



Approximation via Hausdorff operators

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Abstract. Truncating the Fourier transform averaged by means of a generalized Hausdorff operator, we approximate functions and the adjoint to that Hausdorff operator of the given function. We find estimates for the rate of approximation in various metrics in terms of the parameter of truncation and the components of the Hausdorff operator. Explicit rates of approximation of functions and comparison with approximate identities are given in the case of continuous functions from the class $Lip\ \alpha$.

1 Introduction

The classical Hausdorff operator is defined, by means of a kernel φ , as

$$(1.1) \quad (\mathcal{H}_\varphi f)(x) = \int_{\mathbb{R}} \frac{\varphi(t)}{|t|} f\left(\frac{x}{t}\right) dt,$$

and, as is shown first in [11] (see also [17] or [13]), such an operator is bounded in $L^1(\mathbb{R})$ whenever $\varphi \in L^1(\mathbb{R})$.

In the last two decades, various problems related to Hausdorff operators have attracted a lot of attention. The number of publications is growing considerably; to add some of the most notable, we mention [1, 8, 12, 15, 16, 18]. There are two survey papers: [6] and [13]. In the latter, as well as in [14], numerous open problems are given.

The Hausdorff operator (1.1) is expected to have better Fourier analytic properties than f . For example, in general, the inversion formula

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(y) e^{ixy} dy$$

does not hold for $f \in L^1(\mathbb{R})$; in order to “repair” this, one can consider some transformation of the function f or of its Fourier transform. In relation to the Hausdorff operator, we will consider integrals of the form

$$(1.2) \quad \int_{\mathbb{R}} (\mathcal{H}_\varphi \widehat{f})(y) e^{ixy} dy.$$

Received by the editors January 12, 2020; revised July 17, 2020.

Published online on Cambridge Core August 13, 2020.

Alberto Debernardi was supported by the ERC starting grant No. 713927 and the ISF grant No. 447/16.

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AMS subject classification: 41A25, 42A38, 44A15.

Keywords: Hausdorff operators, approximation in Lebesgue spaces, moduli of continuity.

Here we analyze not this Hausdorff operator but a more general one, apparently first considered in [9] (see also [10]). Given an odd function a such that $|a(t)|$ is decreasing, positive, and bijective on $(0, \infty)$ (so that both $|a|$ and $1/|a|$ possess inverse functions in such an interval), we define

$$(1.3) \quad (\mathcal{H}f)(x) = (\mathcal{H}_{\varphi,a}f)(x) = \int_{\mathbb{R}} \varphi(t)|a(t)|f(a(t)x) dt.$$

It is clear that (1.1) corresponds to (1.3) with $a(t) = t^{-1}$, and one can easily derive the corresponding results from the general ones. Moreover, we consider some such particular cases as examples.

There is one more reason for considering general Hausdorff operators: they provide a proper basis for future multidimensional extensions (see, for instance, [3] and [12], where those operators were introduced independently). Such multidimensional operators have been extensively studied in Lebesgue and Hardy spaces. We refer the reader to [12, 15, 16] for further details.

The consideration of these “alternative” transformations such as (1.2) requires the development of a parallel theory to Fourier integrals. In this paper, we address three basic issues of approximation theory applied to (generalized) Hausdorff operators.

- (i) To find the operator T such that the integrals of the type

$$(1.4) \quad \int_{-N}^N (\mathcal{H}_{\varphi,a}\widehat{f})(y)e^{ixy} dy$$

approximate Tf as $N \rightarrow \infty$ (in the L^p norm), for reasonable choices of φ (here some assumptions on f and φ are needed in order for $(\mathcal{H}_{\varphi,a}\widehat{f})(y)$ to be well defined; see the discussion at the beginning of Section 2). As we will see, the operator T is by no means the identity operator, but the dual operator of \mathcal{H} , denoted by \mathcal{H}^* , and formally defined by the relation

$$\int_{\mathbb{R}} \mathcal{H}f(x)g(x) dx = \int_{\mathbb{R}} f(x)\mathcal{H}^*g(x) dx.$$

- (ii) To study the rate of convergence to \mathcal{H}^*f of the partial integrals

$$(1.5) \quad \int_{-N}^N (\mathcal{H}_{\varphi,a}\widehat{f})(y)e^{ixy} dy,$$

as $N \rightarrow \infty$ in the L^p norm, where $1 \leq p \leq \infty$.

- (iii) To modify (1.5) in a way that allows us to derive a method for approximating f in the L^p norm (rather than approximating \mathcal{H}^*f , as in (i) and (ii)).

In particular, the problem of exploiting Hausdorff operators in approximation is raised. Indeed, application of analytic results in approximation seems to be the most convincing proof of their usefulness. This work is the first attempt to understand what kind of approximation problems may appear in the theory of Hausdorff operators and to solve some of them. The results obtained will open new lines in both the theory of Hausdorff operators itself and approximation theory. The difference between Hausdorff means and more typical multiplier (convolution) means, which comes from the difference between dilation invariance for the former and shift invariance for the latter, leads not only to new results but also to novelties in the methods.

The structure of the paper is as follows. In the next section, we begin with certain preliminaries, we formulate the main results. In Section 3, we prove the main results. Section 4 is devoted to presenting some examples of operators and their approximation estimates. After several works on the boundedness of the Hausdorff operators on various function spaces, this paper is the first application of Hausdorff operators to the problems of constructive approximation. In particular, we compare the obtained results with their traditional counterparts (approximate identities given by convolution type operators). Finally, in Section 5, we give concluding remarks, and in particular, we show that some regularity of the kernel φ is needed in order to obtain good approximation estimates.

We denote by

$$\omega(f; \delta)_p = \sup_{|h| \leq \delta} \|f(\cdot + h) - f(\cdot)\|_{L^p(\mathbb{R})}$$

the modulus of continuity in the L^p norm, where $1 \leq p \leq \infty$. If $p = \infty$, then $\omega(f; \delta)_\infty = \omega(f; \delta)$ is the usual modulus of continuity.

We will also write $A \lesssim B$ to denote $A \leq C \cdot B$ for some constant C that does not depend on essential quantities. The symbol $A \asymp B$ means that $A \lesssim B$ and $B \lesssim A$ simultaneously.

2 Main Results

First of all, let us discuss some boundedness properties of the Hausdorff operator in Lebesgue spaces, in order for $\mathcal{H}^*(f)$ (and also the Hausdorff operator in (1.5)) to be well defined. We will always assume that $f \in L^1(\mathbb{R})$, so that \widehat{f} is well defined, and $\widehat{f} \in L^\infty(\mathbb{R})$. On the other hand, a sufficient condition for the operator \mathcal{H}^* to be bounded on $L^p(\mathbb{R})$ is

$$(2.1) \quad \int_{\mathbb{R}} |\varphi(t)| |a(t)|^{1/p} dt < \infty,$$

(moreover, if $\varphi \geq 0$ almost everywhere, then such a condition is also necessary; see the recent paper [4] and also [2]). Similarly, a sufficient condition (and necessary whenever $\varphi \geq 0$ a.e.) for the Hausdorff operator to be bounded on $L^p(\mathbb{R})$ is that

$$(2.2) \quad \int_{\mathbb{R}} |\varphi(t)| |a(t)|^{1/p'} dt < \infty, \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

Summarizing, for $f \in L^1(\mathbb{R}) \cap L^p(\mathbb{R})$ ($1 \leq p \leq \infty$), we have $\widehat{f} \in L^\infty(\mathbb{R}) \cap L^{\max\{2, p'\}}(\mathbb{R})$ (by the usual mapping properties of the Fourier transform), and, moreover, if condition (2.1) holds, then we have $\mathcal{H}^* f \in L^p(\mathbb{R})$. Finally, by (2.2), if

$$(2.3) \quad \int_{\mathbb{R}} |\varphi(t)| \max\{|a(t)|^{1/p}, |a(t)|^{1/2}\} dt < \infty,$$

then $\mathcal{H}_{\varphi, a} \widehat{f}$ is well defined as a function from $L^{\max\{2, p'\}}(\mathbb{R})$. Thus, we will always assume that $f \in L^1(\mathbb{R}) \cap L^p(\mathbb{R})$ and that (2.3) holds. For further results on boundedness (and also Pitt-type inequalities) of Hausdorff operators, we refer the reader to [7].

We give some more observations before stating our main results. It is easy to check by substitution that

$$(2.4) \quad \mathcal{H}^* f(x) = \int_{\mathbb{R}} \varphi(t) f\left(\frac{x}{a(t)}\right) dt.$$

Moreover, since

$$(2.5) \quad \int_{\mathbb{R}} \frac{\sin \frac{s}{t}}{s} ds = \pi, \quad t \in \mathbb{R} \setminus \{0\}$$

we have

$$(2.6) \quad \mathcal{H}^* f(x) = \frac{1}{\pi} \int_{\mathbb{R}} \varphi(t) \int_{\mathbb{R}} f\left(\frac{x}{a(t)s}\right) \frac{\sin a(t)s}{s} ds dt.$$

Let us now define the partial integrals

$$\begin{aligned} (\mathcal{H}_N f)^\vee(x) &= \frac{1}{\pi} \int_{-N}^N \mathcal{H} \widehat{f}(u) e^{iux} du \\ &= \frac{1}{\pi} \int_{-N}^N \int_{\mathbb{R}} \varphi(t) |a(t)| \int_{\mathbb{R}} f(s) e^{-ia(t)su} ds dt e^{iux} du \\ &= \frac{1}{\pi} \int_{\mathbb{R}} \varphi(t) |a(t)| \int_{\mathbb{R}} f(s) \frac{\sin N(x - a(t)s)}{x - a(t)s} ds dt. \end{aligned}$$

By substitution, it is easy to see that

$$(2.7) \quad (\mathcal{H}_N \widehat{f})^\vee(x) = \frac{1}{\pi} \int_{\mathbb{R}} \varphi(t) \int_{\mathbb{R}} f\left(\frac{x}{a(t)} - \frac{s}{N}\right) \frac{\sin(a(t)s)}{s} ds dt.$$

These observations make clear that $(\mathcal{H}_N \widehat{f})^\vee$ is a good candidate to approximate $\mathcal{H}^* f$ (informally, letting $N \rightarrow \infty$ in (2.7) we obtain (2.6)). We will prove that this is actually the case, at least in the L^p setting.

Our main results concerning approximation of adjoint Hausdorff operators read as follows.

Theorem 2.1 For $1 \leq p \leq \infty$, if $f \in L^1(\mathbb{R}) \cap L^p(\mathbb{R})$ and (2.3) holds, we have

$$(2.8) \quad \begin{aligned} \|\mathcal{H}^* f - (\mathcal{H}_N \widehat{f})^\vee\|_{L^p(\mathbb{R})} &\leq C_p \int_{\mathbb{R}} |\varphi(t)| |a(t)|^{1/p} \omega\left(f; \frac{1}{|a(t)|N}\right)_p dt \\ &+ \frac{C_p}{2} \int_{\mathbb{R}} \frac{\omega(f; \frac{|s|}{N})_p}{|s|} \int_{|a^{-1}(1/s)| \leq |t|} |\varphi(t)| |a(t)|^{1/p} dt ds, \end{aligned}$$

where we take the convention $1/p = 0$ if $p = \infty$, and furthermore,

$$C_p = \begin{cases} 2, & \text{if } p = 1, \\ 4, & \text{if } 1 < p < \infty, \\ 2/\pi, & \text{if } p = \infty. \end{cases}$$

The fact that the adjoint Hausdorff operator of a function is approximated may be unsatisfactory in principle, as one would rather approximate the function itself. However, approximating a function instead of its adjoint Hausdorff operator is also possible as a consequence of the following observation. For $\varphi \in L^1(\mathbb{R})$ and $a(t)$ as in the introduction, one has

$$\mathcal{H}_{\varphi,a}^* f(0) = \int_{\mathbb{R}} \varphi(t) f\left(\frac{0}{a(t)}\right) dt = f(0) \int_{\mathbb{R}} \varphi(t) dt.$$

If we denote by $\tau_y f(x) = f(x + y)$ the translation of f by $y \in \mathbb{R}$ and assume that $\int_{\mathbb{R}} \varphi(t) dt = 1$, then

$$\mathcal{H}_{\varphi,a}^* [\tau_y f](0) = f(y).$$

This gives a natural way of approximating f through Hausdorff operators by using the approximant $F_N(y) = (\mathcal{H}_N \widehat{\tau_y f})^\vee(0) = (\mathcal{H}_N [e^{iyx} \widehat{f}(x)])^\vee(0)$. More precisely, we have the following theorem.

Theorem 2.2 Assume $\varphi \in L^1(\mathbb{R})$ and $\int_{\mathbb{R}} \varphi(t) dt = 1$. For $1 \leq p \leq \infty$, if $f \in L^1(\mathbb{R}) \cap L^p(\mathbb{R})$ and (2.3) holds, we have

$$\begin{aligned} \|f - F_N\|_{L^p(\mathbb{R})} &\leq C_p \int_{\mathbb{R}} |\varphi(t)| \omega\left(f; \frac{1}{N|a(t)|}\right)_p dt \\ (2.9) \quad &+ \frac{C_p}{2} \int_{\mathbb{R}} \frac{\omega(f; \frac{|s|}{N})_p}{|s|} \int_{|a^{-1}(1/s)| \leq |t|} |\varphi(t)| dt ds, \end{aligned}$$

where

$$(2.10) \quad F_N(y) = (\mathcal{H}_N \widehat{\tau_y f})^\vee(0) = (\mathcal{H}_N [e^{iyx} \widehat{f}(x)])^\vee(0), \quad y \in \mathbb{R},$$

and

$$C_p = \begin{cases} 2/\pi, & \text{if } p = 1, \infty, \\ 4/\pi, & \text{if } 1 < p < \infty. \end{cases}$$

Remark 2.3 In order for the right-hand sides of (2.8) and (2.9) to be finite, one should assume that φ vanishes at a fast enough rate as $|t| \rightarrow \infty$, or even more, that it has compact support. The latter is the case for the Cesàro operator (where $\varphi = \chi_{(0,1)}$), which we discuss in more detail in Section 4, along with other examples.

3 Proofs

First of all, we give pointwise estimates for

$$|\mathcal{H}^* f(x) - (\mathcal{H}_N \widehat{f})^\vee(x)| \quad \text{and} \quad |f(x) - F_N(x)|,$$

which will be the starting points for all subsequent estimates.

Lemma 3.1 For any $x \in \mathbb{R}$,

$$\begin{aligned}
 & \pi |\mathcal{H}^* f(x) - (\mathcal{H}_N \widehat{f})^\vee(x)| \\
 & \leq \int_{\mathbb{R}} |\varphi(t)a(t)| \int_{|s| \leq 1/|a(t)|} \left| f\left(\frac{x}{a(t)} - \frac{s}{N}\right) - f\left(\frac{x}{a(t)}\right) \right| ds dt \\
 (3.1) \quad & + \int_{\mathbb{R}} \frac{1}{|s|} \int_{|a^{-1}(1/s)| \leq |t|} |\varphi(t)| \left| f\left(\frac{x}{a(t)} - \frac{s}{N}\right) - f\left(\frac{x}{a(t)}\right) \right| dt ds.
 \end{aligned}$$

Proof To prove (3.1), we apply rather straightforward estimates. Indeed,

$$\begin{aligned}
 & \pi |\mathcal{H}^* f(x) - (\mathcal{H}_N \widehat{f})^\vee(x)| \\
 & = \left| \int_{\mathbb{R}} \varphi(t) \int_{\mathbb{R}} \left(f\left(\frac{x}{a(t)} - \frac{s}{N}\right) - f\left(\frac{x}{a(t)}\right) \right) \frac{\sin(a(t)s)}{s} ds dt \right| \\
 & \leq \left| \int_{\mathbb{R}} \varphi(t) \int_{|s| \leq 1/|a(t)|} \left(f\left(\frac{x}{a(t)} - \frac{s}{N}\right) - f\left(\frac{x}{a(t)}\right) \right) \frac{\sin(a(t)s)}{s} ds dt \right| \\
 & \quad + \left| \int_{\mathbb{R}} \varphi(t) \int_{|s| \geq 1/|a(t)|} \left(f\left(\frac{x}{a(t)} - \frac{s}{N}\right) - f\left(\frac{x}{a(t)}\right) \right) \frac{\sin(a(t)s)}{s} ds dt \right| \\
 & \leq \int_{\mathbb{R}} |\varphi(t)a(t)| \int_{|s| \leq 1/|a(t)|} \left| f\left(\frac{x}{a(t)} - \frac{s}{N}\right) - f\left(\frac{x}{a(t)}\right) \right| ds dt \\
 & \quad + \int_{\mathbb{R}} \frac{1}{|s|} \int_{|a^{-1}(1/s)| \leq |t|} |\varphi(t)| \left| f\left(\frac{x}{a(t)} - \frac{s}{N}\right) - f\left(\frac{x}{a(t)}\right) \right| dt ds,
 \end{aligned}$$

as desired. In the last inequality we use that $1/|a|$ possesses an inverse on $(0, \infty)$ (and therefore also on $(-\infty, 0)$, since it is an odd function), and moreover, $(1/|a|)^{-1}(t) = |a(1/t)|^{-1}$ on $(0, \infty)$. ■

Note that by (2.7),

$$(3.2) \quad F_N(x) = (\mathcal{H}_N \widehat{\tau_x f})^\vee(0) = \int_{\mathbb{R}} \varphi(t) \int_{\mathbb{R}} f\left(x - \frac{s}{N}\right) \frac{\sin(a(t)s)}{s} ds dt.$$

Also, by (2.5), we can write, for any $\varphi \in L^1(\mathbb{R})$ with $\int_{\mathbb{R}} \varphi(t) dt = 1$,

$$(3.3) \quad f(x) = \frac{1}{\pi} \int_{\mathbb{R}} \varphi(t) \int_{\mathbb{R}} f(x) \frac{\sin a(t)s}{s} ds dt.$$

Lemma 3.2 For any $x \in \mathbb{R}$,

$$\begin{aligned}
 \pi |f(x) - F_N(x)| & \leq \int_{\mathbb{R}} |\varphi(t)a(t)| \int_{|s| \leq 1/|a(t)|} \left| f\left(x - \frac{s}{N}\right) - f(x) \right| ds dt \\
 & \quad + \int_{\mathbb{R}} \frac{1}{|s|} \int_{|a^{-1}(1/s)| \leq |t|} |\varphi(t)| \left| f\left(x - \frac{s}{N}\right) - f(x) \right| dt ds.
 \end{aligned}$$

Proof By (3.2) and (3.3), we have the equality

$$f(x) - F_N(x) = \frac{1}{\pi} \int_{\mathbb{R}} \varphi(t) \int_{\mathbb{R}} \left(f\left(x - \frac{s}{N}\right) - f(x) \right) \frac{\sin a(t)s}{s} ds dt.$$

The proof now follows the same lines as that of [Lemma 3.1](#), with the only difference being that in the above integral the term, $(f(x - \frac{s}{N}) - f(x))$ is replaced by the term

$$\left(f\left(\frac{x}{a(t)} - \frac{s}{N}\right) - f\left(\frac{x}{a(t)}\right) \right)$$

in [Lemma 3.1](#). ■

We now proceed to the proofs of the main theorems.

Proof of Theorem 2.1 We treat the cases $1 \leq p < \infty$ and $p = \infty$ separately. For the case $p = \infty$, it suffices to estimate the two terms on the right-hand side of (3.1) in the L^∞ norm. For the first one, we have

$$\begin{aligned} & \int_{\mathbb{R}} |\varphi(t)a(t)| \int_{|s| \leq 1/|a(t)|} \left| f\left(\frac{x}{a(t)} - \frac{s}{N}\right) - f\left(\frac{x}{a(t)}\right) \right| ds dt \\ & \leq \int_{\mathbb{R}} |\varphi(t)a(t)| \int_{|s| \leq 1/|a(t)|} \omega\left(f; \frac{|s|}{N}\right) ds dt, \end{aligned}$$

and since $\omega(f; \delta)$ is nondecreasing in δ , we obtain

$$(3.4) \quad \int_{\mathbb{R}} |\varphi(t)a(t)| \int_{|s| \leq 1/|a(t)|} \omega\left(f; \frac{|s|}{N}\right) ds dt \leq 2 \int_{\mathbb{R}} |\varphi(t)| \omega\left(f; \frac{1}{N|a(t)|}\right) dt.$$

As for the second term on the right-hand side of (3.1), we have

$$\begin{aligned} & \int_{\mathbb{R}} \frac{1}{|s|} \int_{|a^{-1}(1/s)| \leq |t|} |\varphi(t)| \left| f\left(\frac{x}{a(t)} - \frac{s}{N}\right) - f\left(\frac{x}{a(t)}\right) \right| dt ds \\ & \leq \int_{\mathbb{R}} \frac{1}{|s|} \int_{|a^{-1}(1/s)| \leq |t|} |\varphi(t)| \omega\left(f; \frac{|s|}{N}\right) dt ds \\ & = \int_{\mathbb{R}} \frac{\omega(f; \frac{|s|}{N})}{|s|} \int_{|a^{-1}(1/s)| \leq |t|} |\varphi(t)| dt ds. \end{aligned}$$

Collecting all the estimates, we get

$$\begin{aligned} \pi |\mathcal{H}^* f(x) - (\mathcal{H}_N \widehat{f})^\vee(x)| & \leq 2 \int_{\mathbb{R}} |\varphi(t)| \omega\left(f; \frac{1}{N|a(t)|}\right) dt \\ & \quad + \int_{\mathbb{R}} \frac{\omega(f; \frac{|s|}{N})}{|s|} \int_{|a^{-1}(1/s)| \leq |t|} |\varphi(t)| dt ds, \end{aligned}$$

where the right-hand side is uniform in x .

Let us now prove the case $1 \leq p < \infty$. Using (3.1), we get

$$\begin{aligned} & \frac{1}{2} \left(\int_{\mathbb{R}} |\mathcal{H}^* f(x) - (\mathcal{H}_N \widehat{f})^\vee(x)|^p dx \right)^{1/p} \\ & \leq \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |\varphi(t)a(t)| \int_{|s| \leq 1/|a(t)|} \left| f\left(\frac{x}{a(t)} - \frac{s}{N}\right) - f\left(\frac{x}{a(t)}\right) \right| ds dt \right)^p dx \right)^{1/p} \end{aligned}$$

$$+ \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{1}{|s|} \int_{|a^{-1}(1/s)| \leq |t|} |\varphi(t)| \left| f\left(\frac{x}{a(t)} - \frac{s}{N}\right) - f\left(\frac{x}{a(t)}\right) \right| dt ds \right)^p dx \right)^{1/p}.$$

Note that if $p = 1$, the factor $\frac{1}{2}$ on the left-hand side can be taken to be 1 (in fact, such a factor appears due to the inequality $(a + b)^p \leq 2^p(a^p + b^p)$, for $a, b \geq 0$ and $p > 1$). On one hand, applying Minkowski's inequality twice, we get

$$\begin{aligned} & \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |\varphi(t)a(t)| \int_{|s| \leq 1/|a(t)|} \left| f\left(\frac{x}{a(t)} - \frac{s}{N}\right) - f\left(\frac{x}{a(t)}\right) \right| ds dt \right)^p dx \right)^{1/p} \\ & \leq \int_{\mathbb{R}} |\varphi(t)a(t)| \left(\int_{\mathbb{R}} \left(\int_{|s| \leq 1/|a(t)|} \left| f\left(\frac{x}{a(t)} - \frac{s}{N}\right) - f\left(\frac{x}{a(t)}\right) \right| ds \right)^p dx \right)^{1/p} dt \\ & \leq \int_{\mathbb{R}} |\varphi(t)a(t)| \int_{|s| \leq 1/|a(t)|} \left(\int_{\mathbb{R}} \left| f\left(\frac{x}{a(t)} - \frac{s}{N}\right) - f\left(\frac{x}{a(t)}\right) \right|^p dx \right)^{1/p} ds dt \\ & = \int_{\mathbb{R}} |\varphi(t)||a(t)|^{1+1/p} \int_{|s| \leq 1/|a(t)|} \left(\int_{\mathbb{R}} \left| f\left(x - \frac{s}{N}\right) - f(x) \right|^p dx \right)^{1/p} ds dt \\ & \leq \int_{\mathbb{R}} |\varphi(t)||a(t)|^{1+1/p} \int_{|s| \leq 1/|a(t)|} \omega\left(f; \frac{|s|}{N}\right)_p ds dt. \end{aligned}$$

Since $\omega(f; \delta)_p$ is nondecreasing in δ , we have

$$\begin{aligned} & \int_{\mathbb{R}} |\varphi(t)||a(t)|^{1+1/p} \int_{|s| \leq 1/|a(t)|} \omega\left(f; \frac{|s|}{N}\right)_p ds dt \\ & \leq 2 \int_{\mathbb{R}} |\varphi(t)||a(t)|^{1/p} \omega\left(f; \frac{1}{|a(t)|N}\right)_p dt. \end{aligned}$$

On the other hand, applying Minkowski's inequality again, we obtain

$$\begin{aligned} & \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{1}{|s|} \int_{|a^{-1}(1/s)| \leq |t|} |\varphi(t)| \left| f\left(\frac{x}{a(t)} - \frac{s}{N}\right) - f\left(\frac{x}{a(t)}\right) \right| dt ds \right)^p dx \right)^{1/p} \\ & \leq \int_{\mathbb{R}} \frac{1}{|s|} \left(\int_{\mathbb{R}} \left(\int_{|a^{-1}(1/s)| \leq |t|} |\varphi(t)| \left| f\left(\frac{x}{a(t)} - \frac{s}{N}\right) - f\left(\frac{x}{a(t)}\right) \right| dt \right)^p dx \right)^{1/p} ds \\ & \leq \int_{\mathbb{R}} \frac{1}{|s|} \int_{|a^{-1}(1/s)| \leq |t|} |\varphi(t)| \left(\int_{\mathbb{R}} \left| f\left(\frac{x}{a(t)} - \frac{s}{N}\right) - f\left(\frac{x}{a(t)}\right) \right|^p dx \right)^{1/p} dt ds \\ & = \int_{\mathbb{R}} \frac{1}{|s|} \int_{|a^{-1}(1/s)| \leq |t|} |\varphi(t)||a(t)|^{1/p} \left(\int_{\mathbb{R}} \left| f\left(x - \frac{s}{N}\right) - f(x) \right|^p dx \right)^{1/p} dt ds \\ & \leq \int_{\mathbb{R}} \frac{\omega\left(f; \frac{|s|}{N}\right)_p}{|s|} \int_{|a^{-1}(1/s)| \leq |t|} |\varphi(t)||a(t)|^{1/p} dt ds. \end{aligned}$$

Collecting all the estimates, we derive

$$\begin{aligned} \frac{1}{2} \|\mathcal{H}^* f - (\mathcal{H}_N \widehat{f})\|_{L^p(\mathbb{R})} &\leq 2 \int_{\mathbb{R}} |\varphi(t)| |a(t)|^{1/p} \omega\left(f; \frac{1}{|a(t)|N}\right) dt \\ &\quad + \int_{\mathbb{R}} \frac{\omega(f; \frac{|s|}{N})_p}{|s|} \int_{|a^{-1}(1/s)| \leq |t|} |\varphi(t)| |a(t)|^{1/p} dt ds, \end{aligned}$$

where the factor 1/2 on the left-hand side is omitted in the case where $p = 1$. The proof is complete. ■

Proof of Theorem 2.2 First of all, note that the case $p = \infty$ follows trivially from Theorem 2.1 and the fact that $\omega(f; \delta) = \omega(\tau_y f; \delta)$ for every $y \in \mathbb{R}$.

We now show the case $1 \leq p < \infty$. By Lemma 3.2,

$$\begin{aligned} &\frac{\pi}{2} \left(\int_{\mathbb{R}} |f(y) - F_N(y)|^p dy \right)^{1/p} \\ &\leq \pi \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |\varphi(t)a(t)| \int_{|s| \leq 1/|a(t)|} \left| f\left(y - \frac{s}{N}\right) - f(y) \right| ds dt \right)^p dy \right)^{1/p} \\ &\quad + \pi \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{1}{|s|} \int_{|a^{-1}(1/s)| \leq |t|} |\varphi(t)| \left| f\left(y - \frac{s}{N}\right) - f(y) \right| dt ds \right)^p dy \right)^{1/p}. \end{aligned}$$

If $p = 1$, the factor $\frac{1}{2}$ on the left-hand side can be omitted, similarly as in the proof of Theorem 2.1. Now, applying Minkowski's inequality twice, we estimate

$$\begin{aligned} &\left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |\varphi(t)a(t)| \int_{|s| \leq 1/|a(t)|} \left| f\left(y - \frac{s}{N}\right) - f(y) \right| ds dt \right)^p dy \right)^{1/p} \\ &\leq \int_{\mathbb{R}} |\varphi(t)a(t)| \left(\int_{\mathbb{R}} \left(\int_{|s| \leq 1/|a(t)|} \left| f\left(y - \frac{s}{N}\right) - f(y) \right| ds \right)^p dy \right)^{1/p} dt \\ &\leq \int_{\mathbb{R}} |\varphi(t)a(t)| \int_{|s| \leq 1/|a(t)|} \left(\int_{\mathbb{R}} \left| f\left(y - \frac{s}{N}\right) - f(y) \right|^p dy \right)^{1/p} ds dt \\ &\leq \int_{\mathbb{R}} |\varphi(t)a(t)| \int_{|s| \leq 1/|a(t)|} \omega\left(f; \frac{|s|}{N}\right)_p ds dt \leq 2 \int_{\mathbb{R}} |\varphi(t)| \omega\left(f; \frac{1}{|a(t)|N}\right)_p dt, \end{aligned}$$

where the last inequality follows from the fact that $\omega(f; \delta)_p$ is increasing in δ . On the other hand, applying Minkowski's inequality again, we obtain

$$\begin{aligned} &\left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{1}{|s|} \int_{|a^{-1}(1/s)| \leq |t|} |\varphi(t)| \left| f\left(y - \frac{s}{N}\right) - f(y) \right| dt ds \right)^p dy \right)^{1/p} \\ &\leq \int_{\mathbb{R}} \frac{1}{|s|} \left(\int_{\mathbb{R}} \left(\int_{|a^{-1}(1/s)| \leq |t|} |\varphi(t)| \left| f\left(y - \frac{s}{N}\right) - f(y) \right| dt \right)^p dy \right)^{1/p} ds \end{aligned}$$

$$\begin{aligned} &\leq \int_{\mathbb{R}} \frac{1}{|s|} \int_{|a^{-1}(1/s)| \leq |t|} |\varphi(t)| \left(\int_{\mathbb{R}} \left| f\left(y - \frac{s}{N}\right) - f(y) \right|^p dy \right)^{1/p} dt ds \\ &\leq \int_{\mathbb{R}} \frac{\omega\left(f; \frac{|s|}{N}\right)_p}{|s|} \int_{|a^{-1}(1/s)| \leq |t|} |\varphi(t)| dt ds. \end{aligned}$$

Putting all the estimates together, we finally obtain (2.9). ■

4 Examples

We now obtain approximations of functions by means of certain specific Hausdorff operators. We shall give bounds for the approximation error explicitly in L^p , $1 \leq p \leq \infty$, in each case, which will follow from Theorem 2.2.

In the first place, we consider a general Hausdorff operator under some assumptions on the kernel φ (besides the assumptions from Theorem 2.2). We suppose without loss of generality that $a(t) > 0$ for $t \in (0, \infty)$, that φ is compactly supported, say on $[-T, T]$, and $\varphi \in L^\infty(\mathbb{R})$ (note that the Cesàro operator, given by $a(t) = 1/t$ and $\varphi = \chi_{(0,1)}$, satisfies these conditions). Then, on one hand,

$$\int_{\mathbb{R}} |\varphi(t)| \omega\left(f; \frac{1}{|a(t)N}\right)_p dt \leq 2T \|\varphi\|_{L^\infty(\mathbb{R})} \int_0^T \omega\left(f; \frac{1}{a(t)N}\right)_p dt.$$

On the other hand,

$$\begin{aligned} \int_{\mathbb{R}} \frac{\omega\left(f; \frac{|s|}{N}\right)_p}{|s|} \int_{|a^{-1}(1/s)| \leq |t|} |\varphi(t)| dt ds &= \int_{\mathbb{R}} \frac{\omega\left(f; \frac{|s|}{N}\right)_p}{|s|} \int_{|a^{-1}(1/s)| \leq |t| \leq T} |\varphi(t)| dt ds \\ &\leq 2T \|\varphi\|_{L^\infty(\mathbb{R})} \int_{0 \leq a^{-1}(1/s) \leq T} \frac{\omega\left(f; \frac{s}{N}\right)_p}{s} ds. \end{aligned}$$

Now, the substitution $s \rightarrow 1/a(t)$ yields

$$\int_{0 \leq a^{-1}(1/s) \leq T} \frac{\omega\left(f; \frac{s}{N}\right)_p}{s} ds = \int_0^T \frac{\omega\left(f; \frac{1}{a(t)N}\right)_p}{a(t)} |a'(t)| dt,$$

so we conclude that for any $1 \leq p \leq \infty$,

$$\begin{aligned} &\|f - F_N\|_{L^p(\mathbb{R})} \\ &\leq 2T \|\varphi\|_{L^\infty(\mathbb{R})} \left(\int_0^T \omega\left(f; \frac{1}{a(t)N}\right)_p dt + \int_0^T \frac{\omega\left(f; \frac{1}{a(t)N}\right)_p}{a(t)} |a'(t)| dt \right) \\ &\leq 4T \|\varphi\|_{L^\infty(\mathbb{R})} \max \left\{ \int_0^T \omega\left(f; \frac{1}{a(t)N}\right)_p dt, \int_0^T \frac{\omega\left(f; \frac{1}{a(t)N}\right)_p}{a(t)} |a'(t)| dt \right\}, \end{aligned}$$

by [Theorem 2.2](#) (recall that $\omega(f; \delta)_\infty = \omega(f; \delta)$). If, furthermore, $a(t) = 1/t$, then for $1 \leq p \leq \infty$,

$$\begin{aligned} \|f - F_N\|_{L^p(\mathbb{R})} &= \left\| f(x) - \int_{\mathbb{R}} \varphi(t) \int_{\mathbb{R}} f\left(x - \frac{s}{N}\right) \frac{\sin \frac{s}{t}}{s} ds dt \right\|_{L^p(\mathbb{R})} \\ (4.1) \qquad \qquad \qquad &\leq 4T^2 \int_0^T \frac{\omega\left(f; \frac{t}{N}\right)_p}{t} dt, \end{aligned}$$

(recall also that in the case $p = 1$, the estimate on the right-hand side can be multiplied by the factor $1/2$). To the best of our knowledge, no approach through Hausdorff operators has been considered in approximation problems so far, and therefore even the basic estimate (4.1) is new in this respect.

4.1 Approximation via the Cesàro Operator

The Cesàro operator \mathcal{C} given by $a(t) = 1/t$ and $\varphi(t) = \chi_{(0,1)}(t)$ [[13](#), [17](#)] is the prototype Hausdorff operator $\mathcal{H}_{\varphi,a}$. In this case, its adjoint operator is

$$\mathcal{C}^* f(x) = \int_0^1 f(tx) dt = \frac{1}{x} \int_0^x f(t) dt,$$

also referred to as the Hardy operator. We have

$$\begin{aligned} (\mathcal{C}_N \widehat{f})^\vee(x) &:= (\mathcal{H}_N \widehat{f})^\vee(x) = \frac{1}{\pi} \int_0^1 \frac{1}{t} \int_{\mathbb{R}} f(s) \frac{\sin N(x - s/t)}{x - s/t} ds dt \\ &= \frac{1}{\pi} \int_0^1 \int_{\mathbb{R}} f\left(tx - \frac{s}{N}\right) \frac{\sin \frac{s}{t}}{s} ds dt. \end{aligned}$$

It readily follows from (4.1) that

$$\begin{aligned} \|f - F_N\|_{L^1(\mathbb{R})} &\leq 2 \int_0^1 \frac{\omega\left(f; \frac{t}{N}\right)_1}{t} dt, \\ \|f - F_N\|_{L^p(\mathbb{R})} &\leq 4 \int_0^1 \frac{\omega\left(f; \frac{t}{N}\right)_p}{t} dt, \end{aligned}$$

and in the case $p = \infty$, we obtain a Dini-type estimate

$$\|f - F_N\|_{L^\infty(\mathbb{R})} \leq 4 \int_0^1 \frac{\omega\left(f; \frac{t}{N}\right)}{t} dt.$$

Note also that for $p = 1$, condition (2.3) does not hold, so we have to restrict ourselves to the case $1 < p \leq \infty$. In particular, we can conclude the following corollary.

Corollary 4.1 *Let $f \in L^1(\mathbb{R}) \cap L^p(\mathbb{R})$ ($1 < p \leq \infty$). Let $a(t) = 1/t$ and $\varphi(t) = \chi_{[0,1]}(t)$.*

- (i) If $1 < p < \infty$ and $\int_0^1 \frac{\omega(f;t)_p}{t} dt < \infty$, then F_N converges to f in $L^p(\mathbb{R})$ as $N \rightarrow \infty$. In particular,

$$\begin{aligned} \|f - F_N\|_{L^p(\mathbb{R})} &= \left\| f(x) - \int_0^1 \int_{\mathbb{R}} f\left(x - \frac{s}{N}\right) \frac{\sin \frac{s}{t}}{s} ds dt \right\|_{L^p(\mathbb{R})} \\ &\leq 4 \int_0^1 \frac{\omega(f; \frac{t}{N})_p}{t} dt. \end{aligned}$$

- (ii) If f is continuous and $\int_0^1 \frac{\omega(f;t)}{t} dt < \infty$, then F_N converges uniformly to f on \mathbb{R} as $N \rightarrow \infty$. In particular,

$$\begin{aligned} \|f - F_N\|_{L^\infty(\mathbb{R})} &= \left\| f(x) - \int_0^1 \int_{\mathbb{R}} f\left(x - \frac{s}{N}\right) \frac{\sin \frac{s}{t}}{s} ds dt \right\|_{L^\infty(\mathbb{R})} \\ &\leq 4 \int_0^1 \frac{\omega(f; \frac{t}{N})}{t} dt. \end{aligned}$$

Remark 4.2 For $0 < q \leq \infty$, $1 \leq p \leq \infty$, and $0 \leq s < 1$, the Besov seminorm (defined via the modulus of continuity) is

$$\|f\|_{B_{p,q}^s(\mathbb{R})} = \left(\int_0^1 (t^{-s} \omega(f;t)_p)^q \frac{dt}{t} \right)^{1/q}.$$

We refer the reader to [19, §5.2.3, Theorem 2] for the description of Besov seminorms in terms of moduli of continuity. Note that in Corollary 4.1, the assumption that

$$\int_0^1 \frac{\omega(f;t)_p}{t} dt < \infty, \quad 1 \leq p \leq \infty,$$

is equivalent to saying that the Besov seminorm $B_{p,1}^0$ of f is finite.

We shall now compare the approximation estimates from Corollary 4.1 with those for approximate identities.

4.2 Comparison: Cesàro Operators and Approximate Identities

Since the Cesàro operator is the prototype example of Hausdorff operator, it is instructive to compare the obtained approximations with the classical ones given by approximate identities for convolutions. A family of functions $\{\mathcal{C}_r\}_{r>0}$ defined on \mathbb{R} is called an *approximate identity* if

- (1) $\sup_r \|\mathcal{C}_r\|_{L^1(\mathbb{R})} < \infty$, and
- (2) for every $\delta > 0$,

$$\int_{|x| \geq \delta} |\mathcal{C}_r(x)| dx \rightarrow 0 \quad \text{as } r \rightarrow \infty.$$

The following is well known [5, Theorem 3.1.6].

Theorem A Let $g \in L^p(\mathbb{R})$, with $1 \leq p < \infty$. If $\{\mathcal{C}_r\}_{r>0}$ is an approximate identity satisfying

$$(4.2) \quad \int_{\mathbb{R}} \mathcal{C}_r(x) dx = 1, \quad r > 0,$$

then

$$\|\mathcal{C}_r * g - g\|_{L^p(\mathbb{R})} \rightarrow 0, \quad \text{as } r \rightarrow \infty.$$

As an example of an approximate identity satisfying (4.2), we have the family of functions

$$\mathcal{C}_r(x) = r\mathcal{C}(rx), \quad r > 0,$$

where $\mathcal{C}(x)$ is the Fejér kernel on the real line,

$$\mathcal{C}(x) = \frac{1}{2\pi} \left(\frac{\sin(x/2)}{x/2} \right)^2.$$

From now on, we assume that the approximate identities we consider satisfy condition (4.2).

Comparing Theorem A and Corollary 4.1, we readily see that the latter requires further assumptions in order to guarantee L^p convergence ($p < \infty$), namely that the seminorm $\|f\|_{B_{p,1}^\alpha(\mathbb{R})}$ is finite (cf. Remark 4.2). However, when restricted to certain classes of functions, the approximation rates become the same, or even better.

As classes of functions, we consider $\text{Lip}_p \alpha = \text{Lip}_p \alpha(\mathbb{R})$ with $0 < \alpha \leq 1$, and $1 \leq p \leq \infty$, which consists of the functions f satisfying

$$\omega(f; \delta)_p \lesssim \delta^\alpha, \quad \delta > 0.$$

Note that $\text{Lip } \alpha = \text{Lip}_\infty \alpha$ is the class of usual Lipschitz- α continuous functions on \mathbb{R} , i.e., those satisfying

$$|f(x) - f(y)| \leq C|x - y|^\alpha, \quad x, y \in \mathbb{R}.$$

For $f \in \text{Lip}_p \alpha$, $0 < \alpha < 1$, and $1 \leq p \leq \infty$, it is known that any approximate identity $\{\mathcal{C}_r\}$ yields the approximation rate

$$(4.3) \quad \|\mathcal{C}_r * f - f\|_{L^p(\mathbb{R})} \lesssim \frac{1}{r^\alpha}, \quad r \rightarrow \infty$$

(see [5, Corollary 3.4.4]), while for $\alpha = 1$, an additional logarithm appears:

$$(4.4) \quad \|\mathcal{C}_r * f - f\|_{L^p(\mathbb{R})} \lesssim \frac{\log r}{r}, \quad r \rightarrow \infty,$$

cf. [5, Problem 3.4.2]. Moreover, both estimates are sharp (see [5, Corollary 3.5.4] and [5, Problem 3.4.2], respectively).

In the case of the Cesàro operator, Corollary 4.1 yields, for any $1 < p < \infty$ and $f \in \text{Lip}_p \alpha$,

$$\|f - F_N\|_{L^p(\mathbb{R})} \lesssim \int_0^1 \frac{\omega(f; \frac{t}{N})_p}{t} dt \lesssim N^{-\alpha} \int_0^1 t^{\alpha-1} dt \asymp N^{-\alpha}, \quad N \rightarrow \infty,$$

while for $f \in \text{Lip } \alpha$,

$$\|f - F_N\|_{L^\infty(\mathbb{R})} \lesssim \int_0^1 \frac{\omega(f; \frac{t}{N})}{t} dt \lesssim N^{-\alpha} \int_0^1 t^{\alpha-1} dt \asymp N^{-\alpha}, \quad N \longrightarrow \infty,$$

with all the estimates valid for the range $0 < \alpha \leq 1$. Note that these approximation rates are the same as those for approximate identities when restricted to functions $f \in \text{Lip}_p \alpha$ with $0 < \alpha < 1$ (compare with (4.3)), and are actually better than their counterparts in the case $\alpha = 1$ (compare with (4.4)), in the sense that the extra logarithm from (4.4) does not appear. Thus, in the case $\alpha = 1$, the ‘‘Hausdorff’’ approximation improves the classical convolution approximations in the sense of rate of convergence.

4.3 Approximation via the Riemann–Liouville Integral

For $\alpha > 0$, the Riemann–Liouville integral is defined as

$$\mathcal{J}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x f(t)(x - t)^\alpha dt = \frac{x^\alpha}{\Gamma(\alpha)} \int_0^x f(t) \left(1 - \frac{t}{x}\right)^\alpha dt.$$

A rescaled version of this operator can be easily obtained as an adjoint Hausdorff operator. Indeed, for $a(t) = 1/t$ and $\varphi_\alpha(t) = (\alpha + 1)(1 - t)^\alpha \chi_{(0,1)}(t)$ (so that $\int_{\mathbb{R}} \varphi_\alpha(t) dt = 1$), we have

$$\begin{aligned} \mathcal{J}^\alpha(f)(x) &:= \mathcal{H}^* f(x) = \int_0^1 f(tx)(1 - t)^\alpha dt = \frac{1}{x} \int_0^1 f(t) \left(1 - \frac{t}{x}\right)^\alpha dt \\ &= \Gamma(\alpha)x^{-\alpha-1} \mathcal{J}^\alpha f(x). \end{aligned}$$

Note that if we formally consider $\alpha = 0$ in the definition of \mathcal{J}^α , we recover the Cesàro operator.

Using Theorem 2.2, we approximate $f(x)$ by

$$\begin{aligned} F_N(x) &= (\mathcal{J}_N^\alpha \widehat{\tau_x f})(0) := (\mathcal{H}_N \widehat{\tau_x f})(0) \\ &= (1 + \alpha) \int_0^1 (1 - t)^\alpha \int_{\mathbb{R}} f\left(x - \frac{s}{N}\right) \frac{\sin(a(t)s)}{s} ds dt; \end{aligned}$$

cf. (3.2). Note that by the observation made in (4.1), we will obtain the same convergence rates via the Riemann–Liouville integral as those we obtain via the Cesàro operator. So, for continuous f , we have

$$\|f - F_N\|_{L^\infty(\mathbb{R})} \lesssim \int_0^1 \frac{\omega(f; \frac{t}{N})}{t} dt,$$

while for $f \in L^p(\mathbb{R})$ with $1 < p < \infty$ (note that for $p = 1$ condition (2.3) does not hold, so we have to exclude such a case), we have

$$\|f - F_N\|_{L^p(\mathbb{R})} \lesssim \int_0^1 \frac{\omega(f; \frac{t}{N})_p}{t} dt$$

by Corollary 4.1 and (4.1).

5 Final Remarks

We conclude with a couple of remarks: first, we show that one can use the same approach to approximate the Hausdorff operator (instead of its adjoint) applied to a function. Secondly, we show that we cannot expect any good approximations of Hausdorff operators if the kernel φ does not decay fast enough at infinity.

5.1 Approximation of Non-adjoint Hausdorff Operators

One can also approximate the Hausdorff operator instead of its adjoint, if one considers the adjoint Hausdorff averages in the approximant. More precisely, it is also possible to approximate $\mathcal{H}f(x)$ by

$$\begin{aligned} (\mathcal{H}_N^* \widehat{f})(x) &= \frac{1}{2\pi} \int_{-N}^N \mathcal{H}^* \widehat{f}(u) e^{iux} du \\ &= \frac{1}{2\pi} \int_{-N}^N \int_{\mathbb{R}} \varphi(t) \int_{\mathbb{R}} f(s) e^{-isu/a(t)} ds dt e^{iux} du \\ &= \frac{1}{\pi} \int_{\mathbb{R}} \varphi(t) \int_{\mathbb{R}} f(s) \frac{\sin N(x - s/a(t))}{x - s/a(t)} ds dt, \end{aligned}$$

which, by substitution, is easily seen to equal

$$\frac{1}{\pi} \int_{\mathbb{R}} \varphi(t) |a(t)| \int_{\mathbb{R}} f\left(a(t)x - \frac{s}{N}\right) \frac{\sin \frac{s}{a(t)}}{s} ds dt.$$

Since for any $t \neq 0$, one has

$$\int_{\mathbb{R}} \frac{\sin \frac{s}{a(t)}}{s} ds = \pi,$$

then

$$\begin{aligned} \mathcal{H}f(x) - (\mathcal{H}_N^* \widehat{f})(x) &= \frac{1}{\pi} \int_{\mathbb{R}} \varphi(t) |a(t)| \int_{\mathbb{R}} \left(f\left(a(t)x - \frac{s}{N}\right) - f(a(t)x) \right) \frac{\sin \frac{s}{a(t)}}{s} ds dt. \end{aligned}$$

A similar estimate to that of [Lemma 3.2](#) can now be proved.

Lemma 5.1 For any $x \in \mathbb{R}$,

$$\begin{aligned} &\pi |\mathcal{H}f(x) - (\mathcal{H}_N^* \widehat{f})(x)| \\ &\leq \int_{\mathbb{R}} \varphi(t) \int_{|s| \leq |a(t)|} \left| f\left(a(t)x - \frac{s}{N}\right) - f(a(t)x) \right| ds dt \\ &\quad + \int_{\mathbb{R}} \frac{1}{s} \int_{|t| \geq |a^{-1}(s)|} \left| f\left(a(t)x - \frac{s}{N}\right) - f(a(t)x) \right| \varphi(t) |a(t)| dt ds. \end{aligned}$$

Proof The proof is essentially the same as that of Lemma 3.2,

$$\begin{aligned}
 & \pi | \mathcal{H}f(x) - (\mathcal{H}_N^* \widehat{f})(x) | \\
 &= \int_{\mathbb{R}} \varphi(t) |a(t)| \int_{|s| \leq |a(t)|} \left(f\left(a(t)x - \frac{s}{N}\right) - f(a(t)x) \right) \frac{\sin \frac{s}{a(t)}}{s} ds dt \\
 & \quad + \int_{\mathbb{R}} \varphi(t) |a(t)| \int_{|s| \geq |a(t)|} \left(f\left(a(t)x - \frac{s}{N}\right) - f(a(t)x) \right) \frac{\sin \frac{s}{a(t)}}{s} ds dt \\
 &\leq \int_{\mathbb{R}} \varphi(t) \int_{|s| \leq |a(t)|} \left| f\left(a(t)x - \frac{s}{N}\right) - f(a(t)x) \right| ds dt \\
 & \quad + \int_{\mathbb{R}} \varphi(t) |a(t)| \int_{|s| \geq |a(t)|} \frac{\left| f\left(a(t)x - \frac{s}{N}\right) - f(a(t)x) \right|}{|s|} ds dt \\
 &\leq \int_{\mathbb{R}} \varphi(t) \int_{|s| \leq |a(t)|} \left| f\left(a(t)x - \frac{s}{N}\right) - f(a(t)x) \right| ds dt \\
 & \quad + \int_{\mathbb{R}} \frac{1}{s} \int_{|t| \geq |a^{-1}(s)|} \left| f\left(a(t)x - \frac{s}{N}\right) - f(a(t)x) \right| \varphi(t) |a(t)| dt ds,
 \end{aligned}$$

as desired. ■

By means of the pointwise estimate from Lemma 5.1, it is possible to obtain approximation results analogous to Theorem 2.1, where the Hausdorff operator, rather than its adjoint, is approximated. The details are essentially the same and are thus omitted.

5.2 A Hausdorff Operator with Slowly Decaying φ : the Bellman Operator

Let us see what happens if we try to approximate an adjoint Hausdorff operator with slowly decaying φ . We consider the particular example of the Bellman operator \mathcal{B} (which is nothing more than the adjoint Cesàro operator \mathcal{C}^*). Its adjoint \mathcal{B}^* is defined by letting $a(t) = 1/t$ and $\varphi(t) = t^{-1}\chi_{(1,\infty)}(t)$ in (2.4):

$$\mathcal{B}^* f(x) = \int_1^\infty \frac{f(tx)}{t} dt = \frac{1}{x} \int_x^\infty \frac{f(t)}{t} dt.$$

It is clear that we cannot use the methods from Section 4 in order to approximate functions, since the hypothesis $\varphi \in L^1(\mathbb{R})$ is not satisfied in this example. What is more, not even the basic assumption (2.3) from Theorem 2.1 is satisfied for any $1 \leq p \leq \infty$. Nevertheless, we now try to use the approximation estimates from Theorem 2.1 (heuristically, since the hypotheses of Theorem 2.1 are not met) just to illustrate their bad behaviour for functions φ that do not decay fast enough. As the approximant for \mathcal{B}^* , we take

$$(\mathcal{B}_N \widehat{f})(x) := \frac{1}{\pi} \int_1^\infty \frac{1}{t^2} \int_{\mathbb{R}} f(s) \frac{\sin N(x - s/t)}{x - s/t} ds dt.$$

For $1 \leq p < \infty$, the estimate from [Theorem 2.1](#) yields

$$\begin{aligned}
 & \| \mathcal{B}^* f - (\mathcal{B}_N \widehat{f}) \|_{L^p(\mathbb{R})} \\
 & \lesssim \int_1^\infty \frac{\omega(f; \frac{t}{N})_p}{t^{1+1/p}} dt + \int_{\mathbb{R}} \frac{\omega(f; \frac{|s|}{N})_p}{|s|} \int_{|s| \leq |t|} \frac{1}{t^{1+1/p}} \chi_{(1, \infty)}(t) dt ds \\
 (5.1) \quad & \asymp \int_0^\infty \frac{\omega(f; \frac{|s|}{N})_p}{|s|} ds,
 \end{aligned}$$

while in the case $p = \infty$,

$$\begin{aligned}
 & \pi \| \mathcal{B}^* f - (\mathcal{B}_N \widehat{f}) \|_{L^\infty(\mathbb{R})} \\
 & \lesssim \int_1^\infty \frac{\omega(f; \frac{t}{N})}{t} dt + \int_{\mathbb{R}} \frac{\omega(f; \frac{|s|}{N})}{|s|} \int_{|s| \leq |t|} \frac{1}{t} \chi_{(1, \infty)}(t) dt ds = \infty;
 \end{aligned}$$

i.e., in this case we cannot guarantee any convergence on the L^p norm by using our estimates, even for well-behaved functions f . As was pointed out in [Remark 2.3](#), this is because in order to obtain useful estimates from [Theorem 2.1](#), one should assume that φ is of compact support, or that it decays fast enough as $|t| \rightarrow \infty$. For the adjoint Cesàro operator, the functions φ has some decay, but it is not fast enough. Also note that the estimate (5.1) is not good, as the right-hand side is infinite for nonconstant functions.

Acknowledgment The authors would like to thank the referee for the comments and kind suggestions, which certainly improved the quality of this paper.

References

- [1] K. Andersen, *Boundedness of Hausdorff operators on $L^p(\mathbb{R}^n)$, $H^1(\mathbb{R}^n)$, and $BMO(\mathbb{R}^n)$* . Acta Sci. Math. (Szeged) 69(2003), 409–418.
- [2] L. Aizenberg and E. Liflyand, *Hardy spaces in Reinhardt domains, and Hausdorff operators*. Illinois J. Math. 53(2009), 1033–1049.
- [3] G. Brown and F. Móricz, *Multivariate Hausdorff operators on the spaces $L^p(\mathbb{R}^n)$* , J. Math. Anal. Appl. 271(2002), 443–454.
- [4] V. Burenkov and E. Liflyand, *Hausdorff operators on Morrey-type spaces*. Kyoto J. Math. 60(2020), 93–106. <http://dx.doi.org/10.1215/21562261-2019-0035>
- [5] P. L. Butzer and R. J. Nessel, *Fourier analysis and approximation. Volume 1: One-dimensional theory*. Pure and Applied Mathematics, 40, Academic Press, New York-London, 1971.
- [6] J. Chen, D. Fan, and S. Wang, *Hausdorff operators on Euclidean spaces*. Appl. Math. J. Chinese Univ. (Ser. B) 28(2014), 548–564. <http://dx.doi.org/10.1007/s11766-013-3228-1>
- [7] M. Dyachenko, E. Nursultanov, and S. Tikhonov, *Hardy-Littlewood and Pitt's inequalities for Hausdorff operators*. Bull. Sci. Math. 147(2018), 40–57. <http://dx.doi.org/10.1016/j.bulsci.2018.06.003>
- [8] Y. Kanjin, *The Hausdorff operators on the real Hardy spaces $H^p(\mathbb{R})$* . Studia Math. 148(2001), 37–45. <http://dx.doi.org/10.4064/sm148-1-4>
- [9] J. C. Kuang, *Generalized Hausdorff operators on weighted Morrey-Herz spaces*. Acta Math. Sinica (Chin. Ser.) 55(2012), 895–902.
- [10] J. C. Kuang, *Generalized Hausdorff operators on weighted Herz spaces*. Mat. Vesnik 66(2014), 19–32.
- [11] C. Georgakis, *The Hausdorff mean of a Fourier-Stieltjes transform*, Proc. Am. Math. Soc. 116(1992), 465–471. <http://dx.doi.org/10.2307/2159753>
- [12] A. Lerner and E. Liflyand, *Multidimensional Hausdorff operator on the real Hardy space*. J. Austr. Math. Soc. 83(2007), 79–86. <http://dx.doi.org/10.1017/S1446788700036399>

- [13] E. Liflyand, *Hausdorff operators on Hardy spaces*. Eurasian Math. J. 4(2013), 101–141.
- [14] E. Liflyand, *Open problems on Hausdorff operators*. In: Complex analysis and potential theory, World. Sci. Publ., Hackensack, NJ, 2007, pp. 280–285.
http://dx.doi.org/10.1142/9789812778833_0030
- [15] E. Liflyand and A. Miyachi, *Boundedness of the Hausdorff operators in H^p spaces, $0 < p < 1$* . Studia Math. 194(2009), 279–292. <http://dx.doi.org/10.4064/sm194-3-4>
- [16] E. Liflyand and A. Miyachi, *Boundedness of multidimensional Hausdorff operators in H^p spaces, $0 < p < 1$* , Trans. Amer. Math. Soc. 371(2019), 4793–4814. <http://dx.doi.org/10.1090/tran/7572>
- [17] E. Liflyand and F. Móricz, *The Hausdorff operator is bounded on the real Hardy space $H^1(\mathbb{R})$* . Proc. Am. Math. Soc. 128(2000), 1391–1396. <http://dx.doi.org/10.1090/S0002-9939-99-05159-X>
- [18] A. R. Mirotin, *Boundedness of Hausdorff operators on real Hardy spaces H^1 over locally compact groups*. J. Math. Anal. Appl. 473(2019), 519–533. <http://dx.doi.org/10.1016/j.jmaa.2018.12.065>
- [19] H. Triebel, *Theory of function spaces*. Monographs in Mathematics, 78, Birkhäuser, Basel, 1983.
<http://dx.doi.org/10.1007/978-3-0346-0416-1>

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