2.2 (Relative bias implies relative low rank). *There exist constants $A(d, H), B(d, H)$, such that if \mathcal{Q} is an (A, B, s) -regular tower of degree $\leq d < \text{char}(\mathbb{F})$ and height $\leq H$, and P is a polynomial of degree $\leq d$ with*

$$\text{bias}_{Z(\mathcal{Q})}(P) \geq q^{-s},$$

then

$$rk_{\mathcal{Q}}(P) \leq A(1 + s)^B.$$

Theorem 2.2 implies that relatively regular collections of polynomials enjoy similar equidistribution properties to those of high rank collections of polynomials. In fact, all the results in [15, 16, 17], [3], which were proved for collections of polynomials of high rank, are also valid for relatively regular collections of polynomials. Some examples of results that transfer seamlessly to the relatively regular setting are given in the Appendix.

The dependence of the constants A, B on the parameters d, H , which can be extracted from our proof, is quite weak and involves recursively defined functions. This is due to the rather involved inductive argument. It is natural to ask whether this result holds with a stronger dependence on d, H .

Question: Can the constants A, B in theorem 2.2 be taken to be “reasonable” functions of d, H ? Say, a tower of exponentials of bounded height as in Milićević’s result [21]?

For a collection of polynomials \mathcal{Q} , we write $I(\mathcal{Q})$ for the ideal they generate. In many applications of the notion of rank (e.g [23], [6]) one is given a collection of polynomials, to which one applies a regularization process, replacing the original collection \mathcal{Q} with a new regular collection \mathcal{Q}' , such that any $\mathcal{Q} \subset I(\mathcal{Q}')$. The issue is that the regularization process is very costly (see [25]). One of the advantages of relative rank is that it allows a very efficient process of regularization.

Theorem 2.3 (Polynomial regularity). *Let A, B, d be given. There exist constants $C(A, B, d), D(A, B, d)$, such that if \mathcal{Q} is a collection of m homogeneous polynomials of positive degrees $\leq d$, then there exists a polynomial tower \mathcal{Q}' of dimension $\leq C(m + s)^D$, degree $\leq d$, and height $\leq d$, such that \mathcal{Q}' is (A, B, s) -regular, and $\mathcal{Q} \subset I(\mathcal{Q}')$.*

Definition 2.4. A collection of polynomials $\mathcal{Q} = (\mathcal{Q}_i)_{i=1}^m \subset \mathbb{F}[x_1, \dots, x_n]$ is s -uniform if

$$\left| \frac{|Z(\mathcal{Q})|}{q^{n-m}} - 1 \right| \leq q^{-s}.$$

As a corollary of Theorem 2.2, we obtain:

Lemma 2.5 (Regular varieties are of the expected size). *There exist constants $A(d, H), B(d, H)$, such that any (A, B, s) -regular polynomial tower \mathcal{Q} of degree $\leq d < \text{char}(\mathbb{F})$ and height $\leq H$ is s -uniform.*

In particular, we obtain the following algebraic consequence:

Theorem 2.6. *Let $d > 0$. Let \mathbb{F} be an algebraically closed field of characteristic zero or $> d$. There exists a constant $A = A(d)$, such that if \mathcal{P} is a collection of c homogeneous polynomials of positive degrees $\leq d$, then there exists a collection \mathcal{Q} of $\leq Ac^A$ homogeneous polynomials of degrees $\leq d$, such that \mathcal{Q} is a regular sequence, and $P_i \in \mathcal{I}(\mathcal{Q})$.*

We prove Theorem 2.6 in the Appendix; it follows from Theorem 2.3 and an adaptation of the results in [15] to the context of relative rank. We also obtain a robust Nullstellensatz result for regular collections.

Theorem 2.7 (Robust Nullstellensatz for regular collections). *There exist constants $A(d, H), B(d, H), s(d)$, such that if $\mathcal{Q} = (Q_{i,j})_{i \in [h], j \in [m_i]}$ is an (A, B, s) -regular polynomial tower of $\deg \leq d < \text{char}(\mathbb{F})$ and height $\leq H$, and P is a polynomial of degree $\leq d$ which vanishes q^{-s} -a.e. on $Z(\mathcal{Q})$, then there exist polynomials $R_{i,j}$ satisfying $\deg(R_{i,j}) + \deg(Q_{i,j}) \leq \deg(P)$ and*

$$P = \sum_{i,j} R_{i,j} Q_{i,j}.$$

In order to prove Theorem 2.2, as in [12, 21], we first reduce it to an analogous statement for multilinear maps. However, the reduction of Theorem 2.2 to its multilinear version is not so straightforward in our case. This will be done by induction on the degree d and will require some intermediate results that appear during the proof of the theorem in the multilinear case. Therefore, we postpone it to Section 9. We now turn to the multilinear case.

2.1. Multilinear definitions and main theorem

Let V_1, \dots, V_k be finite-dimensional vector spaces over the finite field \mathbb{F} . For $I \subset [k]$, we denote $V^I = \prod_{i \in I} V_i$. For $x \in V^{[k]}$, we denote by x_I the projection of x to V^I .

In order to prove our main theorem for multilinear functions, we will need to work with a slightly larger class of functions which we call full multiaffine maps. Recall that a multiaffine map $P : V^I \rightarrow \mathbb{F}$ has a unique presentation $P(x_I) = \sum_{J \subset I} P_J(x_J)$, where $P_J : V^J \rightarrow \mathbb{F}$ is multilinear for all $J \subset I$. By abuse of notation, we will also denote by P the function $V^{[k]} \rightarrow \mathbb{F}$ given by $x \mapsto P(x_I)$.

Definition 2.8. P is called **full** if $P_I \neq 0$. We denote by $\tilde{P} = P_I$ the multilinear part of P .

In [22], a notion of rank for multilinear functions is defined, called *partition rank*.

Definition 2.9. Let $P : V^I \rightarrow \mathbb{F}$ be a multilinear function. The **partition rank** of P is the minimal length of a presentation $P(x) = \sum_{i=1}^r Q_i(x_{I_i}) R_i(x_{I \setminus I_i})$, where $\emptyset \neq I_i \subseteq I$ and $Q_i : V^{I_i} \rightarrow \mathbb{F}, R_i : V^{I \setminus I_i} \rightarrow \mathbb{F}$ are multilinear for all $i \in [r]$. We write $r = \text{prk}(P)$. We extend this definition to full multiaffine maps via $\text{prk}(Q) = \text{prk}(\tilde{Q})$.

The bias of P on $V^{[k]}$ is defined as before, but it has some interesting properties when $P : V^I \rightarrow \mathbb{F}$ is multilinear. Writing $P(x) = A(x_{I \setminus \{i_0\}}) \cdot x_{i_0}$, where $A : V^{I \setminus \{i_0\}} \rightarrow V_{i_0}$ is multilinear for some $i_0 \in [k]$, Fourier analysis yields

$$\text{bias}(P) = |\mathbb{E}_{x \in V^{[k]}} \chi(P(x))| = \mathbb{P}_{x \in V^{I \setminus \{i_0\}}} (A(x) = 0),$$

where $\mathbb{P}_{x \in E}(T) := \frac{|T|}{|E|}$ for a subset $T \subset E$. In particular, the character sum above is always positive and does not depend on our choice of character χ . It is not too difficult to show that multilinear maps with low partition rank exhibit significant bias (see [14, 19]).

Claim 2.10 (Low partition rank implies bias). *If $P : V^I \rightarrow \mathbb{F}$ is multilinear with $\text{prk}(P) = r$, then*

$$\mathbb{E}_{x \in V^{[k]}} \chi(P(x)) \geq q^{-r}.$$

In [21], the following (much more difficult) inequality was proved in the other direction:

Theorem 2.11 (Bias implies low partition rank). *Let \mathbb{F} be a finite field and $0 \leq d < \text{char}(\mathbb{F})$. Suppose $P : V^I \rightarrow \mathbb{F}$ is multilinear with $|I| \leq d$ and $\text{bias}(P) \geq q^{-s}$. Then*

$$\text{prk}(P) \leq A(1 + s)^B,$$

where $A = A(d), B = B(d)$ are constants.

This result immediately extends to full multiaffine maps because a short calculation in [26] (which we will follow in Lemma 4.2) shows that

$$\text{bias}(P) \leq \text{bias}(\tilde{P}),$$

and by definition, they have the same partition rank. We now make the corresponding relative definitions for full multiaffine maps.

Definition 2.12.

1. Let $\mathcal{Q} = (Q_i : V^{I_i} \rightarrow \mathbb{F})_{i \in [m]}$ and $P : V^I \rightarrow \mathbb{F}$ be multilinear maps. The **partition rank of P relative to \mathcal{Q}** is

$$\text{prk}_{\mathcal{Q}}(P) = \min \left\{ \text{prk} \left(P - \sum_{i \in [m], I_i \subset I} Q_i(x_{I_i}) R_i(x_{I \setminus I_i}) \right) : R_i \text{ are multilinear} \right\}.$$

If P, \mathcal{Q} are full multiaffine, we define $\text{prk}_{\mathcal{Q}}(P) = \text{prk}_{\tilde{\mathcal{Q}}}(P)$.

2. The partition rank of a collection of multilinear maps $\mathcal{P} = (P_i : V^I \rightarrow \mathbb{F})_{i \in [c]}$ relative to \mathcal{Q} is

$$\text{prk}_{\mathcal{Q}}(\mathcal{P}) = \min \{ \text{prk}_{\mathcal{Q}}(a \cdot \mathcal{P}) : 0 \neq \mathbf{a} \in \mathbb{F}^c \}.$$

If \mathcal{P}, \mathcal{Q} are full multiaffine with all maps in \mathcal{P} defined on a fixed index set I , set $\text{prk}_{\mathcal{Q}}(\mathcal{P}) = \text{prk}_{\tilde{\mathcal{Q}}}(\tilde{\mathcal{P}})$.

3. A tower

$$\mathcal{Q} = (Q_i)_{i \in [h]}$$

is composed of h collections of full multiaffine functions $Q_i = (Q_{i,j} : V^{I_i} \rightarrow \mathbb{F})_{j \in [m_i]}$, which we call **layers**. The **associated multilinear tower** is $\tilde{\mathcal{Q}} = (\tilde{Q}_i)_{i \in [h]}$. We say the tower is multilinear if $\tilde{\mathcal{Q}} = \mathcal{Q}$. The **degree** of the tower is $\max_i |I_i|$, and the **height** of the tower is h . The **dimension** of the tower is $m = m_1 + \dots + m_h$. For $i \in [h]$, we denote the truncated tower by $\mathcal{Q}_{<i} = (Q_j)_{j \in [i-1]}$.

4. Let $A, B, s > 0$. We say that \mathcal{Q} is (A, B, s) -regular if for each $i \in [h]$, we have

$$\text{prk}_{\mathcal{Q}_{<i}}(Q_i) > A(m_i + m_{i+1} + \dots + m_h + s)^B.$$

Our main theorem for full multiaffine maps is

Theorem 2.13 (Relative bias implies low relative rank (d,H)). *There exist constants $A(d, H), B(d, H)$, such that if $P : V^I \rightarrow \mathbb{F}$ is a full multiaffine map with $|I| \leq d$, \mathcal{Q} is a tower on $V^{[k]}$ of degree $\leq d$ and height $\leq H$, such that:*

1. $\text{bias}_{Z(\mathcal{Q})}(P) \geq q^{-s}$ and
2. \mathcal{Q} is (A, B, s) -regular,

then $\text{rk}_{\mathcal{Q}}(P) \leq A(1 + s)^B$.

We will also prove a relative version of Claim 2.10.

Claim 2.14 (Low relative rank implies relative bias). *There exist constants $A(d, H), B(d, H)$, such that if \mathcal{Q} is an (A, B, s) -regular multilinear tower on $V^{[k]}$ of degree $\leq d$ and height $\leq H$ and $P : V^I \rightarrow \mathbb{F}$ with $|I| \leq d$ is multilinear with $\text{rk}_{\mathcal{Q}}(P) \leq r$, then*

1. $Re[\mathbb{E}_{x \in Z(\mathcal{Q})} \chi(P(x))] \geq q^{-r} - q^{-s}$ and
2. $|Im[\mathbb{E}_{x \in Z(\mathcal{Q})} \chi(P(x))]| \leq q^{-s}$.

Let us now give a high-level sketch of the proof of Theorem 2.13. This will proceed by induction on d , where the base case $d = 1$ follows from basic Fourier analysis. Assuming the theorem holds at degree d , we present some useful consequences in Section 3 and make some reductions in Section 4. After these reductions, the task at hand is to prove the theorem for multilinear $P : V^{[d+1]} \rightarrow \mathbb{F}$ and a multilinear tower \mathcal{Q} on $V^{[d+1]}$ of degree $\leq d$. In Section 5, we show that there are many “derived” towers \mathcal{Q}_t and multilinear functions $\phi_t : V^{[d]} \rightarrow \mathbb{F}^{100s}$, such that P vanishes q^{-s} -a.e. on the variety $Z(\mathcal{Q}_t) \cap \{\phi_t(x_{[d]}) = 0\}$. This is similar in spirit to previous work on this subject, but the derived towers appearing here present a new challenge. In Section 6, we pause to describe the relative regularization process, which is the multilinear analogue of Theorem 2.3. In Section 7, the regularization process is applied to the collections \mathcal{Q}_t, ϕ_t to obtain regular varieties on which P vanishes q^{-s} -a.e. In that section, we also show that this means that P vanishes identically on them, similarly to proposition 5.1 in [10]. A key novel step in our proof consists in then showing that these varieties can be “summed” to obtain a single tower \mathcal{R} of dimension $\leq A(1+s)^B$, such that $P \upharpoonright_{Z(\mathcal{Q} \cup \mathcal{R})} \equiv 0$ and $\mathcal{Q} \cup \mathcal{R}$ is regular. In Section 8, we prove a Nullstellensatz which allows us to conclude that $rk_{\mathcal{Q} \cup \mathcal{R}}(P) = 0$, so $rk_{\mathcal{Q}}(P) \leq \dim(\mathcal{R}) \leq A(1+s)^B$, which completes the inductive step.

3. Some equidistribution lemmas

As mentioned before, the proof of Theorem 2.13 will proceed by induction on d , and the base case $d = 1$ follows from Fourier analysis. For the purposes of the inductive argument, it will be useful to formulate a version of Theorem 2.13 with an auxiliary parameter, which we now do.

Theorem 3.1 (Relative bias implies low relative rank (d, H, l)). *There exist constants $A(d, H, l), B(d, H, l)$, such that if $P : V^I \rightarrow \mathbb{F}$ is a full multi-affine map with $|I| \leq d$, \mathcal{Q} is a tower on $V^{[k]}$ of degree $\leq d$ and height $\leq H$, such that:*

1. $bias_{Z(\mathcal{Q})}(P) \geq q^{-s}$,
2. \mathcal{Q} is (A, B, s) -regular, and
3. $\mathcal{Q}_{\leq H-l}$ is of degree $< d$,

then $rk_{\mathcal{Q}}(P) \leq A(1+s)^B$.

Theorem 2.13 corresponds, of course, to the case $l = H$. We call \mathcal{Q} as above a tower of type (d, H, l) for short. In this section, we assume Theorem 3.1 (d, H, l) holds and explore some of its useful consequences. We warm up with the following simple but important result.

Lemma 3.2 (Regular varieties are of the expected size $(d, H+1, l+1)$). *There exist constants $A(d, H, l), B(d, H, l)$, such that for any $s > 0$ and for any (A, B, s) -regular tower \mathcal{Q} on $V^{[k]}$ of type $(d, h, l+1)$ and dimension m , we have*

$$\left| q^m \frac{|Z(\mathcal{Q})|}{|V^{[k]}|} - 1 \right| \leq q^{-s}.$$

Proof. Let A', B' be the constants from Theorem 3.1 (d, H, l) . If A, B are sufficiently large depending on A', B' , then \mathcal{Q} is $(A', B', s+H+2)$ -regular. So for any $i \in [H+1]$ the tower $\mathcal{Q}_{<i}$ is $(A', B', s+H+m_i)$ -regular of type (d, H, l) and

$$rk_{\mathcal{Q}_{<i}}(\mathcal{Q}_i) > A'(s+H+m_i+2)^{B'}.$$

By Theorem 3.1, it follows that for any $0 \neq a \in \mathbb{F}^{m_i}$, we have

$$bias_{Z(\mathcal{Q}_{<i})}(a \cdot \mathcal{Q}_i) < q^{-(s+H+m_i+1)}.$$

Fourier analysis yields

$$q^{m_i} \frac{|Z(\mathcal{Q}_{\leq i})|}{|Z(\mathcal{Q}_{< i})|} = \sum_{a \in \mathbb{F}^{m_i}} \mathbb{E}_{x \in Z(\mathcal{Q}_{< i})} \chi(a \cdot \mathcal{Q}_i(x)) = 1 + e_i,$$

where $|e_i| \leq q^{-(s+H+1)}$. Then we multiply to get

$$\frac{|Z(\mathcal{Q})|}{|V^{[k]}|} = \prod_{i \in [H+1]} \frac{|Z(\mathcal{Q}_{\leq i})|}{|Z(\mathcal{Q}_{< i})|} = \prod_{i \in [H+1]} (1 + e_i) q^{-m_i} = (1 + e) q^{-m},$$

where $|e| \leq q^{-s}$. □

We now introduce some notation. For a tower \mathcal{Q} on $V^{[k]}$ and a subset $I \subset [k]$, we write $\mathcal{Q}_I = (\mathcal{Q}_i)_{i \in [h], I_i \subset I}$ for the restriction to a tower on V^I . Similarly, we define $Z(\mathcal{Q})_I = Z(\mathcal{Q}_I) \subset V^I$. For $y \in V^I$ and a full multi-affine function $\mathcal{Q} : V^J \rightarrow \mathbb{F}$, the full multi-affine function $\mathcal{Q}(y) : V^{J \setminus I} \rightarrow \mathbb{F}$ is defined by $\mathcal{Q}(y)(z) = \mathcal{Q}(y_{I \cap J}, z)$. Set $\mathcal{Q}(y) = (\mathcal{Q}_i(y))_{i \in [h]}$ for the tower induced on $V^{[k] \setminus I}$, where $\mathcal{Q}_i(y) = (\mathcal{Q}_{i,j}(y) : V^{I_i \setminus I} \rightarrow \mathbb{F})_{j \in [m_i]}$. For the zero locus, we write $Z(\mathcal{Q})(y) = Z(\mathcal{Q}(y))$.

We now state several interrelated lemmas.

Lemma 3.3 (Derivatives(d,H+1,l+1)). *For any C, D , there exist constants $A(C, D, d, H, l)$, $B(C, D, d, H, l)$, such that if \mathcal{Q} is an (A, B, s) -regular tower on $V^{[k]}$ of type $(d, H+1, l+1)$ and $I \subset [k]$, then for q^{-s} -a.e. $y \in Z(\mathcal{Q})_I$, the tower $\mathcal{Q}(y)$ on $V^{[k] \setminus I}$ is (C, D, s) -regular.*

Lemma 3.4 (Fubini(d,H+1,l+1)). *There exist constants $A(d, H, l), B(d, H, l)$, such that if \mathcal{Q} is an (A, B, s) -regular tower on $V^{[k]}$ of type $(d, H+1, l+1)$, $I \subset [k]$ and $g : V^{[k]} \rightarrow \mathbb{C}$ satisfies $|g(x)| \leq 1$ for all $x \in V^{[k]}$, then*

$$\left| \mathbb{E}_{x \in Z(\mathcal{Q})} g(x) - \mathbb{E}_{y \in Z(\mathcal{Q})_I} \mathbb{E}_{z \in Z(\mathcal{Q})(y)} g(y, z) \right| \leq q^{-s}.$$

Lemma 3.5 (Rank-bias(d,H+1,l+1)). *There exist constants $A(d, H, l), B(d, H, l)$, such that if \mathcal{Q} is an (A, B, s) -regular multilinear tower on $V^{[k]}$ of type $(d, H+1, l+1)$ and $P : V^I \rightarrow \mathbb{F}$ is a multilinear map with $|I| \leq d + 1$, then*

1. $|Im[\mathbb{E}_{x \in Z(\mathcal{Q})} \chi(P(x))]| \leq q^{-s}$ and
2. if $rk_{\mathcal{Q}}(P) = r$, then

$$Re[\mathbb{E}_{x \in Z(\mathcal{Q})} \chi(P(x))] \geq q^{-r} - q^{-s}.$$

The above lemmas will be proved together by induction on the height. The implications, assuming Theorem 3.1(d,H,l) throughout, are as follows

$$\text{Rank-bias}(d, H, l+1) + \text{Fubini}(d, H, l+1) \implies \text{Derivatives}(d, H+1, l+1)$$

$$\text{Derivatives}(d, H+1, l+1) \implies \text{Fubini}(d, H+1, l+1)$$

$$\text{Derivatives}(d, H+1, l+1) + \text{Fubini}(d, H+1, l+1) \implies \text{Rank-bias}(d, H+1, l+1).$$

In the base case of height 0, where the tower is empty, Lemmas 3.3 and 3.4 are trivial and Lemma 3.5 is proved in [14, 19]. We assume in the proofs that the tower is indeed of height $H + 1$, otherwise we are done by the inductive hypothesis.

Proof of Lemma 3.3 (Derivatives(d,H+1,l+1)). We are interested in the regularity of $\mathcal{Q}(y)$, which is, by definition, the regularity of its associated multilinear tower. Which multilinear maps appear? For

$Q_{i,j} \in Q_i$, write $Q_{i,j} = \sum_{J \subset I_i} Q_{i,j}^J$ as a sum of multilinear maps $Q_{i,j}^J : V^J \rightarrow \mathbb{F}$, and set $Q'_{i,j} = \sum_{I_i \setminus I \subset J \subset I_i} Q_{i,j}^J$, $Q'_i = \{Q'_{i,j}\}_{j \in m_i}$. Note that the following properties hold:

1. Q, Q' have the same associated multilinear variety (and hence the same regularity),
2. $Z(Q')_I = Z(Q)_I$, and
3. for any $y \in Z(Q)_I$, the multilinear tower associated with $Q(y)$ is $Q'(y)$.

So after replacing Q with Q' , we can assume without loss of generality that $Q(y)$ is a multilinear tower for any $y \in Z(Q)_I$. Write $X = Z(Q)$ for short. Let A', B' be the constants of Lemma 3.3 at height H . We deal separately with two cases.

The first case (and the easier one) is when $I_{H+1} \subset I$. In this case, for any $y \in (X_I)_{<H+1}$, we have $Q(y) = Q_{<H+1}(y)$. For $A = 2^{B'}A', B = B'$, we get that $Q_{<H+1}$ is $(A', B', s+m_{H+1}+1)$ -regular. Then by the inductive hypothesis, we have that for $q^{-(s+m_{H+1}+1)}$ a.e. $y \in (X_I)_{<H+1}$, the tower $Q(y) = Q_{<H+1}(y)$ is (C, D, s) -regular. By Lemma 3.2, we know that if A, B are sufficiently large, then $\frac{|(X_I)_{<H+1}|}{|X_I|} \leq q^{m_{H+1}+1}$. Therefore, $Q(y)$ is regular for q^{-s} -a.e. $y \in X_I$, as desired.

The second case is when $I_{H+1} \not\subset I$. In this case, we have $X_I = (X_I)_{<H+1}$. If A, B are sufficiently large, then Q is $(A', B', 100C(m_{H+1} + s)^D)$ -regular. Let

$$s' = 100C(m_{H+1} + s)^D, \quad \varepsilon = q^{-s'}$$

For the rest of this proof, we will assume, wherever needed, that A, B, A', B', C, D are sufficiently large with respect to d, h, l . By the inductive hypothesis, we have that for ε -a.e. $y \in X_I$, the tower $Q_{<H+1}(y)$ is (C, D, s') -regular. By a union bound, it is enough to show that for $q^{-(s+1)}$ -a.e. $y \in X_I$, we have

$$rk_{Q_{<H+1}(y)}(Q_{H+1}(y)) > C(m_{H+1} + s)^D.$$

Fix $0 \neq a \in \mathbb{F}^{m_{H+1}}$. Applying Theorem 3.1, we have

$$Re \left[\mathbb{E}_{x \in X_{<H+1}} \chi(a \cdot Q_{H+1}(x)) \right] \leq \varepsilon.$$

Lemma 3.4 at height H yields $\mathbb{E}_{y \in X_I} Re \left[\mathbb{E}_{z \in X_{<H+1}(y)} \chi(Q(y, z)) \right] \leq 2\varepsilon$. For ε -a.e. $y \in X_I$, we can apply Lemma 3.5 at height H to get

$$\left| \mathbb{E}_{z \in X_{<H+1}(y)} \chi(Q(y, z)) \right| \leq Re \left[\mathbb{E}_{z \in X_{<H+1}(y)} \chi(Q(y, z)) \right] + \varepsilon.$$

Therefore,

$$\mathbb{E}_{y \in X_I} \left| \mathbb{E}_{z \in X_{<H+1}(y)} \chi(Q(y, z)) \right| < 4\varepsilon.$$

By Markov's inequality, this means that for $2\sqrt{\varepsilon}$ -a.e. $y \in X_I$, we have

$$\left| \mathbb{E}_{z \in X_{<H+1}(y)} \chi(Q(y, z)) \right| < 2\sqrt{\varepsilon}.$$

Then, for $3\sqrt{\varepsilon}$ -a.e. $y \in X_I$, both the above inequality holds and the tower $Q^{<H+1}(y)$ is (C, D, s') -regular. Let $r_a(y) = rk_{Q_{<H+1}(y)}(a \cdot Q_{H+1}(y))$. Applying Lemma 3.5 at height H , we find that for these y , we have

$$q^{-r(y)} < 4\sqrt{\varepsilon} < q^{-C(m_{H+1}+s)^D},$$

which implies $r_a(y) > C(m_{H+1} + s)^D$. Running over all $0 \neq a \in \mathbb{F}^{m_{H+1}}$, we get that for $3\sqrt{\varepsilon}q^{m_{H+1}} \leq q^{-(s+1)}$ -a.e. $y \in X_I$, we have $r(y) > C(m_{H+1} + s)^D$, where $r(y) = \max_{0 \neq a \in \mathbb{F}^{m_{H+1}}} r_a(y)$. This completes the proof. □

Proof of Lemma 3.4 Fubini(d,H+1,l+1). By the triangle inequality,

$$\left| \mathbb{E}_{x \in X} g(x) - \mathbb{E}_{y \in X_I} \mathbb{E}_{z \in X(y)} g(y, z) \right| \leq \mathbb{E}_{y \in X_I} \left| \frac{|X_I| \cdot |X(y)|}{|X|} - 1 \right|.$$

Denoting $f(y) = \frac{|X_I| \cdot |X(y)|}{|X|}$, $\varepsilon = q^{-(s+2)}$, and assuming A, B are sufficiently large, we note the following:

1. $f(y) \geq 0$ for all $y \in X_I$,
2. $\mathbb{E}_{y \in X_I} f(y) = 1$, and
3. by Lemmas 3.3(d,H+1,l+1) and 3.2(d,H+1,l+1), for ε -a.e. $y \in X_I$, we have $1 - \varepsilon < f(y) < 1 + \varepsilon$.

Let $E \subset X_I$ be the set of y for which this last inequality holds, and let $\rho = \frac{|E|}{|X_I|}$ (so $1 - \varepsilon \leq \rho \leq 1$). Then

$$\mathbb{E}_{y \in X_I} |f(y) - 1| = \rho \cdot \mathbb{E}_{y \in E} |f(y) - 1| + (1 - \rho) \mathbb{E}_{y \in E^c} |f(y) - 1|. \tag{3.1}$$

The first term is $\leq \varepsilon$ by the definition of E . For the second term, we have

$$(1 - \rho) \mathbb{E}_{y \in E^c} |f(y) - 1| \leq \varepsilon + (1 - \rho) \mathbb{E}_{y \in E^c} f(y).$$

By the three properties listed above, we get that $(1 - \rho) \mathbb{E}_{y \in E^c} f(y) \leq 2\varepsilon$. Plugging this into inequality (3.1) yields

$$\mathbb{E}_{y \in X_I} |f(y) - 1| \leq 4\varepsilon \leq q^{-s}. \quad \square$$

Proof of Lemma 3.5 rank-bias(d,H+1,l+1). Let $X = Z(\mathcal{Q})$. First we apply Lemma 3.4(d,H+1,l+1) to get

1. $Re[\mathbb{E}_{x \in X} \chi(P(x))] \geq Re[\mathbb{E}_{x \in X_I} \chi(P(x))] - q^{-2s}$ and
2. $|Im[\mathbb{E}_{x \in X} \chi(P(x))]| \leq |Im[\mathbb{E}_{x \in X_I} \chi(P(x))]| + q^{-2s}$,

so we can assume without loss of generality that $I = [k]$. We will prove the lemma by induction on $|I|$. For $|I| = 1$, the lemma follows from basic Fourier analysis. For the general case, we start by showing the second inequality holds. Apply Lemma 3.4 again to get

$$|Im[\mathbb{E}_{x \in X} \chi(P(x))]| \leq \mathbb{E}_{y \in X_I} |Im[\mathbb{E}_{z \in X(y)} \chi(P(y, z))]| + q^{-10s}.$$

Let A', B' be the constants for Lemma 3.5 on an index set of size $|I| - 1$. By Lemma 3.3(d,H+1,l+1), if A, B are sufficiently large then for q^{-10s} -a.e. $y \in X_I$ the multilinear tower $\mathcal{Q}(y)$ is $(A', B', 10s)$ -regular. The induction hypothesis implies that

$$\mathbb{E}_{y \in X_I} |Im[\mathbb{E}_{z \in X(y)} \chi(P(y, z))]| \leq q^{-5s},$$

as desired.

Now for the first inequality, suppose modulo the maps in \mathcal{Q} , we have the low partition rank presentation

$$P(x_{[k]}) = \sum_{i=1}^m l_i(x_1) g_i(x_{[2,k]}) + \sum_{j=m+1}^r q_i(x_{I_j}) r_i(x_{[k] \setminus I_j}),$$

where $I_j \neq \{1\}, [2, k]$. Let $W = \{y \in X_I : l_i(y) = 0 \forall i \in [m]\}$. By Lemma 3.4,

$$\begin{aligned} Re[\mathbb{E}_{x \in X} \chi(P(x))] &\geq \mathbb{E}_{y \in X_I} Re[\mathbb{E}_{z \in X(y)} \chi(P(y, z))] - q^{-10s} \\ &= \frac{|W|}{|X_I|} \mathbb{E}_{y \in W} Re[\mathbb{E}_{z \in X(y)} \chi(P(y, z))] + \left(1 - \frac{|W|}{|X_I|}\right) \mathbb{E}_{y \in X_I \setminus W} Re[\mathbb{E}_{z \in X(y)} \chi(P(y, z))] - q^{-10s}. \end{aligned}$$

Note that:

1. for any $y \in W$, the rank of $P(y, \cdot)$ on $X(y)$ is $\leq r - m$,
2. $|W|/|X_1| \geq q^{-m}$, and
3. for q^{-10s} -a.e. $y \in X_1$, the multilinear tower $\mathcal{Q}(y)$ is $(A', B', 10s)$ -regular.

By these properties and the induction hypothesis for $P(y, \cdot)$ on $X(y)$, we can lower bound the first summand by

$$\frac{|W|}{|X_1|} \mathbb{E}_{y \in W} \operatorname{Re} [\mathbb{E}_{z \in X(y)} \chi(P(y, z))] \geq q^{-r} - q^{-5s},$$

and the second summand by

$$\left(1 - \frac{|W|}{|X_1|}\right) \mathbb{E}_{y \in X_1 \setminus W} \operatorname{Re} [\mathbb{E}_{z \in X(y)} \chi(P(y, z))] \geq -q^{-5s}. \quad \square$$

4. Some reductions

In this section, we reduce the proof of Theorem 2.13 for degree $d + 1$ to the $l = 0$ case of Theorem 3.1.

Lemma 4.1. *Suppose Theorem 3.1(d+1,H,l) holds for $l = 0$ and all H . Then it holds for all l, H .*

Proof. By induction on l , suppose the theorem holds for $(d+1,H,l)$. Let \mathcal{Q} be an (A, B, s) -regular tower on $V^{[k]}$ of type $(d+1,H,l+1)$ and $P : V^I \rightarrow \mathbb{F}$ a full multiaffine map of degree $\leq d + 1$, with $\operatorname{bias}_{Z(\mathcal{Q})}(P) \geq q^{-s}$. Again, we denote $X = Z(\mathcal{Q})$ for short. We saw in the last section that Theorem 3.1(d+1,H,l) implies Lemma 3.4 for $(d+1,H+1,l+1)$, so

$$|\mathbb{E}_{x \in X_I} \chi(P(x))| \geq q^{-2s},$$

that is, we can assume without loss of generality that $I = [k]$. If \mathcal{Q} is of degree $\leq d$, then we are done. Suppose not, then $k = d + 1$, and there is some layer $t \in [H]$ with \mathcal{Q}_t of degree $d + 1$. Let $\mathcal{Q}_{\neq t} = (\mathcal{Q}_i)_{i \in [H] \setminus \{t\}}$ and $X_{\neq t} = Z(\mathcal{Q}_{\neq t})$. By Fourier analysis, we have

$$|\mathbb{E}_{x \in X} \chi(P(x))| \leq \frac{|X_{\neq t}|}{|X|} q^{-m_t} \sum_{a \in \mathbb{F}^{m_t}} |\mathbb{E}_{x \in X_{\neq t}} \chi(P(x) + a \cdot \mathcal{Q}_t(x))|. \quad (4.1)$$

By Lemma 3.2(d+1,H,l+1) (which follows from Theorem 3.1(d+1,H,l)), $\frac{|X_{\neq t}|}{|X|} q^{-m_t} \leq q$. As for the second term, choose $b \in \mathbb{F}^{m_t}$, such that $\operatorname{prk}_{\mathcal{Q}_{\neq t}}(P(x) + b \cdot \mathcal{Q}_t(x))$ is minimal. By subadditivity of relative partition rank, for any $b \neq a \in \mathbb{F}^{m_t}$, we must have

$$\operatorname{prk}_{\mathcal{Q}_{\neq t}}(P(x) + a \cdot \mathcal{Q}_t(x)) \geq \frac{A}{2}(m_t + s)^B.$$

By Theorem 3.1(d+1,H,l) (assuming A, B are sufficiently large), this means that for any $b \neq a \in \mathbb{F}^{m_t}$,

$$|\mathbb{E}_{x \in X_{\neq t}} \chi(P(x) + a \cdot \mathcal{Q}_t(x))| \leq q^{-m_t - 4s}.$$

Plugging these estimates into inequality (4.1) yields

$$q^{-2s-1} \leq |\mathbb{E}_{x \in X_{\neq t}} \chi(P(x) + b \cdot \mathcal{Q}_t(x))|.$$

Now apply Theorem 3.1(d+1,h,l) to conclude that

$$\operatorname{prk}_{\mathcal{Q}}(P(x)) \leq \operatorname{prk}_{\mathcal{Q}_{\neq t}}(P(x) + b \cdot \mathcal{Q}_t(x)) \leq A(1 + s)^B. \quad \square$$

Therefore, all we have to do is prove Theorem 3.1(d+1,H,0) in order to deduce Theorem 2.13 at degree $d + 1$. The following lemma allows us to reduce further to the case that P, Q are multilinear and defined on the same index set. From now on, we assume Theorem 2.13(d,H) holds for all H , and work toward proving Theorem 3.1(d+1,H,0) for all H . We may safely forget about the auxiliary parameter l of Theorem 3.1, it has served its purpose. The following lemma allows us to reduce further to the case where P, Q are multilinear and are defined on the same index set.

Lemma 4.2. *There exist $A(d, h), B(d, h)$, such that if Q is an (A, B, s) -regular tower of degree $\leq d$ and height $\leq h$ on $V^{[k]}$ and $P : V^I \rightarrow \mathbb{F}$ is a full multiaffine map of degree $|I| \leq d + 1$, then*

$$|\mathbb{E}_{x \in X} \chi(P(x))| \leq \text{Re}[\mathbb{E}_{x \in \tilde{X}} \chi(\tilde{P}(x))] + q^{-s},$$

where $X = Z(Q)$ and \tilde{P}, \tilde{X} are the corresponding multilinear map and variety.

Proof. By applying Lemma 3.4, we get

$$|\mathbb{E}_{x \in X} \chi(P(x))| \leq |\mathbb{E}_{x \in X_I} \chi(P(x))| + q^{-2s},$$

so we can assume without loss of generality that $I = [k]$. Now choose some $i \in [k]$, and write $P(x) = A(x_{[k] \setminus \{i\}}) \cdot x_i + b(x_{I \setminus \{i\}})$, where A, b are multiaffine maps. Again, using Lemma 3.4 twice, we have

$$\begin{aligned} |\mathbb{E}_{x \in X} \chi(P(x))| &\leq |\mathbb{E}_{x \in X_{I \setminus \{i\}}} \chi(b(x)) \mathbb{E}_{y \in X(x)} \chi(A(x) \cdot y)| + q^{-10ks} \\ &\leq \mathbb{E}_{x \in X_{I \setminus \{i\}}} |\mathbb{E}_{y \in X(x)} \chi(A(x) \cdot y)| + q^{-10ks} \\ &= \mathbb{E}_{x \in X_{I \setminus \{i\}}} \mathbb{E}_{y \in \widetilde{X(x)}} \chi(A(x) \cdot y) + q^{-10ks} \\ &\leq |\mathbb{E}_{x \in X'} \chi(A(x_{[k] \setminus \{i\}}) \cdot x_i)| + q^{-5ks}, \end{aligned}$$

where X' is the variety associated with X which is linear in the i -th coordinate and $A(x_{[k] \setminus \{i\}}) \cdot x_i$ is the multiaffine map associated with P which is linear in the i -th coordinate. Iterating this inequality over and over for all the other coordinates, we end up with

$$|\mathbb{E}_{x \in X} \chi(P(x))| \leq |\mathbb{E}_{x \in \tilde{X}} \chi(\tilde{P}(x))| + (k + 1)q^{-5ks} \leq \text{Re}[\mathbb{E}_{x \in \tilde{X}} \chi(\tilde{f}(x))] + q^{-s},$$

where the second inequality follows from Lemma 3.5. □

5. Many low rank approximations

In the previous section, we showed that in order to prove Theorem 2.13 for degree $d + 1$, the only new case we have to deal with is when $P : V^{[d+1]} \rightarrow \mathbb{F}$ is multilinear and Q is a multilinear tower on $V^{[d+1]}$ of degree $\leq d$. In this section, we will prove the existence of an abundance of relative low rank approximations, and in the next sections, we will “glue” them together to get a genuine relative low rank presentation. We start with a useful algebraic claim, which we will need here and in later sections.

Claim 5.1. *Let Q be a tower on $V^{[k]}$ and $l \in \mathbb{N}^k$ a vector. Define $Q^{\otimes l}$ to be the tower on $(V_1)^{l_1} \times \dots \times (V_k)^{l_k}$ obtained by multiplying the first coordinate l_1 times, the second l_2 times and so on – that is, a tower of the same height as Q with*

$$\left(Q^{\otimes l}\right)_i = \bigcup_{j \in \prod_{t \in I} [l_t]} Q_i((x_{t,j_t})_{t \in I}).$$

Define $Q^{\times l}$ to be the tower composed of the same equations, but on $V^{[k l_1 \dots l_k]}$, so it has $l_1 \dots l_k$ as many layers as Q (we order the layers corresponding to Q_i arbitrarily within themselves, but keep the order otherwise). Then, if Q is $(A(l_1 \dots l_k)^B, B, s)$ -regular, then both $Q^{\otimes l}$ and $Q^{\times l}$ are (A, B, s) -regular.

Proof. We may assume without loss of generality that \mathcal{Q} is multilinear. Because there are $l_1 \dots l_k$ times as many maps, it is enough to show that the relative ranks are unchanged. Suppose we have a linear combination with

$$rk_{\mathcal{Q}_{<i}} \left(\sum_{\mathbf{j} \in \prod_{t \in I_i} [l_t]} a_{\mathbf{j}} \cdot \mathcal{Q}_i((x_{t,j_t})_{t \in I_i}) \right) < rk_{\mathcal{Q}_{<i}}(\mathcal{Q}_i).$$

For any $\mathbf{j} \in \prod_{t \in I_i} [l_t]$, plugging in $x_{\alpha,\beta} = 0$ for all $\beta \neq j_{\alpha}$, we get

$$rk_{\mathcal{Q}_{<i}}(a_{\mathbf{j}} \cdot \mathcal{Q}_i) < rk_{\mathcal{Q}_{<i}}(\mathcal{Q}_i),$$

which implies that $a_{\mathbf{j}} = 0$. The proof for $\mathcal{Q}^{\times l}$ is identical. □

Now we are ready to prove that we can find an abundance of low rank approximations of P on $Z(\mathcal{Q})$. The following is inspired by lemma 12 in [21].

Proposition 5.2 (Low rank approximations). *There exist constants $A(d, H), B(d, H)$, such that the following holds: If \mathcal{Q} is an (A, B, s) -regular multilinear tower on $V^{[d+1]}$ of degree $\leq d$ and height $\leq H$, $X = Z(\mathcal{Q})$, and $P : V^{[d+1]} \rightarrow \mathbb{F}$ is a multilinear map satisfying $bias_X(P) \geq q^{-s}$, then there exists a set $E \subset (X_{d+1})^{100s}$ of density $\geq q^{-10s}$, such that for all $t \in E$, there is a multilinear function $\phi_t : V^{[d]} \rightarrow \mathbb{F}^{100s}$ with*

$$\mathbb{P}_{x \in X_t \cap \{\phi_t(x_{[d]})=0\}}(P(x) = 0) \geq 1 - q^{-s},$$

where $X_t = \{x \in X : x_{[d]} \in \bigcap_{i=1}^{100s} X(t_i)\}$.

Proof. Choosing a nondegenerate bilinear form on V^{d+1} , we can write $P(x) = A(x_{[d]}) \cdot x_{d+1}$, where $A : V^{[d]} \rightarrow V^{d+1}$ is multilinear. Apply Lemma 3.4 to get

$$q^{-2s} \leq \mathbb{E}_{x_{[d]} \in X_{[d]}} \mathbb{E}_{x_{d+1} \in X(x)} A(x_{[d]}) \cdot x_{d+1} = \mathbb{P}_{x_{[d]} \in X_{[d]}} [A(x_{[d]}) \perp X(x_{[d]})], \tag{5.1}$$

where $v \perp w$ means $v \cdot w = 0$. Now, fix $x_{[d]} \in X_{[d]}$. To test for the event above, choose $t \in (X(x_{[d]}))^l$ at random, where $l = 100s$ and define

$$\phi_t(x_{[d]}) = (A(x_{[d]}) \cdot t_1, \dots, A(x_{[d]}) \cdot t_l).$$

If $A(x_{[d]}) \perp X(x_{[d]})$, then $\phi_t(x) = 0$ for any $t \in (X(x_{[d]}))^l$, and if $A(x_{[d]}) \not\perp X(x_{[d]})$, then

$$\mathbb{P}_{t \in (X(x_{[d]}))^l}(\phi_t(x) = 0) = q^{-l}.$$

Therefore,

$$\mathbb{E}_{x_{[d]} \in X_{[d]}} \mathbb{1}_{A(x_{[d]}) \not\perp X(x_{[d]})} \mathbb{E}_{t \in (X(x_{[d]}))^l} \mathbb{1}_{\phi_t(x)=0} \leq q^{-l}.$$

If A, B are sufficiently large, then by Claim 5.1 and Lemma 3.4, we have

$$\mathbb{E}_{t \in (X_{d+1})^l} \mathbb{E}_{x \in X_t} \mathbb{1}_{A(x_{[d]}) \not\perp X(x_{[d]})} \mathbb{1}_{\phi_t(x)=0} \leq q^{-l} + q^{-100s} \leq q^{-50s}.$$

By Markov's inequality, for q^{-25s} -a.e. $t \in X_{d+1}^l$, we have

$$\mathbb{E}_{x \in X_t} \mathbb{1}_{A(x_{[d]}) \not\perp X(x_{[d]})} \mathbb{1}_{\phi_t(x)=0} \leq q^{-25s}. \tag{5.2}$$

Also, by Lemma 3.4 together with inequality (5.1)

$$\mathbb{E}_{t \in X_{d+1}^l} \mathbb{E}_{x \in X_t} \mathbb{1}_{A(x_{[d]}) \perp X(x_{[d]})} \geq q^{-3s}.$$

Markov’s inequality gives us

$$\mathbb{P}_{t \in X_{d+1}^l} \left[\mathbb{E}_{x \in X_t} \mathbb{1}_{A(x_{[d]}) \perp X(x_{[d]})} \geq q^{-5s} \right] \geq q^{-5s}.$$

So for a collection of $t \in X_{d+1}^l$ of density $\geq q^{-10s}$, we have both

$$\mathbb{E}_{x \in X_t} \mathbb{1}_{A(x_{[d]}) \perp X(x_{[d]})} \geq q^{-5s}$$

and inequality (5.2). For these t , we have

$$\begin{aligned} \mathbb{P}_{x \in X_t \cap \{\phi_t(x_{[d]})=0\}} (P(x) = 0) &\geq \mathbb{P}_{x \in X_t} (A(x_{[d]}) \perp X(x_{[d]}) | \phi_t(x) = 0) \\ &= \frac{\mathbb{P}_{x \in X_t} (A(x_{[d]}) \perp X(x_{[d]}))}{\mathbb{P}_{x \in X_t} (A(x_{[d]}) \perp X(x_{[d]}) + \mathbb{P}_{x \in X_t} (A(x_{[d]}) \not\perp X(x_{[d]}), \phi_t(x) = 0)} \\ &\geq \frac{q^{-5s}}{q^{-5s} + \mathbb{P}_{x \in X_t} (A(x_{[d]}) \not\perp X(x_{[d]}), \phi_t(x) = 0)} \geq \frac{q^{-5s}}{q^{-5s} + q^{-25s}} \geq 1 - q^{-s}. \end{aligned}$$

□

6. Regularization

In order to improve our approximation from the last section to a genuine low rank representation, we will have to take the $100s$ maps $\phi_t : V^{[d]} \rightarrow \mathbb{F}^{100s}$ and replace them by a tower which is relatively regular together with the maps of X_t . In this section, we describe an efficient algorithm for doing this. The following notion will come in handy:

Definition 6.1. Let $\mathcal{R} = (\mathcal{R}_i)_{i \in [h]}$, $\mathcal{Q} = (\mathcal{Q}_j)_{j \in [h]}$ be towers on $V^{[k]}$ and $m_i = |\mathcal{R}_i|$. We say that \mathcal{R} is (A, B, s) -regular relative to \mathcal{Q} if for all $i \in [h]$, we have

$$rk_{\mathcal{Q} \cup \mathcal{R}_{<i}}(\mathcal{R}_i) > A(m_i + m_{i+1} + \dots + m_h + s)^B.$$

Note that if \mathcal{Q}, \mathcal{R} are two towers, such that \mathcal{R} is (A, B, s) -regular relative to \mathcal{Q} and \mathcal{Q} is $(A, B, s+m)$ -regular (m is the dimension of \mathcal{R}), then $\mathcal{Q} \cup \mathcal{R}$ is (A, B, s) -regular with the layers ordered by having \mathcal{Q} on the bottom and \mathcal{R} on top.

Claim 6.2 (Regular decomposition). *For any A, B , there exist constants $C(A, B, d, k), D(A, B, d, k)$, such that the following holds: If \mathcal{Q} is a tower on $V^{[k]}$ and \mathcal{R} is a collection of full multiaffine maps of degree $\leq d$, then there exist towers $\mathcal{S}_1, \dots, \mathcal{S}_l$, such that:*

1. \mathcal{S}_i is (A, B, s) -regular relative to \mathcal{Q} ,
2. \mathcal{S}_i is of degree $\leq d$ and height $\leq 2^k$,
3. $\dim(\mathcal{S}_i) \leq C(s + \dim(\mathcal{R}))^D$, and
4. we have a decomposition $Z(\mathcal{Q} \cup \mathcal{R}) = \bigsqcup_{i=1}^l Z(\mathcal{Q} \cup \mathcal{S}_i)$.

In addition, if \mathcal{Q}, \mathcal{R} are multilinear, then we can take \mathcal{S}_1 to be multilinear.

Note that Theorem 2.3 is the above claim for homogeneous polynomials in the simplest case where \mathcal{Q} is empty, and the proof we will now give of Claim 6.2 may be easily modified to prove Theorem 2.3.

Proof. First, arrange \mathcal{R} in a tower $\mathcal{R} = (\mathcal{R}_i : V^{l_i} \rightarrow \mathbb{F})_{i \in [2^k]}$, with the higher degree maps above the lower degree ones. As usual, denote $m_i = |\mathcal{R}_i|$. To the i -th layer we assign a number n_i , according to

a formula we will give later. We describe an algorithm for producing \mathcal{S}_i and then show that it has the desired properties. If for every i

$$rk_{\mathcal{Q} \cup \mathcal{R}_{<i}}(\mathcal{R}_i) > A(n_i + s)^B,$$

then we stop. Otherwise, there is some $0 \neq a \in \mathbb{F}^{m_i}$, such that

$$a \cdot \mathcal{R}_i = \sum_{j=1}^t q_j(x_{I_j})r_j(x_{I_i \setminus I_j}) + \sum_{J \subseteq I_i} u_J(x_J) + \sum_{f \in \mathcal{Q} \cup \mathcal{R}_{<i}, I_f \subset I_i} f(x_{I_f})h_f(x_{I_i \setminus I_f}),$$

where $t \leq A(n_i + s)^B$. Assume without loss of generality that $a_{m_i} \neq 0$. Let \mathcal{R}' be the tower obtained from \mathcal{R} by deleting R_{i,m_i} . In $Z(\mathcal{Q} \cup \mathcal{R}')$, the value of R_{i,m_i} is determined by the q_j, r_j, u_J . We add these maps to the lower layers of \mathcal{R}' and call the new tower \mathcal{R}'' . We can then split

$$Z(\mathcal{Q} \cup \mathcal{R}) = \bigsqcup_{i=1}^c (Z(\mathcal{Q}) \cap \{\mathcal{R}'' = b_i\}),$$

where $b_1, \dots, b_c \in \mathbb{F}^{\mathcal{R}''}$ are the values causing the maps in \mathcal{R} to vanish on $Z(\mathcal{Q}) \cap \{\mathcal{R}'' = b_i\}$. Now we do the same thing for each of the towers $\mathcal{R}'' - b_i$ (with the same n_i we chose at the start) and so on. Because at each stage we replace a map by maps of lower degree (and when handling a linear map, we just delete it), this algorithm eventually terminates. We now give the definition of n_i and check that the \mathcal{S}_i we end up with satisfy requirements 1–4. Set

$$n_{2^k} = m_{2^k}, n_i = m_i + n_{i+1} [2A(s + n_{i+1})^B + 2^d].$$

We prove inductively that the total number of polynomials passing through layer i and above during this process is $\leq n_i$. For $i = 2^k$, this is clear, since during regularization we only add polynomials to lower layers. Now suppose this bound holds for i . Any polynomials added to layer $i - 1$ come from regularizing the higher layers. Each polynomial upstairs can contribute at most $2A(s + n_i)^B + 2^d$ when regularizing, so the bound at layer $i - 1$ is proved, assuming the bound at layer i . Therefore, conditions 1 and 3 are satisfied. Conditions 2 and 4 clearly hold. If \mathcal{Q}, \mathcal{R} are multilinear, then we can take $b_1 = 0$ at every stage, leaving us with \mathcal{S}_1 multilinear. \square

7. Fixing the approximation

In this section, we will take the many approximations that we get from Proposition 5.2 and “glue” them together to get a single tower \mathcal{R} , such that P vanishes identically on $Z(\mathcal{Q} \cup \mathcal{R})$. A central tool will be the following proposition (cf. proposition 5.1 in [10]):

Proposition 7.1 (Almost surely vanishing implies vanishing). *There exist constants $A(d, H), B(d, H), s(d)$, such that if \mathcal{Q} is an (A, B, s) -regular tower on $V^{[d+1]}$ of degree $\leq d$ and height $\leq H$ and $f : V^{[d+1]} \rightarrow \mathbb{F}$ is a polynomial map (not even necessarily multiaffine) of degree $\leq d + 1$ satisfying*

$$\mathbb{P}_{x \in Z(\mathcal{Q})}(f(x) = 0) \geq 1 - q^{-s},$$

then $f(x) = 0$ for all $x \in Z(\mathcal{Q})$.

Let $\mathcal{Q} = (\mathcal{Q}_i)_{i \in [h]}$, where $\deg(\mathcal{Q}_i) = d_i$, and write $X = Z(\mathcal{Q})$ for short. In order to prove this lemma, we will need to count parallelepipeds based at any point $x \in X$. Given $x \in X$ and an integer l , for each layer $i \in [h]$, we define a layer $(\mathcal{Q}_{x,i})_i$ of maps $(V_1)^l \times \dots \times (V_{d+1})^l \rightarrow \mathbb{F}$ by

$$(\mathcal{Q}_{x,i})_i := \{\mathcal{Q}_i(x + \omega \cdot t) : \omega \in \{0, 1\}^l, 0 < |\omega| \leq d_i\},$$

and $\mathcal{Q}_{x,l}$ is the resulting tower. Writing $X_{x,l} = Z(\mathcal{Q}_{x,l})$, note that

$$X_{x,l} := \{t_1, \dots, t_l \in V^{[d+1]} : x + \omega \cdot t \in X \forall \omega \in \{0, 1\}^l\}.$$

We will also need to count parallelepipeds containing an additional point of the form $x + t_1$. Given $t_1 \in X - x$ and $c \in [l]$, set

$$X_{x,l,c}(t_1) = \{t_2, \dots, t_l \in V^{[d+1]} : (t_1 - \dots - t_c, t_2, \dots, t_l) \in X_{x,l}\}.$$

Lemma 7.2. *If \mathcal{Q} is $(Cl^{(d+1)D}, D, s)$ -regular, then $\mathcal{Q}_{x,l}$ is an (C, D, s) -regular tower of the same degree and height. Also, if C, D are sufficiently large depending on A, B, d, H, l , then for q^{-s} a.e. $t_1 \in X - x$, we have $X_{x,l,c}(t_1) = Z(\mathcal{R})$, where \mathcal{R} is an (A, B, s) -regular tower of the same degree as \mathcal{Q} , height $\leq Hl^{d+1}$ and*

$$\dim(\mathcal{R}) = \dim(\mathcal{Q}_{x,l}) - \dim(\mathcal{Q}).$$

Proof. As usual, let $\tilde{\mathcal{Q}}$ denote the multilinear tower corresponding to \mathcal{Q} . The multilinear tower corresponding to $\mathcal{Q}_{x,l}$ is then $\tilde{\mathcal{Q}}_{0,l}$, so we can assume without loss of generality that \mathcal{Q} is multilinear and $x = 0$. By Claim 5.1, the tower $\mathcal{Q}^{\otimes(l, \dots, l)}$ is (A, B, s) -regular. The maps in $\mathcal{Q}_{x,l}$ are linear combinations of the maps in $\mathcal{Q}^{\otimes(l, \dots, l)}$, so we just need to verify that they are linearly independent. Let $Q \in \mathcal{Q}$ be a map of degree e , assume without loss of generality that $Q : V^{[e]} \rightarrow \mathbb{F}$, and suppose (to get a contradiction) that we have a nontrivial linear combination

$$\sum_{\omega \in \{0,1\}^l, 0 < |\omega| \leq e} a_\omega Q(\omega \cdot t) = 0. \tag{7.1}$$

Let ω_0 be the largest element with respect to the lexicographic ordering on $\{0, 1\}^l$, such that $a_{\omega_0} \neq 0$, and choose $i_1, \dots, i_e \in \omega_0$, such that for every other ω with $a_\omega \neq 0$, we have $\{i_1, \dots, i_e\} \not\subset \omega$. Then the coefficient of $Q(t_{i_1}(1), \dots, t_{i_e}(e))$ in the left-hand side of equation (7.1) is $a_{\omega_0} \neq 0$, contradiction. Now we turn to the second part of the lemma. Since invertible affine transformations do not affect regularity, the variety

$$\{(t_1, \dots, t_l) : (t_1 - \dots - t_c, t_2, \dots, t_l) \in X_{x,l}\}$$

is the zero locus of a (C, D, s) -regular tower on $(V_1)^l \times \dots \times (V_{d+1})^l$. This tower remains regular if we split up the layers into at most l^{d+1} layers so that it is a tower on

$$V_1 \times \dots \times V_1 \times \dots \times V_{d+1} \times \dots \times V_{d+1}.$$

Then, by Lemma 3.3, we get that for q^{-s} -a.e. $t_1 \in X - x$ the variety

$$\{(t_2, \dots, t_l) : (t_1 - \dots - t_c, t_2, \dots, t_l) \in X_{x,l}\}$$

is the zero locus of a tower \mathcal{R} which has the required properties. □

We now use this to prove the proposition from the start of this section.

Proof of Proposition 7.1. Let $G = \{x \in X : f(x) = 0\}$, and $B = X \setminus G$ be the good and bad points of X , respectively. Fix $x \in X$. Since f is a polynomial of degree $\leq d + 1$, for any $t_1, \dots, t_{d+2} \in V^{[d+1]}$, we have

$$\sum_{\omega \in \{0,1\}^{d+2}} (-1)^{|\omega|} f(x + \omega \cdot t) = 0.$$

This means that if we can find $t \in X_{x,d+2}$ with $x + \omega \cdot t \in G$ for all $0 \neq \omega \in \{0, 1\}^{d+2}$, then $x \in G$ as well. Since the previous claim (together with Lemma 3.2) implies, in particular, that $X_{x,d+2} \neq \emptyset$, it is enough to show the following bound for any fixed $0 \neq \omega \in \{0, 1\}^{d+2}$

$$\mathbb{E}_{t \in X_{x,d+2}} 1_B(x + \omega \cdot t) \leq 2^{-(d+2)}.$$

By reordering the t_j , assume without loss of generality that $\omega = (1^c, 0^{d+2-c})$. By replacing t_1 with $t_1 - t_2 - \dots - t_c$, the left-hand side becomes

$$\mathbb{E}_{t, (t_1 - t_2 - \dots - t_c, t_2, \dots, t_{d+2}) \in X_{x,d+2}} 1_B(x + t_1).$$

By Lemma 7.2, the conclusion of Lemma 3.4 holds (the proof is the same), meaning

$$\mathbb{E}_{t, (t_1 - t_2 - \dots - t_c, t_2, \dots, t_{d+2}) \in X_{x,d+2}} 1_B(x + t_1) \leq \mathbb{E}_{t_1 \in X_{-x}} 1_B(x + t_1) + q^{-s} \leq q^{1-s}.$$

This proves the proposition for $s \geq 3 + d$. □

We now prove a claim which will allow us to glue the various approximations we get in Lemma 5.2 together.

Claim 7.3. *There exist $A(d, H), B(d, H), s(d)$, such that if $f : V^{[d+1]} \rightarrow \mathbb{F}$ is multilinear, $\mathcal{Q}, \mathcal{G}_1, \mathcal{G}_2$ are multilinear towers on $V^{[d+1]}$ satisfying $f \upharpoonright_{Z(\mathcal{Q} \cup \mathcal{G}_j)} = 0$ and $\mathcal{Q} \cup \mathcal{G}_1 \cup \mathcal{G}_2$ is (A, B, s) -regular of degree $\leq d$ and height $\leq H$, then for any $i \in [d + 1]$, we have*

$$f \upharpoonright_{\{x \in Z(\mathcal{Q}) : x_{[d+1] \setminus \{i\}} \in Z(\mathcal{G}_1 \cup \mathcal{G}_2)_{[d+1] \setminus \{i\}}\}} \equiv 0.$$

Proof. Denote $Z = Z(\mathcal{Q})$, $Z_j = Z(\mathcal{G}_j)$. For any $x \in (Z \cap Z_1 \cap Z_2)_{[d+1] \setminus \{i\}}$, we have $f(x, \cdot) \upharpoonright_{(Z \cap Z_j)(x)} = 0$ for $j = 1, 2$ so by multilinearity,

$$f(x, \cdot) \upharpoonright_{(Z \cap Z_1)(x) + (Z \cap Z_2)(x)} = 0.$$

By Lemma 3.3, for q^{-2s} -a.e. such x , the linear equations appearing in $\mathcal{Q}(x) \cup \mathcal{G}_1(x) \cup \mathcal{G}_2(x)$ are linearly independent, in which case

$$(Z \cap Z_1)(x) + (Z \cap Z_2)(x) = Z(x).$$

By Lemma 3.4,

$$\mathbb{P}_{\{x \in Z : x_{[d+1] \setminus \{i\}} \in (Z_1 \cap Z_2)_{[d+1] \setminus \{i\}}\}} (f(x) = 0) \geq 1 - q^{-s}.$$

Applying Proposition 7.1, we get that this holds for any x in the above variety. □

Corollary 7.4. *There exist $A(d, H), B(d, H), s(d)$, such that when $f : V^{[d+1]} \rightarrow \mathbb{F}$ is multilinear, $\mathcal{Q}, \mathcal{G}_1, \dots, \mathcal{G}_{2^l}$ are multilinear towers on $V^{[d+1]}$, such that the \mathcal{G}_j only depend on the first l coordinates, $\mathcal{Q} \cup \mathcal{G}_1 \cup \dots \cup \mathcal{G}_{2^l}$ is (A, B, s) -regular of degree $\leq d$ and height $\leq H$, and*

$$f \upharpoonright_{Z(\mathcal{Q}) \cap Z(\mathcal{G}_j)} \equiv 0 \forall j \in [2^l],$$

then $f \upharpoonright_{Z(\mathcal{Q})} \equiv 0$.

Proof. By induction on l . For $l = 0$, this is clear. Suppose it holds up to some l and we are given $\mathcal{G}_1, \dots, \mathcal{G}_{2^{l+1}}$ depending on the first $l + 1$ coordinates. Applying Claim 7.3 to each pair $\mathcal{G}_i, \mathcal{G}_{i+1}$, we get that

$$f \upharpoonright_{\{x \in Z(\mathcal{Q}) : x_{[d+1] \setminus \{l\}} \in Z(\mathcal{G}_i \cup \mathcal{G}_{i+1})_{[d+1] \setminus \{l\}}\}} \equiv 0.$$

Then apply the inductive hypothesis to

$$\mathcal{Q}, (\mathcal{G}_1 \cup \mathcal{G}_2)_{[l]}, \dots, (\mathcal{G}_{2^{l+1}-1} \cup \mathcal{G}_{2^l+1})_{[l]}. \quad \square$$

We summarize our results thus far with the following proposition, which brings us very close to proving Theorem 2.13.

Proposition 7.5. *There exist constants $C(A, B, d, H), D(A, B, d, H)$, such that the following holds: If $P : V^{[d+1]} \rightarrow \mathbb{F}$ is multilinear, \mathcal{Q} is a multilinear tower of degree $\leq d$ and height $\leq H$, such that*

1. $\text{bias}_{Z(\mathcal{Q})}(P) \geq q^{-s}$ and
2. \mathcal{Q} is (C, D, s) -regular,

then there exists a multilinear tower \mathcal{R} of degree $\leq d$ and height $\leq 2^{d+1}$ satisfying

1. $\dim(\mathcal{R}) \leq C(1+s)^D$,
2. $\mathcal{R} \cup \mathcal{Q}$ is (A, B, s) -regular, and
3. $P \upharpoonright_{Z(\mathcal{R} \cup \mathcal{Q})} \equiv 0$.

Proof. Denote $X = Z(\mathcal{Q})$, and let E be the set from Proposition 5.2. Given $t \in E^{2^{d+1}}$, let $Y_i = \{x \in V^{[d+1]} : \phi_{t_i}(x_{[d]}) = 0\}$ – so P vanishes q^{-s} -a.e. on $X_{t_i} \cap Y_i$. Applying Lemma 6.2, there are towers $\mathcal{R}_{i,1}, \dots, \mathcal{R}_{i,l}$ which are (A, B, s) -regular relative to the tower defining X_{t_i} , satisfy $\dim(\mathcal{R}_{i,j}) \leq F(1+s)^G$ for some constants $F(A, B, d), G(A, B, d)$, and such that

$$X_{t_i} \cap Y_i = \bigsqcup_{j=1}^l (X_{t_i} \cap Y_{i,j}),$$

where $Y_{i,j} = Z(\mathcal{R}_{i,j})$. By averaging, there is some $j \in [l]$, such that P vanishes q^{-s} -a.e. on $X_{t_i} \cap Y_{i,j}$ – replace Y_i by this $Y_{i,j}$. Now suppose that X_{t_i} is $(A, B, s + F(1+s)^G)$ -regular (we will soon explain why we can choose t , such that this happens), and note that $X_{t_i} \cap Y_i$ is (A, B, s) -regular. Applying Lemma 4.2, we can replace Y_i by \tilde{Y}_i and still have $\mathbb{P}_{x \in X_{t_i} \cap Y_i} \geq 1 - q^{1-s}$. By Lemma 7.1, $P \upharpoonright_{X_{t_i} \cap \tilde{Y}_i} \equiv 0$. Setting $Y = Y_1 \cap \dots \cap Y_{2^{d+1}}$, we have $P \upharpoonright_{X_t \cap Y} \equiv 0$ for all $t \in [2^{d+1}]$. After applying Lemma 6.2 to Y relative to $X_t := X_{t_1} \cap \dots \cap X_{t_{2^{d+1}}}$ and assuming X_t is $(A, B, s + F(2^{d+1}F(1+s)^G)^G)$ -regular, we get that $X_t \cap Y$ is (A, B, s) -regular. By Corollary 7.4, we conclude that $P \upharpoonright_{X \cap Y} \equiv 0$ as desired.

Now we need to justify our assumptions on the regularity of X_t . Writing $n = 100s$, Claims 3.3 and 5.1 imply that if \mathcal{Q} is (C, D, s) -regular for sufficiently large $C(A, B, d, H), D(A, B, d, H)$, then for $q^{-2^{d+2}n}$ -a.e. $t \in \left(X_{d+1}^n\right)^{2^{d+1}}$, the variety X_t is $(A, B, s + F(2^{d+1}F(1+s)^G)^G)$ -regular as desired (in which case X_{t_i} , which is only defined by some of the equations, also must have the regularity we wanted). By Lemma 5.2, E has density at least q^{-n} in X_{d+1}^n , so $E^{2^{d+1}}$ has density at least $q^{-2^{d+1}n}$ in $\left(X_{d+1}^n\right)^{2^{d+1}}$, which means that there must be some $t \in E^{2^{d+1}}$ for which X_t has the desired regularity. \square

8. Nullstellensatz

In this section, we finish the proof of Theorem 2.13 by proving a Nullstellensatz for regular towers. This is similar to section 3 of [21], but by restricting our attention to regular towers, we can prove a stronger result for them. We start with the following definition.

Definition 8.1. We give $V^{[k]}$ a graph structure where two points $x, y \in V^{[k]}$ are connected by an edge if there exists some $i \in [k]$, such that $x_{[k] \setminus \{i\}} = y_{[k] \setminus \{i\}}$, that is, they differ in at most one coordinate.

We will need the following technical lemma.

Lemma 8.2. *There exist constants $A(d, H), B(d, H)$, such that if \mathcal{Q} is an (A, B, s) -regular tower on $V^{[d+1]}$ of degree $\leq d$ and height $\leq H$, $X = Z(\mathcal{Q})$ and $I \subset [d + 1]$, then, for q^{-s} -a.e. $x \in X$, for a.e. $y \in X \cap X(x_{\geq I})$ (with $X(x_{\geq I}) := \cap_{I \subset J} (X(x_J) \times V^J)$), the points x_I, y_I are connected by a path in $X(y_{[d+1] \setminus I})$.*

Proof. A sufficient condition for there to be a path in $X(y_{[d+1] \setminus I})$ between x_I, y_I is that there exists z , such that for all $K \subset I$, we have

$$(x_K, z_{I \setminus K}), (y_K, z_{I \setminus K}) \in X(y_{[d+1] \setminus I}).$$

Take the following variety

$$Y = \{(x_K, z_{I \setminus K}, y_{[d+1] \setminus I}), (y_K, z_{I \setminus K}, y_{[d+1] \setminus I}), (x_J, y_{[d+1] \setminus J}) \in X : K \subset I \subset J\}.$$

The equations defining it are partial to those of $\mathcal{Q}^{\times 3}$, so by Claim 5.1, we have $Y = Z(\mathcal{G})$ for a (C, D, s) -regular tower of degree $\leq d$ and height $\leq 3^{d+1}H$ as long as \mathcal{Q} is $(C(3^{d+1})^D, D, s)$ -regular. By Lemmas 3.3 and 3.4, for q^{-s} -a.e. $x \in X$, for q^{-s} -a.e. $y \in X \cap X(x_{\geq I})$, the variety $Y(x, y)$ (which is composed of the z 's we want) is (C', D', s) -regular. In particular, by Lemma 3.2, it is nonempty. \square

Now we are ready to prove:

Theorem 8.3 (Nullstellensatz). *There exist constants $A(d, H), B(d, H), s(d)$, such that if \mathcal{Q} is an (A, B, s) -regular multilinear tower on $V^{[d+1]}$ of deg $\leq d$ and height $\leq H$ and $P : V^{[d+1]} \rightarrow \mathbb{F}$ is a multilinear map which vanishes q^{-s} -a.e. on $Z(\mathcal{Q})$, then $rk_{\mathcal{Q}}(P) = 0$.*

The proof of this theorem (like everything else in this paper) will be by induction. For the inductive step, we will use the following simple corollary many times.

Corollary 8.4. *If $R_1, R_2 : V^I \rightarrow \mathbb{F}$ are multilinear maps, $\mathcal{Q}_1 \cup \mathcal{Q}_2$ is a multilinear tower on $V^{[d+1]}$ for which the conclusion of Theorem 8.3 holds, and*

$$\mathbb{P}_{x \in Z(\mathcal{Q}_1 \cup \mathcal{Q}_2)} [R_1(x) = R_2(x)] \geq 1 - q^{-s},$$

then there exists a multilinear map $R : V^I \rightarrow \mathbb{F}$ with $R \upharpoonright_{Z(\mathcal{Q}_1)} \equiv R_1 \upharpoonright_{Z(\mathcal{Q}_1)}$ and $R \upharpoonright_{Z(\mathcal{Q}_2)} \equiv R_2 \upharpoonright_{Z(\mathcal{Q}_2)}$.

Proof of corollary. By Theorem 8.3, $rk_{\mathcal{Q}_1 \cup \mathcal{Q}_2}(R_1 - R_2) = 0$. Therefore, there exist multilinear maps $f, g : V^I \rightarrow \mathbb{F}$ with $rk_{\mathcal{Q}_1}(f) = rk_{\mathcal{Q}_2}(g) = 0$ and $R_1 - R_2 = f + g$. Setting $R = R_1 - f = R_2 + g$ does the job. \square

Proof of Theorem 8.3. By Lemma 7.1, we have that $P \upharpoonright_{Z(\mathcal{Q})} = 0$. The proof will proceed by induction on the degree of \mathcal{Q} , and then on the number of layers of maximal degree. For the base case $\mathcal{Q} = \emptyset$, this is trivial. For the inductive step, choose some $\mathcal{Q}_0 \in \mathcal{Q}$ with a maximal size index set I , and write $\mathcal{Q}' = \mathcal{Q} \setminus \{\mathcal{Q}_0\}$. Apply Lemma 8.2 to the variety $X = Z(\mathcal{Q}') \cap \{\mathcal{Q}_0 = 1\}$. Choose $x^1, \dots, x^{2^{d+1}} \in X$, such that for each j , x^j satisfies the conclusion of Lemma 8.2 with q^{-s} -a.e. $y \in X \cap X(x^j_{\geq I})$ and such that $\mathcal{Q} \cup \mathcal{Q}(x_{\geq I})$ is (C, D, s) -regular, where $\mathcal{Q}(x_{\geq I}) := \bigcup_{j=1}^{2^{d+1}} \mathcal{Q}(x^j_{\geq I})$. The reason we can choose such x^j 's is that the first condition holds $2^{d+1}q^{-s}$ -a.s. by a union bound, and the second one holds (as long as A, B are sufficiently large) q^{-s} -a.s. by Claim 5.1 and Lemma 3.3. Let $\mathcal{Q}^j = \mathcal{Q}' \cup \mathcal{Q}(x^j_{\geq I})$, and define $R^j(y_{[d+1] \setminus I}) = P(x^j_I, y_{[d+1] \setminus I})$. We claim that

$$\mathbb{P}_{y \in Z(\mathcal{Q}^j)} [P(y) = R^j(y_{[d+1] \setminus I})\mathcal{Q}_0(y_I)] \geq 1 - q^{-s}.$$

Note that

$$Z(\mathcal{Q}^j) = \bigsqcup_{a \in \mathbb{F}} Z(\mathcal{Q}^j) \cap \{\mathcal{Q}_0 = a\},$$

and both sides vanish on $Z(Q)$ by assumption, so by scaling one of the coordinates, it is enough to prove that

$$\mathbb{P}_{y \in Z(Q^j) \cap \{Q_0=1\}} [P(y) = R^j(y_{[d+1] \setminus I})Q(y_I)] \geq 1 - q^{-s}.$$

Recall that by our definitions, $Z(Q^j) \cap \{Q_0 = 1\} = X \cap X(x_{\geq I}^j)$. Now, for any $y \in X \cap X(x_{\geq I}^j)$, we have equality at the point $(x_I^j, y_{[d+1] \setminus I}) \in Z(Q^j)$. Also, for q^{-s} -a.e. such y , the points x_I^j, y_I are connected in $X(y_{[d+1] \setminus I})$. Each time two points in X differ by a single coordinate in I , the values of both sides do not change, because the difference of the two points is in $Z(Q)$. This means that

$$\mathbb{P}_{y \in X \cap X(x_{\geq I}^j)} [P(y) = R^j(y_{[d+1] \setminus I})Q_0(y_I)] \geq 1 - q^{-s}$$

as desired, implying the same for $Z(Q^j)$.

By Lemma 7.1, we get that $P(y) = R^j(y_{[d+1] \setminus I})Q(y_I)$ for all $y \in Z(Q^j)$. The following claim will complete the inductive step.

Claim 8.5. *There exist constants $C(d, H), D(d, H)$, such that if $\mathcal{F}, \mathcal{G}^j$ for $j \in [2^k]$ are towers on $V^{[d+1]}$, of degree $\leq d$ and height $\leq H$, such that the \mathcal{G}^j depend only on the first k coordinates, and $Q : V^I \rightarrow \mathbb{F}, R^j : V^{[d+1] \setminus I} \rightarrow \mathbb{F}$ are multilinear maps, where*

1. $P(y) = R^j(y_{[d+1] \setminus I})Q(y_I) \forall y \in Z(\mathcal{F} \cup \mathcal{G}^j)$,
2. $\mathcal{F} \cup \{Q\} \cup \bigcup_{j \in [2^k]} \mathcal{G}^j$ is (C, D, s) -regular, and
3. the Nullstellensatz holds for $\mathcal{F} \cup \bigcup_{j \in [2^k]} \mathcal{G}^j$ and towers contained in it,

then there exists a multilinear $R : V^{[d+1] \setminus I} \rightarrow \mathbb{F}$, such that $P(y) = R(y_{[d+1] \setminus I})Q(y_I)$ for all $y \in Z(\mathcal{F})$.

Proof of claim. We prove this by induction on k . For $k = 0$, there is nothing to prove. Now suppose we have proved it up to k and \mathcal{G}^j are all defined on the first $k + 1$ coordinates. By looking at $Z(\mathcal{F} \cup \mathcal{G}^i \cup \mathcal{G}^{i+1})$, we get $R^i(y_{[d+1] \setminus I})Q(y_I) = R^{i+1}(y_{[d+1] \setminus I})Q(y_I)$ on $Z(\mathcal{F} \cup \mathcal{G}^i \cup \mathcal{G}^{i+1})$. By Lemma 3.3 (together with Lemma 3.2), for q^{-s} -a.e. $y_{[d+1] \setminus I} \in Z(\mathcal{F} \cup \mathcal{G}^i \cup \mathcal{G}^{i+1})_{[d+1] \setminus I}$, there exists y_I , such that $y \in Z(\mathcal{F} \cup \mathcal{G}^i \cup \mathcal{G}^{i+1}) \cap \{Q(y_I) = 1\}$, in which case, $R^i(y_{[d+1] \setminus I}) = R^{i+1}(y_{[d+1] \setminus I})$. By Corollary 8.4, there exists S^i , such that $P \upharpoonright_{Z(\mathcal{F} \cup \mathcal{G}^i)} = S^i(x_{[d+1] \setminus I})Q(x_I)$ for $j = i, i + 1$. Applying Lemma 7.3, we get that $P \upharpoonright_{Z(\mathcal{F} \cup (\mathcal{G}^i \cup \mathcal{G}^{i+1})_{[k]})} = S^i(x_{[d+1] \setminus I})Q(x_I)$. Now apply the induction hypothesis to the collection $(\mathcal{G}^i \cup \mathcal{G}^{i+1})_{[k]}$, and we are done. □

Applying the above claim with $\mathcal{F} = Q'$ and $\mathcal{G}^j = Q(x_{\geq I}^j)$ (note that the equations coming from $x_{\geq I}^j$ are of $\text{deg} < |I|$, so the induction hypothesis applies to $Q' \cup \bigcup_{j \in [2^{d+1}]} Q^j$ and towers partial to it), we get that $P(y) = R(y_{[d+1] \setminus I})Q(y_I)$ on $Z(Q')$. By the inductive hypothesis, $\text{rk}_{Q'}(P(y) - R(y_{[d+1] \setminus I})Q(y_I)) = 0$, and this completes the inductive step. □

9. Reduction to the multilinear case

In this section, we show that Theorems 2.2 and 2.7 follow from their multilinear analogues – Theorems 2.13 and 8.3. In order to do this, we will show that both relative bias and relative rank behave well when passing from polynomials to corresponding multilinear maps. We start by recalling the polarization identity, which associates a multilinear map to a polynomial. The following is well known.

Claim 9.1. *Given a polynomial P on V of degree $d < \text{char}(\mathbb{F})$, the map $\bar{P} : V^d \rightarrow \mathbb{F}$ defined by $\bar{P}(x_1, \dots, x_d) = \frac{1}{d!} \nabla_{x_1} \dots \nabla_{x_d} P(0)$ is multilinear and satisfies $\bar{P}(x, \dots, x) = \bar{P}(x)$ (the homogeneous part of P of highest degree).*

Definition 9.2. Given a polynomial tower Q on V , of degree $\leq d < \text{char}(\mathbb{F})$ and height $\leq H$, there exists a symmetric multilinear tower $Q(k)$ of degree $\leq d$ and height $\leq 2^k H$ on V^k corresponding to it. For a layer Q_i of degree e and a subset $E \subset [k]$ of size e , we get $Q(k)_{i,E} : V^E \rightarrow \mathbb{F}$ given by

\bar{Q}_i . Conversely, given a multilinear tower Q on V^k , it induces a polynomial tower $Q \circ D$ on V by precomposition with the diagonal map $Q \circ D = Q(x, \dots, x)$.

Claim 9.3. *If Q is an $(A2^{k^B}, B, s)$ -regular polynomial tower, then $Q(k)$ is (A, B, s) -regular. Conversely, if $Q(k)$ is $(A2^{k^B}, B, s)$ -regular and $k \geq \text{deg}(Q)$, then Q is (A, B, s) -regular.*

Proof. Assume without loss of generality that Q is homogeneous. The polynomials of degree d_i in Q give $\binom{k}{d_i} \leq 2^k$ polynomials in $Q(k)$, so we just need to check that

$$\text{prk}_{\bar{Q}_{<i}}(\bar{Q}_i) \geq \text{rk}_{Q_{<i}}(Q_i).$$

This follows by plugging in the diagonal for any relative low partition rank combination. For the converse, we need to check that

$$\text{prk}_{\bar{Q}_{<i}}(\bar{Q}_i) \leq 2^k \text{rk}_{Q_{<i}}(Q_i).$$

This follows from the polarization identity for any low relative polynomial rank combination. □

We recall Theorem 2.2:

Theorem 9.4 (Relative bias implies relative low rank). *There exist constants $A(d, H), B(d, H)$, such that if Q is an (A, B, s) -regular tower of degree $\leq d < \text{char}(\mathbb{F})$ and height $\leq H$ and P is a polynomial of degree $\leq d$ with*

$$\text{bias}_{Z(Q)}(P) \geq q^{-s},$$

then

$$\text{rk}_Q(P) \leq A(1 + s)^B.$$

We call Q as above a tower of type (d, H) for short. The proof will be by induction on the degree d . For $d = 1$, it follows from classical Fourier analysis. Now, we assume the theorem holds for degree d and explore some consequences.

Lemma 9.5 (Regular varieties are of the expected size (d, H)). *There exist constants $A(d, H), B(d, H)$, such that for any $s > 0$ and for any (A, B, s) -regular polynomial tower Q of type (d, H) defined on V of degree $\leq d$, height $\leq H$ and dimension m , we have*

$$\left| q^m \frac{|Z(Q)|}{|V|} - 1 \right| \leq q^{-s}.$$

Proof. Identical to that of Lemma 3.2. □

Now, we will prove a lemma analogous to Lemma 3.3. For a polynomial tower Q and $t \in V$, we denote

$$\Delta_t Q = \{ \Delta_t Q : Q \in \mathcal{Q} \}, \quad Q_t = Q \cup \Delta_t Q.$$

Lemma 9.6 (Derivatives). *For any C, D , there exist constants $A(C, D, d, H), B(C, D, d, H)$, such that if Q is an (Ak^B, B, s) -regular polynomial tower of type (d, H) , then for q^{-s} -a.e. $t_1, \dots, t_k \in Z(Q(1))$, the tower $Q_{t_1} \cup \dots \cup Q_{t_k}$ is (C, D, s) -regular.*

Proof. By Claim 9.3, it is enough that $(Q_{t_1} \cup \dots \cup Q_{t_k})(d)$ is $(C2^{dD}, D, s)$ -regular. By the same claim, $Q(d+1)$ is (Ek^F, F, s) -regular as long as A, B are large. By Claim 5.1 applied with multiplicity vector $l = (1, \dots, 1, k)$, the tower $Q' = (Q(d+1))^{\otimes l}$ is (E, F, s) -regular. Then by Lemma 3.3 for E, F sufficiently large, it follows that for q^{-s} -a.e. $t_1, \dots, t_k \in Z(Q(1))$, the tower $Q'(t_1, \dots, t_k)$ is (C, D, s) -regular. $Q'(t_1, \dots, t_k) = (Q_{t_1} \cup \dots \cup Q_{t_k})(d)$, so we are done. □

Corollary 9.7 (Fubini). *There are constants $A(d, H), B(d, H)$, such that if \mathcal{Q} is an (A, B, s) -regular polynomial tower of type (d, H) and f is a one-bounded function, then we have*

$$|\mathbb{E}_{x,y \in Z(\mathcal{Q})} f(x, y) - \mathbb{E}_{t \in Z(\mathcal{Q}(1))} \mathbb{E}_{x \in Z(\mathcal{Q}_t)} f(x, x+t)| \leq q^{-s}.$$

Proof. Identical to the proof of Lemma 3.4. □

For a function $f : V \rightarrow \mathbb{C}$, and $t \in V$, denote the (multiplicative) derivative by $D_t f(x) = f(x+t)\bar{f}(x)$. For $t = (t_1 \dots t_k) \in V^k$, write $\mathcal{Q}_t = (\dots (\mathcal{Q}_{t_1})_{t_2} \dots)_{t_k}$.

Lemma 9.8 (Cauchy-Schwarz). *There exist $A(d, H, k), B(d, H, k)$, such that if \mathcal{Q} is an (A, B, s) -regular polynomial tower of degree $\leq d$ and height $\leq H$, and $f : V \rightarrow \mathbb{C}$ is a one-bounded function, then*

$$|\mathbb{E}_{x \in Z(\mathcal{Q})} f(x)|^{2^k} \leq \text{Re}(\mathbb{E}_{t \in Z(\mathcal{Q}(k))} \mathbb{E}_{x \in Z(\mathcal{Q}_t)} D_{t_1} \dots D_{t_k} f(x)) + q^{-s}.$$

Proof. By induction on k . For $k = 1$, this follows from Corollary 9.7. Suppose it holds up to $k - 1$. Then if A, B are sufficiently large, we can apply Corollary 9.7 to get

$$\begin{aligned} |\mathbb{E}_{x \in Z(\mathcal{Q})} f(x)|^{2^k} &= |\mathbb{E}_{x,y \in Z(\mathcal{Q})} f(x)\bar{f}(y)|^{2^{k-1}} \leq |\mathbb{E}_{t \in Z(\mathcal{Q}(1))} \mathbb{E}_{x \in Z(\mathcal{Q}_t)} D_t f(x)|^{2^{k-1}} + q^{-4s} \\ &\leq \mathbb{E}_{t \in Z(\mathcal{Q}(1))} |\mathbb{E}_{x \in Z(\mathcal{Q}_t)} D_t f(x)|^{2^{k-1}} + q^{-4s}. \end{aligned}$$

If A, B are sufficiently large, then by Lemma 9.6, the inductive hypothesis applies to q^{-4s} -a.e. \mathcal{Q}_t . Therefore,

$$\begin{aligned} \mathbb{E}_{t \in Z(\mathcal{Q}(1))} |\mathbb{E}_{x \in Z(\mathcal{Q}_t)} D_t f(x)|^{2^{k-1}} \\ \leq \text{Re}(\mathbb{E}_{t \in Z(\mathcal{Q}(1))} \mathbb{E}_{s \in Z(\mathcal{Q}_t(k-1))} \mathbb{E}_{x \in Z((\mathcal{Q}_t)_s)} D_s D_t f(x)) + q^{-2s}. \end{aligned}$$

Finally, using the fact that $\mathcal{Q}_t(k - 1) = (\mathcal{Q}(k))(t)$ and applying Lemma 3.4 for $\mathcal{Q}(k)$, we get that

$$\begin{aligned} \text{Re}(\mathbb{E}_{t \in Z(\mathcal{Q}(1))} \mathbb{E}_{s \in Z(\mathcal{Q}_t(k-1))} \mathbb{E}_{x \in Z(\mathcal{Q}_{(t,s)})} D_s D_t f(x)) \\ \leq \text{Re}(\mathbb{E}_{t \in Z(\mathcal{Q}(k))} \mathbb{E}_{x \in Z(\mathcal{Q}_t)} D_{t_1} \dots D_{t_k} f(x)) + q^{-2s}, \end{aligned}$$

which completes the inductive step. □

Corollary 9.9. *Theorem 2.2 holds for polynomials of degree $\leq d + 1$ on towers of degree $\leq d$.*

Proof. When $\text{deg}(P) \leq d$, we are already assuming this holds, so we deal with the case $\text{deg}(P) = d + 1$. If $\text{bias}_{Z(\mathcal{Q})}(P) \geq q^{-s}$, then by Lemma 9.8, we get that $\text{bias}_{Z(\mathcal{Q}(d+1))}(\bar{P}) \geq q^{-2s}$. By Claim 9.3, the tower $\mathcal{Q}(d + 1)$ is regular, so we can apply Theorem 2.13 to deduce that $\text{prk}_{\mathcal{Q}(d+1)}(\bar{P}) \leq A(1 + s)^B$. Plugging in the diagonal, we get that $\text{rk}_{\mathcal{Q}}(P) \leq A(1 + s)^B$. □

To deal with towers of degree $d + 1$, we introduce an auxiliary parameter like we did in the proof of Theorem 2.13:

Theorem 9.10 (Relative bias implies relative low rank). *There exist constants $A(d, H, l), B(d, H, l)$, such that if \mathcal{Q} is an (A, B, s) -regular tower of degree $\leq d + 1 < \text{char}(\mathbb{F})$ and height $\leq H$, such that $\mathcal{Q}_{\leq H-l}$ is of degree $\leq d$, and P is a polynomial of degree $\leq d + 1$ with*

$$\text{bias}_{Z(\mathcal{Q})}(P) \geq q^{-s},$$

then

$$\text{rk}_{\mathcal{Q}}(P) \leq A(1 + s)^B.$$

A tower \mathcal{Q} as above is called a tower of type $(d + 1, H, l)$ for short. We have just seen this theorem holds for towers of type $(d + 1, H, 0)$. Theorem 2.2 is for towers of type $(d + 1, H, H)$.

Proof of Theorem 9.10. By induction on l . We have just seen the base case $l = 0$. Suppose the theorem holds for towers of type $(d + 1, H, l)$ and \mathcal{Q} is now of type $(d + 1, H, l + 1)$. The inductive assumption implies Lemma 9.5 for \mathcal{Q} and $\mathcal{Q}_{<H}$ (as in the proof of Lemma 3.2), so we get

$$\sum_{a \in \mathbb{F}^{m_H}} \text{bias}_{Z(\mathcal{Q}_{<H})}(P + a \cdot \mathcal{Q}_H) \geq q^{-2s}.$$

By subadditivity of relative rank, there is at most a single $a_0 \in \mathbb{F}^{m_H}$ with $\text{rk}_{\mathcal{Q}_{<H}}(P + a \cdot \mathcal{Q}_H) \leq A/2(m_H + s)^B$, and if $\text{deg}(P) < \text{deg}(\mathcal{Q}_H)$, then $a_0 = 0$. By the theorem applied to $\mathcal{Q}_{<H}$, we get that for all $a \neq a_0$,

$$\text{bias}_{Z(\mathcal{Q}_{<H})}(P + a \cdot \mathcal{Q}_H) \leq q^{-4s - m_H},$$

so $\text{bias}_{Z(\mathcal{Q}_{<H})}(P + a_0 \cdot \mathcal{Q}_H) \geq q^{-4s}$. This gives

$$\text{rk}_{\mathcal{Q}}(P) \leq \text{rk}_{Z(\mathcal{Q}_{<H})}(P + a_0 \cdot \mathcal{Q}_H) \leq A(1 + s)^B. \quad \square$$

We are also ready to prove Theorem 2.7:

Theorem 9.11 (Robust Nullstellensatz for regular collections). *There exist constants $A(d, H), B(d, H), s(d)$, such that if $\mathcal{Q} = (Q_{i,j})_{i \in [h], j \in [m_i]}$ is an (A, B, s) -regular polynomial tower of $\text{deg} \leq d < \text{char}(\mathbb{F})$ and height $\leq H$, and P is a polynomial of degree $\leq d$ which vanishes q^{-s} -a.e. on $Z(\mathcal{Q})$, then there exist polynomials $R_{i,j}$ satisfying $\text{deg}(R_{i,j}) + \text{deg}(Q_{i,j}) \leq \text{deg}(P)$ and*

$$P = \sum_{i,j} R_{i,j} Q_{i,j}.$$

Proof. By induction on $\text{deg}(P)$. For constant P , there is nothing to prove, since we know by Lemma 9.5 that $Z(\mathcal{Q}) \neq \emptyset$. Suppose the theorem holds up to degree $k - 1$ and $\text{deg}(P) = k$. Suppose P vanishes q^{-s} -a.e. on $Z(\mathcal{Q})$. By Lemmas 3.4 and 9.8, \bar{P} vanishes q^{-s} -a.e. on $Z(\mathcal{Q}(k))$. By Theorem 8.3, $\text{rk}_{\mathcal{Q}(k)}(\bar{P}) = 0$. Plugging in the diagonal, this implies that $\text{rk}_{\mathcal{Q}}(\bar{P}) = 0$ so there exist $R_{i,j}$ with $\bar{P} = \sum R_{i,j} \bar{Q}_{i,j}$. The polynomial $P' = P - \sum R_{i,j} Q_{i,j}$ has $\text{deg}(P') \leq k - 1$ and also vanishes q^{-s} -a.e. on $Z(\mathcal{Q})$. Applying the inductive hypothesis finishes the proof. \square

A. Adaptations of results involving rank to relative rank

We formulate and prove the relevant parts of the Theorems in [15, 17], replacing rank with relative rank. Let \mathbb{F} be a field. For an algebraic \mathbb{F} -variety \mathbb{X} , we write $X(\mathbb{F}) := \mathbb{X}(\mathbb{F})$. To simplify notations, we often write X instead of $X(\mathbb{F})$. In particular, we write $V := \mathbb{V}(\mathbb{F})$ when \mathbb{V} is a vector space and write \mathbb{F}^N for $\mathbb{A}^N(\mathbb{F})$.

For a \mathbb{F} -vector space \mathbb{V} , we denote by $\mathcal{P}_{\bar{d}}(\mathbb{V})$ the algebraic variety of tuples polynomials $\bar{P} = (P_1, \dots, P_c)$ on \mathbb{V} of degrees $\leq \bar{d}$ and by $\mathcal{P}_{\bar{d}}(V)$ the set of tuples of polynomials functions $\bar{P} : V \rightarrow \mathbb{F}^c$ of degree $\leq \bar{d}$. We always assume that $d < \text{char}(\mathbb{F})$, so the restriction map $\mathcal{P}_{\bar{d}}(\mathbb{V})(\mathbb{F}) \rightarrow \mathcal{P}_{\bar{d}}(V)$ is a bijection.

We recall some of the definitions in [15].

Definition A.1.

- For $m \geq 1$ and a \mathbb{F} -vector space V , we denote by $\text{Aff}_m(\mathbb{V})$ the algebraic variety of affine maps $\phi : \mathbb{A}^m \rightarrow \mathbb{V}$ and write $\text{Aff}_m(V) := \text{Aff}_m(\mathbb{V})(\mathbb{F})$.
- We define an algebraic morphism $\tilde{\kappa}_{\bar{P}} : \text{Aff}_m(\mathbb{V}) \rightarrow \mathcal{P}_{\bar{d}}(\mathbb{A}^m)$ by $\tilde{\kappa}_{\bar{P}}(\phi) := \bar{P} \circ \phi$, and denote by $\kappa_{\bar{P}}$ the corresponding map $\text{Aff}_m(V) \rightarrow \mathcal{P}_{\bar{d}}(\mathbb{F}^m)$.

Definition A.2. A map $\kappa : M \rightarrow N$ between finite sets is ε -uniform, where $\varepsilon > 0$, if for all $n \in N$, we have

$$||N||\kappa^{-1}(n) - |M| \leq \varepsilon|M|.$$

Theorem A.3. *There exist constants $A(d, h), B(d, h)$, such that for any $m, s \geq 1$, any finite field $\mathbb{F} = \mathbb{F}_q$, and any (Am^B, B, s) -regular tower \bar{P} of degree $\leq d < \text{char}(\mathbb{F})$ and height $\leq h$, the map $\kappa_{\bar{P}} : \text{Aff}_m(\mathbb{V}) \rightarrow \mathcal{P}_{\bar{d}}(\mathbb{A}^m)$ is q^{-s} uniform.*

As a corollary, we obtain (as in [15])

Theorem A.4. *For any $m \geq 1$, any algebraically closed field \mathbb{F} of characteristic zero or $> d$, any \mathbb{F} -vector space V , and any $(Am^B, B, 2)$ -regular tower \bar{P} of degree $\leq d$ and height $\leq h$, where A, B are the constants of Theorem A.3, we have*

1. *The map $\kappa_{\bar{P}}$ is surjective,*
2. *All fibers of the morphism $\bar{\kappa}_{\bar{P}}$ are of the same dimension,*
3. *The morphism $\bar{\kappa}_{\bar{P}}$ is flat, and*
4. *$\bar{P} \subset \mathbb{F}[V^{\vee}]$ is a regular sequence.*

Let $\epsilon > 0$, we say that a property holds for ϵ -a.e. $s \in S$ if it holds for all but $(1 - \epsilon)|S|$ of the elements in S .

As in [15], Theorem A.3 follows from the following result:

Theorem A.5. *Let $d \geq 0$. There exist $A(d, h), B(d, h)$, such that for any $s, m > 0$, the following holds: Let $\mathbb{F} = \mathbb{F}_q$ be a finite field of characteristic $> d$ and let \bar{P} be an (Am^B, B, s) -regular tower of degree $\leq d$ and height $\leq h$ composed of polynomials (P_1, \dots, P_c) of degrees (d_1, \dots, d_c) on $V = \mathbb{F}^n$. Then:*

1. *For any collection of polynomials $\bar{R} = (R_i)_{1 \leq i \leq c}$, with $R_i : \mathbb{F}^m \rightarrow \mathbb{F}$ of degree d_i , there exists an affine map $w : \mathbb{F}^m \rightarrow \mathbb{F}^n$, such that $\bar{P}(w(x)) = \bar{R}(x)$. Furthermore, if we denote by $n_{\bar{R}}$ the number of such affine maps, then for any \bar{R}_1, \bar{R}_2 as above, $|1 - n_{\bar{R}_1}/n_{\bar{R}_2}| < q^{-s}$.*
2. *If \bar{P} is homogeneous, then for any homogeneous collection $\bar{R} = (R_i)_{1 \leq i \leq c}$, $R_i : \mathbb{F}^m \rightarrow \mathbb{F}$ of degree d_i , there exists a linear map $w : \mathbb{F}^m \rightarrow \mathbb{F}^n$, such that $\bar{P}(w(x)) = \bar{R}(x)$. Furthermore, if we denote by $n_{\bar{R}}$ the number of such linear maps, then for any \bar{R}_1, \bar{R}_2 as above, $|1 - n_{\bar{R}_1}/n_{\bar{R}_2}| < q^{-s}$.*

Proof. We prove by induction on d . For $d = 1$, the statement is clear. Assume $d > 1$. We are given a filtered collection of polynomials $\mathcal{P} = \{P^{e,f}\}_{1 \leq e \leq h, 0 \leq f \leq J_e}$. We first make the following observation. Let P be of degree d . $P(t) = \sum_{I \in \mathcal{I}_d} a_I t_I$, where $\mathcal{I}_d(n)$ is the set of ordered tuples $I = (i_1, \dots, i_d)$ with $1 \leq i_1 \leq \dots \leq i_d \leq n$, and $t_I = t_{i_1} \dots t_{i_d}$. Note that for any polynomials $l(t)$ of degrees $< d$, we have that $P(t) + l(t)$ is of the same rank as P . We can write

$$P(w(x)) = \sum_{I \in \mathcal{I}_d(n)} a_I w^I(x) = \sum_{I \in \mathcal{I}_d(n)} a_I \sum_{l_1, \dots, l_d=1}^m w_{l_1}^{i_1} \dots w_{l_d}^{i_d} x_{l_1} \dots x_{l_d},$$

where $w^I = \prod_{i \in I} w^i$. For $(l_1, \dots, l_d) \in \mathcal{I}_d(m)$, the term $x_{l_1} \dots x_{l_d}$ has as coefficient

$$Q_{(l_1, \dots, l_d)}(w) = \sum_{\sigma \in S_d} \sum_{I \in \mathcal{I}(n)} a_I w_{l_{\sigma(1)}}^{i_1} \dots w_{l_{\sigma(d)}}^{i_d}.$$

Observe that restricted to the subspace $w_{l_1} = \dots = w_{l_d}$, we can write the above as

$$Q_{(l_1, \dots, l_d)}(w) = d!P(w_{l_1}) + R(w),$$

where $w_j = (w_j^1, \dots, w_j^n)$, and $R(w)$ is of lower degree in w_{l_1} .

Now, consider the filtered system \mathcal{P} . It gives rise to a filtered system

$$\{Q_{(l_1, \dots, l_e)}^{e,f}(w)\}_{1 \leq e \leq h, 0 \leq f \leq J_e, (l_1, \dots, l_{de}) \in \mathcal{I}_{de}(m)}.$$

Given an element $Q_{(l_1, \dots, l_d)}^{e,f}$ at level e of degree d , we restrict to a subspace as above and get $Q^{e,f}(w) = d!P(w_{l_1}) + R^{e,f}(w)$ and $R(w)$ is of lower degree in w_{l_1} . But the same holds for all lower degree polynomials as well – restricted to this subspace, they are of the form $j!P^{e,f}(w_{l_1}) + R^{e,f}(w)$, where $R^{e,f}$ of lower degree in w_{l_1} . Similarly, this holds for a linear combination of elements at level h . This is sufficient since the relative rank can only decrease when restricting to a subspace. So we just need to make sure that the original relative rank beats the new number of polynomials at each level, which is $\leq m^{O(d)}$. □

We formulate a number of additional results that were proved in the context of high rank collections of polynomials and whose proofs can be easily adapted to the new notion of relative rank.

We recall the definition of weakly polynomial function from [15].

Definition A.6.

1. Let V be a \mathbb{F} -vector space and $X \subset V$. We say that a function $f : X \rightarrow \mathbb{F}$ is *weakly polynomial* of degree $\leq a$ if the restriction $f|_L$ to any affine subspace $L \subset X$ is a polynomial of degree $\leq a$.
2. X satisfies \star^a if any weakly polynomial function of degree $\leq a$ on X is a restriction of a polynomial function of degree $\leq a$ on V .

Theorem A.7. *For any $d, a \geq 1$, there exist constants $A(d, h, a), B(d, h, a)$, such that the following holds: for any field \mathbb{F} which is either finite with $|\mathbb{F}| > ad$ or algebraically closed of characteristic zero or $> d$, a \mathbb{F} -vector space V , an $(A, B, 2)$ -regular tower \mathcal{P} of degree $\leq d$, and height $\leq h$, the subset $Z(\mathcal{P}) \subset V$ has the property \star_a .*

Combined with the polynomial cost regularization of Theorem 2.3, we obtain:

Theorem A.8. *For any $d, a \geq 1$, there exists $A = A(d, a)$, such the following holds: for any field \mathbb{F} which is either finite with $|\mathbb{F}| > ad$ or algebraically closed of characteristic zero or $> d$, a \mathbb{F} -vector space V , a collection of polynomials $\bar{P} = (P_i)_{1 \leq i \leq c}$ with $\deg P_i \leq d$, there exists a collection of polynomials \mathcal{Q} , such that $\bar{P} \subset \mathcal{I}(\mathcal{Q})$, such that $Z(\mathcal{Q}) \subset V$ has the property \star_a , and $|\mathcal{Q}| \leq Ac^A$.*

Finally, we can also adapt the results from [17] to the setting of relative rank: Let V be a vector space over a field \mathbb{F} . An m -cube in a vector space V is a collection $(u|\bar{v}), u \in V, \bar{v} \in V^m$ of 2^m points $\{u + \sum_{i=1}^m \omega_i v_i\}, \omega_i \in \{0, 1\}$.

For any map $f : V \rightarrow H$ where H is an Abelian group, we denote by f_m the map from the set $C_m(V)$ of m -cubes to H given by

$$f_m(u|\bar{v}) = \sum_{\bar{\omega} \in \{0,1\}^m} (-1)^{|\bar{\omega}|} f(u + \sum_{i=1}^m \omega_i v_i),$$

where $|\bar{\omega}| = \sum_{i=1}^m \omega_i$. For a subset $X \subset V$, we denote $C_m(X)$ the set of m -cubes in V with all vertices in X . Note that in the case that $H = \mathbb{F}$, where \mathbb{F} a prime field, functions $f : V \rightarrow \mathbb{F}$, such that f_m vanishes on $C_m(V)$ are precisely polynomials of degree $< m$.

Theorem A.9. *Let $m, d > 0$. There exists $C = C(d, m, h)$, such that the following holds: For any $0 < \epsilon$, a finite field \mathbb{F} and an \mathbb{F} -vector space V , a $(C, C, 2)$ -regular tower \mathcal{P} of degree $\leq d$ and height $\leq h$, and any $f : X = Z(\mathcal{P}) \rightarrow H$ with $f_m(c) = 0$ for ϵ -a.e. $c \in C_m(X)$, there exists a function $h : X \rightarrow H$, such that $h_m \equiv 0$, and $h(x) = f(x)$ on $C\epsilon$ -a.e. $x \in X$.*

Acknowledgments. The first author would like to thank the second author for introducing him to this question and its various applications. He would also like to thank his wife, Noa, for her constant support and encouragement. The authors were supported by European Research Council grant ErgComNum 682150 and Israel Science Foundation grant 2112/20.

Competing interest. The authors have no competing interest to declare.

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