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# $\times a$ and $\times b$ empirical measures, the irregular set and entropy

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*Abstract.* For integers $a$ and $b \geq 2$, let $T_a$ and $T_b$ be multiplication by $a$ and $b$ on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. The action on $\mathbb{T}$ by $T_a$ and $T_b$ is called $\times a$, $\times b$ action and it is known that, if $a$ and $b$ are multiplicatively independent, then the only $\times a$, $\times b$ invariant and ergodic measure with positive entropy of $T_a$ or $T_b$ is the Lebesgue measure. However, it is not known whether there exists a non-trivial $\times a$, $\times b$ invariant and ergodic measure. In this paper, we study the empirical measures of $x \in \mathbb{T}$ with respect to the $\times a$, $\times b$ action and show that the set of $x$ such that the empirical measures of $x$ do not converge to any measure has Hausdorff dimension one and the set of $x$ such that the empirical measures can approach a non-trivial $\times a$, $\times b$ invariant measure has Hausdorff dimension zero. Furthermore, we obtain some equidistribution result about the $\times a$, $\times b$ orbit of $x$ in the complement of a set of Hausdorff dimension zero.

Key words: multiplication on the circle, irregular set, entropy, Hausdorff dimension, equidistribution
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## 1. *Introduction and main theorems*

In this paper, we write $\mathbb{Z}_{\geq 0}$ for the set of integers equal to or larger than zero and $\mathbb{N}$ for the set of positive integers. Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and, for $a \in \mathbb{Z}$ with $a \geq 2$, define $T_a : \mathbb{T} \to \mathbb{T}$ by

$$T_a(x) = ax, \quad x \in \mathbb{T}.$$

We take $a, b \in \mathbb{Z}$ such that $a, b \geq 2$. Since $T_a$ and $T_b$ are commutative, they define the $\mathbb{Z}^2_{\geq 0}$-action on $\mathbb{T}$ and we call it the $\times a$, $\times b$ *action*. Here we notice that, if $\log a / \log b \in \mathbb{Q}$, then $a = c^k$ and $b = c^l$ for some $c \geq 2$, $k, l \in \mathbb{N}$, and the $\times a$, $\times b$ action derives from the $\times c$ action by the single map $T_c$. Therefore, we are interested in the case when $a$ and $b$ are *multiplicatively independent*, that is, $\log a / \log b \notin \mathbb{Q}$.

There is a distinction between the $\times a$ action by the single map $T_a$ and the $\times a$, $\times b$ action about the closed invariant subsets. It is well known that the $\times a$ action has many invariant closed subsets of $\mathbb{T}$. However, H. Furstenberg showed that $\times a$, $\times b$ invariant (that is, invariant under $T_a$ and $T_b$) closed subsets are very restricted.

PROPOSITION 1.1. [6, Theorem IV.1] *Suppose $a$ and $b$ are multiplicatively independent, that is,* $\log a/\log b \notin \mathbb{Q}$. *Let $X \subset \mathbb{T}$ be a non-empty, closed and $\times a$, $\times b$ invariant subset. Then $X = \mathbb{T}$ or $X$ is a finite set in $\mathbb{Q}/\mathbb{Z}$.*

He also conjectured the measure-theoretic version of Proposition 1.1. We write $M(\mathbb{T})$ for the set of Borel probability measures on $\mathbb{T}$ and $M_{\times a,\times b}(\mathbb{T})$ for the set of $\times a$, $\times b$ invariant Borel probability measures on $\mathbb{T}$, that is, the set of $\mu \in M(\mathbb{T})$ such that $\mu$ is invariant under $T_a$ and $T_b$. Furthermore, we write $E_{\times a,\times b}(\mathbb{T})$ for the set of $\times a$, $\times b$ invariant and ergodic probability measures on $\mathbb{T}$, that is, the set of $\mu \in M_{\times a,\times b}(\mathbb{T})$ such that $\mu$ is ergodic with respect to the $\mathbb{Z}^2_{\geq 0}$-action by $T_a$ and $T_b$. The Lebesgue measure on $\mathbb{T}$ is denoted by $m_{\mathbb{T}}$. We notice that $m_{\mathbb{T}} \in E_{\times a,\times b}(\mathbb{T})$.

*Conjecture 1.2.* Suppose $a$ and $b$ are multiplicatively independent. Let $\mu \in E_{\times a,\times b}(\mathbb{T})$. Then $\mu = m_{\mathbb{T}}$ or $\mu$ is an atomic measure equidistributed on a $\times a$, $\times b$ periodic orbit on $\mathbb{Q}/\mathbb{Z}$.

This problem has been open for a long time. However, the following theorem was shown by Rudolph in [11] when $a$ and $b$ are relatively prime and by Johnson in [9] when $a$ and $b$ are multiplicatively independent. For a $T$-invariant probability measure $\mu$ ($T = T_a$ or $T_b$), we write $h_\mu(T)$ for the measure-theoretic entropy of $T$ with respect to $\mu$.

THEOREM 1.3. (The Rudolph–Johnson Theorem) *Suppose $a$ and $b$ are multiplicatively independent. Let $\mu \in E_{\times a,\times b}(\mathbb{T})$ such that $h_\mu(T_a) > 0$ or $h_\mu(T_b) > 0$. Then $\mu = m_{\mathbb{T}}$.*

By Theorem 1.3, if there exists some non-trivial $\times a$, $\times b$ invariant and ergodic probability measure $\mu$, then $h_\mu(T_a) = h_\mu(T_b) = 0$. There are distinct proofs of Theorem 1.3 and stronger results in [4, 7, 8], although the positive entropy assumption is crucial in all of them.

For $x \in \mathbb{T}$, let $\delta_x$ be the probability measure supported on the one point set $\{x\}$. For each $N \in \mathbb{N}$, we write $\delta^N_{\times a,\times b,x} \in M(\mathbb{T})$ for the *$N$-empirical measure of $x$ (with respect to the $\times a$, $\times b$ action)*, that is,

$$\delta^N_{\times a,\times b,x} = \frac{1}{N^2}\sum_{m,n=0}^{N-1} \delta_{T_a^m T_b^n x}.$$

If we give $M(\mathbb{T})$ the weak* topology, then $M(\mathbb{T})$ is a compact and metrizable space. It is easily seen that any accumulation point in $M(\mathbb{T})$ of $\delta^N_{\times a,\times b,x}$ ($N \in \mathbb{N}$), that is, $\mu \in M(\mathbb{T})$ such that $\delta^{N_k}_{\times a,\times b,x} \to \mu$ in $M(\mathbb{T})$ as $k \to \infty$ for some divergent subsequence $\{N_k\}_{k=1}^\infty$ in $\mathbb{N}$, is $\times a$, $\times b$ invariant. If $\mu \in E_{\times a,\times b}(\mathbb{T})$, then, by Birkhoff's ergodic theorem,

$$\delta^N_{\times a,\times b,x} \xrightarrow[N\to\infty]{} \mu \quad \text{for } \mu\text{-almost every } x.$$

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We refer the reader to [10] for Birkhoff's ergodic theorem for $\mathbb{Z}^2_{\geq 0}$-actions. In this paper, we study two types of subsets of $\mathbb{T}$ about the behavior of $\delta^N_{\times a,\times b,x}$ as $N \to \infty$: the set of $x$ such that $\delta^N_{\times a,\times b,x}$ does not converge to any invariant measure, which is called the *irregular set* for the empirical measure, and the set of $x$ such that $\delta^N_{\times a,\times b,x}$ accumulates to some invariant probability measure that has the given upper bound of entropy. Our main results give an estimate of the Hausdorff dimension of these sets.

We give the first main result in this paper about the irregular set. We write $J$ for the irregular set. We notice that, by Birkhoff's ergodic theorem, $\mu(J) = 0$ for any $\mu \in M_{\times a,\times b}(\mathbb{T})$. However, in general, the irregular set can be either small or large. For example, it is clear that, if an action on a compact metric space is uniquely ergodic, then its irregular set is empty. On the other hand, the following fact holds for the $\times a$ action by the single map $T_a$. For a Hölder continuous function $\varphi : \mathbb{T} \to \mathbb{R}$, we write $J_\varphi$ for the irregular set for $\varphi$, that is, the set of $x \in \mathbb{T}$ such that the Birkhoff average $N^{-1} \sum_{n=0}^{N-1} \varphi(T_a^n x)$ $(N \in \mathbb{N})$ does not converge as $N \to \infty$. If $\varphi$ is not cohomologous to a constant, then $\dim_H J_\varphi = 1$ and hence the irregular set for the empirical measure has Hausdorff dimension one. We remark that this fact holds under more general situations (see [1]). Under these situations, there exist many distinct invariant and ergodic measures which have sufficiently large dimension, and hence many subsets with large Hausdorff dimension on which the Birkhoff average converges to distinct values. Since $\times a$, $\times b$ invariant and ergodic measures on $\mathbb{T}$ are restricted by Theorem 1.3, the situation of the $\times a$, $\times b$ action is different from what we mentioned above. However, it is shown that the irregular set is a subset of $\mathbb{T}$ with large Hausdorff dimension. In [3], it is shown that the set of $x \in \mathbb{T}$ such that the $\times 2$, $\times 3$ empirical measures by another way of taking averages do not converge to $m_{\mathbb{T}}$ has positive Hausdorff dimension. Our theorem below is a stronger result.

THEOREM 1.4. *Let $J$ be the set of $x \in \mathbb{T}$ such that $\delta^N_{\times a,\times b,x}$ $(N \in \mathbb{N})$ does not converge to any $\times a$, $\times b$ invariant probability measure as $N \to \infty$. Then*

$$\dim_H J = 1.$$

We notice that Theorem 1.4 is shown without the hypothesis that $a$ and $b$ are multiplicatively independent. It is remarkable that Theorem 1.4 immediately leads to the following result that is stronger than itself, which is about the irregular sets for Fourier basis functions. For $k \in \mathbb{Z}$, we write $e_k(x) = e^{2k\pi i x}$ $(x \in \mathbb{T})$ and, as above, $J_{e_k}$ for the irregular set for $e_k$, that is, the set of $x \in \mathbb{T}$ such that the Birkhoff average $N^{-2} \sum_{m,n=0}^{N-1} e_k(T_a^m T_b^n x)$ $(N \in \mathbb{N})$ does not converge as $N \to \infty$.

COROLLARY 1.5. *For $k \in \mathbb{Z} \setminus \{0\}$,*

$$\dim_H J_{e_k} = 1.$$

We prove Theorem 1.4 and Corollary 1.5 in §2.

Next we give the second main result. As we said above, if $a$ and $b$ are multiplicatively independent, it is conjectured that there exist no non-trivial $\times a, \times b$ invariant and ergodic measures (Conjecture 1.2). This problem seems to be very difficult; however, by Theorem 1.3, those non-trivial invariant measures have entropy zero. We expect that the

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set of $x \in \mathbb{T}$ such that $\delta^N_{\times a, \times b, x}$ approaches a non-trivial measure as $N \to \infty$ is a small subset of $\mathbb{T}$. The following theorem and corollary answer this expectation.

THEOREM 1.6. *Let* $0 < t < \min\{\log b, (\log a)^2/\log b\}$ *and let* $K_t$ *be the set of* $x \in \mathbb{T}$ *such that* $\delta^N_{\times a, \times b, x}$ $(N \in \mathbb{N})$ *accumulates to some* $\mu \in M_{\times a, \times b}(\mathbb{T})$ *such that* $h_\mu(T_a) \le t$. *Then*

$$\dim_H K_t \le \frac{2\sqrt{\log b}\sqrt{t}}{\log a + \sqrt{\log b}\sqrt{t}}.$$

We notice that Theorem 1.6 is shown without the hypothesis that $a$ and $b$ are multiplicatively independent. By taking $\bigcap_{t>0} K_t$ and applying Theorem 1.3, we obtain the following corollary.

COROLLARY 1.7. *Suppose* $a$ *and* $b$ *are multiplicatively independent. Let* $K$ *be the set of* $x \in \mathbb{T}$ *such that* $\delta^N_{\times a, \times b, x}$ $(N \in \mathbb{N})$ *accumulates to some* $\mu \in E_{\times a, \times b}(\mathbb{T})$ *such that* $\mu \neq m_{\mathbb{T}}$. *Then*

$$\dim_H K = 0.$$

If $a$ and $b$ are multiplicatively independent, Theorems 1.6 and 1.3 lead to the result about the distributions of the $\times a$, $\times b$ orbits. For $0 < t \le 1$ and $x \in \mathbb{T}$, we say that the $\times a$, $\times b$ orbit $\{a^m b^n x\}_{m,n \in \mathbb{Z}_{\ge 0}}$ of $x$ is *t-semiequidistributed* if

$$\liminf_{N\to\infty} \frac{1}{N^2} \sum_{m,n=0}^{N-1} f(a^m b^n x) \ge t \int_{\mathbb{T}} f \, dm_{\mathbb{T}}$$

for any $f \in C(\mathbb{T})$ such that $f \ge 0$ on $\mathbb{T}$ and

$$\liminf_{N\to\infty} \frac{1}{N^2} |\{(m,n) \in \mathbb{Z}^2 \mid 0 \le m, n < N, a^m b^n x \in U\}| \ge t \cdot m_{\mathbb{T}}(U)$$

for any open subset $U \subset \mathbb{T}$. It is easy to see that the latter statement follows from the former. This property says that the orbit $\{a^m b^n x\}_{m,n \in \mathbb{Z}_{\ge 0}}$ includes an equidistributed portion of the ratio that is at least $t$. Then we have the following theorem.

THEOREM 1.8. *Suppose* $a$ *and* $b$ *are multiplicatively independent. Let* $0 < t < \min\{\log b, (\log a)^2/\log b\}$ *and let* $K_t \subset \mathbb{T}$ *be as above. Then, for each* $x \in \mathbb{T} \setminus K_t$, *the orbit* $\{a^m b^n x\}_{m,n \in \mathbb{Z}_{\ge 0}}$ *is* $t/\log a$*-semiequidistributed.*

If $t > 0$ is small, by Theorem 1.6, we have that $\dim_H K_t \le O(\sqrt{t})$ and Theorem 1.8 implies that, for $x \in \mathbb{T}$, the orbit $\{a^m b^n x\}_{m,n \in \mathbb{Z}_{\ge 0}}$ is $t/\log a$-semiequidistributed if $x$ is in the complement of the set of small Hausdorff dimension about $\sqrt{t}$. In particular, by taking $X = \bigcup_{t>0}(\mathbb{T} \setminus K_t)$, we have the following corollary.

COROLLARY 1.9. *Suppose* $a$ *and* $b$ *are multiplicatively independent. Then there exists* $X \subset \mathbb{T}$ *such that* $\dim_H(\mathbb{T} \setminus X) = 0$ *and, for any* $x \in X$, *the* $\times a$, $\times b$ *orbit* $\{a^m b^n x\}_{m,n \in \mathbb{Z}_{\ge 0}}$ *of* $x$ *is* $s$*-semiequidistributed for some* $s = s(x) > 0$.

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We notice that the $\times a$ action on $\mathbb{T}$ by the single $T_a$ does not exhibit this property, since there exists a $\times a$ invariant Cantor set $C \subset \mathbb{T}$ such that $0 < \dim_H C < 1$. We will prove Theorems 1.6 and 1.8 in §3.

## 2. *Proof of Theorem 1.4 and Corollary 1.5*

In this section, we prove Theorem 1.4 and Corollary 1.5. First, we see that Theorem 1.4 leads immediately to Corollary 1.5.

*Proof of Corollary 1.5.* We assume that Theorem 1.4 holds. Since the linear space spanned by $\{e_k\}_{k\in\mathbb{Z}}$ over $\mathbb{C}$ is dense in the Banach space of $\mathbb{C}$-valued continuous functions on $\mathbb{T}$ with the supremum norm and $J_{e_0} = \emptyset$, it can be seen that $J = \bigcup_{k\in\mathbb{Z}\setminus\{0\}} J_{e_k}$. Hence, using Theorem 1.4,

$$1 = \dim_H J = \sup_{k\in\mathbb{Z}\setminus\{0\}} \dim_H J_{e_k} \tag{2.1}$$

For $k \in \mathbb{Z} \setminus \{0\}$, $T_k : \mathbb{T} \ni x \mapsto kx \in \mathbb{T}$ is commutative with $T_a$ and $T_b$ and $e_k = e_1 \circ T_k$. Therefore, we have $J_{e_k} = T_k^{-1} J_{e_1}$. Moreover, it can be seen that $\dim_H T_k^{-1} J_{e_1} = \dim_H J_{e_1}$. From these and equation (2.1), it follows that

$$1 = \dim_H J_{e_1} = \dim_H J_{e_k},$$

which completes the proof. □

Next, we prove Theorem 1.4. We develop the method in [3] and construct subsets of $J$ which have Hausdorff dimension arbitrarily near one. We need the notion of homogeneous Moran sets. We refer the reader to [5] for the definition and the results about homogeneous Moran sets. We remark that we change the definition a little from [5] for our use. It can be seen that the same results hold.

Let $\{n_k\}_{k=1}^{\infty}$ be a sequence of positive integers and let $\{c_k\}_{k=1}^{\infty}$ be a sequence of positive numbers satisfying that $n_k c_k \le 1$ $(k = 1, 2, \ldots)$ and $c_k < c$ $(k = 1, 2, \ldots)$ for some $0 < c < 1$. Let $D_0 = \{\emptyset\}$, $D_k = \{(i_1, \ldots, i_k) \mid 1 \le i_j \le n_j,\ j = 1, \ldots, k\}$ for each $k = 1, 2, \ldots$ and $D = \bigcup_{k\ge 0} D_k$. If $\sigma = (\sigma_1, \ldots, \sigma_k) \in D_k$ and $\tau = (\tau_1, \ldots, \tau_m) \in D_m$, we write $\sigma * \tau = (\sigma_1, \ldots, \sigma_k, \tau_1, \ldots, \tau_m) \in D_{k+m}$.

*Definition 2.1.* A collection $\mathscr{F} = \{J_\sigma\}_{\sigma\in D}$ of closed intervals of $\mathbb{T}$ has *homogeneous Moran structure* about $\{n_k\}_{k=1}^{\infty}$ and $\{c_k\}_{k=1}^{\infty}$ if it satisfies the following.

(i) $J_\emptyset = \mathbb{T}$.

(ii) For each $k = 0, 1, \ldots$ and $\sigma \in D_k$, $J_{\sigma * i}$ $(i = 1, \ldots, n_{k+1})$ are subintervals of $J_\sigma$ and $\mathring{J}_{\sigma * i}$ $(i = 1, \ldots, n_{k+1})$ are pairwise disjoint (where $\mathring{A}$ denotes the interior of $A$ with respect to the usual topology of $\mathbb{T}$).

(iii) For each $k = 1, 2, \ldots, \sigma \in D_{k-1}$ and $1 \le i \le n_k$,

$$c_k = \frac{|J_{\sigma * i}|}{|J_\sigma|}$$

(where $|A|$ denotes the length of a interval $A$ of $\mathbb{T}$).

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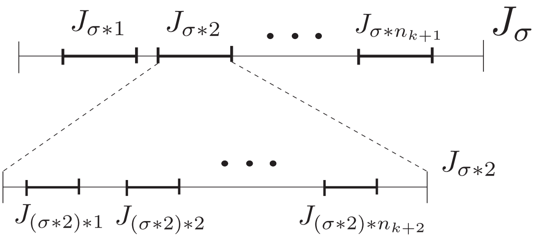


FIGURE 1. Homogeneous Moran structure.

We illustrate homogeneous Moran structure in Figure 1. If $\mathscr{F}$ is a collection of closed intervals having homogeneous Moran structure, we write

$$E(\mathscr{F}) = \bigcap_{k\geq 0} \bigcup_{\sigma \in D_k} J_\sigma$$

and call $E(\mathscr{F})$ the *homogeneous Moran set* determined by $\mathscr{F}$.

We write $\mathscr{M}(\{n_k\}, \{c_k\})$ for the set of homogeneous Moran sets determined by some collection $\mathscr{F}$ of closed intervals having homogeneous Moran structure about $\{n_k\}_{k=1}^\infty$ and $\{c_k\}_{k=1}^\infty$. Then we have the following estimate of Hausdorff dimension of homogeneous Moran sets.

THEOREM 2.1. [5, Theorem 2.1] *Let*

$$s_1 = \liminf_{k\to\infty} \frac{\log n_1 \cdots n_k}{-\log c_1 \cdots c_k}, \quad s_2 = \liminf_{k\to\infty} \frac{\log n_1 \cdots n_k}{-\log c_1 \cdots c_k c_{k+1} n_{k+1}}.$$

*Then, for any* $E \in \mathscr{M}(\{n_k\}, \{c_k\})$,

$$s_2 \leq \dim_H E \leq s_1.$$

We begin the proof of Theorem 1.4. We take arbitrary $0 < r < 1$ near 1. It is sufficient to construct a subset $E$ of $J$ with Hausdorff dimension $\geq r$.

We first construct divergent subsequences $\{N_k\}_{k=1}^\infty$ and $\{L_k\}_{k=1}^\infty$ in $\mathbb{N}$ by induction. We take a countable subset $\{\psi_i\}_{i=1}^\infty \subset C(\mathbb{T})$ so that $0 < \psi_i \leq 1$ on $\mathbb{T}$ for each $i$ and, for a sequence $\{\mu_n\}_{n=1}^\infty \subset M(\mathbb{T})$ and $\mu \in M(\mathbb{T})$, $\mu_n \to \mu$ as $n \to \infty$ is equivalent to $\int_\mathbb{T} \psi_i \, d\mu_n \to \int_\mathbb{T} \psi_i \, d\mu$ as $n \to \infty$ for any $i$. For each $d \in \mathbb{N}$, we write $I_{d,j} = [j/d, (j+1)/d] \bmod \mathbb{Z}$ for $j = 0, \ldots, d-1$ and $I_d = \{I_{d,j} \mid j = 0, \ldots, d-1\}$. We remark that $I_{ab}$ is a common Markov partition of $\mathbb{T}$ with respect to $T_a, T_b$ and $T_{ab}$. We put $N_0 = L_0 = 0$. Let $k > 0$ and suppose that $N_i, L_i$ are determined for $i = 0, \ldots, k-1$ so that $L_{i-1} < N_i < \lfloor rL_i \rfloor < L_i$ for $1 \leq i < k$. For $N \in \mathbb{N}$, we define

$$X_{k,N} = \left\{ x \in \mathbb{T} \,\middle|\, \left| \frac{1}{N^2} \sum_{m,n=0}^{N-1} \psi_i(T_a^m T_b^n x) - \int_\mathbb{T} \psi_i \, dm_\mathbb{T} \right| < \frac{1}{3k}, 1 \leq i \leq k \right\}. \quad (2.2)$$

Then, by Birkhoff's ergodic theorem for $m_\mathbb{T} \in E_{\times a, \times b}(\mathbb{T})$,

$$m_\mathbb{T}(X_{k,N}) > r$$

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for sufficiently large $N \in \mathbb{N}$. We take $l_k \in \mathbb{N}$ so that

$$|\psi_i(x) - \psi_i(y)| < \frac{1}{3k}, \quad i = 1, \ldots, k \tag{2.3}$$

for any $x, y \in \mathbb{T}$ such that $|x - y| \leq (ab)^{-l_k}$. We take $N_k \in \mathbb{N}$ such that $N_k > L_{k-1} + l_k$, $m_{\mathbb{T}}(X_{k,N_k}) > r$,

$$\frac{N_k^2 - (N_k - L_{k-1} - l_k)^2}{N_k^2} < \frac{1}{6k}, \tag{2.4}$$

and

$$\frac{\sum_{i=0}^{k-1}(N_i + L_i)}{N_k} < \frac{1}{k}.$$

Let $x \in X_{k,N_k}$. For $y \in \mathbb{T}$, suppose that $T_{ab}^{L_{k-1}}x$ and $T_{ab}^{L_{k-1}}y$ are contained in the same element of $I_{(ab)^{N_k-L_{k-1}}}$. Then, for any $L_{k-1} \leq m, n < N_k - l_k$, $T_a^m T_b^n x$ and $T_a^m T_b^n y$ are contained in the same element of $I_{(ab)^{l_k}}$. From the definition of $X_{k,N_k}$ (2.2) and inequalities (2.3) and (2.4), we have, for $1 \leq i \leq k$,

$$\begin{aligned}
&\left|\frac{1}{N_k^2}\sum_{m,n=0}^{N_k-1}\psi_i(T_a^m T_b^n y) - \int_{\mathbb{T}}\psi_i\, dm_{\mathbb{T}}\right| \\
&\quad\leq \left|\frac{1}{N_k^2}\sum_{m,n=0}^{N_k-1}\psi_i(T_a^m T_b^n y) - \frac{1}{N_k^2}\sum_{m,n=0}^{N_k-1}\psi_i(T_a^m T_b^n x)\right| \\
&\qquad+ \left|\frac{1}{N_k^2}\sum_{m,n=0}^{N_k-1}\psi_i(T_a^m T_b^n x) - \int_{\mathbb{T}}\psi_i\, dm_{\mathbb{T}}\right| \\
&\quad\leq \left|\frac{1}{N_k^2}\sum_{m,n=L_{k-1}}^{N_k-l_k-1}\psi_i(T_a^m T_b^n y) - \frac{1}{N_k^2}\sum_{m,n=L_{k-1}}^{N_k-l_k-1}\psi_i(T_a^m T_b^n x)\right| \\
&\qquad+ 2\cdot\frac{N_k^2 - (N_k - L_{k-1} - l_k)^2}{N_k^2} + \frac{1}{3k} \\
&\quad< \frac{1}{k}.
\end{aligned}$$

We take $L_k \in \mathbb{N}$ so that $\lfloor rL_k \rfloor > N_k$ and

$$\frac{\sum_{i=0}^{k-1}(N_i + L_i) + N_k}{L_k} < \frac{1}{k}.$$

As a result, we obtain divergent subsequences $\{N_k\}_{k=1}^{\infty}$ and $\{L_k\}_{k=1}^{\infty}$ in $\mathbb{N}$ such that:

(i)
$$L_{k-1} < N_k < \lfloor rL_k \rfloor < L_k, \quad k = 1, 2, \ldots,$$
where we write $L_0 = 0$;

(ii) for $k = 1, 2, \ldots, m_{\mathbb{T}}(X_{k,N_k}) > r$;

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(iii) for $k = 1, 2, \ldots,$ if $x \in X_{k,N_k}$ and $y \in \mathbb{T}$ satisfies that $T_{ab}^{L_{k-1}} x$ and $T_{ab}^{L_{k-1}} y$ are contained in the same element of $I_{(ab)^{N_k - L_{k-1}}}$, then

$$\left| \frac{1}{N_k^2} \sum_{m,n=0}^{N_k - 1} \psi_i(T_a^m T_b^n y) - \int_{\mathbb{T}} \psi_i \, dm_{\mathbb{T}} \right| < \frac{1}{k}$$

for $1 \leq i \leq k$; and

(iv)

$$\lim_{k \to \infty} \frac{\sum_{i=1}^{k-1}(N_i + L_i)}{N_k} = 0, \quad \lim_{k \to \infty} \frac{\sum_{i=1}^{k-1}(N_i + L_i) + N_k}{L_k} = 0.$$

Next, we construct a subset $E$, as mentioned above. We write $\Omega = \{0, 1, \ldots, ab - 1\}^{\mathbb{Z}_{\geq 0}}$ and $\pi : \Omega \to \mathbb{T}$ for the coding map about the Markov partition $I_{ab}$ with respect to $T_{ab}$, that is, for $\omega = (\omega_0, \omega_1, \ldots) \in \Omega$, $x = \pi(\omega) \in \mathbb{T}$ is the element such that $\{x\} = \bigcap_{i=0}^{\infty} T_{ab}^{-i} I_{ab,\omega_i}$. For $k = 1, 2, \ldots,$ we define

$$\Lambda_k = \{\omega \in \Omega \mid \omega_i = \omega'_i, L_{k-1} \leq i < N_k \text{ for some } \omega' \in \pi^{-1} X_{k,N_k}\}.$$

For $L \leq N \in \mathbb{Z}_{\geq 0}$, we call a subset $C \subset \Omega$ a cylinder set on $[L, N]$ if $C = C_{L,N}(\omega') = \{\omega \in \Omega \mid \omega_i = \omega'_i, L \leq i \leq N\}$ for some $\omega' \in \Omega$. Then $\Lambda_k$ can be written as the finite and disjoint union of cylinder sets on $[L_{k-1}, N_k - 1]$, that is,

$$\Lambda_k = \bigsqcup_{C \in \mathscr{C}_k} C,$$

where $\mathscr{C}_k = \{C_{L_{k-1}, N_k - 1}(\omega') \mid \omega' \in \pi^{-1} X_{k,N_k}\}$. We have $\pi(\Lambda_k) = \bigcup_{C \in \mathscr{C}_k} \pi(C) \supset X_{k,N_k}$, $m_{\mathbb{T}}(\pi(C)) = (ab)^{L_{k-1} - N_k}$ for each $C \in \mathscr{C}_k$ and $\pi(C)$ and $\pi(C')$ intersect only on $\mathbb{Q}/\mathbb{Z} \subset \mathbb{T}$ if $C, C' \in \mathscr{C}_k$ and $C \neq C'$. Hence, by property (ii) of $\{N_k\}_{k=1}^{\infty}$,

$$r < m_{\mathbb{T}}(X_{k,N_k}) \leq m_{\mathbb{T}}(\pi(\Lambda_k)) = \sum_{C \subset \mathscr{C}_k} m_{\mathbb{T}}(\pi(C)) = |\mathscr{C}_k| (ab)^{L_{k-1} - N_k}$$

and

$$|\mathscr{C}_k| > r(ab)^{N_k - L_{k-1}}. \tag{2.5}$$

We define

$$\Lambda = \{\omega \in \Omega \mid \omega \in \Lambda_k \text{ and } \omega_i = 0, \lfloor r L_k \rfloor \leq i < L_k \text{ for any } k = 1, 2, \ldots\}$$

and

$$E = \pi(\Lambda).$$

We show that this $E$ is a subset of $J$ such that $\dim_H E \geq r$.

PROPOSITION 2.2. *We have*

$$E \subset J.$$

*Proof.* Let $x \in E$ and take $\omega \in \Lambda$ such that $x = \pi(\omega)$. For each $k \geq 1$, since $\omega \in \Lambda_k$, we can take $\omega' \in \Omega$ such that $x' = \pi(\omega') \in X_{k,N_k}$ and $\omega_i = \omega'_i$ for $L_{k-1} \leq i < N_k$. Then it

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follows that $T_{ab}^{L_{k-1}}x'$ and $T_{ab}^{L_{k-1}}x$ are contained in the same element of $I_{(ab)^{N_k-L_{k-1}}}$ and, from property (iii) of $\{N_k\}_{k=1}^{\infty}$,

$$\left|\frac{1}{N_k^2}\sum_{m,n=0}^{N_k-1}\psi_i(T_a^m T_b^n x)-\int_{\mathbb{T}}\psi_i\,dm_{\mathbb{T}}\right|<\frac{1}{k}$$

for $1\le i\le k$. Hence,

$$\frac{1}{N_k^2}\sum_{m,n=0}^{N_k-1}\psi_i(T_a^m T_b^n x)\xrightarrow[k\to\infty]{}\int_{\mathbb{T}}\psi_i\,dm_{\mathbb{T}}$$

for any $i$. This fact implies that

$$\delta_{\times a,\times b,x}^{N_k}\xrightarrow[k\to\infty]{}m_{\mathbb{T}}. \tag{2.6}$$

Next, we show that $\delta_{\times a,\times b,x}^{L_k}$ does not converge to $m_{\mathbb{T}}$ as $k\to\infty$. We take $l\in\mathbb{N}$ such that $(ab)^{-l}<2^{-1}(1-r)^2$ and $\varphi\in C(\mathbb{T})$ such that $0\le\varphi\le 1$ on $\mathbb{T}$, $\varphi=1$ on $[0,(ab)^{-l}]$ mod $\mathbb{Z}$ and $(ab)^{-l}\le\int_{\mathbb{T}}\varphi\,dm_{\mathbb{T}}<2^{-1}(1-r)^2$. For sufficiently large $k$,

$$\lfloor rL_k\rfloor\le rL_k<L_k-l,\quad \frac{2(1-r)lL_k-l^2}{L_k^2}<\frac{1}{2}(1-r)^2.$$

Furthermore, since $\omega\in\Lambda$, it follows that $T_{ab}^i x\in[0,(ab)^{-1}]$ mod $\mathbb{Z}$ for any $\lfloor rL_k\rfloor\le i<L_k$, and hence $T_a^m T_b^n x\in[0,(ab)^{-l}]$ mod $\mathbb{Z}$ for any $\lfloor rL_k\rfloor\le m,n<L_k-l$. Then

$$\begin{aligned}\frac{1}{L_k^2}\sum_{m,n=0}^{L_k-1}\varphi(T_a^m T_b^n x)&\ge\frac{1}{L_k^2}\sum_{m,n=\lfloor rL_k\rfloor}^{L_k-l-1}\varphi(T_a^m T_b^n x)\\&=\frac{(L_k-l-\lfloor rL_k\rfloor)^2}{L_k^2}\\&\ge\frac{((1-r)L_k-l)^2}{L_k^2}\\&=(1-r)^2-\frac{2(1-r)lL_k-l^2}{L_k^2}\\&>\frac{1}{2}(1-r)^2.\end{aligned}$$

Hence,

$$\begin{aligned}\liminf_{k\to\infty}\frac{1}{L_k^2}\sum_{m,n=0}^{L_k-1}\varphi(T_a^m T_b^n x)&\ge\frac{1}{2}(1-r)^2\\&>\int_{\mathbb{T}}\varphi\,dm_{\mathbb{T}}.\end{aligned}$$

This implies that $\delta_{\times a,\times b,x}^{L_k}$ does not converge to $m_{\mathbb{T}}$ as $k\to\infty$. This and (2.6) imply that $x\in J$, and this completes the proof. □

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PROPOSITION 2.3. *We have*

$$\dim_H E \geq r.$$

*Proof.* We show that $E$ is a homogeneous Moran set and use Theorem 2.1. Let $k = 1, 2, \ldots$. First, we notice that, for $\omega \in \Lambda$, $\omega \in C$ for some $C \in \mathscr{C}_k$: the subfamily of cylinder sets on $[L_{k-1}, N_k - 1]$. We define

$$n_{k,1} = |\mathscr{C}_k|, \quad c_{k,1} = (ab)^{-(N_k - L_{k-1})}.$$

Second, we notice that, for $\omega \in \Lambda$, $\omega_i$ is arbitrary for $N_k \leq i < \lfloor rL_k \rfloor$. For each $N_k \leq i < \lfloor rL_k \rfloor$, we define

$$n_{k,2,i} = ab, \quad c_{k,2,i} = (ab)^{-1}.$$

And finally, we notice that, for $\omega \in \Lambda$, $\omega_i = 0$ for $\lfloor rL_k \rfloor \leq i < L_k$. We define

$$n_{k,3} = 1, \quad c_{k,3} = (ab)^{-(L_k - \lfloor rL_k \rfloor)}.$$

We write

$$\{n_l\}_{l=1}^{\infty} = \{n_{1,1}, \ldots, n_{k-1,3}, n_{k,1}, n_{k,2,N_k}, \ldots, n_{k,2,\lfloor rL_k \rfloor - 1}, n_{k,3}, n_{k+1,1}, \ldots\},$$
$$\{c_l\}_{l=1}^{\infty} = \{c_{1,1}, \ldots, c_{k-1,3}, c_{k,1}, c_{k,2,N_k}, \ldots, c_{k,2,\lfloor rL_k \rfloor - 1}, c_{k,3}, c_{k+1,1}, \ldots\}.$$

Then, by the definition of $E$, it is seen that $E \in \mathscr{M}(\{n_l\}, \{c_l\})$. Hence, by Theorem 2.1,

$$\dim_H E \geq s_2 = \liminf_{l \to \infty} \frac{\log n_1 \cdots n_l}{-\log c_1 \cdots c_l c_{l+1} n_{l+1}}. \tag{2.7}$$

We estimate the right-hand side of (2.7).

Suppose $n_l = n_{k,1}$ and $c_l = c_{k,1}$. Then $n_{l+1} = n_{k,2,N_k} = ab$ and $c_{l+1} = c_{k,2,N_k} = (ab)^{-1}$. From inequality (2.5), it follows that

$$\begin{aligned} n_1 \cdots n_l &= \prod_{j=1}^{k-1} (n_{j,1} n_{j,2,N_j} \cdots n_{j,2,\lfloor rL_j \rfloor - 1} n_{j,3}) \cdot n_{k,1} \\ &= \prod_{j=1}^{k-1} (|\mathscr{C}_j| (ab)^{\lfloor rL_j \rfloor - N_j}) \cdot |\mathscr{C}_k| \\ &> \prod_{j=1}^{k-1} (r(ab)^{\lfloor rL_j \rfloor - L_{j-1}}) \cdot r(ab)^{N_k - L_{k-1}}, \end{aligned}$$

and

$$\begin{aligned} c_1 \cdots c_l &= \prod_{j=1}^{k-1} (c_{j,1} c_{j,2,N_j} \cdots c_{j,2,\lfloor rL_j \rfloor - 1} c_{j,3}) \cdot c_{k,1} \\ &= \prod_{j=1}^{k-1} ((ab)^{-(L_j - L_{j-1})}) \cdot (ab)^{-(N_k - L_{k-1})} \\ &= (ab)^{-N_k}. \end{aligned}$$

Hence,

$$\begin{aligned}
&\frac{\log n_1 \cdots n_l}{-\log c_1 \cdots c_l c_{l+1} n_{l+1}} \\
&\quad \geq \frac{\log\{\prod_{j=1}^{k-1}(r(ab)^{\lfloor rL_j\rfloor - L_{j-1}}) \cdot r(ab)^{N_k - L_{k-1}}\}}{-\log\{(ab)^{-N_k}\}} \\
&\quad = \frac{\sum_{j=1}^{k-1}\{\log r + (\lfloor rL_j\rfloor - L_{j-1})\log(ab)\} + \log r + (N_k - L_{k-1})\log(ab)}{N_k \log(ab)} \\
&\quad = \frac{k\log r}{N_k \log(ab)} + \frac{\sum_{j=1}^{k-1}(\lfloor rL_j\rfloor - L_{j-1})}{N_k} + \frac{N_k - L_{k-1}}{N_k}. \qquad (2.8)
\end{aligned}$$

From property (iv) of $\{N_k\}_{k=1}^{\infty}$ and $\{L_k\}_{k=1}^{\infty}$, the right-hand side converges to 1 as $k \to \infty$.

Suppose $n_l = n_{k,2,i}$ and $c_l = c_{k,2,i}$ for some $N_k \leq i < \lfloor rL_k\rfloor$. Then,

$$n_{l+1} = \begin{cases} n_{k,2,i+1} = ab, & i \neq \lfloor rL_k\rfloor - 1, \\ n_{3,k} = 1, & i = \lfloor rL_k\rfloor - 1, \end{cases} \qquad c_{l+1} = \begin{cases} c_{k,2,i+1} = (ab)^{-1}, & i \neq \lfloor rL_k\rfloor - 1, \\ c_{3,k} = (ab)^{-(L_k - \lfloor rL_k\rfloor)}, & i = \lfloor rL_k\rfloor - 1. \end{cases}$$

From inequality (2.5), it follows that

$$\begin{aligned}
n_1 \cdots n_l &= \prod_{j=1}^{k-1}(n_{j,1} n_{j,2,N_j} \cdots n_{j,2,\lfloor rL_j\rfloor - 1} n_{j,3}) \cdot n_{k,1} n_{k,2,N_k} \cdots n_{k,2,i} \\
&> \prod_{j=1}^{k-1}(r(ab)^{\lfloor rL_j\rfloor - L_{j-1}}) \cdot r(ab)^{N_k - L_{k-1}} \cdot (ab)^{i - N_k + 1} \\
&= \prod_{j=1}^{k-1}(r(ab)^{\lfloor rL_j\rfloor - L_{j-1}}) \cdot r(ab)^{i - L_{k-1} + 1},
\end{aligned}$$

and

$$\begin{aligned}
c_1 \cdots c_l &= \prod_{j=1}^{k-1}(c_{j,1} c_{j,2,N_j} \cdots c_{j,2,\lfloor rL_j\rfloor - 1} c_{j,3}) \cdot c_{k,1} c_{k,2,N_k} \cdots c_{k,2,i} \\
&= (ab)^{-N_k} \cdot (ab)^{-(i - N_k + 1)} \\
&= (ab)^{-(i+1)}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\frac{\log n_1 \cdots n_l}{-\log c_1 \cdots c_l c_{l+1} n_{l+1}} \\
&\quad \geq \frac{\log\{\prod_{j=1}^{k-1}(r(ab)^{\lfloor rL_j\rfloor - L_{j-1}}) \cdot r(ab)^{i - L_{k-1} + 1}\}}{-\log\{(ab)^{-(i+1)} \cdot c_{l+1} n_{l+1}\}} \\
&\quad = \frac{\sum_{j=1}^{k-1}\{\log r + (\lfloor rL_j\rfloor - L_{j-1})\log(ab)\} + \log r + (i - L_{k-1} + 1)\log(ab)}{(i+1)\log(ab) - \log(c_{l+1} n_{l+1})}. \qquad (2.9)
\end{aligned}$$

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If $i < \lfloor rL_k \rfloor - 1$, then $c_{l+1}n_{l+1} = 1$ and the right-hand side of (2.9) is

$$\frac{\sum_{j=1}^{k-1}\{\log r + (\lfloor rL_j \rfloor - L_{j-1})\log(ab)\} + \log r + (i - L_{k-1} + 1)\log(ab)}{(i+1)\log(ab)}$$
$$\geq \frac{k\log r}{N_k \log(ab)} + \frac{\sum_{j=1}^{k-1}(\lfloor rL_j \rfloor - L_{j-1})}{\lfloor rL_k \rfloor} - \frac{L_{k-1}}{N_k} + 1. \tag{2.10}$$

From property (iv) of $\{N_k\}_{k=1}^{\infty}$ and $\{L_k\}_{k=1}^{\infty}$, the right-hand side converges to 1 as $k \to \infty$. If $i = \lfloor rL_k \rfloor - 1$, then $c_{l+1}n_{l+1} = (ab)^{-(L_k - \lfloor rL_k \rfloor)}$ and the right-hand side of (2.9) is

$$\frac{\sum_{j=1}^{k-1}\{\log r + (\lfloor rL_j \rfloor - L_{j-1})\log(ab)\} + \log r + (\lfloor rL_k \rfloor - L_{k-1})\log(ab)}{L_k \log(ab)}$$
$$\geq \frac{k\log r}{L_k \log(ab)} + \frac{\sum_{j=1}^{k-1}(\lfloor rL_j \rfloor - L_{j-1})}{L_k} + \frac{\lfloor rL_k \rfloor - L_{k-1}}{L_k}. \tag{2.11}$$

From property (iv) of $\{N_k\}_{k=1}^{\infty}$ and $\{L_k\}_{k=1}^{\infty}$, the right-hand side converges to $r$ as $k \to \infty$.

Suppose $n_l = n_{k,3}$ and $c_l = c_{k,3}$. Then $n_{l+1} = n_{k+1,1} = |\mathscr{C}_{k+1}|$, $c_{l+1} = c_{k+1,1} = (ab)^{-(N_{k+1} - L_k)}$ and, from inequality (2.5),

$$c_{l+1}n_{l+1} > r.$$

From (2.5) again, it follows that

$$\begin{aligned} n_1 \cdots n_l &= \prod_{j=1}^{k}(n_{j,1}n_{j,2,N_j} \cdots n_{j,2,\lfloor rL_j \rfloor - 1}n_{j,3}) \\ &> \prod_{j=1}^{k}(r(ab)^{\lfloor rL_j \rfloor - L_{j-1}}), \end{aligned}$$

and

$$\begin{aligned} c_1 \cdots c_l &= \prod_{j=1}^{k}(c_{j,1}c_{j,2,N_j} \cdots c_{j,2,\lfloor rL_j \rfloor - 1}c_{j,3}) \\ &= (ab)^{-L_k}. \end{aligned}$$

Hence,

$$\begin{aligned} &\frac{\log n_1 \cdots n_l}{-\log c_1 \cdots c_l c_{l+1} n_{l+1}} \\ &\quad \geq \frac{\log\{\prod_{j=1}^{k}(r(ab)^{\lfloor rL_j \rfloor - L_{j-1}})\}}{-\log\{(ab)^{-L_k} \cdot c_{l+1}n_{l+1}\}} \\ &\quad \geq \frac{\sum_{j=1}^{k}\{\log r + (\lfloor rL_j \rfloor - L_{j-1})\log(ab)\}}{L_k \log(ab) - \log r} \\ &\quad = \frac{k\log r}{L_k \log(ab) - \log r} + \frac{\sum_{j=1}^{k-1}(\lfloor rL_j \rfloor - L_{j-1})\log(ab)}{L_k \log(ab) - \log r} + \frac{(\lfloor rL_k \rfloor - L_{k-1})\log(ab)}{L_k \log(ab) - \log r}. \end{aligned} \tag{2.12}$$

From property (iv) of $\{N_k\}_{k=1}^{\infty}$ and $\{L_k\}_{k=1}^{\infty}$, the right-hand side converges to $r$ as $k \to \infty$.

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From inequalities (2.8), (2.9), (2.10), (2.11) and (2.12),

$$s_2 = \liminf_{l\to\infty} \frac{\log n_1 \cdots n_l}{-\log c_1 \cdots c_l c_{l+1} n_{l+1}} \geq r$$

and, using inequality (2.7), we complete the proof. □

By Propositions 2.2 and 2.3, we have $1 \geq \dim_H J \geq \dim_H E \geq r$ and $0 < r < 1$ is arbitrary. Hence, we have $\dim_H J = 1$ and this completes the proof of Theorem 1.4.

## 3. *Proof of Theorems 1.6 and 1.8*

In this section, we prove Theorems 1.6 and 1.8. First, we prove Theorem 1.8 as the proof is more elementary than that of Theorem 1.6.

*Proof of Theorem 1.8.* Suppose $a$ and $b$ are multiplicatively independent. Let $0 < t < \min\{\log b, (\log a)^2/\log b\}$ and let $x \in \mathbb{T} \setminus K_t$. Assume that there exists $f \in C(\mathbb{T})$ such that $f \geq 0$ on $\mathbb{T}$ and

$$\liminf_{N\to\infty} \frac{1}{N^2} \sum_{m,n=0}^{N-1} f(a^m b^n x) < \frac{t}{\log a} \int_{\mathbb{T}} f \, dm_{\mathbb{T}}.$$

We can take $0 < \varepsilon < 1$ and some divergent subsequence $\{N_k\}_{k=1}^{\infty}$ in $\mathbb{N}$ such that

$$\frac{1}{N_k^2} \sum_{m,n=0}^{N_k-1} f(a^m b^n x) < \frac{t}{\log a} \int_{\mathbb{T}} f \, dm_{\mathbb{T}} - \varepsilon \tag{3.1}$$

for each $k$. Furthermore, since $M(\mathbb{T})$ is compact with respect to the weak* topology, we can take $\{N_k\}_{k=1}^{\infty}$ so that $\delta_{\times a,\times b,x}^{N_k}$ converges to some $\mu \in M(\mathbb{T})$ as $k \to \infty$. Then $\mu \in M_{\times a,\times b}(\mathbb{T})$ and $\mu$ is an accumulation point of $\delta_{\times a,\times b,x}^{N}$ $(N \in \mathbb{N})$. Since $x \in \mathbb{T} \setminus K_t$, we have $h_\mu(T_a) > t$. Here, we decompose $\mu$ into $\times a$, $\times b$ ergodic components. There exists a Borel probability measure $\tau$ on the compact and metrizable space $M_{\times a,\times b}(\mathbb{T})$ such that $\tau(E_{\times a,\times b}(\mathbb{T})) = 1$ and

$$\int_{\mathbb{T}} \varphi \, d\mu = \int_{E_{\times a,\times b}(\mathbb{T})} \int_{\mathbb{T}} \varphi \, d\nu d\tau(\nu)$$

for any $\varphi \in C(\mathbb{T})$. By the upper semicontinuity of $h_\nu(T_a)$, it can be seen that

$$h_\mu(T_a) = \int_{E_{\times a,\times b}(\mathbb{T})} h_\nu(T_a) \, d\tau(\nu)$$

and, by Theorem 1.3, $h_\nu(T_a) = 0$ for any $\nu \in E_{\times a,\times b}(\mathbb{T}) \setminus \{m_{\mathbb{T}}\}$. Hence,

$$t < h_\mu(T_a) = \tau(\{m_{\mathbb{T}}\}) h_{m_{\mathbb{T}}}(T_a) = \tau(\{m_{\mathbb{T}}\}) \log a. \tag{3.2}$$

Letting $k \to \infty$ in inequality (3.1), it follows from (3.2) that

$$\begin{aligned} \frac{t}{\log a} \int_{\mathbb{T}} f \, dm_{\mathbb{T}} - \varepsilon &\geq \int_{\mathbb{T}} f \, d\mu \\ &= \int_{E_{\times a,\times b}(\mathbb{T})} \int_{\mathbb{T}} f \, d\nu d\tau(\nu) \end{aligned}$$

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$$\geq \tau(\{m_{\mathbb{T}}\}) \int_{\mathbb{T}} f \, dm_{\mathbb{T}}$$
$$\geq \frac{t}{\log a} \int_{\mathbb{T}} f \, dm_{\mathbb{T}}$$

and this is a contradiction. Hence,

$$\liminf_{N\to\infty} \frac{1}{N^2} \sum_{m,n=0}^{N-1} f(a^m b^n x) \geq \frac{t}{\log a} \int_{\mathbb{T}} f \, dm_{\mathbb{T}}$$

for any $f \in C(\mathbb{T})$ such that $f \geq 0$ on $\mathbb{T}$.

Let $U \subset \mathbb{T}$ be an open subset. For any $0 < \varepsilon < 1$, there exists $f \in C(\mathbb{T})$ such that $0 \leq f \leq 1$ on $\mathbb{T}$, $f = 0$ on $\mathbb{T} \setminus U$ and $\int_{\mathbb{T}} f \, dm_{\mathbb{T}} \geq m_{\mathbb{T}}(U) - \varepsilon$. Then, by the statement above, it follows that

$$\begin{aligned}\liminf_{N\to\infty} \frac{1}{N^2} |\{(m, n) \in \mathbb{Z}^2 \mid 0 \leq m, n < N, a^m b^n x \in U\}| &\geq \liminf_{N\to\infty} \frac{1}{N^2} \sum_{m,n=0}^{N-1} f(a^m b^n x)\\ &\geq \frac{t}{\log a} \int_{\mathbb{T}} f \, dm_{\mathbb{T}}\\ &\geq \frac{t}{\log a}(m_{\mathbb{T}}(U) - \varepsilon).\end{aligned}$$

By letting $\varepsilon \to 0$, we get

$$\liminf_{N\to\infty} \frac{1}{N^2} |\{(m, n) \in \mathbb{Z}^2 \mid 0 \leq m, n < N, a^m b^n x \in U\}| \geq \frac{t}{\log a} \cdot m_{\mathbb{T}}(U),$$

and this completes the proof. □

Next, we prove Theorem 1.6. The following argument can be thought of as an extension of that in [2] for the $\mathbb{Z}^2_{\geq 0}$-action by $T_a$ and $T_b$. Let $k \in \mathbb{N}$. We have that $p = (p_1, \ldots, p_k) \in \mathbb{R}^k$ is a *k-distribution* if $\sum_{i=1}^k p_i = 1$ and $p_i \geq 0$. For such a $p$, we write $H(p) = -\sum_{i=1}^k p_i \log p_i$ for the entropy of $p$. If $N \in \mathbb{N}$ and $c = (c_1, \ldots, c_N) \in \{1, \ldots, k\}^N$, we define the $k$-distribution $\mathrm{dist}(c) = (p_1, \ldots, p_k)$, where $p_i = N^{-1}|\{n \in \{1, \ldots, N\} \mid c_n = i\}|$.

LEMMA 3.1. *For $k, N \in \mathbb{N}$ and $t > 0$, let*

$$R(k, N, t) = \{c \in \{1, \ldots, k\}^N \mid H(\mathrm{dist}(c)) \leq t\}.$$

*Then, fixing k and t,*

$$\limsup_{N\to\infty} \frac{1}{N} \log |R(k, N, t)| \leq t.$$

*Proof.* See [2, Lemma 4]. □

Suppose that $\beta = \{\beta_1, \ldots, \beta_k\}$ is a finite cover of $\mathbb{T}$. For $x \in \mathbb{T}$ and $N \in \mathbb{N}$, we say that $(\beta_{i_0}, \ldots, \beta_{i_{N-1}}) \in \beta^N$ is an *N-choice for x with respect to $T_a$ and $\beta$* if $T_a^n x \in \beta_{i_n}$ for $0 \leq n < N$. Then $(\beta_{i_0}, \ldots, \beta_{i_{N-1}})$ gives a $k$-distribution $q(\beta_{i_0}, \ldots, \beta_{i_{N-1}}) =$

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$\mathrm{dist}(i_0, \ldots, i_{N-1})$. We write $\mathrm{Dist}_\beta(x, N)$ for the set of such $k$-distributions obtained for all $N$-choices for $x$.

Suppose that $B = \{B_i\}$ is a finite cover of $\mathbb{T}$. For $E \subset \mathbb{T}$, we write $E \prec B$ if $E \subset B_i$ for some $B_i \in B$ and, for a family of subsets $E = \{E_j\}$, $E \prec B$ if $E_j \prec B$ for any $E_j \in E$. For a map $T : \mathbb{T} \to \mathbb{T}$, $l \in \mathbb{N}$ and a family of subsets $E = \{E_j\}$, we define $T^{-l}E = \{T^{-l}E_j\}$.

LEMMA 3.2. *Let $B = \{B_i\}$ be a finite open cover of $\mathbb{T}$ such that every $B_i \in B$ is an open interval on $\mathbb{T}$ such that $|B_i| < 1/(1+a)$ and, for each $M \in \mathbb{N}$, let $\beta_M$ be a finite cover of $\mathbb{T}$ such that $\beta_M \prec T_a^{-l}B$ for $0 \le l < M$. For $0 < t < \log a$, we define $Q(t, \{\beta_M\}_{M\in\mathbb{N}})$ as the set of $x \in \mathbb{T}$ satisfying the following: for any $0 < \varepsilon < 1$ and $M_0 \in \mathbb{N}$, there exists $M \ge M_0$ such that,*

$$\text{for infinitely many } N \in \mathbb{N},\ \frac{1}{M}H(q) \le t + \varepsilon \quad \text{for some } q \in \bigcup_{0\le n< tN/\log b} \mathrm{Dist}_{\beta_M}(T_b^n x, N).$$

*Then,*

$$\dim_H Q(t, \{\beta_M\}_{M\in\mathbb{N}}) \le \frac{2t}{\log a + t}.$$

*Proof.* For each $M \in \mathbb{N}$, let $\beta_M = \{\beta_{M,1}, \ldots, \beta_{M,k_M}\}$, $k_M = |\beta_M|$. We take $0 < \varepsilon < 3^{-1}(\log a - t)$. By Lemma 3.1, there exists $N_{\varepsilon,M} \in \mathbb{N}$ such that

$$|R(k_M, N, M(t+\varepsilon))| \le e^{NM(t+2\varepsilon)} \tag{3.3}$$

for any $N \ge N_{\varepsilon,M}$. We take $M_0 \in \mathbb{N}$ such that $M_0 \ge t^{-1}\log b$. Since $H(p)$ is uniformly continuous in a $k_M$-distribution $p$, we can see that, for any $x \in Q(t, \{\beta_M\}_{M\in\mathbb{N}})$, there exists $M \ge M_0$ such that

$$\text{for infinitely many } N \in \mathbb{N},\ \frac{1}{M}H(q) \le t + \varepsilon$$
$$\text{for some } q \in \bigcup_{0\le n<tMN/\log b} \mathrm{Dist}_{\beta_M}(T_b^n x, MN).$$

Indeed, we obtain this by adding some $0 \le l < M$ to $N$ in the definition of $Q(t, \{\beta_M\}_{M\in\mathbb{N}})$ for $\varepsilon/2$. For each $M \in \mathbb{N}$, we take $N'_{\varepsilon,M} \in \mathbb{N}$ such that

$$N'_{\varepsilon,M} \ge N_{\varepsilon,M}, \quad M^2(k_M)^M \sum_{N\ge N'_{\varepsilon,M}} Ne^{-\varepsilon MN} < \frac{1}{2^M}. \tag{3.4}$$

For each $M, N \in \mathbb{N}$ and $x \in \mathbb{T}$, we take an $MN$-choice $(\beta_{M,i_0(x)}, \ldots, \beta_{M,i_{MN-1}(x)})$ for $x$ with respect to $T_a$ and $\beta_M$ such that

$$H(q(\beta_{M,i_0(x)}, \ldots, \beta_{M,i_{MN-1}(x)})) = \min_{q\in\mathrm{Dist}_{\beta_M}(x,MN)} H(q). \tag{3.5}$$

For $0 \le l < M$, we define a $k_M$-distribution

$$q_{M,l}(x, N) = \mathrm{dist}(i_l(x), i_{M+l}(x), \ldots, i_{M(N-1)+l}).$$

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Then $q(\beta_{M,i_0(x)}, \ldots, \beta_{M,i_{MN-1}(x)}) = M^{-1} \sum_{l=0}^{M-1} q_{M,l}(x, N)$. Hence, by the concavity of $H(p)$ in a $k_M$-distribution $p$, we have $H(q_{M,l}(x, N)) \le H(q(\beta_{M,i_0(x)}, \ldots, \beta_{M,i_{MN-1}(x)}))$ for some $0 \le l < M$, depending on $M$, $N$ and $x$.

For $M \ge M_0$, $N \ge N'_{\varepsilon,M}$ and $n \in \mathbb{Z}$ with $0 \le n < tMN/\log b$ and $0 \le l < M$, we define

$$S(M, N, n, l) = \{x \in \mathbb{T} \mid H(q_{M,l}(T_b^n x, N)) \le M(t + \varepsilon)\}.$$

Then

$$Q(t, \{\beta_M\}_{M\in\mathbb{N}}) \subset \bigcup_{\substack{M\ge M_0, N\ge N'_{\varepsilon,M}\\ 0\le n<tMN/\log b, 0\le l<M}} S(M, N, n, l).$$

Let $M \ge M_0$, $N \ge N'_{\varepsilon,M}$, $0 \le n < tMN/\log b$, $0 \le l < M$ and $x \in S(M, N, n, l)$. For the $MN$-choice $(\beta_{M,i_0(T_b^n x)}, \ldots, \beta_{M,i_{MN-1}(T_b^n x)})$ for $T_b^n x$ with respect to $T_a$ and $\beta_M$ as in (3.5),

$$(i_l(T_b^n x), i_{M+l}(T_b^n x), \ldots, i_{M(N-1)+l}(T_b^n x)) \in R(k_M, N, M(t + \varepsilon)).$$

We define

$$\begin{aligned} A_{M,l}(T_b^n x, N) = \{y \in \mathbb{T} \mid\ & T_a^j y \in \beta_{M,i_j(T_b^n x)} \text{ for } 0 \le j < l, \\ & T_a^{Mr+l} y \in \beta_{M,i_{Mr+l}(T_b^n(x))} \text{ for } 0 \le r < N\}. \end{aligned}$$

Then, by the assumption of $B$ and $\beta_M$, $A_{M,l}(T_b^n x, N) \prec T_a^{-j} B$ for $0 \le j < MN$. Hence, by the assumption that $|B_i| < 1/(a + 1)$ for each $B_i \in B$, we have $\mathrm{diam} A_{M,l}(T_b^n x, N) < a^{-MN+1}$, where $\mathrm{diam} A$ denotes the diameter $\sup_{x,y\in A} |x - y|$ of $A \subset \mathbb{T}$ with respect to the standard metric of $\mathbb{T}$. We divide $A_{M,l}(T_b^n x, N)$ into $A_{M,l}(T_b^n x, N) = \bigsqcup_{s=0}^{b-1} A_{M,l}^s(T_b^n x, N)$, where $A_{M,l}^s(T_b^n x, N) = A_{M,l}(T_b^n x, N) \cap ([s/b, (s + 1)/b) \bmod \mathbb{Z})$. Then $x \in T_b^{-n} A_{M,l}(T_b^n x, N) = \bigsqcup_{s=0}^{b-1} T_b^{-n} A_{M,l}^s(T_b^n x, N)$. For each $s = 0, \ldots, b - 1$, we get the $b^n$ components of $T_b^{-n} A_{M,l}^s(T_b^n x, N)$, which we write as $E_{M,l}^{s,u}(x, N, n)$, $u = 1, \ldots, b^n$, satisfying

$$\mathrm{diam} E_{M,l}^{s,u}(x, N, n) < b^{-n} a^{-MN+1}. \tag{3.6}$$

We define

$$\begin{aligned} E(M_0) = \{E_{M,l}^{s,u}(x, N, n) \mid\ & M \ge M_0, N \ge N'_{\varepsilon,M}, 0 \le n < tMN/\log b, 0 \le l < M, \\ & x \in S(M, N, n, l), s = 0, \ldots, b - 1, u = 1, \ldots, b^n\}. \end{aligned}$$

Then $E(M_0)$ is a cover of $Q(t, \{\beta_M\}_{M\in\mathbb{N}})$ such that $\mathrm{diam} E(M_0) \le a^{-M_0+1}$. Fix $M \ge M_0$, $N \ge N'_{\varepsilon,M}$, $0 \le n < tMN/\log b$ and $0 \le l < M$. The number of $A_{M,l}(T_b^n x, N)$ ($x \in S(M, N, n, l)$) is bounded by $|\beta_M|^l |R(k_M, N, M(t + \varepsilon))| = (k_M)^l |R(k_M, N, M(t + \varepsilon))|$. Hence, the number of $E_{M,l}^{s,u}(x, N, n)$ ($x \in S(M, N, n, l)$, $s = 0, \ldots, b - 1$, $u = 1, \ldots, b^n$) is bounded by $b^{n+1}(k_M)^l |R(k_M, N, M(t + \varepsilon))|$. We put $\lambda = (\log a - t - 3\varepsilon)/(\log a + t)$. Since $\log a - t > 3\varepsilon$, we have $\lambda > 0$. We also have $1 - \lambda = ((1 + \lambda)t + 3\varepsilon)/\log a > 0$. Using inequalities (3.3) and (3.6),

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$$
\begin{aligned}
&\sum_{E\in E(M_0)} (\operatorname{diam} E)^{1-\lambda}\\
&\le \sum_{\substack{M\ge M_0, N\ge N'_{\varepsilon,M},\\ 0\le n<tMN/\log b, 0\le l<M}} b^{n+1}(k_M)^l |R(k_M, N, M(t+\varepsilon))| (b^{-n}a^{-MN+1})^{1-\lambda}\\
&\le b\sum_{M\ge M_0} M(k_M)^M \sum_{\substack{N\ge N'_{\varepsilon,M},\\ 0\le n<tMN/\log b}} b^{\lambda n}|R(k_M, N, M(t+\varepsilon))| a^{(-MN+1)((1+\lambda)t+3\varepsilon)/\log a}\\
&\le b\sum_{M\ge M_0} M(k_M)^M \sum_{\substack{N\ge N'_{\varepsilon,M},\\ 0\le n<tMN/\log b}} b^{\lambda n} e^{MN(t+2\varepsilon)} e^{(-MN+1)((1+\lambda)t+3\varepsilon)}\\
&= e^{(1+\lambda)t+3\varepsilon} b\sum_{M\ge M_0} M(k_M)^M \sum_{\substack{N\ge N'_{\varepsilon,M},\\ 0\le n<tMN/\log b}} b^{\lambda n} e^{-MN(\lambda t+\varepsilon)}\\
&\le e^{(1+\lambda)t+3\varepsilon} b\sum_{M\ge M_0} M(k_M)^M \sum_{N\ge N'_{\varepsilon,M}} \bigg(\frac{tMN}{\log b}+1\bigg) b^{tMN\lambda/\log b} e^{-MN(\lambda t+\varepsilon)}\\
&\le \frac{2te^{(1+\lambda)t+3\varepsilon}b}{\log b}\sum_{M\ge M_0} M^2(k_M)^M \sum_{N\ge N'_{\varepsilon,M}} Ne^{-MN\varepsilon}\\
&\le \frac{2te^{(1+\lambda)t+3\varepsilon}b}{\log b}\sum_{M\ge M_0}\frac{1}{2^M}.
\end{aligned}
$$

The last inequality is due to (3.4). When $M_0 \to \infty$, the right-hand side converges to zero. This implies that $\dim_H Q(t, \{\beta_M\}_{M\in\mathbb{N}}) \le 1-\lambda = (2t+3\varepsilon)/(\log a + t)$. By $\varepsilon \to 0$, we obtain the lemma. □

Before starting a proof of Theorem 1.6, we prepare a notion. Suppose that $\beta = \{\beta_1, \ldots, \beta_k\}$ is a finite cover of $\mathbb{T}$. For $x \in \mathbb{T}$ and $N \in \mathbb{N}$, we say that $(\beta_{i_{m,n}})_{0\le m,n<N} \in \beta^{\{(m,n)|0\le m,n<N\}}$ is an *$N$-choice for $x$ with respect to $T_a, T_b$ and $\beta$* if $T_a^m T_b^n x \in \beta_{i_{m,n}}$ for $0 \le m, n < N$. Then $(\beta_{i_{m,n}})_{0\le m,n<N}$ gives a $k$-distribution $q((\beta_{i_{m,n}})_{0\le m,n<N}) = \operatorname{dist}((i_{m,n})_{0\le m,n<N})$. We notice that, if $\underline{\beta} = (\beta_{i_{m,n}})_{0\le m,n<N}$ is an $N$-choice for $x$ with respect to $T_a$, $T_b$ and $\beta$, then, for $0 \le n < N$, $\underline{\beta}_n = (\beta_{i_{0,n}}, \ldots, \beta_{i_{N-1,n}})$ is an $N$-choice for $T_b^n x$ with respect to $\beta$ and $T_a$, and $q(\underline{\beta}) = N^{-1}\sum_{n=0}^{N-1} q(\underline{\beta}_n)$.

*Proof of Theorem 1.6.* Let $B$ be a finite open cover of $\mathbb{T}$ as in Lemma 3.2 and let $\alpha$ be a finite Borel partition of $\mathbb{T}$ such that $\overline{\alpha_i} \prec B$ for each $\alpha_i \in \alpha$. For each $M \in \mathbb{N}$, we write $\alpha_M = \bigvee_{i=0}^{M-1} T_a^{-i}\alpha = \{\alpha_{M,1}, \ldots, \alpha_{M,k_M}\}$, $k_M = |\alpha_M|$ and take a finite open cover $\beta_M = \{\beta_{M,1}, \ldots, \beta_{M,k_M}\}$ of $\mathbb{T}$ such that $\alpha_{M,i} \subset \beta_{M,i}$ and $\beta_{M,i} \prec T_a^{-l}B$ $(0 \le l < M)$ for each $i = 1, \ldots, k_M$. Let $0 < t < \min\{\log a, \log b\}$. If we show that $K_{t^2/\log b} \subset Q(t, \{\beta_M\}_{M\in\mathbb{N}})$, then, by Lemma 3.2, we have $\dim_H K_{t^2/\log b} \le \dim_H Q(t, \{\beta_M\}_{M\in\mathbb{N}}) \le 2t/(\log a + t)$ and, by putting $t' = t^2/\log b$, we obtain the theorem. We show that $K_{t^2/\log b} \subset Q(t, \{\beta_M\}_{M\in\mathbb{N}})$.

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Let $x \in K_{t^2/\log b}$ and take $\mu \in M_{\times a,\times b}(\mathbb{T})$ such that $h_\mu(T_a) \le t^2/\log b$ and $\delta^N_{\times a,\times b,x}(N \in \mathbb{N})$ accumulates to $\mu$. We take a divergent subsequence $\{N_j\}_{j=1}^\infty$ in $\mathbb{N}$ such that $\delta^{N_j}_{\times a,\times b,x} \to \mu$ as $j \to \infty$. We take $0 < \varepsilon < 1$. Since $h_\mu(T_a, \alpha) = \lim_{M\to\infty} M^{-1} H_\mu(\alpha_M) \le h_\mu(T_a) \le t^2/\log b$,

$$\frac{1}{M} H_\mu(\alpha_M) < \frac{t^2}{\log b} + \frac{t\varepsilon}{\log b}$$

for sufficiently large $M \in \mathbb{N}$. We fix such an $M$.

We write $q(\mu, \alpha_M) = (\mu(\alpha_{M,1}), \ldots, \mu(\alpha_{M,k_M}))$: a $k_M$-distribution and notice that $H(q(\mu, \alpha_M)) = H_\mu(\alpha_M) < M(t^2/\log b + t\varepsilon/\log b)$. We take a sufficiently small $\eta > 0$ so that, for a $k_M$-distribution $q$,

$$|q - q(\mu, \alpha_M)| < \eta \quad \text{implies that } H(q) < M\bigg(\frac{t^2}{\log b} + \frac{t\varepsilon}{\log b}\bigg), \tag{3.7}$$

where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^{k_M}$. For each $i = 1, \ldots, k_M$, we take a compact subset $C_i$ such that $C_i \subset \alpha_{M,i}$ and $\mu(\alpha_{M,i} \setminus C_i) < \eta/2\sqrt{k_M}k_M$. Then we take an open subset $V_i$ such that $C_i \subset V_i \subset \beta_{M,i}$ and $V_i$ $(i = 1, \ldots, k_M)$ are pairwise disjoint. Since $\delta^{N_j}_{\times a,\times b,x} \to \mu$ as $j \to \infty$ with respect to the weak* topology,

$$\delta^{N_j}_{\times a,\times b,x}(V_i) > \mu(C_i) - \frac{\eta}{2\sqrt{k_M}k_M}, \quad i = 1, \ldots, k_M,$$

and hence

$$\delta^{N_j}_{\times a,\times b,x}(V_i) > \mu(\alpha_{M,i}) - \frac{\eta}{\sqrt{k_M}k_M}, \quad i = 1, \ldots, k_M$$

for sufficiently large $j$.

For $j$ as above, we take an $N_j$-choice $\underline{\beta_M} = (\beta_{i_{m,n}})_{0\le m,n<N_j}$ for $x$ with respect to $T_a$, $T_b$ and $\beta_M$ such that $i_{m,n} = i$ whenever $T_a^m T_b^n x \in V_i$. Then, when we write $q(\underline{\beta_M}) = (q_1, \ldots, q_{k_M})$, we get

$$q_i \ge \delta^{N_j}_{\times a,\times b,x}(V_i) > \mu(\alpha_{M,i}) - \frac{\eta}{\sqrt{k_M}k_M}, \quad i = 1, \ldots, k_M.$$

Since $q(\underline{\beta_M})$ and $q(\mu, \alpha_M) = (\mu(\alpha_{M,1}), \ldots, \mu(\alpha_{M,k_M}))$ are $k_M$-distributions, this implies that

$$|q_i - \mu(\alpha_{M,i})| < \frac{\eta}{\sqrt{k_M}}, \quad i = 1, \ldots, k_M.$$

Hence, by (3.7),

$$H(q(\underline{\beta_M})) < M\bigg(\frac{t^2}{\log b} + \frac{t\varepsilon}{\log b}\bigg). \tag{3.8}$$

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Now, since $0 < t < \log b$,

$$\begin{aligned} q(\underline{\beta_M}) &= \frac{1}{N_j}\sum_{n=0}^{N_j-1} q(\underline{\beta_M}_n) \\ &= \frac{\lfloor tN_j/\log b\rfloor + 1}{N_j}\bigg\{\frac{1}{\lfloor tN_j/\log b\rfloor + 1}\sum_{0\le n<tN_j/\log b} q(\underline{\beta_M}_n)\bigg\} \\ &\quad + \frac{N_j - \lfloor tN_j/\log b\rfloor - 1}{N_j}\bigg\{\frac{1}{N_j - \lfloor tN_j/\log b\rfloor - 1}\sum_{tN_j/\log b\le n<N_j} q(\underline{\beta_M}_n)\bigg\}. \end{aligned}$$

Hence, by the concavity of $H(p)$ in a $k_M$-distribution $p$ and (3.8),

$$\begin{aligned} &\frac{t}{\log b} H\bigg(\frac{1}{\lfloor tN_j/\log b\rfloor + 1}\sum_{0\le n<tN_j/\log b} q(\underline{\beta_M}_n)\bigg) \\ &\quad\le \frac{\lfloor tN_j/\log b\rfloor + 1}{N_j} H\bigg(\frac{1}{\lfloor tN_j/\log b\rfloor + 1}\sum_{0\le n<tN_j/\log b} q(\underline{\beta_M}_n)\bigg) \\ &\qquad + \frac{N_j - \lfloor tN_j/\log b\rfloor - 1}{N_j} H\bigg(\frac{1}{N_j - \lfloor tN_j/\log b\rfloor - 1}\sum_{tN_j/\log b\le n<N_j} q(\underline{\beta_M}_n)\bigg) \\ &\quad\le H(q(\underline{\beta_M})) \\ &\quad< M\bigg(\frac{t^2}{\log b} + \frac{t\varepsilon}{\log b}\bigg) \end{aligned}$$

and

$$H\bigg(\frac{1}{\lfloor tN_j/\log b\rfloor + 1}\sum_{0\le n<tN_j/\log b} q(\underline{\beta_M}_n)\bigg) < M(t+\varepsilon).$$

Using the concavity of $H(p)$ again,

$$H(q(\underline{\beta_M}_n)) < M(t+\varepsilon)$$

for some $0 \le n < tN_j/\log b$. Since $q(\underline{\beta_M}_n) \in \mathrm{Dist}_{\beta_M}(T_b^n x, N_j)$, this shows that $x$ satisfies the condition in Lemma 3.2 for $N_j$ and $M$. Since this is satisfied for infinitely many $N_j$ $(j \in \mathbb{N})$, for sufficiently large $M \in \mathbb{N}$ and for arbitrary $0 < \varepsilon < 1$, we have $x \in Q(t, \{\beta_M\}_{M\in\mathbb{N}})$. Then we have $K_{t^2/\log b} \subset Q(t, \{\beta_M\}_{M\in\mathbb{N}})$ and this completes the proof. □

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