On Riemann–Hilbert problem and multiple high-order pole solutions to the cubic Camassa–Holm equation

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In this work, the Riemann–Hilbert (RH) problem is employed to study the multiple high-order pole solutions of the cubic Camassa–Holm (cCH) equation with the term characterizing the effect of linear dispersion under zero boundary conditions and nonzero boundary conditions. Under the reflectionless situation, we generalize the residue theorem and obtain the multiple high-order pole solutions of cCH equation by solving an algebraic system. During the process of establishing the solution of RH problem, to simplify the calculations involving the implicitly expressed of variables (x, t) in the solution, we introduce a new scale (y, t) to ensure the solution of RH problem is explicitly expressed with respect to it. Finally, the exact solutions are obtained for cases involving one high-order pole and N high-order poles.

1. Introduction

It is widely recognized that the Camassa–Holm (CH) equation

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \qquad (1.1)$$

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is derived by Camassa and Holm [8] through an asymptotic expansion of the Hamiltonian for Euler equation in the shallow water regime. This equation has garnered considerable interest among scholars due to its ability to describe various physical phenomena. As a model of shallow water waves, the CH equation (Equation 1.1) possesses an integrable infinite dimensional bi-Hamiltonian structure [11, 16], which is a generalization of the Korteweg–de Vries (KdV) equation [39]. In addition to this, it is also provided with an infinite hierarchy of local conservation laws [3, 13, 35, 42] and the peaked solitons [12, 37]. As research on the CH equation continues to advance, numerous generalized forms of the equation have been proposed and classified [32]. A modified Camassa–Holm (mCH) equation was firstly defined with the method of geometric by Górka and Reyes [18]. The local well-posedness for the Cauchy problem of this equation has been investigated [27]. The Fokas–Olver–Rosenau–Qiao equation which is also known as the mCH equation with cubic nonlinearity has sparked wide research. For convenience, due to the cubic nonlinearity of the model, we call this equation the cubic Camassa–Holm (cCH) equation to distinguish the already existing mCH equation [18] throughout the article.

The cCH equation takes on the form

$$m_t + ((u^2 - u_x^2)m)_x = 0, \quad m = u - u_{xx}. \tag{1.2}$$

It was discovered by Fokas [14], Olver and Rosenau [33] and derived by Fuchssterner [17], Qiao [36], and Novikov [32]. It is derived as an integrable system by a conventional method of the tri-Hamiltonian duality [14, 17]. This allows for a deeper understanding of its integrability and mathematical properties. For instance, it is completely integrable and admits a Lax pair [6, 7, 34]. It is worth mentioning that the multiple soliton solutions of cCH equation with the solution approaching a nonzero constant at both infinities of the space variable has been solved by Riemann–Hilbert (RH) method [5, 43]. The smooth and singular multisoliton solutions of cCH equation with linear dispersion were solved in [31] by the bilinear transformation method under the rapidly decreasing boundary condition. The integrable semi-discretization problem and the N-soliton solutions of this equation also have been investigated in [38] through Hirota's bilinear approach. In addition, the peaked soliton solutions, kink-peakon interactional solutions, and weak kink solutions have been studied [21, 22, 26, 44]. Furthermore, the multi-soliton solutions of the cCH equation have been calculated by the Darboux transformation [45] and the RH approach has been developed for studying the cCH equation with step-like initial data [25]. Based on its physical background, plenty of important results have been given by scholars [9, 15, 19, 23, 30].

In this work, our focus is on the cCH equation including a linear dispersion term

$$\begin{cases} m_t + ((u^2 - u_x^2)m)_x + \nu u_x = 0, \quad m = u - u_{xx}, \\ u(x,0) = u_0(x), \end{cases} \tag{1.3}$$

with zero boundary conditions (ZBCs)

$$\lim_{x \to \pm\infty} u_0(x) = 0, \tag{1.4}$$

and nonzero boundary conditions (NZBCs)

$$\lim_{x \to \pm\infty} u_0(x) = u_\pm, \quad u_\pm > 0. \tag{1.5}$$

For convince, we assume that $u_\pm = 1$. ν is a positive constant which characterizes the effect of the linear dispersion and the function $u = u(x,t)$ is a function of time variable t and spatial variable x. The subscripts x and t appended to the functions m and u mean the partial differentiation of them.

Equation (Equation 1.3) was initially introduced by Fokas [14]. Later on, the Lax integrability bi-Hamiltonian structure and conservation laws of it was firstly proposed by Qiao, Xia, and Li [44]. Subsequently, Wang and Qiao [43] established the RH problem of the FQXL model

$$m_t = \tfrac{1}{2}k_1 \left[m \left(u^2 - u_x^2 \right) \right]_x + \tfrac{1}{2}k_2 \left(2mu_x + m_x u \right), \ m = u - u_{xx}, \tag{1.6}$$

which is in fact equivalent to Eq. (Equation 1.3) when letting $k_1 = -2$ and $k_2 = 0$. The RH problem with some important properties of the cCH equation has been investigated in references [46, 48], then by adopting nonlinear steepest descent method and $\bar{\partial}$ steepest descent method, the long-time asymptotic behaviours of the solution with Schwartz initial data and weighted Sobolev initial data have been studied respectively. In an effort to obtain more properties of soliton solutions, this article goes deeply into the high-order pole solutions for the cCH equation based on the RH problem constructed in these two references. Due to the high-order negative powers at the poles, particularly for negative order equations, it can be challenging to construct solutions using general residue conditions. So this issue has not yet been fully resolved. Additionally, studying high-order solitons and high-order dispersion in relation to the cCH equation can lead to the discovery of new phenomena and important properties, making it a research area of great significance.

The inverse scattering transform (IST) method was introduced as a specialized calculation adapted to the initial-value problem for the KdV equation [10]. Through extended by some researchers, it can be used to solve the Fourier transform [2, 20, 49]. This method has been recognized as an important tool for solving various problems in mathematics, physics, and other fields. Meanwhile, it is particularly useful in obtaining the exact solutions for nonlinear wave equations by performing a series of linear computations. The RH problem, as a simplified modern version of IST, is a forceful tool to deal with the initial-boundary value problems of integrable systems. Its characteristic is that the solution of the corresponding RH problem can recover the reconstruction formula of the solution to the differential equation. There are numerous equations solved by RH approach when the space variable of solutions approach to zero or nonzero constant at infinities. Such as, the nonlinear Schrödinger (NLS) equation [4, 40, 47], the coupled modified KdV equation [24, 28, 41], the derivative NLS equation [51], the nonlocal reverse-space-time NLS equation [1], and the nonlocal real reverse-space-time mKdV equation [29].

The production and propagation of solitons play a crucial role in optical fiber communication. It is well-known that high-order soliton solutions are primarily obtained from high-order poles. The high-order soliton can be applied to describe a weak bound state of solitons and may appear in the study of train propagation of

solitons with almost equivalent velocities and amplitudes. Therefore, studying high-order solitons and high-order dispersion is of utmost importance in this field. Taking advantage the RH method, through a standard dressing procedure, the soliton matrices of the high-order poles for the third-order flow Gerdjikov–Ivanov equation are constructed in [52]. Nevertheless, when dealing with the RH problem with N-order poles, it is necessary to clearly give the corresponding residue conditions which are very complex and difficulty.

Recently, a new method can solve the RH problem with high-order poles are developed [50]. By using Laurent expansion and Taylor series, we generalize the residue theorem and get out of the complicated calculations, then the multiple high-order pole solitons are established. Hence, comparing to the work of [43], we not only provide the multi-soliton solutions of (Equation 1.3) but also consider the multiple high-order pole solutions with both ZBCs and NZBCs. In addition to study Ablowitz-Kaup-Newell-Suger (AKNS) hierarchy, we extend the RH method to study the non-AKNS-type integrable cCH equation with ZBCs and NZBCs and give the exact expressions of soliton solutions with one high-order pole and multiple high-order poles. We generalize the results presented in [5] after taking into account the impact of additional a linear dispersion term νu_x. The Lax pair in [5] is the simplest special case of our Lax pair. When the parameter ν is equal to zero and the orders of the poles are all equal to one, the solutions we obtained under NZBCs can degenerate into the solution in [5]. These distinctions make our work more general and involve more complex analysis. It is worth noting that there are some differences from that on cCH equation (Equation 1.2) as mentioned in the following aspects.

(I) Since the cCH equation we have researched has a linear dispersion term νu_x, the Lax pair has a more complex form regarding the negative power of spectral parameter. Especially for the NZBCs, due to the deformation of the potential function, the Lax pair undergoes corresponding deformation as well. Then the suitable uniformization variable μ should be introduced. This gives rise to new spectral singularities that pose challenges for subsequent analysis.

(II) The cCH equation belongs to the family of negative-order equations and the uniformization variable μ is introduced, there are some singularities in the Lax pair. Under ZBCs, the cCH equation exhibits singularities at points $\mu = 0$, $\mu = \infty$ and branch cut points $\mu = \pm i$. Under NZBCs, new singularities arise at points $\mu = \pm 1$. Analysing the asymptotic properties of these points is crucial because they significantly impact the construction of the RH problem.

(III) Under NZBCs, based on the relationship $y(x,t) = x - \int_x^{+\infty}(\hat{m}(\xi,t)-1)d\xi$ between variables x and y and considering the condition $\hat{X}_0(x,t,\pm i) \equiv 0$, we study the scattering data $a(\mu)$ and $b(\mu)$ about the above relationship. Then the implicit expression of the potential function u is given by the solution of RH problem.

(IV) Under NZBCs, in order to give the explicit expression of the potential function u of cCH equation, we need to use the compatibility condition $\tilde{U}_t - \tilde{V}_y + [\tilde{U}, \tilde{V}] = 0$ and expand $\tilde{U}, \tilde{V}$ at singular points.

(V) The function $a(\mu)$ is a Wronskian of analytic Jost solutions, which vanishes to N-order at the N-order zero points, we obtain a linear expression for the Nth derivative of Jost solutions. Then by utilizing Laurent expansion of $e^{\pm 2it\theta(\mu)}$, the multiple high-order pole solutions of cCH equation under ZBCs and NZBCs are established.

The outline of this work is arranged as follows. In §2, the spectrum problem of the cCH equation with ZBCs is analysed in detail. By introducing the Jost solutions, the analytical, asymptotic, and symmetric properties are gained. The cCH equation has singularities at points $\mu = 0$, $\mu = \infty$, and $\mu = \pm i$, through introducing suitable transformations to research these singularities, we can construct RH problem by a new scale (y, t) and recover the potential function at point $\mu = i$. Then, the expression of the solution with one N-order pole and multiple N-order poles are given, respectively. Finally, we give some examples analysing the dynamic behaviours of one-soliton solutions and two-order pole solutions. We compare the solution obtained by the RH method and the bilinear method, providing the expression for the solutions, and analysing them in conjunction with the figures.

In §3, by the same processing method, the RH problem of the cCH equation with NZBCs is established. Note that in addition to the above singularities, the points $\mu = \pm 1$ also are singularities, we need to consider the asymptotic properties of the eigenfunctions and scattering data at points $\mu = \pm 1$. After that, the uniqueness of the solution about cCH equation with the new scale (y, t) is given. On the basis of proving the theory that the data of the RH problem are independent of some specific initial data of the cCH equation, we give the parametric representation of the solution. Finally, the expression of the solutions with one N-order pole and multiple N-order poles are derived, respectively. Furthermore, we analyse the dynamic behaviours of solutions by selecting special parameters.

2. RH problem and high-order pole under ZBCs

2.1. Spectral analysis

It is well-known that cCH equation (Equation 1.3) admits the following Lax pair

$$\phi_x = U\phi, \quad \phi_t = V\phi, \tag{2.1}$$

where $U = U(x, t; \lambda)$ and $V = V(x, t; \lambda)$ are 2×2 matrices defined by

$$U = \begin{pmatrix} -\frac{Q}{2} & \frac{\lambda m}{2} \\ -\frac{\lambda m}{2} & \frac{Q}{2} \end{pmatrix}, \quad V = \begin{pmatrix} \frac{Q}{\lambda^2} + \frac{QR}{2} & -\frac{u - Qu_x}{\lambda} - \frac{\lambda Rm}{2} \\ \frac{u + Qu_x}{\lambda} + \frac{\lambda Rm}{2} & -\frac{Q}{\lambda^2} - \frac{QR}{2} \end{pmatrix}, \tag{2.2}$$

with

$$Q = Q(\nu, \lambda) = \sqrt{1 - \frac{\nu \lambda^2}{2}}, \qquad R = u^2 - u_x^2. \tag{2.3}$$

It can be seen that under the following transformations

$$x = \tilde{x}, \quad t = \frac{2}{\nu}\tilde{t}, \quad u(x,t) = \sqrt{\frac{\nu}{2}}\tilde{u}(\tilde{x}, \tilde{t}), \tag{2.4}$$

Eq. (Equation 1.3) turns to

$$\tilde{m}_{\tilde{t}} + (\tilde{m}\tilde{R})_{\tilde{x}} + 2\tilde{u}_{\tilde{x}} = 0, \quad \tilde{m} = \tilde{u} - \tilde{u}_{\tilde{x}\tilde{x}}, \quad \tilde{R} = \tilde{u}^2 - \tilde{u}_{\tilde{x}}^2. \tag{2.5}$$

Without loss of generality, we pick $\nu = 2$. To get rid of the multi-value problem of square roots, we introduce a new transformation

$$Q = \frac{i}{2}(\mu - \frac{1}{\mu}), \quad \lambda = \frac{1}{2}(\mu + \frac{1}{\mu}). \tag{2.6}$$

After doing a gauge transformation

$$\psi = P^{-1}\phi e^{J}, \tag{2.7}$$

with

$$P = P(x,t) = \sqrt{\frac{q+1}{2q}}\begin{pmatrix} 1 & -\frac{im}{q+1} \\ -\frac{im}{q+1} & 1 \end{pmatrix}, \tag{2.8}$$

$$\delta = x - \int_x^{+\infty}(q-1)dy - \frac{2t}{\lambda^2}, \quad q = \sqrt{m^2 + 1}, \tag{2.9}$$

where we define $J = \frac{1}{2}Q\delta\sigma_3$. The asymptotic condition is shown as follows

$$\psi(x,t,\mu) \sim I, \quad x \to \pm\infty.$$

Equation (Equation 1.3) has the following form Lax pair

$$\psi_x + [J_x, \psi] = X\psi, \tag{2.10a}$$

$$\psi_t + [J_t, \psi] = T\psi, \tag{2.10b}$$

where

$$X = \frac{im_x}{2q^2}\sigma_1 + \frac{m}{2\mu q}\begin{pmatrix} -im & 1 \\ -1 & im \end{pmatrix},$$

$$T = \frac{im_t}{2q^2}\sigma_1 - \frac{mR}{2\mu q}\begin{pmatrix} -im & 1 \\ -1 & im \end{pmatrix} + \frac{(\mu^2-1)u_x}{\mu^2+1}\sigma_1 - \frac{2\mu u}{(\mu^2+1)q}\begin{pmatrix} -im & 1 \\ -1 & im \end{pmatrix}$$
$$+ \frac{2i\mu(\mu^2-1)}{(\mu^2+1)^2}\begin{pmatrix} \frac{1}{q}-1 & -\frac{im}{q} \\ \frac{im}{q} & 1-\frac{1}{q} \end{pmatrix},$$

there $[A, B] = AB - BA$ and the notations $J = J(x, t, \mu)$, $X = X(x, t, \mu)$, $T = T(x, t, \mu)$ are 2×2 matrices. The σ_1 is Pauli matrix and the Pauli matrices are that

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The two Jost solutions of (Equation 2.10a) are completely determined by Volterra integrable equations

$$\psi_\pm(x, t, \mu) = I + \int_{\pm\infty}^x e^{-\frac{i}{4}(\mu - \frac{1}{\mu})\int_y^x q d\tau \hat{\sigma}_3} [X(y, t, \mu)\psi_\pm(y, t, \mu)] \, dy, \qquad (2.11)$$

where $e^{A\hat{\sigma}_3}B = e^{A\sigma_3}Be^{-A\sigma_3}$. They have the analytic properties, symmetry properties, and asymptotic properties shown in the following three propositions.

PROPOSITION 2.1. *Providing that* $u_0(x) \in L^1(\mathbb{R}^\pm)$, *the Jost eigenfunctions* $\psi_\pm(x, t, \mu)$ *have the following analytic properties, that is,* $\psi_{-,1}(x, t, \mu)$ *and* $\psi_{+,2}(x, t, \mu)$ *are analytic in* $\mathbb{C}^+$, $\psi_{+,1}(x, t, \mu)$ *and* $\psi_{-,2}(x, t, \mu)$ *are analytic in* $\mathbb{C}^-$, *there* $\psi_{\pm,j}(x, t, \mu)$ $(j = 1, 2)$ *represent the j-th column of* $\psi_\pm(x, t, \mu)$, $\mathbb{C}^+ = \{\mu \in \mathbb{C} | Im\mu > 0\}$ *and* $\mathbb{C}^- = \{\mu \in \mathbb{C} | Im\mu < 0\}$.

PROPOSITION 2.2. *The Jost eigenfunctions* $\psi_\pm(x, t, \mu)$ *have the following symmetry conditions*

$$\psi_\pm(\mu) = \sigma_1 \psi_\pm^*(-\frac{1}{\mu^*})\sigma_1 = \sigma_2 \psi_\pm^*(\mu^*)\sigma_2 = \frac{1}{P^2}\sigma_2 \psi_\pm(-\frac{1}{\mu})\sigma_2. \qquad (2.12)$$

PROPOSITION 2.3. *The Jost eigenfunctions* $\psi_\pm(x, t, \mu)$ *possess asymptotic behaviour in the* μ-*plane*

$$\psi_\pm(x, t, \mu) = I + \mathcal{O}(\frac{1}{\mu}), \quad \mu \to \pm\infty, \qquad (2.13)$$

and according to the symmetry relations of $\psi_\pm(x, t, \mu)$, *there are the same relations when* $\mu \to 0$.

It can be seen that $\phi_\pm(x, t, \mu)$ are two fundamental matrix solutions of Lax pair (Equation 2.1). As a result, there exists a matrix $S(\mu)$ which independent of variables x and t satisfies

$$\phi_-(x, t, \mu) = \phi_+(x, t, \mu)S(\mu), \qquad (2.14)$$

then on the basis of transformation (Equation 2.7), the relation (Equation 2.14) can be written in the following form

$$\psi_-(x, t, \mu)e^{-J(x,t,\mu)} = \psi_+(x, t, \mu)e^{-J(x,t,\mu)}S(\mu), \qquad (2.15)$$

where $S(\mu)$ is a 2×2 matrix. We can derive the symmetries of $S(\mu)$ from proposition 2.2 as follows

$$S(\mu) = S^*(\frac{1}{\mu^*}) = \sigma_3 S(-\frac{1}{\mu})\sigma_3 = \sigma_2 S^*(\mu^*)\sigma_2. \tag{2.16}$$

According to symmetry relation (Equation 2.12), we can define the matrix $S(\mu)$ as the following form

$$S(\mu) = \begin{pmatrix} a(\mu) & -b^*(\mu^*) \\ b(\mu) & a^*(\mu^*) \end{pmatrix}. \tag{2.17}$$

Because $tr(U) = 0$, based on Abel theorem, we have $\det(\psi_\pm) = 1$, then $\det(S(\mu)) = 1$. By directly calculating Eq. (Equation 2.15), $a(\mu)$ and $b(\mu)$ can be expressed as

$$a(\mu) = Wr(\psi_{-1}, \psi_{+2}), \quad b(\mu) = Wr(\psi_{+1}, \psi_{-1}). \tag{2.18}$$

We define the reflection coefficient

$$\rho(\mu) = \frac{b(\mu)}{a(\mu)}. \tag{2.19}$$

Taking into account the properties of the Jost eigenfunctions $\psi_\pm(x, t, \mu)$, we obtain the corresponding properties to the elements $a(\mu)$ and $b(\mu)$ of the scattering matrix $S(\mu)$ as follows.

PROPOSITION 2.4. *The scattering data $a(\mu)$ and $b(\mu)$ satisfy the following properties*

(i) *$a(\mu)$ can be analytically continued in $\mathbb{C}^+$, and $b(\mu)$ is continuous in $\mathbb{R}$;*
(ii) *$a(\mu) = 1 + \mathcal{O}(\frac{1}{\mu})$ and $b(\mu) \to 0$ as $\mu \to \infty$, and from the symmetry relations of $\psi_\pm(x, t, \mu)$, there are the same relations when $\mu \to 0$;*
(iii) *the reflection coefficient $\rho(\mu)$ possesses the symmetry relations*

$$\rho(\mu) = \rho^*(\frac{1}{\mu^*}) = -\rho(-\frac{1}{\mu}) = -\rho^*(-\mu^*). \tag{2.20}$$

It is noticed that, since the branch cut points $\mu = \pm i$ exist in the extended complex μ-plane, the asymptotic behaviour of eigenfunctions $\psi_\pm(x, t, \mu)$ ought to be considered as $\mu \to \pm i$. Introducing a new Jost function

$$\tilde{\psi}_\pm(x, t, \mu) = \phi_\pm(x, t, \mu)e^{(\frac{Q}{2}x - \frac{Q}{\lambda^2}t)\sigma_3}, \tag{2.21}$$

then we have

$$\tilde{\psi}_\pm(x, t, \mu) \sim I, \quad x \to \pm\infty. \tag{2.22}$$

The Lax pair (Equation 2.1) can be converted to

$$(\tilde{\psi}_\pm)_x = -\frac{Q}{2}[\sigma_3, \tilde{\psi}_\pm] + \tilde{X}\tilde{\psi}_\pm, \qquad (2.23a)$$

$$(\tilde{\psi}_\pm)_t = \frac{Q}{\lambda^2}[\sigma_3, \tilde{\psi}_\pm] + \tilde{T}\tilde{\psi}_\pm, \qquad (2.23b)$$

with

$$\tilde{X} = \begin{pmatrix} 0 & \frac{\lambda m}{2} \\ -\frac{\lambda m}{2} & 0 \end{pmatrix},$$

$$\tilde{T} = \frac{R}{2}\begin{pmatrix} Q & -\lambda m \\ \lambda m & -Q \end{pmatrix} + \frac{u - Qu_x}{\lambda}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Taking into account the asymptotic expansion of $\tilde{\psi}$ at $\mu \to i$, we have

$$\tilde{\psi} = I + \tilde{\psi}_1(\mu - i) + \mathcal{O}((\mu - i)^2), \quad \mu \to i, \qquad (2.24)$$

with

$$\tilde{\psi}_1 = \begin{pmatrix} 0 & -\frac{1}{2}(u + u_x) \\ -\frac{1}{2}(u - u_x) & 0 \end{pmatrix}. \qquad (2.25)$$

According to the transformations (Equation 2.7) and (Equation 2.21), we have

$$\psi_\pm(x, t, \mu) = P^{-1}(x, t)\tilde{\psi}_\pm(x, t, \mu)e^{\frac{1}{2}Q(\mu)\int_{\pm\infty}^x (q-1)dy}. \qquad (2.26)$$

Taking the limits of (Equation 2.26) as $\mu \to i$, we obtain the asymptotic of $a(\mu)$

$$a(\mu) = e^{-\frac{1}{2}\int_{-\infty}^{+\infty}(q-1)dx}(1 + \mathcal{O}((\mu - i)^2)), \quad \mu \to i. \qquad (2.27)$$

The analytic properties of $\psi_\pm(x, t, \mu)$ stated above allow rewriting the relation (Equation 2.15) as a sectionally meromorphic matrix.

2.2. RH problem

Define a piecewise meromorphic 2×2-matrix valued function

$$M(\mu) = M(x, t; \mu) := \begin{cases} \left(\frac{\psi_{-,1}(x,t;\mu)}{a(\mu)}, \psi_{+,2}(x, t; \mu)\right), & \text{as } \mu \in \mathbb{C}^+, \\ \left(\psi_{+,1}(x, t; \mu), \frac{\psi_{-,2}(x,t;\mu)}{a^*(\mu^*)}\right), & \text{as } \mu \in \mathbb{C}^-. \end{cases} \qquad (2.28)$$

RHP 2.5. *Find a matrix $M(\mu) = M(x, t, \mu)$ satisfies the following conditions*

(i) Analyticity: the matrix $M(\mu)$ is a meromorphic function in $\mathbb{C}\backslash\mathbb{R}$.

(ii) *Asymptotic behaviour:*

$$M(\mu) = I + \mathcal{O}(\frac{1}{\mu}), \quad \mu \to \infty, \tag{2.29}$$

$$M(\mu) = P^{-1}[I + \tilde{\psi}_1(\mu - i)]e^{\frac{1}{2}\int_{+\infty}^{x}(q-1)dy\sigma_3} + \mathcal{O}((\mu - i)^2), \quad \mu \to i. \tag{2.30}$$

(iii) *Jump condition:*

$$M_+(\mu) = M_-(\mu)G(\mu), \quad \mu \in \mathbb{R}, \tag{2.31}$$

where

$$G(\mu) = \begin{pmatrix} 1 + |\rho(\mu)|^2 & e^{-Q\delta}\rho^*(\mu) \\ e^{Q\delta}\rho(\mu) & 1 \end{pmatrix}. \tag{2.32}$$

(iv) *Symmetries:*
$$M(\mu) = \sigma_3 M^*(-\mu^*)\sigma_3 = \sigma_2 M^*(\mu^*)\sigma_2 = P^{-2}\sigma_1 M^*(-\tfrac{1}{\mu^*})\sigma_1.$$

Since the function $\delta(x, t, \mu)$ has integral term, which directly lead to the difficulty to solve Eq. (Equation 1.3), a new space variable is introduced as follows

$$y(x, t) = x - \int_x^{+\infty}(q(s, t) - 1)ds. \tag{2.33}$$

Defining the function $\tilde{M}(\mu)$ and phase function $\theta(\mu)$ on the new scale $y(x, t)$

$$\tilde{M}(\mu) = \tilde{M}(y, t, \mu) = M(x(y, t), t, \mu), \tag{2.34}$$

$$\theta(\mu) = \frac{1}{2}iQ(\mu)\left(\frac{y}{t} - \frac{2}{\lambda^2(\mu)}\right), \tag{2.35}$$

then, the RH problem 2.5 is transformed as follows.

RHP 2.6. *Find a matrix $\tilde{M}(\mu) = \tilde{M}(y, t, \mu)$ satisfies the following conditions*

(i) *Analyticity: the matrix $\tilde{M}(\mu)$ is a meromorphic function in $\mathbb{C}\backslash\mathbb{R}$.*
(ii) *Asymptotic behaviour:*

$$\tilde{M}(\mu) = I + \mathcal{O}(\frac{1}{\mu}), \quad \mu \to \infty, \tag{2.36}$$

$$\tilde{M}(\mu) = P^{-1}[I + \tilde{\psi}_1(\mu - i)]e^{\frac{1}{2}\int_{+\infty}^{x}(q-1)dy\sigma_3} + \mathcal{O}((\mu - i)^2), \quad \mu \to i. \tag{2.37}$$

(iii) Jump condition:

$$\tilde{M}_+(\mu) = \tilde{M}_-(\mu)V(\mu), \quad \mu \in \mathbb{R}, \tag{2.38}$$

where

$$V(\mu) = \begin{pmatrix} 1 + |\rho(\mu)|^2 & e^{2it\theta(\mu)}\rho^*(\mu) \\ e^{-2it\theta(\mu)}\rho(\mu) & 1 \end{pmatrix}. \tag{2.39}$$

(iv) Symmetries:

$$\tilde{M}(\mu) = \sigma_3\tilde{M}^*(-\mu^*)\sigma_3 = \sigma_2\tilde{M}^*(\mu^*)\sigma_2 = P^{-2}\sigma_1\tilde{M}^*(-\tfrac{1}{\mu^*})\sigma_1.$$

Using the asymptotic property of function $\tilde{M}(\mu)$ as $\mu \to i$, the relationship between the solution of cCH equation and the RH problem is obtained

$$u(x,t) = \lim_{\mu \to i} \frac{1}{\mu - i}\left(1 - \frac{(\tilde{M}_{11}(\mu) + \tilde{M}_{21}(\mu))(\tilde{M}_{12}(\mu) + \tilde{M}_{22}(\mu))}{(\tilde{M}_{11}(i) + \tilde{M}_{21}(i))(\tilde{M}_{12}(i) + \tilde{M}_{22}(i))}\right), \tag{2.40}$$

with

$$x(y,t) = y - \ln\left(\frac{\tilde{M}_{12}(i) + \tilde{M}_{22}(i)}{\tilde{M}_{11}(i) + \tilde{M}_{21}(i)}\right), \tag{2.41}$$

where $\tilde{M}_{ij}(\mu)$ $(i,j = 1,2)$ represents the element in raw i and column j of $\tilde{M}(\mu)$.

2.3. Single high-order pole solutions

In order to obtain a pure soliton solution, this section we will consider the reflectionless situation, i.e., $b(\mu) = 0$. Then $V(\mu) = 0$ for $\mu \in \mathbb{R}$. We assume that $\mu_0 \in \mathbb{C}^+$ is the only one N-order zero point of the scattering data $a(\mu)$, then the N-order zero points $\left\{-\mu_0^*, -\frac{1}{\mu_0}, \frac{1}{\mu_0^*}\right\} \in \mathbb{C}^+$ are also obtained from the symmetry of $a(\mu)$. Additionally, $\left\{\mu_0^*, -\mu_0, -\frac{1}{\mu_0^*}, \frac{1}{\mu_0}\right\} \in \mathbb{C}^-$ are the N-order zero points of the scattering data $a^*(\mu^*)$.

The discrete spectrum is the set $\mathcal{Z} = \left\{\pm\mu_0, \pm\mu_0^*, \pm\frac{1}{\mu_0}, \pm\frac{1}{\mu_0^*}\right\}$, which can be shown in Figure 1. After taking $v_1 = \mu_0$, $v_2 = \frac{1}{\mu_0^*}$, the scattering data $a(\mu)$ can be expressed as

$$a(\mu) = \prod_{j=1}^{2}(\mu - v_j)^N(\mu + v_j^*)^N a_0(\mu), \tag{2.42}$$

where $a_0(\mu) \neq 0$ for all $\mu \in \mathbb{C}^+$. Therefore, from the definition of $\tilde{M}(y,t,\mu)$, it can be seen that $\mu = v_j$ and $\mu = -v_j^*$ $(j = 1,2)$ are N-order pole points of $\tilde{M}_{11}$. Simultaneously, $\mu = v_j^*$ and $\mu = -v_j$ $(j = 1,2)$ are N-order pole points of $\tilde{M}_{12}$.
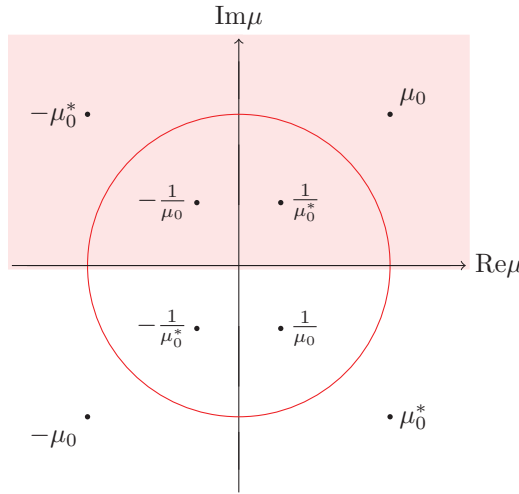
Figure 1. Distribution of the discrete spectrum μ.

Based on the definitions of (Equation 2.14) and (Equation 2.15), under the new scale $y(x,t)$, we define $\breve{\phi}_\pm(y,t,\mu) = \phi_\pm(x,t,\mu)$ and $\breve{\psi}_\pm(y,t,\mu) = \psi_\pm(x,t,\mu)$ and obtain the following relationships

$$\breve{\phi}_-(y,t,\mu) = \breve{\phi}_+(y,t,\mu)S(\mu), \tag{2.43}$$

and

$$\breve{\psi}_-(y,t,\mu)e^{it\theta(\mu)\sigma_3} = \breve{\psi}_+(y,t,\mu)e^{it\theta(\mu)\sigma_3}S(\mu), \tag{2.44}$$

where $\theta(\mu)$ is defined in (Equation 2.35). Under the assumption that $\mu = v_j$ ($j = 1, 2$) are N-order zero points of $a(\mu)$, from the following two relations, $a(\mu)$ can be represented as a Wronskian of analytic Jost solutions

$$a(\mu) = Wr(\breve{\phi}_{-1}, \breve{\phi}_{+2}), \tag{2.45}$$

then at zero points v_j ($j = 1, 2$), $a(\mu)$ vanishes to N-order, there exist complex constants $b_{j,s}$ ($j = 1, 2$) ($s = 1, 2, \cdots, N$) satisfy the follows

$$\frac{\partial^m[\breve{\phi}_+(y,t,v_j)]_2}{\partial\mu^m} = \sum_{l=0}^{m} \begin{pmatrix} m \\ l \end{pmatrix} b_{j,m-l+1} \frac{\partial^l[\breve{\phi}_-(y,t,v_j)]_1}{\partial\mu^l}, \tag{2.46}$$

and

$$\frac{\partial^m[\breve{\psi}_+(y,t,v_j)]_2}{\partial\mu^m} = \sum_{l=0}^{m} \begin{pmatrix} m \\ l \end{pmatrix} b_{j,m-l+1} \frac{\partial^l[\breve{\psi}_-(y,t,v_j)]_1 e^{2it\theta(v_j)}}{\partial\mu^l}, \tag{2.47}$$

with $m = 0, 1, \cdots, N - 1$.

On the basis of the definition of $M(\mu)$ defined by (Equation 2.28) and the asymptotic behaviour in RH problem 2.6, we set

$$\tilde{M}_{11}(y,t,\mu) = 1 + \sum_{j=1}^{2}\sum_{s=1}^{N}\left(\frac{1}{(\mu-v_j)^s} + \frac{(-1)^s}{(\mu+v_j^*)^s}\right)F_{j,s}(y,t), \qquad (2.48\text{a})$$

$$\tilde{M}_{12}(y,t,\mu) = \sum_{j=1}^{2}\sum_{s=1}^{N}\left(\frac{1}{(\mu-v_j^*)^s} + \frac{(-1)^{s+1}}{(\mu+v_j)^s}\right)G_{j,s}(y,t), \qquad (2.48\text{b})$$

where $F_{j,s}(y,t)$ and $G_{j,s}(y,t)$ are undetermined functions. Making use of Taylor series expansion, one can obtain the following relationships

$$e^{-2it\theta(\mu)} = \sum_{l=0}^{+\infty}\zeta_{j,l}(y,t)(\mu-v_j)^l, \quad e^{2it\theta(\mu)} = \sum_{l=0}^{+\infty}\zeta_{j,l}^*(y,t)(\mu-v_j^*)^l, \qquad (2.49)$$

where the element $\zeta_{j,l}(y,t)$ is

$$\zeta_{j,l}(y,t) = \lim_{\mu\to v_j}\frac{1}{l!}\frac{\partial^l}{\partial\mu^l}e^{-2it\theta(\mu)}, \quad \zeta_{j,l}^*(y,t) = \lim_{\mu\to v_j^*}\frac{1}{l!}\frac{\partial^l}{\partial\mu^l}e^{2it\theta(\mu)}.$$

It can be obtained that the coefficient of item $(\mu-v_j)^{-s}$ of function $\tilde{M}_{11}(y,t,\mu)$ is $F_{j,s}(y,t)$. Now we extend the residue theorem by combining (Equation 2.44) and (Equation 2.47) and obtain the following relations

$$F_{j,s} = \sum_{l=s}^{N}\sum_{m=0}^{l-s}\sum_{p=1}^{2}\sum_{q=1}^{N}\binom{q+m-1}{m}\times$$
$$r_{j,l}\zeta_{j,l-s-m}\left\{\frac{(-1)^m}{(v_j-v_p^*)^{q+m}} + \frac{(-1)^{m+q+1}}{(v_j+v_p)^{q+m}}\right\}G_{p,q}, \qquad (2.50)$$

where

$$r_{j,l} = \lim_{\mu\to v_j}\frac{b_{j,l}}{(N-l)!}\frac{\partial^{N-l}}{\partial(\mu-v_j)^{N-l}}\frac{(\mu-v_j)^N}{a(\mu)}. \qquad (2.51)$$

Likewise, the coefficient of item $(\mu-v_j^*)^{-s}$ of function $\tilde{M}_{12}(y,t,\mu)$ is $G_{j,s}(y,t)$. Through the same method, we have

$$G_{j,s} = -\sum_{l=s}^{N}r_{j,l}^*\zeta_{j,l-s}^* - \sum_{l=s}^{N}\sum_{m=0}^{l-s}\sum_{p=1}^{2}\sum_{q=1}^{N}\binom{q+m-1}{m}\times$$
$$r_{j,l}^*\zeta_{j,l-s-m}^*\left\{\frac{(-1)^m}{(v_j^*-v_p)^{q+m}} + \frac{(-1)^{m+q}}{(v_j^*+v_p^*)^{q+m}}\right\}F_{p,q}. \qquad (2.52)$$

Introducing the new notations for $j, p = 1, 2$

$$|\eta_j\rangle = (\eta_{j1}, \cdots, \eta_{jN})^T, \quad \eta_{js} = -\sum_{l=s}^{N} r_{j,l}^* \zeta_{j,l-s}^*(y,t), \tag{2.53}$$

then defining the $N \times N$ matrices $\vartheta_{j,p} = [\vartheta_{j,p}]_{s,q}$ and $\tilde{\vartheta}_{j,p} = [\tilde{\vartheta}_{j,p}]_{s,q}$ for $s, q = 1, 2, \cdots, N$ as follows

$$\vartheta_{j,p} = \sum_{l=s}^{N} \sum_{m=0}^{l-s} \binom{q+m-1}{m} r_{j,l} \zeta_{j,l-s-m} \left\{ \frac{(-1)^m}{(v_j - v_p^*)^{q+m}} + \frac{(-1)^{m+q+1}}{(v_j + v_p)^{q+m}} \right\}, \tag{2.54}$$

$$\tilde{\vartheta}_{j,p} = \sum_{l=s}^{N} \sum_{m=0}^{l-s} \binom{q+m-1}{m} r_{j,l}^* \zeta_{j,l-s-m}^* \left\{ \frac{(-1)^m}{(v_j^* - v_p)^{q+m}} + \frac{(-1)^{m+q}}{(v_j^* + v_p^*)^{q+m}} \right\}, \tag{2.55}$$

$$|F_p\rangle = (F_{p,1}, \cdots, F_{p,N})^T, \quad |G_p\rangle = (G_{p,1}, \cdots, G_{p,N})^T, \tag{2.56}$$

$$\vartheta = \begin{pmatrix} \vartheta_{1,1} & \vartheta_{1,2} \\ \vartheta_{2,1} & \vartheta_{2,2} \end{pmatrix}, \quad \tilde{\vartheta} = \begin{pmatrix} \tilde{\vartheta}_{1,1} & \tilde{\vartheta}_{1,2} \\ \tilde{\vartheta}_{2,1} & \tilde{\vartheta}_{2,2} \end{pmatrix}, \quad I_\sigma = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \tag{2.57}$$

where I is a $N \times N$ identity matrix. Equations (Equation 2.50)–(Equation 2.52) can be written as a system of linear equations

$$\begin{aligned} I|F_p\rangle - \vartheta_{j,p}|G_p\rangle &= 0, \\ \tilde{\vartheta}_{j,p}|F_p\rangle + I|G_p\rangle &= |\eta_j\rangle. \end{aligned} \tag{2.58}$$

Then taking

$$|F\rangle = (|F_1\rangle, |F_2\rangle)^T, \quad |G\rangle = (|G_1\rangle, |G_2\rangle)^T, \tag{2.59}$$

$$|\eta\rangle = (|\eta_1\rangle, |\eta_2\rangle)^T. \tag{2.60}$$

The system (Equation 2.58) has the following solution

$$|F\rangle = (I_\sigma + \vartheta\tilde{\vartheta})^{-1}\vartheta|\eta\rangle, \quad |G\rangle = (I_\sigma + \tilde{\vartheta}\vartheta)^{-1}|\eta\rangle. \tag{2.61}$$

The new expression of (Equation 2.48a) is derived as

$$\tilde{M}_{11}(y,t,\mu) = \frac{\det(I_\sigma + \vartheta\tilde{\vartheta} + |\eta\rangle\langle\Upsilon(\mu)|\vartheta)}{\det(I_\sigma + \vartheta\tilde{\vartheta})}, \tag{2.62}$$

$$\tilde{M}_{12}(y,t,\mu) = \frac{\det(I_\sigma + \tilde{\vartheta}\vartheta + |\eta\rangle\langle\tilde{\Upsilon}(\mu)|)}{\det(I_\sigma + \tilde{\vartheta}\vartheta)} - 1, \tag{2.63}$$

with

$$
\begin{aligned}
\langle\Upsilon(\mu)| &= (\langle\Upsilon_1(\mu)|,\langle\Upsilon_2(\mu)|), \quad \langle\Upsilon_j(\mu)| = (\Upsilon_{j1},\dots,\Upsilon_{jN}), \\
\langle\tilde{\Upsilon}(\mu)| &= (\langle\tilde{\Upsilon}_1(\mu)|,\langle\tilde{\Upsilon}_2(\mu)|), \quad \langle\tilde{\Upsilon}_j(\mu)| = (\tilde{\Upsilon}_{j1},\dots,\tilde{\Upsilon}_{jN}), \\
\langle\Upsilon'(\mu)| &= (\langle\Upsilon'_1(\mu)|,\langle\Upsilon'_2(\mu)|), \quad \langle\Upsilon'_j(\mu)| = (\Upsilon'_{j1},\dots,\Upsilon'_{jN}), \\
\langle\tilde{\Upsilon}'(\mu)| &= (\langle\tilde{\Upsilon}'_1(\mu)|,\langle\tilde{\Upsilon}'_2(\mu)|), \quad \langle\tilde{\Upsilon}'_j(\mu)| = (\tilde{\Upsilon}'_{j1},\dots,\tilde{\Upsilon}'_{jN}),
\end{aligned}
\tag{2.64}
$$

$$
\Upsilon_{js} = \frac{1}{(\mu - v_j)^s} + \frac{(-1)^s}{(\mu + v_j^*)^s}, \quad \tilde{\Upsilon}_{js} = \frac{1}{(\mu - v_j^*)^s} + \frac{(-1)^{s+1}}{(\mu + v_j)^s},
$$

$$
\Upsilon'_{js} = \frac{-s}{(\mu - v_j)^{s+1}} + \frac{s(-1)^{s+1}}{(\mu + v_j^*)^{s+1}}, \quad \tilde{\Upsilon}'_{js} = \frac{-s}{(\mu - v_j^*)^{s+1}} + \frac{s(-1)^s}{(\mu + v_j)^{s+1}},
$$

where we differentiate the components of vector $\langle\Upsilon(\mu)|$ from variable μ to obtain vector $\langle\Upsilon'(\mu)|$, the vector $\langle\tilde{\Upsilon}'(\mu)|$ can also be gained from the same way. In the following content, we will continue to use this definition. On the basis of the symmetry of $\tilde{M}(\mu) = \sigma_2 \tilde{M}^*(\mu^*)\sigma_2$, we have

$$
\tilde{M}_{22}(y,t,\mu) = \frac{\det(I_\sigma + \tilde{\vartheta}^*\vartheta^* + |\eta^*\rangle\langle\Upsilon^*(\mu^*)|\vartheta^*)}{\det(I_\sigma + \tilde{\vartheta}^*\vartheta^*)},
\tag{2.65}
$$

$$
\tilde{M}_{21}(y,t,\mu) = -\frac{\det(I_\sigma + \vartheta^*\tilde{\vartheta}^* + |\eta^*\rangle\langle\tilde{\Upsilon}^*(\mu^*)|)}{\det(I_\sigma + \vartheta^*\tilde{\vartheta}^*)} + 1.
\tag{2.66}
$$

Hence the precise expression formula for the solution of cCH equation with single high-order pole under the ZBCs can be obtained in theorem 2.7.

THEOREM 2.7 *The solution of cCH equation with single high-order pole under the ZBCs can be obtained as follows*

$$
u(x,t) = -\left(\frac{\tilde{M}'_{11}(i) + \tilde{M}'_{21}(i)}{\tilde{M}_{11}(i) + \tilde{M}_{21}(i)} + \frac{\tilde{M}'_{12}(i) + \tilde{M}'_{22}(i)}{\tilde{M}_{12}(i) + \tilde{M}_{22}(i)}\right),
\tag{2.67}
$$

with

$$
\tilde{M}_{11}(i) = \frac{\det(I_\sigma + \vartheta\tilde{\vartheta} + |\eta\rangle\langle\Upsilon(i)|\vartheta)}{\det(I_\sigma + \vartheta\tilde{\vartheta})},
$$

$$
\tilde{M}_{21}(i) = -\frac{\det(I_\sigma + \vartheta^*\tilde{\vartheta}^* + |\eta^*\rangle\langle\tilde{\Upsilon}^*(-i)|)}{\det(I_\sigma + \vartheta^*\tilde{\vartheta}^*)} + 1,
$$

$$
\tilde{M}_{12}(i) = \frac{\det(I_\sigma + \tilde{\vartheta}\vartheta + |\eta\rangle\langle\tilde{\Upsilon}(i)|)}{\det(I_\sigma + \tilde{\vartheta}\vartheta)} - 1,
$$

$$
\tilde{M}_{22}(i) = \frac{\det(I_\sigma + \tilde{\vartheta}^*\vartheta^* + |\eta^*\rangle\langle\Upsilon^*(-i)|\vartheta^*)}{\det(I_\sigma + \tilde{\vartheta}^*\vartheta^*)},
$$

$$
\tilde{M}'_{11}(i) = \frac{\det(I_\sigma + \vartheta\tilde{\vartheta} + |\eta\rangle\langle\Upsilon'(i)|\vartheta)}{\det(I_\sigma + \vartheta\tilde{\vartheta})} - 1,
$$

$$\tilde{M}'_{21}(i) = -\frac{\det(I_\sigma + \vartheta^*\tilde{\vartheta}^* + |\eta^*\rangle\langle\tilde{\Upsilon}'^*(-i)|)}{\det(I_\sigma + \vartheta^*\tilde{\vartheta}^*)} + 1,$$

$$\tilde{M}'_{12}(i) = \frac{\det(I_\sigma + \tilde{\vartheta}\vartheta + |\eta\rangle\langle\tilde{\Upsilon}'(i)|)}{\det(I_\sigma + \tilde{\vartheta}\vartheta)} - 1,$$

$$\tilde{M}'_{22}(i) = \frac{\det(I_\sigma + \tilde{\vartheta}^*\vartheta^* + |\eta^*\rangle\langle\Upsilon'^*(-i)|\vartheta^*)}{\det(I_\sigma + \tilde{\vartheta}^*\vartheta^*)} - 1,$$

where $x = x(y,t)$ is defined in (Equation 2.41), and the elements $|\eta\rangle$, I_σ, ϑ, $\tilde{\vartheta}$, $\langle\Upsilon(i)|$, $\langle\tilde{\Upsilon}(i)|$, $\langle\Upsilon'(i)|$, and $\langle\tilde{\Upsilon}'(i)|$ are respectively defined in (Equation 2.53), (Equation 2.57), and (Equation 2.64).

2.3.1. One-soliton solution

Next, we construct the simple one-soliton solution and compare the expressions of the solutions obtained by RH approach and Hirota's bilinear method. In this case, the zero points of scattering data are one-order, and there is no need to calculate the high-order derivative formula of the Jost eigenfunctions and the complex Taylor expansions, which degenerates into the residue conditions of RH matrix. In order to compare with the solution obtained by bilinear method, we need to process the RH matrix and use the above construction process to obtain the simplest expression of solution.

For one-soliton solution, we have $N = 1$. Supposing $v_1 = \mu_0$ is a first-order zero point of the scattering data $a(\mu)$, then the symmetrical relationships indicate that $\left\{-\mu_0^*, -\frac{1}{\mu_0}, \frac{1}{\mu_0^*}\right\} \in \mathbb{C}^+$ also are zero points of $a(\mu)$. Then from Eqs. (Equation 2.44), (Equation 2.45), and (Equation 2.47), we use the symmetry relationships in RH problem 2.6 to calculate the following residue conditions

$$\begin{aligned}
\operatorname*{Res}_{\mu=\mu_0} \tilde{M}^{(1)}(y,t,\mu) &= c_0 e^{-2it\theta(\mu_0)}\tilde{M}^{(2)}(y,t,\mu_0), \\
\operatorname*{Res}_{\mu=\frac{1}{\mu_0^*}} \tilde{M}^{(1)}(y,t,\mu) &= -\frac{c_0^*}{\mu_0^{*2}}e^{2it\theta^*(\mu_0)}\tilde{M}^{(2)}(y,t,\frac{1}{\mu_0^*}), \\
\operatorname*{Res}_{\mu=-\mu_0^*} \tilde{M}^{(1)}(y,t,\mu) &= -\frac{c_0}{\mu_0^2}e^{2it\theta^*(\mu_0)}\tilde{M}^{(2)}(y,t,-\mu_0^*), \\
\operatorname*{Res}_{\mu=-\frac{1}{\mu_0}} \tilde{M}^{(1)}(y,t,\mu) &= -\frac{c_0}{\mu_0^2}e^{-2it\theta(\mu_0)}\tilde{M}^{(2)}(y,t,-\frac{1}{\mu_0}).
\end{aligned} \qquad (2.68)$$

To simplify the calculation process, we redefine the RH matrix as

$$\tilde{N}_1(\mu) = \tilde{M}_{11}(\mu) + \tilde{M}_{21}(\mu), \quad \tilde{N}_2(\mu) = \tilde{M}_{12}(\mu) + \tilde{M}_{22}(\mu). \qquad (2.69)$$

The characteristic function satisfies $\psi_\pm(\mu) = \sigma_1\psi_\pm(-\mu)\sigma_1$, which means that $\tilde{M}(\mu) = \sigma_1\tilde{M}(-\mu)\sigma_1$, then $\tilde{N}_1(\mu) = \tilde{N}_2(-\mu)$. Calculating from formulas

(Equation 2.48a) and combining with the symmetries of matrix $\tilde{M}(\mu)$, we obtain

$$
\begin{aligned}
\tilde{N}_1(\mu) =& 1 + \frac{c_0 e^{-2it\theta(\mu_0)}}{\mu - \mu_0}\tilde{N}_1(-\mu_0) + \frac{c_0^* e^{2it\theta^*(\mu_0)}}{\mu + \mu_0^*}\tilde{N}_1(\mu_0^*) + \\
& \frac{-\frac{c_0}{\mu_0^2}e^{-2it\theta(\mu_0)}}{\mu + \frac{1}{\mu_0}}\tilde{N}_1(\frac{1}{\mu_0}) + \frac{-\frac{c_0^*}{\mu_0^{*2}}e^{2it\theta^*(\mu_0)}}{\mu - \frac{1}{\mu_0^*}}\tilde{N}_1(-\frac{1}{\mu_0^*}),
\end{aligned}
\tag{2.70}
$$

there $c_0 = \frac{b_0}{\dot{a}(\mu)}$ is some complex number, we can set $c_0 = |c_0|e^{i\phi}$ ($\phi \in (0, \pi)$). The linear equation system shown in formula (Equation 2.58) can be simplified as

$$
\begin{aligned}
\tilde{N}_1(-\mu_0) =& 1 + \frac{c_0 e^{-2it\theta(\mu_0)}}{-\mu_0 - \mu_0}\tilde{N}_1(-\mu_0) + \frac{c_0^* e^{2it\theta^*(\mu_0)}}{-\mu_0 + \mu_0^*}\tilde{N}_1(\mu_0^*) + \\
& \frac{-\frac{c_0}{\mu_0^2}e^{-2it\theta(\mu_0)}}{-\mu_0 + \frac{1}{\mu_0}}\tilde{N}_1(\frac{1}{\mu_0}) + \frac{-\frac{c_0^*}{\mu_0^{*2}}e^{2it\theta^*(\mu_0)}}{-\mu_0 - \frac{1}{\mu_0^*}}\tilde{N}_1(-\frac{1}{\mu_0^*}),
\end{aligned}
\tag{2.71}
$$

$$
\begin{aligned}
\tilde{N}_1(\mu_0^*) =& 1 + \frac{c_0 e^{-2it\theta(\mu_0)}}{\mu_0^* - \mu_0}\tilde{N}_1(-\mu_0) + \frac{c_0^* e^{2it\theta^*(\mu_0)}}{\mu_0^* + \mu_0^*}\tilde{N}_1(\mu_0^*) + \\
& \frac{-\frac{c_0}{\mu_0^2}e^{-2it\theta(\mu_0)}}{\mu_0^* + \frac{1}{\mu_0}}\tilde{N}_1(\frac{1}{\mu_0}) + \frac{-\frac{c_0^*}{\mu_0^{*2}}e^{2it\theta^*(\mu_0)}}{\mu_0^* - \frac{1}{\mu_0^*}}\tilde{N}_1(-\frac{1}{\mu_0^*}),
\end{aligned}
\tag{2.72}
$$

$$
\begin{aligned}
\tilde{N}_1(\frac{1}{\mu_0}) =& 1 + \frac{c_0 e^{-2it\theta(\mu_0)}}{\frac{1}{\mu_0} - \mu_0}\tilde{N}_1(-\mu_0) + \frac{c_0^* e^{2it\theta^*(\mu_0)}}{\frac{1}{\mu_0} + \mu_0^*}\tilde{N}_1(\mu_0^*) + \\
& \frac{-\frac{c_0}{\mu_0^2}e^{-2it\theta(\mu_0)}}{\frac{1}{\mu_0} + \frac{1}{\mu_0}}\tilde{N}_1(\frac{1}{\mu_0}) + \frac{-\frac{c_0^*}{\mu_0^{*2}}e^{2it\theta^*(\mu_0)}}{\frac{1}{\mu_0} - \frac{1}{\mu_0^*}}\tilde{N}_1(-\frac{1}{\mu_0^*}),
\end{aligned}
\tag{2.73}
$$

$$
\begin{aligned}
\tilde{N}_1(-\frac{1}{\mu_0^*}) =& 1 + \frac{c_0 e^{-2it\theta(\mu_0)}}{-\frac{1}{\mu_0^*} - \mu_0}\tilde{N}_1(-\mu_0) + \frac{c_0^* e^{2it\theta^*(\mu_0)}}{-\frac{1}{\mu_0^*} + \mu_0^*}\tilde{N}_1(\mu_0^*) + \\
& \frac{-\frac{c_0}{\mu_0^2}e^{-2it\theta(\mu_0)}}{-\frac{1}{\mu_0^*} + \frac{1}{\mu_0}}\tilde{N}_1(\frac{1}{\mu_0}) + \frac{-\frac{c_0^*}{\mu_0^{*2}}e^{2it\theta^*(\mu_0)}}{-\frac{1}{\mu_0^*} - \frac{1}{\mu_0^*}}\tilde{N}_1(-\frac{1}{\mu_0^*}).
\end{aligned}
\tag{2.74}
$$

After solving this linear system, inputting the results into formula (Equation 2.70), we can obtain the solution of $\tilde{N}_1(\mu)$, then substituting it into the following formula

$$
u(x, t) = -\left(\frac{\tilde{N}_1'(i)}{\tilde{N}_1(i)} + \frac{\tilde{N}_1'(-i)}{\tilde{N}_1(-i)} \right),
\tag{2.75}
$$

where

$$x(y,t) = y - \ln\left(\frac{\tilde{N}_1(-i)}{\tilde{N}_1(i)}\right), \tag{2.76}$$

we can obtain the solution of the cCH equation.

Next, we discuss the solution when scattering data $a(\mu)$ has only two zero points through two examples, where the discrete spectrum has four points.

Case 1: Supposing $v_1 = \mu_0 = e^{i\tau} \in \mathbb{C}^+$ ($\tau \in (0,\pi)$), $v_2 = \frac{1}{\mu_0^*} = v_1$. Then $|v_1| = 1$, and we further assume that $\mu_0 = r_1 + ir_2$. It can be obtained that $\theta(e^{i\tau}) = \theta(-e^{-i\tau})$. Next we define $\chi_1 \triangleq \theta(e^{i\tau}) = -\frac{i\sin\tau y}{2t} + \frac{i\sin\tau}{\cos^2\tau}$, hence $e^{-2it\chi_1}$ is a real number about variables y and t. Thus algebraic system (Equation 2.70)–(Equation 2.74) have the following forms

$$\tilde{N}_1(\mu) = 1 + \frac{|c_0|e^{i\phi}e^{\chi_1}}{\mu - e^{i\tau}}\tilde{N}_1(-e^{i\tau}) + \frac{|c_0|e^{-i\phi}e^{\chi_1}}{\mu + e^{-i\tau}}\tilde{N}_1(e^{-i\tau}), \tag{2.77}$$

$$\begin{aligned}
\tilde{N}_1(-e^{i\tau}) &= 1 + \frac{|c_0|e^{i\phi}e^{\chi_1}}{-2e^{i\tau}}\tilde{N}_1(-e^{i\tau}) + \frac{|c_0|e^{-i\phi}e^{\chi_1}}{e^{-i\tau} - e^{i\tau}}\tilde{N}_1(e^{-i\tau}), \\
\tilde{N}_1(e^{-i\tau}) &= 1 + \frac{|c_0|e^{i\phi}e^{\chi_1}}{e^{-i\tau} - e^{i\tau}}\tilde{N}_1(-e^{i\tau}) + \frac{|c_0|e^{-i\phi}e^{\chi_1}}{2e^{-i\tau}}\tilde{N}_1(e^{-i\tau}),
\end{aligned} \tag{2.78}$$

then we have the solutions

$$\begin{aligned}
\tilde{N}_1(-e^{i\tau}) &= \frac{1}{H_1}\left(1 - \frac{r_1}{2ir_2}|c_0|e^{\chi_1 - i\phi + i\tau}\right), \\
\tilde{N}_1(e^{-i\tau}) &= \frac{1}{H_1}\left(1 - \frac{r_1}{2ir_2}|c_0|e^{\chi_1 + i\phi - i\tau}\right),
\end{aligned} \tag{2.79}$$

where

$$H_1 = 1 + \frac{r_1^2}{4r_2^2}|c_0|e^{2\chi_1} + i|c_0|e^{\chi_1}\sin(\phi - \tau), \tag{2.80}$$

then we have

$$\tilde{N}_1(\mu) = 1 + \frac{1}{H_1}\left(\frac{|c_0|e^{\phi + \chi_1} - \frac{r_1|c_0|^2e^{2\chi_1 + \phi}}{2ir_2e^{-i(\tau - \phi)}}}{\mu - e^{i\tau}} + \frac{|c_0|e^{\chi_1 - \phi} - \frac{r_1|c_0|^2e^{2\chi_1 - \phi}}{2ir_2e^{i\tau}}}{\mu + e^{i(\tau - \phi)}}\right), \tag{2.81}$$

after assuming the parameter values are $|c_0| = \frac{2r_2\,\mathrm{sgn}\,(r_1)}{r_1}$ and $\phi - \tau = \frac{\pi}{2}$, the expression of solution can be obtained as follows

$$u(x,t) = -\frac{4r_2}{r_1^3}\,\mathrm{sgn}\,(r_1)\frac{\cosh(\chi_1)}{cosh(2\chi_1) + \frac{1+r_2^2}{r_1^2}}, \tag{2.82}$$
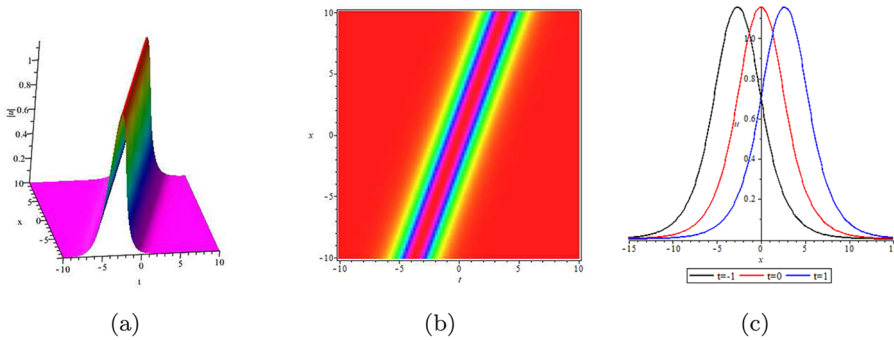
Figure 2. (a)–(c) describe the local structure, density, and intensity profiles with different times of one-soliton solution $|u|$. Parameters $r_1 = \frac{\sqrt{3}}{2}$, $r_2 = \frac{1}{2}$, $\tau = \frac{\pi}{6}$.

with

$$x = x(y,t) = y - \ln\left(\frac{1 + \frac{1+r_2}{1-r_2}e^{2\chi_1}}{1 + \frac{1-r_2}{1+r_2}e^{2\chi_1}}\right). \tag{2.83}$$

REMARK 2.8. By comparing with the soliton solution of cCH equation obtained in reference [31], it can be seen that the solution constructed by the RH method is exactly the same as the solution obtained by Hirota's bilinear method after special parameter selection. Both of the above methods are used to study the construction of soliton solution for cCH equation under ZBCs. The difference is that [31] constructs multiple soliton solutions, while our article constructs arbitrary-order pole solutions. Similarly, when the orders of the zero points degenerate to one, the expressions of solutions obtained in these two articles are the same for multiple zero points case.

REMARK 2.9. According to the detailed analysis of literature [31], it can be seen that when the parameters take different values, we can obtain smooth soliton solutions and symmetric singular soliton solutions. When the selected zero point μ_0 is complex and its modulus is not equal to zero, the breather solutions can be obtained.

From expression (Equation 2.83), it can be seen that when $x \to \pm\infty$, there exists $y \to \pm\infty$. The solution (Equation 2.82) indicates that $r_1 = 0$ ($\tau = \frac{\pi}{2}$) is a singularity point and $cosh(2\chi_1) + \frac{1+r_2^2}{r_1^2}$ has no zero point; otherwise, the following relationship will presence

$$r_1^2(e^{-2yr_2 + \frac{2tr_2}{r_1^2}} + e^{2yr_2 - \frac{2tr_2}{r_1^2}}) = -2(1 + r_2^2), \tag{2.84}$$

since the point $\mu_0 \in \mathbb{C}^+$, the right-hand side of the equation is always negative and the left-hand side is always positive, which leads to a contradiction. According to our analysis of constructing the RH problem, the points $\mu = 0, \pm i$ are singularities which corresponding to the obtained solution. There $\mu_0 \in \mathbb{C}^+$ and it has only one singularity point $\mu_0 = i$.

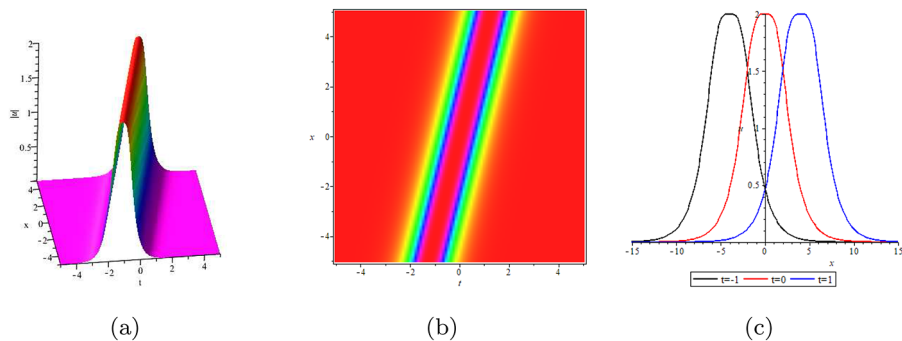(a)   (b)   (c)

Figure 3. (a)–(c) describe the local structure, density, and intensity profiles with different times of one-soliton solution $|u|$. Parameters $r_1 = \frac{\sqrt{2}}{2}$, $r_2 = \frac{\sqrt{2}}{2}$, $\tau = \frac{\pi}{4}$.

(a)   (b)   (c)
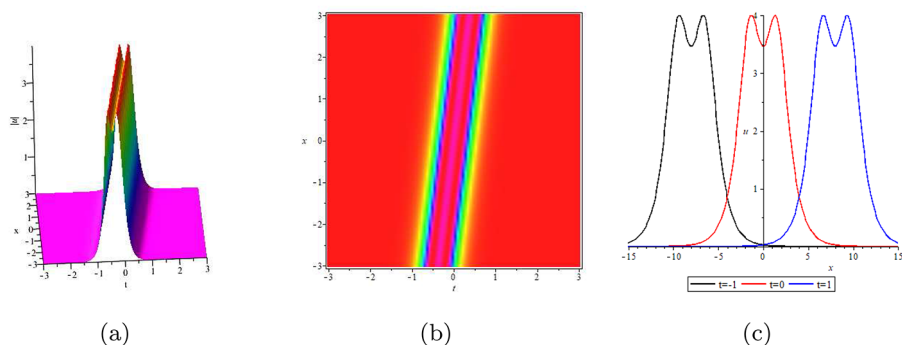
Figure 4. (a)–(c) describe the local structure, density, and intensity profiles with different times of one-soliton solution $|u|$. Parameters $r_1 = \frac{1}{2}$, $r_2 = \frac{\sqrt{3}}{2}$, $\tau = \frac{\pi}{3}$.

To demonstrate the characteristics of solitons, we present wave propagations including images of local structure, density, and intensity distribution in figures 2–4 after selecting different parameters. It can be seen that the single smooth soliton solutions are obtained at these values. Specifically, figure 2(a) shows a Hump soliton solution. figure 3(a) is a critical situation where the soliton exhibits a top horizontal characteristic. Finally, figure 4(a) shows the M-shape soliton solution. From these figures, it can be seen that as the values of τ increasing, the peak of the soliton first appears in a horizontal state and then sinks inward.

Case 2: Supposing $v_1 = \mu_0 = ir \in i\mathbb{R}$, $v_2 = \frac{1}{\mu_0^*}$. It can be obtained that $\theta(\mu_0) = \theta(\frac{1}{\mu_0^*})$. Next we define $\chi_2 \triangleq \theta(\mu_0) = -\frac{i}{4} \frac{r^2+1}{r} \left(\frac{y}{t} + \frac{8r^2}{(r^2-1)^2} \right)$, hence $e^{-2it\chi_2}$ is a real number about variables y and t. Thus algebraic system (Equation 2.70)–(Equation 2.74) have the following forms

$$\tilde{N}_1(\mu) = 1 + \frac{|c_0|e^{i\phi}e^{\chi_2}}{\mu - ir} \tilde{N}_1(-ir) + \frac{\frac{|c_0|}{r^2}e^{-i\phi}e^{\chi_2}}{\mu + \frac{1}{ir}} \tilde{N}_1(\frac{1}{ir}), \qquad (2.85)$$

$$\tilde{N}_1(-ir) = 1 + \frac{|c_0|e^{i\phi}e^{\chi_2}}{-2ir}\tilde{N}_1(-ir) + \frac{\frac{|c_0|}{r^2}e^{-i\phi}e^{\chi_2}}{-ir + \frac{1}{ir}}\tilde{N}_1(\frac{1}{ir}),$$

$$\tilde{N}_1(\frac{1}{ir}) = 1 + \frac{|c_0|e^{i\phi}e^{\chi_2}}{\frac{1}{ir} - ir}\tilde{N}_1(-ir) + \frac{\frac{|c_0|}{r^2}e^{-i\phi}e^{\chi_2}}{\frac{2}{ir}}\tilde{N}_1(\frac{1}{ir}),$$

(2.86)

the solutions are

$$\tilde{N}_1(-ir) = \frac{1}{H_2}(1 - \frac{1-r^2}{2ir(1+r^2)}|c_0|e^{\chi_2+\phi}),$$

$$\tilde{N}_1(\frac{1}{ir}) = \frac{1}{H_2}(1 + \frac{1-r^2}{2ir(1+r^2)}|c_0|e^{\chi_2+\phi}),$$

(2.87)

with

$$H_2 = 1 - \frac{(r^2-1)^2}{4r^2(r^2+1)^2}|c_0|^2e^{2\chi_2} - \frac{i|c_0|e^{\chi_2+\phi}}{r}.$$

(2.88)

Substitute formula (Equation 2.87) into (Equation 2.85) and take an appropriate value for c_0 to calculate the solution as follows

$$u(x,t) = \frac{16r^2(1+r^2)}{(1-r^2)^3}\frac{\sinh(\chi_2)}{cosh(2\chi_2) + \frac{(1+r^2)^2+4r^2}{(1-r^2)^2}},$$

(2.89)

with

$$x = x(y,t) = y - \ln\left(\frac{1 + \frac{(1+r)^2}{(1-r)^2}e^{2\chi_2}}{1 + \frac{(1-r)^2}{(1+r)^2}e^{2\chi_2}}\right).$$

(2.90)

The solution (Equation 2.89) indicates that $r=1$ is a singularity point and $cosh(2\chi_2) + \frac{(1+r^2)^2+4r^2}{(1-r^2)^2}$ has no zero point; otherwise, the following relationship will presence

$$(1-r^2)^2(e^{2\chi_2} + e^{-2\chi_2}) = -2(1+r^2)^2 - 8r^4,$$

(2.91)

this is clearly impossible.

REMARK 2.10. After comparison, when selecting appropriate parameters, the expression of solution (Equation 2.89) solved by RH approach is the same as the solution (3.9a) solved by Hirota's bilinear method in reference [31]. Then the antisymmetric singular soliton can be obtained.

2.3.2. Two-order pole solution
For the $N=2$ case, it corresponds to the solution with a two-order pole. The expression of $u(x,t)$ defined in theorem 2.7.

Case 3: Assuming that $v_1 = \mu_0 = e^{i\tau}$ be one two-order zero point of the scattering data $a(\mu)$, then $v_2 = e^{i\tau} = v_1$ is also the two-order zero point of $a(\mu)$. The discrete spectrum is the set $\mathcal{Z} = \{e^{i\tau}, -e^{-i\tau}, e^{-i\tau}, -e^{i\tau}\}$. We choose some parameters to simulate the solution, then the different dynamic models can be obtained in Figures (5)–(6).

Setting $\tau = \frac{\pi}{6}$ and $\{r_{1,1} = 2, r_{1,2} = 1, r_{2,1} = 1, r_{2,2} = 2\}$. On the basis of the construction process of single high-order pole solutions, the following relationships are calculated.

$$\zeta_{1,0} = \zeta_{2,0} = e^{\frac{4t}{3} - \frac{y}{2}}, \quad \zeta_{1,1} = \zeta_{2,1} = \frac{-8i(40t - 9y)e^{\frac{4t}{3} - \frac{y}{2}}(i\sqrt{3} - 1)}{(i\sqrt{3} + 3)^3(\sqrt{3} + i)^2},$$

$$\vartheta_{1,1} = \vartheta_{1,3} = \frac{5\sqrt{3}e^{\frac{4t}{3} - \frac{y}{2}}}{9}\left(\sqrt{3}\left(\frac{21 + 27y}{40} - \frac{3i}{2} - 3t\right) + it - \frac{9i(y - 1)}{40} + \frac{9}{10}\right),$$

$$\vartheta_{1,2} = \vartheta_{1,4} = -\frac{2880e^{\frac{4t}{3} - \frac{y}{2}}}{(i\sqrt{3} + 3)^3(\sqrt{3} + i)^4}\left(\sqrt{3}\left(\frac{6 - y}{40} - \frac{i}{10} + \frac{t}{9}\right) + it - \frac{9i(y + 1)}{40} - \frac{1}{2}\right),$$

$$\vartheta_{3,1} = \vartheta_{3,3} = \frac{10\sqrt{3}e^{\frac{4t}{3} - \frac{y}{2}}}{9}\left(\sqrt{3}\left(\frac{21 + 27y}{40} - \frac{3i}{8} - 3t\right) + it - \frac{9i(y - 1)}{40} + \frac{9}{40}\right),$$

$$\vartheta_{3,2} = \vartheta_{3,4} = -\frac{5760e^{\frac{4t}{3} - \frac{y}{2}}}{(i\sqrt{3} + 3)^3(\sqrt{3} + i)^4}\left(\sqrt{3}\left(\frac{6 - y}{40} - \frac{i}{40} + \frac{t}{9}\right) + it - \frac{9i(y + 2)}{40} - \frac{1}{8}\right),$$

$$\vartheta_{2,1} = \vartheta_{2,3} = -\frac{e^{\frac{4t}{3} - \frac{y}{2}}(i\sqrt{3} - 2)}{\sqrt{3} + i}, \quad \vartheta_{2,2} = \vartheta_{2,4} = -\frac{e^{\frac{4t}{3} - \frac{y}{2}}(2i\sqrt{3} + 3)}{(\sqrt{3} + i)^2},$$

$$\vartheta_{4,1} = \vartheta_{4,3} = -\frac{e^{\frac{4t}{3} - \frac{y}{2}}(2i\sqrt{3} - 4)}{\sqrt{3} + i}, \quad \vartheta_{4,2} = \vartheta_{4,4} = -\frac{e^{\frac{4t}{3} - \frac{y}{2}}(4i\sqrt{3} + 6)}{(\sqrt{3} + i)^2},$$

$$\eta_{11} = -2e^{\frac{4t}{3} - \frac{y}{2}} - \frac{8ie^{\frac{4t}{3} - \frac{y}{2}}(1 + i\sqrt{3})(40t - 9y)}{(i\sqrt{3} - 3)^3(i - \sqrt{3})^2}, \quad \eta_{12} = -e^{\frac{4t}{3} - \frac{y}{2}},$$

$$\eta_{21} = -e^{\frac{4t}{3} - \frac{y}{2}} - \frac{16ie^{\frac{4t}{3} - \frac{y}{2}}(1 + i\sqrt{3})(40t - 9y)}{(i\sqrt{3} - 3)^3(i - \sqrt{3})^2}, \quad \eta_{22} = -2e^{\frac{4t}{3} - \frac{y}{2}}.$$

Substitute the above results into the expression of solution (Equation 2.67) to obtain the two-order pole solution, and the propagation phenomenon of soliton is shown in figure 5. It can be seen that the energies of two waves for solution decrease after collision; thus, it is not an elastic collision that occurs. According to the previous analysis, the singularity of the solution is $\mu_0 = i$.

Using a similar analysis, when the value is set to $\mu_0 = \frac{3\pi}{4}$, the soliton dynamics behaviour shown in figure 6. figure 6(b) presents the density image of the wave propagation process, while figure 6(c) shows the intensity distribution at different times.
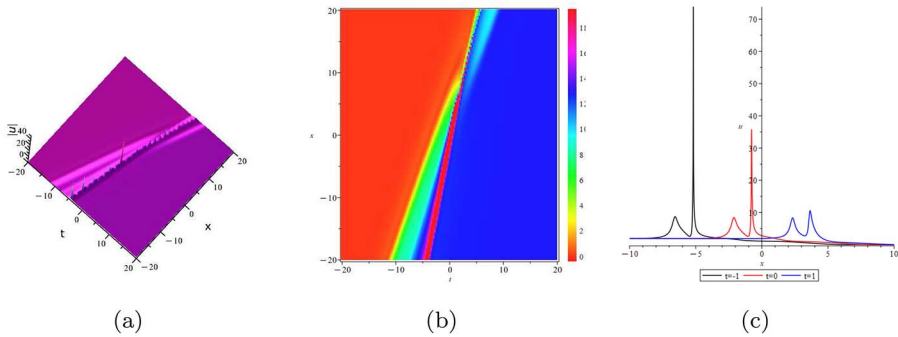
Figure 5. (a)–(c) describe the local structure, density, and intensity profiles with different times of the soliton solutions $|u|$ with one two-order pole. Parameters $r_{1,1} = 2$, $r_{1,2} = 1$, $r_{2,1} = 1$, $r_{2,2} = 2$, $\tau = \frac{\pi}{6}$.
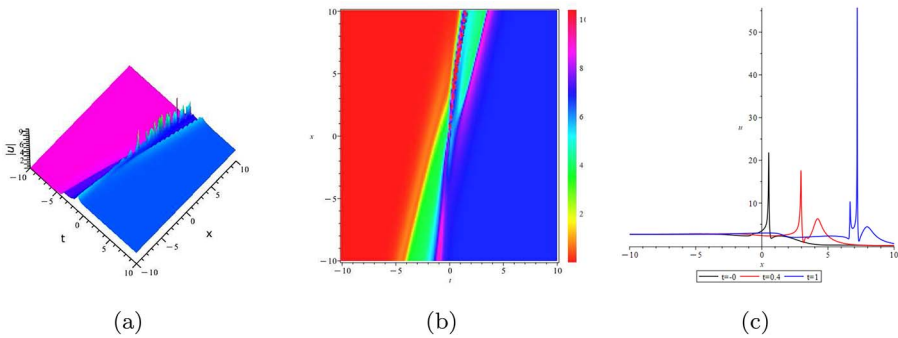

Figure 6. (a)–(c) describe the local structure, density, and intensity profiles with different times of the soliton solutions $|u|$ with one two-order pole. Parameters $r_{1,1} = 2$, $r_{1,2} = 1$, $r_{2,1} = 1$, $r_{2,2} = 2$, $\tau = \frac{3\pi}{4}$.

2.4. Multiple high-order pole solutions

Now the general circumstance that $a(\mu)$ has N high-order zero points $\mu_1, \mu_2, \ldots, \mu_N$, $\mu_k \in \mathbb{C}^+$ for $k = 1, 2, \ldots, N$ and the corresponding powers are $n_1, n_2, \ldots, n_N$, respectively. Assume that $v_1^k = \mu_k$ and $v_2^k = \frac{1}{\mu_k^*}$. Similar to the case of one high-order pole, from the definition of $\tilde{M}(y, t, \mu)$, it can be seen that $\mu = v_j^k$ and $\mu = -v_j^{k*}$ $(j = 1, 2)$ are n_k-order pole points of $\tilde{M}_{11}$. Simultaneously, $\mu = v_j^{k*}$ and $\mu = -v_j^k$ $(j = 1, 2)$ are n_k-order pole points of $\tilde{M}_{12}$.

Under the assumption that $\mu = v_j^k$ $(j = 1, 2)$ are n_k-order zero points of $a(\mu)$ for $k = 1, 2, \ldots, N$, utilizing the Wronskian $a(\mu) = Wr(\breve{\phi}_{-1}, \breve{\phi}_{+2})$, there exist complex constants $b_{j,s}^k$ $(s = 1, 2, \cdots, n_N)$ satisfy the follows

$$\frac{\partial^m [\tilde{\phi}_+(y, t, v_j^k)]_2}{\partial \mu^m} = \sum_{l=0}^{m} \binom{m}{l} b_{j,m-l+1}^k \frac{\partial^l [\tilde{\phi}_-(y, t, v_j^k)]_1}{\partial \mu^l}, \qquad (2.92)$$

and

$$\frac{\partial^m [\tilde{\psi}_+(y,t,v_j^k)]_2}{\partial \mu^m} = \sum_{l=0}^{m} \begin{pmatrix} m \\ l \end{pmatrix} b_{j,m-l+1}^k \frac{\partial^l [\tilde{\psi}_-(y,t,v_j^k)]_1 e^{2it\theta(v_j^k)}}{\partial \mu^l}, \qquad (2.93)$$

with $m = 0, 1, \ldots, n_{N-1}$. Utilizing the parallel approach in above, we define the $n_k \times n_k$ matrices $\Theta_{kd}^{j,p} = [\Theta_{kd}^{j,p}]_{s,q}$ and $\tilde{\Theta}_{kd}^{j,p} = [\tilde{\Theta}_{kd}^{j,p}]_{s,q}$ for $s, q = 1, 2, \cdots, n_k$ and introduce the following notations for $j, p = 1, 2$

$$|\eta_j^k\rangle = (\eta_{j1}^k, \cdots, \eta_{jn_k}^k)^T, \quad \eta_{js}^k = -\sum_{l=s}^{n_k} r_{j,l}^{k*} \zeta_{j,l-s}^{k*}(y,t), \qquad (2.94)$$

$$r_{j,l}^k = \lim_{\mu \to v_j^k} \frac{b_{j,l}^k}{(n_k-l)!} \frac{\partial^{n_k-l}}{\partial(\mu-v_j^k)^{n_k-l}} \frac{(\mu-v_j^k)^{n_k}}{a(\mu)}, \qquad (2.95)$$

$$|\Gamma\rangle = (|\Gamma^1\rangle, \cdots, |\Gamma^N\rangle)^T, \quad |\Gamma^k\rangle = (|\eta_1^k\rangle, |\eta_2^k\rangle)^T, \qquad (2.96)$$

$$\Theta_{kd}^{j,p} = \sum_{l=s}^{n_k} \sum_{m=0}^{l-s} \begin{pmatrix} q+m-1 \\ m \end{pmatrix} r_{j,l}^k \zeta_{j,l-s-m}^k \left\{ \frac{(-1)^m}{(v_j^k - v_p^{k*})^{q+m}} + \frac{(-1)^{m+q+1}}{(v_j^k + v_p^k)^{q+m}} \right\}, \qquad (2.97)$$

$$\tilde{\Theta}_{kd}^{j,p} = \sum_{l=s}^{n_k} \sum_{m=0}^{l-s} \begin{pmatrix} q+m-1 \\ m \end{pmatrix} r_{j,l}^{k*} \zeta_{j,l-s-m}^{k*} \left\{ \frac{(-1)^m}{(v_j^{k*} - v_p^k)^{q+m}} + \frac{(-1)^{m+q}}{(v_j^{k*} + v_p^{k*})^{q+m}} \right\}, \qquad (2.98)$$

$$\Theta = \begin{pmatrix} [\Theta_{11}^{j,p}] & \cdots & [\Theta_{1N}^{j,p}] \\ \vdots & \ddots & \vdots \\ [\Theta_{N1}^{j,p}] & \cdots & [\Theta_{NN}^{j,p}] \end{pmatrix}, \quad \tilde{\Theta} = \begin{pmatrix} [\tilde{\Theta}_{11}^{j,p}] & \cdots & [\tilde{\Theta}_{1N}^{j,p}] \\ \vdots & \ddots & \vdots \\ [\tilde{\Theta}_{N1}^{j,p}] & \cdots & [\tilde{\Theta}_{NN}^{j,p}] \end{pmatrix}, \qquad (2.99)$$

$$I_\epsilon = \begin{pmatrix} I_{\epsilon_1} & & \\ & \ddots & \\ & & I_{\epsilon_N} \end{pmatrix}, \quad I_{\epsilon_k} = \begin{pmatrix} I & \\ & I \end{pmatrix}_{2n_k \times 2n_k} \quad k = 1, \cdots, N, \qquad (2.100)$$

$$\langle Y(\mu)| = (\langle Y_1^1(\mu)|, \langle Y_2^1(\mu)|, \cdots, \langle Y_1^N(\mu)|, \langle Y_2^N(\mu)|),$$
$$\langle \tilde{Y}(\mu)| = (\langle \tilde{Y}_1^1(\mu)|, \langle \tilde{Y}_2^1(\mu)|, \cdots, \langle \tilde{Y}_1^N(\mu)|, \langle \tilde{Y}_2^N(\mu)|),$$
$$\langle Y'(\mu)| = (\langle Y_1^{1'}(\mu)|, \langle Y_2^{1'}(\mu)|, \cdots, \langle Y_1^{N'}(\mu)|, \langle Y_2^{N'}(\mu)|), \qquad (2.101)$$
$$\langle \tilde{Y}'(\mu)| = (\langle \tilde{Y}_1^{1'}(\mu)|, \langle \tilde{Y}_2^{1'}(\mu)|, \cdots, \langle \tilde{Y}_1^{N'}(\mu)|, \langle \tilde{Y}_2^{N'}(\mu)|),$$

$$\langle Y_j^k(\mu)| = (Y_{j1}^k, \cdots, Y_{jn_k}^k), \quad Y_{js}^k = \frac{1}{(\mu-v_j^k)^s} + \frac{(-1)^s}{(\mu+v_j^{k*})^s},$$

$$\langle \tilde{Y}_j^k(\mu)| = (\tilde{Y}_{j1}^k, \cdots, \tilde{Y}_{jn_k}^k), \quad \tilde{Y}_{js}^k = \frac{1}{(\mu - v_j^{k*})^s} + \frac{(-1)^{s+1}}{(\mu + v_j^k)^s},$$

$$\langle Y_j^{k\prime}(\mu)| = (Y_{j1}^{k\prime}, \cdots, Y_{jn_k}^{k\prime}), \quad Y_{js}^{k\prime} = \frac{-s}{(\mu - v_j^k)^{s+1}} + \frac{s(-1)^{s+1}}{(\mu + v_j^{k*})^{s+1}},$$

$$\langle \tilde{Y}_j^{k\prime}(\mu)| = (\tilde{Y}_{j1}^{k\prime}, \cdots, \tilde{Y}_{jn_k}^{k\prime}), \quad \tilde{Y}_{js}^{k\prime} = \frac{-s}{(\mu - v_j^{k*})^{s+1}} + \frac{s(-1)^s}{(\mu + v_j^k)^{s+1}}.$$

Similar to the theorem 2.7, we give the solution of cCH equation with multiple high-order poles as follows.

THEOREM 2.11 *The solution of cCH equation with multiple high-order poles under the ZBCs can be obtained as follows*

$$u(x,t) = -\left(\frac{\tilde{M}_{11}'(i) + \tilde{M}_{21}'(i)}{\tilde{M}_{11}(i) + \tilde{M}_{21}(i)} + \frac{\tilde{M}_{12}'(i) + \tilde{M}_{22}'(i)}{\tilde{M}_{12}(i) + \tilde{M}_{22}(i)} \right), \qquad (2.102)$$

with

$$\tilde{M}_{11}(i) = \frac{\det(I_\epsilon + \Theta\tilde{\Theta} + |\Gamma\rangle\langle Y(i)|\Theta)}{\det(I_\epsilon + \Theta\tilde{\Theta})},$$

$$\tilde{M}_{21}(i) = -\frac{\det(I_\epsilon + \Theta^*\tilde{\Theta}^* + |\Gamma^*\rangle\langle \tilde{Y}^*(-i)|)}{\det(I_\epsilon + \Theta^*\tilde{\Theta}^*)} + 1,$$

$$\tilde{M}_{12}(i) = \frac{\det(I_\epsilon + \tilde{\Theta}\Theta + |\Gamma\rangle\langle \tilde{Y}(i)|)}{\det(I_\epsilon + \tilde{\Theta}\Theta)} - 1,$$

$$\tilde{M}_{22}(i) = \frac{\det(I_\epsilon + \tilde{\Theta}^*\Theta^* + |\Gamma^*\rangle\langle Y^*(-i)|\Theta^*)}{\det(I_\epsilon + \tilde{\Theta}^*\Theta^*)},$$

$$\tilde{M}_{11}'(i) = \frac{\det(I_\epsilon + \Theta\tilde{\Theta} + |\Gamma\rangle\langle Y'(i)|\Theta)}{\det(I_\epsilon + \Theta\tilde{\Theta})} - 1,$$

$$\tilde{M}_{21}'(i) = -\frac{\det(I_\epsilon + \Theta^*\tilde{\Theta}^* + |\Gamma^*\rangle\langle \tilde{Y}'^*(-i)|)}{\det(I_\epsilon + \Theta^*\tilde{\Theta}^*)} + 1,$$

$$\tilde{M}_{12}'(i) = \frac{\det(I_\epsilon + \tilde{\Theta}\Theta + |\Gamma\rangle\langle \tilde{Y}'(i)|)}{\det(I_\epsilon + \tilde{\Theta}\Theta)} - 1,$$

$$\tilde{M}_{22}'(i) = \frac{\det(I_\epsilon + \tilde{\Theta}^*\Theta^* + |\Gamma^*\rangle\langle Y'^*(-i)|\Theta^*)}{\det(I_\epsilon + \tilde{\Theta}^*\Theta^*)} - 1,$$

and $x = x(y,t)$ can be obtained by (Equation 2.41), the elements $|\Gamma\rangle$, I_ϵ, Θ, $\tilde{\Theta}$, $\langle Y(i)|$, $\langle \tilde{Y}(i)|$, $\langle Y'(i)|$ and $\langle \tilde{Y}'(i)|$ are defined in (Equation 2.96), (Equation 2.99), (Equation 2.100), and (Equation 2.101).

3. RH problem and high-order pole under NZBCs

3.1. Spectral analysis

After doing the transformations $u(x,t) = \hat{u}(x-t,t)+1$, $\hat{m} = \hat{u} - \hat{u}_{xx} + 1$ and $\hat{R} = \hat{u}^2 - \hat{u}_x^2 + 2\hat{u}$, the Lax pair (Equation 2.1) admits the forms

$$\hat{\phi}_x = \hat{U}\hat{\phi}, \quad \hat{\phi}_t = \hat{V}\hat{\phi}, \tag{3.1}$$

$$\hat{U} = \begin{pmatrix} -\frac{Q}{2} & \frac{\lambda\hat{m}}{2} \\ -\frac{\lambda\hat{m}}{2} & \frac{Q}{2} \end{pmatrix}, \hat{V} = \begin{pmatrix} \frac{Q}{\lambda^2} + \frac{Q\hat{R}}{2} & -\frac{\hat{u}-Q\hat{u}_x+1}{\lambda} - \frac{\lambda\hat{R}\hat{m}}{2} \\ \frac{\hat{u}+Q\hat{u}_x+1}{\lambda} + \frac{\lambda\hat{R}\hat{m}}{2} & -\frac{Q}{\lambda^2} - \frac{Q\hat{R}}{2} \end{pmatrix}. \tag{3.2}$$

Using the similar approach with ZBCs, we have the gauge transformation

$$\hat{\psi} = F^{-1}\hat{\phi}e^{\hat{J}}, \tag{3.3}$$

with

$$F = \begin{pmatrix} 1 & \frac{\lambda}{\sqrt{Q^2-\lambda^2}+Q} \\ \frac{\lambda}{\sqrt{Q^2-\lambda^2}+Q} & 1 \end{pmatrix},$$

where we define $\hat{J} = p\sigma_3$. $\hat{\psi}$ satisfies the asymptotic conditions

$$\hat{\psi}(x,t,\lambda) \sim I, \quad x \to \pm\infty.$$

The parameter $p(x,t,\lambda)$ is defined as follows

$$p = \frac{\sqrt{1-2\lambda^2}}{2}\left(x - \int_x^{+\infty}(\hat{m}(\xi,t)-1)d\xi - \frac{2}{\lambda^2}t\right). \tag{3.4}$$

Equation (Equation 1.3) admits the following form Lax pair

$$\hat{\psi}_x + [\hat{J}_x, \hat{\psi}] = \hat{X}\hat{\psi}, \tag{3.5a}$$

$$\hat{\psi}_t + [\hat{J}_t, \hat{\psi}] = \hat{T}\hat{\psi}, \tag{3.5b}$$

with

$$\hat{X} = \frac{\lambda(\hat{m}-1)Q}{2\sqrt{1-2\lambda^2}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \frac{(\hat{m}-1)Q^2}{2\sqrt{1-2\lambda^2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\hat{T} = \frac{1}{2\sqrt{1-2\lambda^2}}\left(\lambda\hat{R}(\hat{m}-1) + \frac{2\hat{u}}{\lambda}\right)Q\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \frac{\hat{u}_x Q}{\lambda}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$- \frac{1}{\sqrt{1 - 2\lambda^2}} \left(\hat{u} + \frac{1}{2}((\hat{m} - 1)\hat{R} + 2\hat{u})Q^2 - \lambda^2\hat{u} \right) \sigma_3,$$

where $\hat{J} = \hat{J}(x, t, \lambda)$, $\hat{X} = \hat{X}(x, t, \lambda)$, $\hat{T} = \hat{T}(x, t, \lambda)$ are 2×2 matrices.

A new spectral parameter k satisfied $2\lambda^2 = 4k^2 + 1$ is introduced, and to avoid multi-value problem, we define

$$\lambda = -\frac{1}{2\sqrt{2}} \left(\mu + \frac{1}{\mu} \right), \quad k = \frac{1}{4} \left(\mu - \frac{1}{\mu} \right), \tag{3.6}$$

then we have

$$p = -\frac{i(\mu^2 - 1)}{4\mu} \left(\int_x^{+\infty} (\hat{m}(\xi, t) - 1)d\xi - x + \frac{16\mu^2}{(\mu^2 + 1)^2}t \right), \tag{3.7}$$

and

$$\hat{X}(x, t, \mu) = \frac{i((\mu^2 + 1)^2 - 8\mu^2)(\hat{m} - 1)}{8\mu(\mu^2 - 1)}\sigma_3$$
$$+ \frac{i(\mu^2 + 1)\sqrt{8\mu^2 - (\mu^2 + 1)^2}(\hat{m} - 1)}{8|\mu|(\mu^2 - 1)} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

the two Jost solutions of ([Equation 3.5a](#)) with the parameter μ are completely determined by Volterra integrable equations

$$\hat{\psi}_\pm(x, t, \mu) = I + \int_{\pm\infty}^x e^{\frac{i(\mu^2 - 1)}{4\mu} \int_x^\xi \hat{m}(\tau, t)d\tau \hat{\sigma}_3} \left[\hat{X}(\xi, t, \mu)\hat{\psi}_\pm(\xi, t, \mu) \right] d\xi. \tag{3.8}$$

The Jost eigenfunctions $\hat{\psi}_\pm(x, t, \mu)$ have the analytic properties, symmetry properties, and asymptotic properties shown in the following three propositions.

PROPOSITION 3.1. *Providing that $u_0(x) - 1 \in L^1(\mathbb{R}^\pm)$, the Jost eigenfunctions $\hat{\psi}_\pm(x, t, \mu)$ have the following analytic properties. $\hat{\psi}_{-,1}(x, t, \mu)$ and $\hat{\psi}_{+,2}(x, t, \mu)$ are analytic in $\mathbb{C}^+$ and continuously extended to $\mathbb{C}^+ \cup \mathbb{R}$ except the points $\mu = \pm 1$, $\hat{\psi}_{+,1}(x, t, \mu)$ and $\hat{\psi}_{-,2}(x, t, \mu)$ are analytic in $\mathbb{C}^-$ and continuously extended to $\mathbb{C}^- \cup \mathbb{R}$ except the points $\mu = \pm 1$, there $\hat{\psi}_{\pm,j}(x, t, \mu)$ $(j = 1, 2)$ represent the j-th column of $\hat{\psi}_\pm(x, t, \mu)$, $\mathbb{C}^+ = \{\mu \in \mathbb{C} | Im\mu > 0\}$ and $\mathbb{C}^- = \{\mu \in \mathbb{C} | Im\mu < 0\}$.*

PROPOSITION 3.2. *The Jost eigenfunctions $\hat{\psi}_\pm(x, t, \mu)$ have the following symmetry conditions*

$$\hat{\psi}_\pm(\mu) = \sigma_1\hat{\psi}_\pm^*(\mu^*)\sigma_1 = \sigma_2\hat{\psi}_\pm(-\mu)\sigma_2 = \sigma_1\hat{\psi}_\pm(\frac{1}{\mu})\sigma_1. \tag{3.9}$$

PROPOSITION 3.3. *The Jost eigenfunctions* $\hat{\psi}_\pm(x,t,\mu)$ *possess asymptotic behaviour in the* μ-*plane*

$$
\begin{aligned}
\hat{\psi}_\pm(x,t,\mu) &= I + \mathcal{O}(\frac{1}{\mu}), \quad \mu \to \infty, \\
\hat{\psi}_\pm(x,t,\mu) &= I + \mathcal{O}(\mu), \quad \mu \to 0, \\
\hat{\psi}_\pm(x,t,\mu) &= \frac{i}{4(\mu-1)}\alpha_\pm(x,t)\Sigma_1 + \mathcal{O}(1), \quad \mu \to 1, \\
\hat{\psi}_\pm(x,t,\mu) &= -\frac{i}{4(\mu+1)}\alpha_\pm(x,t)\Sigma_2 + \mathcal{O}(1), \quad \mu \to -1,
\end{aligned}
\tag{3.10}
$$

there $\alpha_\pm(x,t) \in \mathbb{R}$, $\Sigma_1 = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$ *and* $\Sigma_2 = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$.

REMARK 3.4. It should be noted that the Lax pair (Equation 2.1) for the cCH equation under ZBCs has singularities at points $\mu = 0$, $\mu = \infty$ and branch cut points $\mu = \pm i$ in the extended complex μ-plane. Besides, the Lax pair (Equation 3.1) for the cCH equation under NZBCs in addition to possessing the above singularities also has the new singularities at points $\mu = \pm 1$. Therefore, the asymptotic behaviour of these points should be controlled.

Because $\hat{\phi}_\pm(x,t,\mu)$ are two fundamental matrix solutions of the Lax pair (Equation 3.1), there exists a linear relation between them.

$$
\hat{\phi}_-(x,t,\mu) = \hat{\phi}_+(x,t,\mu)S(\mu),
\tag{3.11}
$$

using the transformation (Equation 3.3), matrix $S(\mu)$ can be introduced as follows

$$
\hat{\psi}_-(x,t,\mu) = \hat{\psi}_+(x,t,\mu)e^{-p(x,t,\mu)\sigma_3}S(\mu)e^{p(x,t,\mu)\sigma_3},
\tag{3.12}
$$

there $\mu \in \mathbb{R}$, $\mu \neq \pm 1$. On the basis of symmetry conditions of $\hat{\psi}_\pm(x,t,\mu)$, the matrix $S(\mu)$ can be expressed as

$$
S(\mu) = \begin{pmatrix} a(\mu) & b^*(\mu^*) \\ b(\mu) & a^*(\mu^*) \end{pmatrix}.
\tag{3.13}
$$

Taking into account the properties of the Jost eigenfunctions $\hat{\psi}_\pm(x,t,\mu)$, we obtain the corresponding properties to the elements $a(\mu)$ and $b(\mu)$ of the scattering matrix $S(\mu)$ as follows.

PROPOSITION 3.5. *The scattering data* $a(\mu)$ *and* $b(\mu)$ *satisfy the following relations*

  (i) $a(\mu)$ *is analytic in* $\mathbb{C}^+$ *and continuously extended to* $\mathbb{C}^+ \cup \mathbb{R}$ *except the points* $\mu = \pm 1$, $a(\mu) \to 1$ *as* $\mu \to \infty$,
  (ii) $b(\mu)$ *is continuous in* $\mu \in \mathbb{R}\backslash\{-1,0,1\}$, $b(\mu) \to 0$ *as* $\mu \to \infty$,
  (iii) $a(\mu) = \frac{i\beta}{4(\mu-1)} + \mathcal{O}(1)$, $b(\mu) = \frac{i\beta}{4(\mu-1)} + \mathcal{O}(1)$, $\mu \to 1$, $\beta \in \mathbb{R}$,

(iv) $a(\mu) = \frac{i\beta}{4(\mu+1)} + \mathcal{O}(1), \quad b(\mu) = -\frac{i\beta}{4(\mu+1)} + \mathcal{O}(1), \quad \mu \to -1, \, \beta \in \mathbb{R},$
(v) $a^*(\mu^*) = a(-\mu) = a(\frac{1}{\mu}), \quad b^*(\mu^*) = -b(-\mu) = b(\frac{1}{\mu}),$
(vi) $|a(\mu)|^2 - |b(\mu)|^2 = 1, \quad \mu \in \mathbb{R}, \, \mu \neq \pm 1.$

REMARK 3.6. It should be noted that $\beta \neq 0$ of proposition 3.5 is generic and when $\beta = 0$, we have $a_1^2 = 1 + b_1^2$ with $a(\pm 1) = a_1$, $b(\pm 1) = \pm b_1$, $a_1, b_1 \in \mathbb{R}$. According to (Equation 3.12), $\alpha_+(x,t) = (a_1 - b_1)\alpha_-(x,t)$.

The analytic properties of $\hat{\psi}_\pm(x,t,\mu)$ stated above allow rewriting the relation (Equation 3.12) as a sectionally meromorphic matrix.

3.2. RH problem

A piecewise meromorphic 2×2-matrix valued function $M(x,t,\mu)$ is introduced as follows

$$M(x,t;\mu) = \begin{cases} \left(\dfrac{\hat{\psi}_{-,1}(x,t;\mu)}{a(\mu)}, \hat{\psi}_{+,2}(x,t;\mu) \right), & \text{as } Im\mu > 0, \\[3mm] \left(\hat{\psi}_{+,1}(x,t;\mu), \dfrac{\hat{\psi}_{-,2}(x,t;\mu)}{a^*(\mu^*)} \right), & \text{as } Im\mu < 0. \end{cases} \tag{3.14}$$

At the same time, we define the reflection coefficient

$$r(\mu) = \frac{b(\mu)}{a(\mu)}, \quad \mu \in \mathbb{R}. \tag{3.15}$$

RHP 3.7. *Find a matrix* $M(\mu) = M(x,t,\mu)$ *satisfies the following conditions*

(i) Analyticity: the matrix $M(\mu)$ *is a meromorphic function in* $\mathbb{C}\backslash\mathbb{R}$.
(ii) Jump condition:

$$M_+(x,t,\mu) = M_-(x,t,\mu)G(x,t,\mu), \quad \mu \in \mathbb{R}, \mu \neq \pm 1, \tag{3.16}$$

where

$$G(x,t,\mu) = e^{-p(x,t,\mu)\sigma_3}G_0(\mu)e^{p(x,t,\mu)\sigma_3}, \tag{3.17}$$

and

$$G_0(\mu) = \begin{pmatrix} 1 - r(\mu)r^*(\mu^*) & -r^*(\mu^*) \\ r(\mu) & 1 \end{pmatrix}, \tag{3.18}$$

$p(x,t,\mu)$ *is shown in (Equation 3.7).*
(iii) Normalization condition:

$$M(x,t,\mu) \to I, \quad \mu \to \infty. \tag{3.19}$$

(iv) Singularity condition:

$$M(x,t,\mu) = \begin{cases} \frac{i\alpha_+(x,t)}{4(\mu-1)} \begin{pmatrix} -D & 1 \\ -D & 1 \end{pmatrix} + \mathcal{O}(1), & \mu \to 1, \\[2em] -\frac{i\alpha_+(x,t)}{4(\mu+1)} \begin{pmatrix} D & 1 \\ -D & -1 \end{pmatrix} + \mathcal{O}(1), & \mu \to -1, \end{cases} \tag{3.20}$$

where $\alpha_+(x,t) \in \mathbb{R}$ *and* $Im\mu > 0$,

$$D := \begin{cases} 0, & \beta \neq 0, \\[1em] \frac{a_1+b_1}{a_1}, & \beta = 0, \end{cases} \tag{3.21}$$

$a_1 = a(1)$, $b_1 = b(1)$ *and* $\beta := -2i \lim_{\mu \to 1} (\mu - 1)a(\mu)$.

(v) Symmetry conditions:

$$M(\mu) = \sigma_1 M^*(\mu^*)\sigma_1 = \sigma_2 M(-\mu)\sigma_2 = \sigma_1 M(\frac{1}{\mu})\sigma_1. \tag{3.22}$$

(vi) $\det M(x,t,\mu) \equiv 1$.

We substitute Eqs. ([Equation 3.6](#)) into Lax pair ([Equation 3.5a](#)) and take the new transformation

$$\hat{\psi}_0 = \hat{\varphi}e^{\hat{J}_0}, \tag{3.23}$$

with

$$p_0 = \frac{i(\mu^2 - 1)}{4\mu}x - \frac{4i(\mu^2 - 1)\mu}{(\mu^2 + 1)^2}t, \tag{3.24}$$

there $\hat{J}_0 = p_0\sigma_3$, the Lax pair can be converted to

$$\hat{\psi}_{0x} + [\hat{J}_{0x}, \hat{\psi}_0] = \hat{X}_0\hat{\psi}_0, \tag{3.25a}$$

$$\hat{\psi}_{0t} + [\hat{J}_{0t}, \hat{\psi}_0] = \hat{T}_0\hat{\psi}_0, \tag{3.25b}$$

with

$$\hat{X}_0(x,t,\mu) = -\frac{(\mu^2 + 1)\sqrt{8\mu^2 - (\mu^2 + 1)^2}(\hat{m} - 1)}{8|\mu|(\mu^2 - 1)}\sigma_2$$
$$+ \left(\frac{i((\mu^2 + 1)^2 - 8\mu^2)(\hat{m} - 1)}{8\mu(\mu^2 - 1)} - \frac{i(\hat{m} - 1)(\mu^2 - 1)}{4\mu} \right)\sigma_3,$$

$$\hat{T}_0(x,t,\mu) = \frac{i(\mu^2-1)}{4\mu}\hat{R}\hat{m}\sigma_3 + \frac{i(\mu^2-1)}{2\mu}\hat{u}\sigma_3 + \hat{T}(x,t,\mu).$$

Then the Jost solutions of Equation 3.25a are expressed as

$$\hat{\psi}_{0\pm}(x,t,\mu) = I + \int_{\pm\infty}^{x} e^{-\frac{i(\mu^2-1)}{4\mu}(x-\xi)\hat{\sigma}_3}\left[\hat{X}_0(\xi,t,\mu)\hat{\psi}_{0\pm}(\xi,t,\mu)\right]d\xi. \qquad (3.26)$$

Utilizing the transformation (Equation 3.23), there exist matrices $r_\pm(\mu)$ satisfying

$$\hat{\psi}_\pm(x,t,\mu) = \hat{\psi}_{0\pm}(x,t,\mu)e^{-\hat{J}_0}r_\pm(\mu)e^{\hat{J}}. \qquad (3.27)$$

Thus $r_\pm(\mu) = e^{\hat{J}_0(\pm\infty,t,\mu)-\hat{J}(\pm\infty,t,\mu)}$. On the basis of (Equation 3.7) and (Equation 3.24),

$$r_+(\mu) = I, \quad r_-(\mu) = e^{\frac{i(\mu^2-1)}{4\mu}\int_{-\infty}^{+\infty}(\hat{m}(\xi,t)-1)d\xi\sigma_3}. \qquad (3.28)$$

Because $\hat{X}_0(x,t,\pm i) \equiv 0$, making use of (Equation 3.26) and $a(\mu) = \det(\hat{\psi}_{-,1},\hat{\psi}_{+,2})$, we have

$$M(x,t,i) = \begin{pmatrix} e^{\frac{1}{2}\int_x^{+\infty}(\hat{m}(\xi,t)-1)d\xi} & 0 \\ 0 & e^{-\frac{1}{2}\int_x^{+\infty}(\hat{m}(\xi,t)-1)d\xi} \end{pmatrix}, \qquad (3.29)$$

with symmetry of $M(x,t,\mu)$, we can obtain $M(x,t,-i)$. Besides, the symmetry conditions (Equation 3.22) indicate $M(i)$ is a diagonal matrix with real entries, taking into account $\det M \equiv 1$, there exists $f(x,t) \in \mathbb{R}$ such that $M_{11}(x,t,i) = f(x,t)$ and $M_{22}(x,t,i) = \frac{1}{f(x,t)}$.

Based on the new space variable $y(x,t) = x - \int_x^{+\infty}(\hat{m}(\xi,t)-1)d\xi$ and $\tilde{M}(y,t,\mu) = M(x,t,\mu)$, we have a new RH problem.

RHP 3.8. *Find a matrix $\tilde{M}(\mu) = \tilde{M}(y,t,\mu)$ satisfies the following conditions*

(i) *Analyticity: the matrix $\tilde{M}(\mu)$ is a meromorphic function in $\mathbb{C}\backslash\mathbb{R}$.*
(ii) *Jump condition:*

$$\tilde{M}_+(y,t,\mu) = \tilde{M}_-(y,t,\mu)\tilde{G}(y,t,\mu), \quad \mu \in \mathbb{R}, \mu \neq \pm 1, \qquad (3.30)$$

where

$$\tilde{G}(y,t,\mu) = e^{-\tilde{p}(y,t,\mu)\sigma_3}G_0(\mu)e^{\tilde{p}(y,t,\mu)\sigma_3}, \qquad (3.31)$$

$$\tilde{p}(y,t,\mu) = \frac{i(\mu^2-1)}{4\mu}y - \frac{4i(\mu^2-1)\mu}{(\mu^2+1)^2}t, \qquad (3.32)$$

$G_0(\mu)$ *defined by (Equation 3.18).*

(iii) Normalization condition:

$$\tilde{M}(y,t,\mu) \to I, \quad \mu \to \infty. \qquad (3.33)$$

(iv) Singularity condition:

$$\tilde{M}(y,t,\mu) = \begin{cases} \frac{i\tilde{\alpha}_+(y,t)}{4(\mu-1)} \begin{pmatrix} -D & 1 \\ -D & 1 \end{pmatrix} + \mathcal{O}(1), & \mu \to 1, \\[4mm] -\frac{i\tilde{\alpha}_+(y,t)}{4(\mu+1)} \begin{pmatrix} D & 1 \\ -D & -1 \end{pmatrix} + \mathcal{O}(1), & \mu \to -1, \end{cases} \qquad (3.34)$$

with some $\tilde{\alpha}_+(y,t) \in \mathbb{R}$ and $Im\mu > 0$.

(v) Symmetry conditions:

$$\tilde{M}(\mu) = \sigma_1 \tilde{M}^*(\mu^*)\sigma_1 = \sigma_2 \tilde{M}(-\mu)\sigma_2 = \sigma_1 \tilde{M}(\frac{1}{\mu})\sigma_1. \qquad (3.35)$$

(vi) $\det \tilde{M}(y,t,\mu) \equiv 1.$

The implicit expression of the potential function u under NZBCs is constructed by using the solution of RH problem.

THEOREM 3.9 *Consider that $\tilde{M}(y,t,\mu)$ is a solution of RH problem 3.8, according to the definition of $\tilde{M}(y,t,\mu)$ at $\mu = i$, we make $\hat{\Phi}_1(x(y,t),t) = \tilde{\Phi}_1(y,t) = \tilde{M}_{11}(y,t,i) + \tilde{M}_{21}(y,t,i)$ and $\hat{\Phi}_2(x(y,t),t) = \tilde{\Phi}_2(y,t) = \tilde{M}_{12}(y,t,i) + \tilde{M}_{22}(y,t,i)$, then the solution $\hat{u}(x,t)$ of Cauchy problem (Equation 1.3) has the following representation*

$$\left(\hat{R} + \frac{2\hat{u}}{\hat{m}} \right)(\hat{m} + 1) = \partial_t \ln \frac{\tilde{\Phi}_1(y,t)}{\tilde{\Phi}_2(y,t)}, \qquad (3.36)$$

with

$$\hat{R} = \hat{u}^2 - \hat{u}_x^2 + 2\hat{u}, \quad \hat{m} = \hat{u} - \hat{u}_{xx} + 1.$$

Proof. Considering the new notations $\tilde{R}(y,t) = \hat{R}(x(y,t),t)$, $\tilde{m}(y,t) = \hat{m}(x(y,t),t)$, $\tilde{u}(y,t) = \hat{u}(x(y,t),t)$ and $\tilde{u}_x(y,t) = \hat{u}_x(x(y,t),t)$, with regard to $x(y(x,t),t) = x$, we have

$$\partial_t(x(y(x,t),t)) = x_y(y,t)y_t(x,t) + x_t(y,t) = 0. \qquad (3.37)$$

Since $y(x,t) = x - \int_x^{+\infty}(\hat{m}(\xi,t) - 1)d\xi$, then

$$x_y(y,t) = \frac{1}{\tilde{m}(y,t)}. \qquad (3.38)$$

Based on $\hat{m}_t = -(\hat{m}\hat{R})_x - 2\hat{u}_x$ and unite (Equation 3.37)–(Equation 3.38), we have

$$x_t(y,t) = \tilde{R}(y,t) + \frac{2\tilde{u}(y,t)}{\tilde{m}(y,t)}. \tag{3.39}$$

By (Equation 3.29), we have

$$\hat{\Phi}_1(x,t) = e^{\frac{1}{2}\int_x^{+\infty}(\tilde{m}(\xi,t)-1)d\xi}, \quad \hat{\Phi}_2(x,t) = e^{-\frac{1}{2}\int_x^{+\infty}(\tilde{m}(\xi,t)-1)d\xi}, \tag{3.40}$$

then

$$x = y(x,t) + \ln\frac{\tilde{\Phi}_1(y,t)}{\tilde{\Phi}_2(y,t)}. \tag{3.41}$$

Differential equation (Equation 3.41) with respect to t,

$$x_t = -(\tilde{R}(y,t)\tilde{m}(y,t) + 2\tilde{u}(y,t)) + \partial_t \ln\frac{\tilde{\Phi}_1(y,t)}{\tilde{\Phi}_2(y,t)}, \tag{3.42}$$

making further calculation, we can get (Equation 3.36), then the theorem 3.9 can be proved. □

Next, in order to obtain the solution of Lax pair (Equation 3.1) represented by the new variable $y(x,t)$, using the gauge transformation $\hat{\varphi} = F^{-1}\hat{\phi}$, there we have $\tilde{\varphi}(y,t,\lambda) = \hat{\varphi}(x,t,\lambda)$. Taking (Equation 3.38) and (Equation 3.39) into consideration, the Lax pair (Equation 3.1) turns into

$$\tilde{\varphi}_y = \tilde{U}\tilde{\varphi}, \quad \tilde{\varphi}_t = \tilde{V}\tilde{\varphi}, \tag{3.43}$$

there $\tilde{U} = \tilde{X} - ik\sigma_3$ and $\tilde{V} = \tilde{T} + \frac{2ik}{\lambda^2}\sigma_3$. Then we change the parameters to μ and expanding $\tilde{X}$ and $\tilde{T}$ at the singularities $\mu = \pm 1, \pm i$

$$\tilde{\varphi}_y + \frac{i(\mu^2 - 1)}{4\mu}\sigma_3\tilde{\varphi} = \tilde{X}_1\tilde{\varphi}, \tag{3.44a}$$

$$\tilde{\varphi}_t - \frac{4i(\mu^2 - 1)\mu}{(\mu^2 + 1)^2}\sigma_3\tilde{\varphi} = \tilde{T}_1\tilde{\varphi}, \tag{3.44b}$$

with

$$\tilde{X}_1 = -\frac{iv_1}{\mu - 1}\Sigma_1 + \frac{iv_1}{\mu + 1}\Sigma_2 + v_1\sigma_2,$$

$$\tilde{T}_1 = -\frac{iv_2}{\mu - 1}\Sigma_1 + \frac{iv_2}{\mu + 1}\Sigma_2 + \frac{v_3\Delta_1 + v_4\Delta_2}{\mu - i} + \frac{v_3\Delta_2 + v_4\Delta_1}{\mu + i},$$

where $\Delta_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\Delta_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, Σ_1 and Σ_2 are shown in proposition 3.3. Besides, we also have $v_1(y,t) = -\frac{\tilde{m}-1}{2\tilde{m}}$, $v_2(y,t) = \tilde{u}$, $v_3(y,t) = -\tilde{u} - \tilde{u}_x$, and $v_4(y,t) = \tilde{u} - \tilde{u}_x$.

PROPOSITION 3.10. *Assume $\tilde{M}(y,t,\mu)$ is a solution of RH problem 3.8, then*

$$\tilde{\varphi}(y,t,\mu) = \tilde{M}(y,t,\mu)e^{-\tilde{p}(y,t,\mu)\sigma_3} \tag{3.45}$$

satisfies the Lax pair (Equation 3.44a), there $\tilde{p}(y,t,\mu)$ is shown in (Equation 3.32).

Proof.

(i) First of all, we think about the y differential part of the Lax pair (Equation 3.44a). From (Equation 3.30) and (Equation 3.45), we can obtain

$$\tilde{\varphi}_y\tilde{\varphi}^{-1} = \tilde{M}_y\tilde{M}^{-1} - \tilde{p}_y\tilde{M}\sigma_3\tilde{M}^{-1}, \tag{3.46}$$

then the possible subsistent singularities are $\mu = 0, 1, -1, \infty$. Now we estimate $\tilde{\varphi}_y\tilde{\varphi}^{-1}$ at these points. On the basis of the symmetry conditions (Equation 3.35), expanding $\tilde{M}(y,t,\mu)$ with μ at $\mu \to \infty$

$$\tilde{M} = I + \frac{1}{\mu}[w\sigma_3 + \tilde{w}(\Delta_1 + \Delta_2)] + \mathcal{O}(\mu^{-2}), \quad \mu \to \infty, \tag{3.47}$$

where $w = w(y,t) \in i\mathbb{R}$, $\tilde{w} = \tilde{w}(y,t) \in \mathbb{R}$. Considering the expansion of $\tilde{p}_y$ at $\mu \to \infty$, $\mu \to 0$, and $\mu \to \pm 1$, we have

$$\begin{aligned}
\tilde{\varphi}_y\tilde{\varphi}^{-1} &= -\frac{i\mu}{4}\sigma_3 - \frac{\tilde{w}}{2}\sigma_2 + \mathcal{O}(\mu^{-1}), \quad \mu \to \infty. \\
\tilde{\varphi}_y\tilde{\varphi}^{-1} &= \frac{i}{4\mu}\sigma_3 + \frac{\tilde{w}}{2}\sigma_2 + \mathcal{O}(\mu), \quad \mu \to 0. \\
\tilde{\varphi}_y\tilde{\varphi}^{-1} &= -\frac{i\alpha_1}{\mu-1}\Sigma_1 + \mathcal{O}(1), \quad \mu \to 1, \\
\tilde{\varphi}_y\tilde{\varphi}^{-1} &= \frac{i\alpha_1}{\mu+1}\Sigma_2 + \mathcal{O}(1), \quad \mu \to -1.
\end{aligned} \tag{3.48}$$

Uniting (Equation 3.48), since $\tilde{M}(y,t,i)$ is a diagonal matrix,

$$-\frac{i\alpha_1}{i-1}\Sigma_1 + \frac{i\alpha_1}{i+1}\Sigma_2 - \frac{\tilde{w}}{2}\sigma_2$$

is also diagonal, then $\frac{\tilde{w}(y,t)}{2} = -\alpha_1(y,t)$, and the y differential part of the Lax pair (Equation 3.44a) can be attained, where $v_1(y,t) = \alpha_1(y,t)$.

(ii) Secondly, using the similar method, we think over the t differential part of the Lax pair (Equation 3.44a). Due to

$$\tilde{p}_t(\mu) = \frac{2}{(\mu-i)^2} - \frac{2i}{\mu-i} + \mathcal{O}(1), \quad \mu \to i, \tag{3.49a}$$

$$\tilde{p}_t(\mu) = -\frac{2}{(\mu+i)^2} - \frac{2i}{\mu+i} + \mathcal{O}(1), \quad \mu \to -i, \qquad (3.49b)$$

then the possible subsistent singularities are $\mu = 0, 1, -1, i, -i, \infty$. Now estimating $\tilde{\varphi}_t \tilde{\varphi}^{-1} = \tilde{M}_t \tilde{M}^{-1} - \tilde{p}_t \tilde{M} \sigma_3 \tilde{M}^{-1}$ at these points at $\mu \to \infty$, $\mu \to 0$ and $\mu \to \pm 1$ we have $\tilde{p}_t(\mu) = \mathcal{O}(\mu^{-1})$, such that,

$$
\begin{aligned}
\tilde{\varphi}_t \tilde{\varphi}^{-1} &= \mathcal{O}(\mu^{-1}), \quad \mu \to \infty. \\
\tilde{\varphi}_t \tilde{\varphi}^{-1} &= \mathcal{O}(\mu), \quad \mu \to 0. \\
\tilde{\varphi}_t \tilde{\varphi}^{-1} &= -\frac{i\alpha_2}{\mu-1}\Sigma_1 + \mathcal{O}(1), \quad \mu \to 1, \\
\tilde{\varphi}_t \tilde{\varphi}^{-1} &= \frac{i\alpha_2}{\mu+1}\Sigma_2 + \mathcal{O}(1), \quad \mu \to -1.
\end{aligned} \qquad (3.50)
$$

Subsequently, on the basis of symmetries (Equation 3.35), estimating $\tilde{M}(\mu)$ at $\mu \to i$,

$$\tilde{M} = \begin{pmatrix} f_1 & 0 \\ 0 & f_1^{-1} \end{pmatrix} + \begin{pmatrix} 0 & f_2 \\ f_3 & 0 \end{pmatrix} (\mu - i) + \mathcal{O}((\mu-i)^2). \qquad (3.51)$$

We have

$$\tilde{\varphi}_t \tilde{\varphi}^{-1} = -\frac{2}{(\mu-i)^2}\sigma_3 + \frac{2}{\mu-i}\begin{pmatrix} i & 2f_2 f_1 \\ -2f_3 f_1^{-1} & -i \end{pmatrix} + \mathcal{O}(1). \qquad (3.52)$$

By symmetry, at $\mu \to -i$,

$$\tilde{\varphi}_t \tilde{\varphi}^{-1} = \frac{2}{(\mu+i)^2}\sigma_3 + \frac{2}{\mu+i}\begin{pmatrix} i & -2f_3 f_1^{-1} \\ 2f_2 f_1 & -i \end{pmatrix} + \mathcal{O}(1). \qquad (3.53)$$

There $f_j = f_j(y,t)$ $(j = 1, 2, 3)$. Uniting (Equation 3.50)–(Equation 3.53), we have $g_1(y,t) = 2f_2(y,t)f_1(y,t)$ and $g_2(y,t) = -2f_3(y,t)f_1^{-1}(y,t)$ are holomorphic functions in the whole complex μ-plane and vanished as $\mu \to \infty$. After that, the t differential part of the Lax pair (Equation 3.44a) can be attained, there $v_2(y,t) = \alpha_2(y,t)$.

$$\square$$

For the sake of testifying the compatibility conditions $\tilde{U}_t - \tilde{V}_y + [\tilde{U}, \tilde{V}] = 0$ of Lax pair (Equation 3.43) can be deduced Eq. (Equation 1.3) with variables (y,t), we have the theorem 3.11 and theorem 3.12.

THEOREM 3.11 *Equation (Equation 1.3) with variables (y,t) can be shown as follows*

$$\tilde{m}_t^{-1}(y,t) = 2\tilde{u}_x(y,t) - \frac{2\tilde{u}\tilde{m}_x}{\tilde{m}^3}(y,t) + \frac{2\tilde{u}_x}{\tilde{m}^2}(y,t), \qquad (3.54)$$

with

$$\tilde{m}(y,t) = \tilde{u}(y,t) - \tilde{u}_{xx}(y,t) + 1, \quad x_y(y,t) = \tilde{m}^{-1}(y,t).$$

Proof. According to equations $\hat{m}_t = -(\hat{m}\hat{R})_x - 2\hat{u}_x$, $x_t(y,t) = \tilde{R}(y,t) + \frac{2\tilde{u}(y,t)}{\tilde{m}(y,t)}$ and $\tilde{R}_x = 2\tilde{m}\tilde{u}_x$,

$$
\begin{aligned}
\tilde{m}_t(y,t) &= \tilde{m}_x(x(y,t),t)x_t(y,t) + \tilde{m}_t(x(y,t),t) \\
&= \left(-\frac{2\tilde{u}\tilde{m}_x}{\tilde{m}^3} + \frac{2\tilde{u}_x}{\tilde{m}^2} \right)(-\tilde{m}^2) + 2\tilde{u}_x(-\tilde{m}^2),
\end{aligned}
$$

then we have Eq. (Equation 3.54), and the theorem 3.11 can be proved. $\qquad\square$

THEOREM 3.12 *It can be demonstrable that the functions $\alpha_1(y,t)$, $\alpha_2(y,t)$, $g_1(y,t)$, and $g_2(y,t)$ of proposition 3.10 content the following algebraic system*

$$\alpha_{1t} + (g_1 + g_2) = 0, \tag{3.55a}$$

$$\alpha_2 + (g_1 - g_2) = 0, \tag{3.55b}$$

$$(g_1 - g_2)_y - (1 + 2\alpha_1)(g_1 + g_2) = 0, \tag{3.55c}$$

$$(g_1 + g_2)_y + 2\alpha_1 - (1 + 2\alpha_1)(g_1 - g_2) = 0, \tag{3.55d}$$

then the compatibility conditions of Lax pair (Equation 3.43) can be satisfied.

Proof. By the definitions about $\alpha_1(y,t)$, $\alpha_2(y,t)$, $g_1(y,t)$, and $g_2(y,t)$ which are shown in proposition 3.10, there $g_1(y,t) = 2f_2(y,t)f_1(y,t)$, $g_2(y,t) = -2f_3(y,t)f_1^{-1}(y,t)$. Subsequently, estimating the compatibility conditions $\tilde{U}_t - \tilde{V}_y + [\tilde{U}, \tilde{V}] = 0$ at these points $\mu = 0, 1, -1, i, -i, \infty$, the theorem 3.12 can be proved. $\square$

PROPOSITION 3.13. *In compliance with proposition 3.10 and theorem 3.11, the relations of $\tilde{m}$ and $\tilde{\mu}$ with notations $\{\alpha_1, \alpha_2, g_1, g_2\}$ can be available as follows*

$$\tilde{m} = \frac{1}{1 + 2\alpha_1}, \quad \tilde{u} = \alpha_2, \tag{3.56}$$

with the conditions $x_y = 1 + 2\alpha_1$ and $(\alpha_1\alpha_2)_x = -\frac{\alpha_{2x}}{2}$, we can derive Eq. (Equation 3.54) from (Equation 3.55a)–(Equation 3.55d).

Proof. Actually, (Equation 3.55a) can be obtained by (Equation 3.54), and from (Equation 3.55c) we have

$$\tilde{u}_x = \tilde{u}_y x_y^{-1} = -(g_1 + g_2).$$

According to

$$\tilde{m} = \tilde{u} - (\tilde{u}_x)_y \tilde{m} + 1,$$

we have

$$\frac{1}{1 + 2\alpha_1} = -(g_1 - g_2) + (g_1 + g_2)_y \frac{1}{1 + 2\alpha_1} + 1,$$

which is Eq. (Equation 3.55d). □

Combining Eqs. (Equation 3.56) and (Equation 3.55b), we can further obtain the relationship between the solution of cCH equation and the RH problem as shown in (Equation 3.82).

3.3. Single high-order pole solutions

In order to obtain a pure soliton solution, this section we will consider the reflectionless situation, i.e., $b(\mu) = 0$ ($\mu \in \mathbb{R}$). Then $\tilde{G}(\mu) = 0$ for $\mu \in \mathbb{R}$. Let $z_0 \in \mathbb{C}^+$ be the N-order zero point of the scattering data $a(\mu)$, then $\left\{ -z_0^*, -\frac{1}{z_0}, \frac{1}{z_0^*} \right\} \in \mathbb{C}^+$ are also the N-order zero points of the scattering data $a(\mu)$. Furthermore, $\left\{ z_0^*, -z_0, -\frac{1}{z_0^*}, \frac{1}{z_0} \right\} \in \mathbb{C}^-$ are the N-order zero points of the scattering data $a^*(\mu^*)$. The discrete spectrum is the set $\mathcal{X} = \left\{ \pm z_0, \pm z_0^*, \pm \frac{1}{z_0}, \pm \frac{1}{z_0^*} \right\}$, which is same as the case in figure 1. Taking $w_1 = z_0$, $w_2 = \frac{1}{z_0^*}$, the scattering data $a(\mu)$ has the following form

$$a(\mu) = \prod_{j=1}^{2} (\mu - w_j)^N (\mu + w_j^*)^N a_0(\mu), \tag{3.57}$$

where $a_0(\mu) \neq 0$ for all $\mu \in \mathbb{C}^+$. Therefore, from the definition of $\tilde{M}(y, t, \mu)$, it can be seen that $\mu = w_j$ and $\mu = -w_j^*$ ($j = 1, 2$) are N-order pole points of $\tilde{M}_{11}$. Simultaneously, $\mu = w_j^*$ and $\mu = -w_j$ ($j = 1, 2$) are N-order pole points of $\tilde{M}_{12}$. According to the definitions of (Equation 3.11) and (Equation 3.12), under the new scale $y(x, t)$, we define $\check{\phi}_\pm(y, t, \mu) = \hat{\phi}_\pm(x, t, \mu)$ and $\check{\psi}_\pm(y, t, \mu) = \hat{\psi}_\pm(x, t, \mu)$ and obtain the following relationships

$$\check{\phi}_-(y, t, \mu) = \check{\phi}_+(y, t, \mu) S(\mu), \tag{3.58}$$

and

$$\check{\psi}_-(y, t, \mu) e^{-it\theta(\mu)\sigma_3} = \check{\psi}_+(y, t, \mu) e^{-it\theta(\mu)\sigma_3} S(\mu), \tag{3.59}$$

there we redefine relation $\tilde{p}(\mu) = it\theta(\mu)$, $\tilde{p}(\mu)$ is shown in (Equation 3.32). Under the assumption that $\mu = w_j$ ($j = 1, 2$) are N-order zero points of $a(\mu)$, from the

following two relations, we have a Wronskian form

$$a(\mu) = Wr(\check{\phi}_{-1}, \check{\phi}_{+2}), \tag{3.60}$$

then $a(\mu)$ vanishes to N-order at zero points w_j $(j = 1, 2)$, there exist complex constants $b_{j,s}$ $(j = 1, 2)$ $(s = 1, 2, \cdots, N)$ satisfy the follows

$$\frac{\partial^m [\check{\phi}_+(y, t, w_j)]_2}{\partial \mu^m} = \sum_{l=0}^{m} \begin{pmatrix} m \\ l \end{pmatrix} b_{j,m-l+1} \frac{\partial^l [\check{\phi}_-(y, t, w_j)]_1}{\partial \mu^l}, \tag{3.61}$$

and

$$\frac{\partial^m [\check{\psi}_+(y, t, w_j)]_2}{\partial \mu^m} = \sum_{l=0}^{m} \begin{pmatrix} m \\ l \end{pmatrix} b_{j,m-l+1} \frac{\partial^l [\check{\psi}_-(y, t, w_j)]_1 e^{2it\theta(w_j)}}{\partial \mu^l}, \tag{3.62}$$

with $m = 0, 1, \ldots, N - 1$. From the normalization condition of RH problem 3.8, one can set

$$\tilde{M}_{11} = 1 + \sum_{j=1}^{2} \sum_{s=1}^{N} \left(\frac{1}{(\mu - w_j)^s} + \frac{(-1)^s}{(\mu + w_j^*)^s} \right) F_{j,s} - \frac{i\tilde{\alpha}_+(y, t)}{4(\mu - 1)} D - \frac{i\tilde{\alpha}_+(y, t)}{4(\mu + 1)} D, \tag{3.63a}$$

$$\tilde{M}_{12} = \sum_{j=1}^{2} \sum_{s=1}^{N} \left(\frac{1}{(\mu - w_j^*)^s} + \frac{(-1)^{s+1}}{(\mu + w_j)^s} \right) G_{j,s} + \frac{i\tilde{\alpha}_+(y, t)}{4(\mu - 1)} - \frac{i\tilde{\alpha}_+(y, t)}{4(\mu + 1)}, \tag{3.63b}$$

where D is shown in (Equation 3.21) and $\tilde{\alpha}_+(y, t) \in \mathbb{R}$. Besides, $F_{j,s} = F_{j,s}(y, t)$ and $G_{j,s} = G_{j,s}(y, t)$ are undetermined functions. Once these functions are solved, the solution of cCH equation will be acquired. To address this problem, making use of Taylor series expansion, one has

$$e^{2it\theta(\mu)} = \sum_{l=0}^{+\infty} \zeta_{j,l}(y, t)(\mu - w_j)^l, \tag{3.64}$$

$$\zeta_{j,l}(y, t) = \lim_{\mu \to w_j} \frac{1}{l!} \frac{\partial^l}{\partial \mu^l} e^{2it\theta(\mu)}. \tag{3.65}$$

It can be obtained that the coefficient of item $(\mu - w_j)^{-s}$ of function $\tilde{M}_{11}(y, t, \mu)$ is $F_{j,s}(y, t)$. Now we extend the residue theorem by combining (Equation 3.59) and

([Equation 3.62](#)) and obtain the following relations

$$
\begin{aligned}
F_{j,s} = (-1)^m \frac{i\tilde{\alpha}_+(y,t)}{4} \sum_{l=s}^{N} \sum_{m=0}^{l-s} \left(\frac{r_{j,l}\zeta_{j,l-s-m}}{(w_j-1)^{m+1}} - \frac{r_{j,l}\zeta_{j,l-s-m}}{(w_j+1)^{m+1}} \right) \\
+ \sum_{l=s}^{N} \sum_{m=0}^{l-s} \sum_{p=1}^{2} \sum_{q=1}^{N} \binom{q+m-1}{m} r_{j,l}\zeta_{j,l-s-m} \times \\
\left\{ \frac{(-1)^m}{(w_j-w_p^*)^{q+m}} + \frac{(-1)^{m+q+1}}{(w_j+w_p)^{q+m}} \right\} G_{p,q},
\end{aligned}
\tag{3.66}
$$

where

$$
r_{j,l} = \lim_{\mu \to v_j} \frac{b_{j,l}}{(N-l)!} \frac{\partial^{N-l}}{\partial(\mu-w_j)^{N-l}} \frac{(\mu-w_j)^N}{a(\mu)}.
\tag{3.67}
$$

Likewise, the coefficient of item $(\mu-w_j^*)^{-s}$ of function $\tilde{M}_{12}(y,t,\mu)$ is $G_{j,s}(y,t)$. Through the same method, we have

$$
\begin{aligned}
G_{j,s} = (-1)^{m+1} \frac{i\tilde{\alpha}_+(y,t)}{4} D \sum_{l=s}^{N} \sum_{m=0}^{l-s} \left(\frac{r_{j,l}^*\zeta_{j,l-s-m}^*}{(w_j^*-1)^{m+1}} + \frac{r_{j,l}^*\zeta_{j,l-s-m}^*}{(w_j^*+1)^{m+1}} \right) \\
+ \sum_{l=s}^{N} \sum_{m=0}^{l-s} + \sum_{l=s}^{N} \sum_{m=0}^{l-s} \sum_{p=1}^{2} \sum_{q=1}^{N} \binom{q+m-1}{m} \times \\
r_{j,l}^*\zeta_{j,l-s-m}^* \left\{ \frac{(-1)^m}{(w_j^*-w_p)^{q+m}} + \frac{(-1)^{m+q}}{(w_j^*+w_p^*)^{q+m}} \right\} F_{p,q},
\end{aligned}
\tag{3.68}
$$

The system ([Equation 2.58](#)) has the following solution For the sake of convenience, selecting the suitable notations for $j,p=1,2$,

$$
|\eta_j^1\rangle = (\eta_{j1}^1, \ldots, \eta_{jN}^1)^T, \quad |\eta_j^2\rangle = (\eta_{j1}^2, \ldots, \eta_{jN}^2)^T,
\tag{3.69}
$$

$$
\begin{aligned}
\eta_{js}^1 &= \frac{i\tilde{\alpha}_+(y,t)}{4} \sum_{l=s}^{N} \sum_{m=0}^{l-s} \left(\frac{(-1)^m}{(w_j-1)^{m+1}} - \frac{(-1)^m}{(w_j+1)^{m+1}} \right) r_{j,l}\zeta_{j,l-s-m}, \\
\eta_{js}^2 &= \sum_{l=s}^{N} \sum_{m=0}^{l-s} r_{j,l}^*\zeta_{j,l-s-m}^* \\
&\quad - \frac{i\tilde{\alpha}_+(y,t)}{4} D \sum_{l=s}^{N} \sum_{m=0}^{l-s} \left(\frac{(-1)^m}{(w_j^*-1)^{m+1}} + \frac{(-1)^m}{(w_j^*+1)^{m+1}} \right) r_{j,l}^*\zeta_{j,l-s-m}^*,
\end{aligned}
$$

then defining the $N \times N$ matrices $\Omega_{j,p} = [\Omega_{j,p}]_{s,q}$ and $\tilde{\Omega}_{j,p} = [\tilde{\Omega}_{j,p}]_{s,q}$ for $s, q = 1, 2, \cdots, N$ as follows

$$\Omega_{j,p} = \sum_{l=s}^{N} \sum_{m=0}^{l-s} \left(\begin{array}{c} q+m-1 \\ m \end{array} \right) r_{j,l} \zeta_{j,l-s-m} \left\{ \frac{(-1)^m}{(w_j - w_p^*)^{q+m}} + \frac{(-1)^{m+q+1}}{(w_j + w_p)^{q+m}} \right\},$$
(3.70)

$$\tilde{\Omega}_{j,p} = \sum_{l=s}^{N} \sum_{m=0}^{l-s} \left(\begin{array}{c} q+m-1 \\ m \end{array} \right) r_{j,l}^* \zeta_{j,l-s-m}^* \left\{ \frac{(-1)^m}{(w_j^* - w_p)^{q+m}} + \frac{(-1)^{m+q}}{(w_j^* + w_p^*)^{q+m}} \right\},$$
(3.71)

$$|A_1\rangle = (|\eta_1^1\rangle, |\eta_2^1\rangle)^T, \quad |A_2\rangle = (-|\eta_1^2\rangle, -|\eta_2^2\rangle)^T,$$
(3.72)

$$|F_p\rangle = (F_{p,1}, \ldots, F_{p,N})^T, \quad |G_p\rangle = (G_{p,1}, \ldots, G_{p,N})^T,$$
(3.73)

$$\Omega = \left(\begin{array}{cc} \Omega_{1,1} & \Omega_{1,2} \\ \Omega_{2,1} & \Omega_{2,2} \end{array} \right), \quad \tilde{\Omega} = \left(\begin{array}{cc} \tilde{\Omega}_{1,1} & \tilde{\Omega}_{1,2} \\ \tilde{\Omega}_{2,1} & \tilde{\Omega}_{2,2} \end{array} \right), \quad I_\varepsilon = \left(\begin{array}{cc} -I & 0 \\ 0 & -I \end{array} \right), \quad (3.74)$$

where I is a $N \times N$ identity matrix and the definition of Ω matrix and $\tilde{\Omega}$ matrix is the same. Combining functions (Equation 3.66)–(Equation 3.68), we have

$$\begin{aligned} I|F_p\rangle - \Omega_{j,p}|G_p\rangle &= |\eta_j^1\rangle, \\ \tilde{\Omega}_{j,p}|F_p\rangle - I|G_p\rangle &= -|\eta_j^2\rangle. \end{aligned}$$
(3.75)

After taking

$$|K_1\rangle = (|F_1\rangle, |F_2\rangle)^T, \quad |K_2\rangle = (|G_1\rangle, |G_2\rangle)^T,$$
(3.76)

by straight calculation, we have

$$\begin{aligned} |K_1\rangle &= (I_\varepsilon + \Omega\tilde{\Omega})^{-1}\Omega|A_2\rangle - (I_\varepsilon + \Omega\tilde{\Omega})^{-1}|A_1\rangle, \\ |K_2\rangle &= -(I_\varepsilon + \tilde{\Omega}\Omega)^{-1}\tilde{\Omega}|A_1\rangle + (I_\varepsilon + \tilde{\Omega}\Omega)^{-1}|A_2\rangle. \end{aligned}$$
(3.77)

The new expression of Equation 3.63a are derived as

$$\begin{aligned} \tilde{M}_{11} = 1 &- \frac{i\tilde{\alpha}_+(y,t)}{4} D \left(\frac{1}{\mu - 1} + \frac{1}{\mu + 1} \right) \\ &+ \frac{\det(I_\varepsilon + \Omega\tilde{\Omega} + \Omega|A_2\rangle\langle E(\mu)|) - \det(I_\varepsilon + \Omega\tilde{\Omega} + |A_1\rangle\langle E(\mu)|)}{\det(I_\varepsilon + \Omega\tilde{\Omega})}, \end{aligned}$$
(3.78)

$$\begin{aligned} \tilde{M}_{12} = &\frac{i\tilde{\alpha}_+(y,t)}{4} \left(\frac{1}{\mu - 1} - \frac{1}{\mu + 1} \right) \\ &+ \frac{\det(I_\varepsilon + \tilde{\Omega}\Omega + |A_2\rangle\langle \tilde{E}(\mu)|) - \det(I_\varepsilon + \tilde{\Omega}\Omega + \tilde{\Omega}|A_1\rangle\langle \tilde{E}(\mu)|)}{\det(I_\varepsilon + \tilde{\Omega}\Omega)}, \end{aligned}$$
(3.79)

with

$$
\begin{aligned}
\langle E(\mu)| &= (\langle E_1(\mu)|, \langle E_2(\mu)|), \quad \langle E_j(\mu)| = (E_{j1}, \dots, E_{jN}), \\
\langle \tilde{E}(\mu)| &= (\langle \tilde{E}_1(\mu)|, \langle \tilde{E}_2(\mu)|), \quad \langle \tilde{E}_j(\mu)| = (\tilde{E}_{j1}, \dots, \tilde{E}_{jN}), \\
\langle E'(\mu)| &= (\langle E'_1(\mu)|, \langle E'_2(\mu)|), \quad \langle E'_j(\mu)| = (E'_{j1}, \dots, E'_{jN}), \\
\langle \tilde{E}'(\mu)| &= (\langle \tilde{E}'_1(\mu)|, \langle \tilde{E}'_2(\mu)|), \quad \langle \tilde{E}'_j(\mu)| = (\tilde{E}'_{j1}, \dots, \tilde{E}'_{jN}),
\end{aligned} \tag{3.80}
$$

$$
E_{js} = \frac{1}{(\mu - w_j)^s} + \frac{(-1)^s}{(\mu + w_j^*)^s}, \quad \tilde{E}_{js} = \frac{1}{(\mu - w_j^*)^s} + \frac{(-1)^{s+1}}{(\mu + w_j)^s},
$$

$$
E'_{js} = \frac{-s}{(\mu - w_j)^{s+1}} + \frac{s(-1)^{s+1}}{(\mu + w_j^*)^{s+1}}, \quad \tilde{E}'_{js} = \frac{-s}{(\mu - w_j^*)^{s+1}} + \frac{s(-1)^s}{(\mu + w_j)^{s+1}}.
$$

According to the symmetry conditions (Equation 3.22), we get that

$$
\begin{aligned}
\tilde{M}_{21} = &-\frac{i\tilde{\alpha}_+(y,t)}{4}\left(\frac{1}{\mu - 1} - \frac{1}{\mu + 1}\right) \\
&+ \frac{\det(I_\varepsilon + \Omega^*\tilde{\Omega}^* + |A_2^*\rangle\langle\tilde{E}^*(\mu^*)|) - \det(I_\varepsilon + \Omega^*\tilde{\Omega}^* + \tilde{\Omega}^*|A_1^*\rangle\langle\tilde{E}^*(\mu^*)|)}{\det(I_\varepsilon + \Omega^*\tilde{\Omega}^*)}.
\end{aligned} \tag{3.81}
$$

Hence the precise expression formula for the solution of cCH equation with single high-order pole under the NZBCs can be obtained in theorem 3.14.

THEOREM 3.14 *The solution of cCH equation with single high-order pole under the nonzero condition can be derived as the form*

$$
\hat{u}(x,t) = -2(\tilde{M}'_{12}(i)\tilde{M}_{11}(i) + \tilde{M}'_{21}(i)\tilde{M}_{11}^{-1}(i)), \tag{3.82}
$$

$$
x = x(y,t) = y + 2\ln \tilde{M}_{11}(i),
$$

with

$$
\tilde{M}_{11}(i) = 1 - \frac{\tilde{\alpha}_+(y,t)D}{4} + \frac{\det(I_\varepsilon + \Omega\tilde{\Omega} + \Omega|A_2\rangle\langle E(i)|) - \det(I_\varepsilon + \Omega\tilde{\Omega} + |A_1\rangle\langle E(i)|)}{\det(I_\varepsilon + \Omega\tilde{\Omega})},
$$

$$
\tilde{M}'_{12}(i) = \frac{\tilde{\alpha}_+(y,t)}{4} + \frac{\det(I_\varepsilon + \tilde{\Omega}\Omega + |A_2\rangle\langle\tilde{E}'(i)|) - \det(I_\varepsilon + \tilde{\Omega}\Omega + \tilde{\Omega}|A_1\rangle\langle\tilde{E}'(i)|)}{\det(I_\varepsilon + \tilde{\Omega}\Omega)},
$$

$$
\begin{aligned}
\tilde{M}'_{21}(i) = &-\frac{\tilde{\alpha}_+(y,t)}{4} \\
&+ \frac{\det(I_\varepsilon + \Omega^*\tilde{\Omega}^* + |A_2^*\rangle\langle\tilde{E}'^*(-i)|) - \det(I_\varepsilon + \Omega^*\tilde{\Omega}^* + \tilde{\Omega}^*|A_1^*\rangle\langle\tilde{E}'^*(-i)|)}{\det(I_\varepsilon + \Omega^*\tilde{\Omega}^*)},
\end{aligned}
$$

where D is shown in (Equation 3.21), $\tilde{\alpha}_+(y,t) \in \mathbb{R}$, and the elements $|A_1\rangle$, $|A_2\rangle$, I_ε, Ω, $\tilde{\Omega}$, $\langle E(i)|$, $\langle\tilde{E}(i)|$, $\langle E'(i)|$, $\langle\tilde{E}'(i)|$ are defined in (Equation 3.72), (Equation 3.74), and (Equation 3.80).

3.3.1. One-soliton solution

Next, to construct the simple one-soliton solution of cCH equation under NZBCs, we consider the specific data as follows.

Case 4: For this case, we consider the situation $N = 1$, Assuming $w_1 = z_0 = e^{i\lambda} \in \mathbb{C}^+$, $(\lambda \in (0, \pi))$ is one-order pole of scattering data $a(\mu)$, then we have $w_2 = \frac{1}{z_0^*} = w_1$. It can be obtained that $\theta(e^{i\lambda}) = \theta(-e^{-i\lambda})$. Next we define $\theta_1 \triangleq \theta(e^{i\lambda}) = \frac{i \sin \lambda y}{2t} - \frac{2i \sin \lambda}{\cos^2 \lambda}$, hence $e^{2it\theta_2}$ is real. Suppose that $D = 1$, $\tilde{\alpha}_+ = 1$, $r_{1,1} = 2$, $r_{2,1} = 1$. Then the parameters satisfy the following relationships, there $\Omega_{1,1} = \Omega_{1,2} = \Omega_{2,1} = \Omega_{2,2}$, $\tilde{\Omega}_{1,1} = \tilde{\Omega}_{1,2} = \tilde{\Omega}_{2,1} = \tilde{\Omega}_{2,2}$, $\eta_{21}^1 = \frac{1}{2}\eta_{11}^1$, $\eta_{21}^2 = \frac{1}{2}\eta_{11}^2$, $\zeta_{1,0} = \zeta_{2,0}$.

$$\zeta_{1,0} = e^{-\frac{4\sin\lambda}{\cos^2\lambda e^{2i\lambda}}\left(\left(t+\frac{y}{8}\right)e^{2i\lambda}+\frac{y}{16}(e^{4i\lambda}+1)\right)},$$

$$\Omega_{1,1} = 2e^{-\frac{4\sin\lambda}{\cos^2\lambda e^{2i\lambda}}\left(\left(t+\frac{y}{8}\right)e^{2i\lambda}+\frac{y}{16}(e^{4i\lambda}+1)\right)}\left(\frac{1}{e^{i\lambda}-e^{-i\lambda}+\frac{1}{2e^{i\lambda}}}\right),$$

$$\tilde{\Omega}_{1,1} = 2e^{-\frac{((16t+2y)e^{-2i\lambda}+y(e^{-4i\lambda}+1))\sin\lambda}{4\cos^2\lambda e^{2i\lambda}}}\left(\frac{1}{e^{-i\lambda}-e^{i\lambda}-\frac{1}{2e^{-i\lambda}}}\right),$$

$$\eta_{11}^1 = \frac{i}{2}\left(\frac{1}{e^{i\lambda}-1}-\frac{1}{e^{i\lambda}+1}\right)e^{-\frac{4\sin\lambda}{\cos^2\lambda e^{2i\lambda}}\left(\left(t+\frac{y}{8}\right)e^{2i\lambda}+\frac{y}{16}(e^{4i\lambda}+1)\right)},$$

$$\eta_{11}^2 = \left(2-\frac{i}{2}\left(\frac{1}{e^{i\lambda}-1}+\frac{1}{e^{i\lambda}+1}\right)\right)e^{-\frac{4\sin\lambda}{\cos^2\lambda e^{2i\lambda}}\left(\left(t+\frac{y}{8}\right)e^{2i\lambda}+\frac{y}{16}(e^{4i\lambda}+1)\right)},$$

substituting the above results into the formula (Equation 3.82) we can obtain the one-soliton solution of cCH equation under NZBCs.

Supposing $\lambda = \frac{\pi}{6}$, the expression for the solution is

$$\hat{u}(x,t) = -2\left(T_2^1 T_1^1 + \frac{T_3^1}{T_1^1}\right), \tag{3.83}$$

$$x = x(y,t) = y + 2\ln T_1^1,$$

$$T_1^1 = \frac{12ie^{-\frac{8t}{3}-\frac{y}{2}}+153i\sqrt{3}e^{-\frac{16t}{3}-y}+20i\sqrt{3}-246e^{-\frac{16t}{3}-y}-20}{(i\sqrt{3}-1)\left(27e^{-\frac{16t}{3}-\frac{y}{2}}+18ie^{-\frac{16t}{3}-\frac{y}{2}}\sqrt{3}+4\right)},$$

$$T_2^1 = \frac{141ie^{-\frac{16t}{3}-\frac{y}{2}}\sqrt{3}-6e^{-\frac{8t}{3}-\frac{y}{2}}+216e^{-\frac{16t}{3}-\frac{y}{2}}+32}{27e^{-\frac{16t}{3}-\frac{y}{2}}+18ie^{-\frac{16t}{3}-\frac{y}{2}}\sqrt{3}+4},$$

$$T_3^1 = -\frac{18(i\sqrt{3}-1)e^{-\frac{8t}{3}-\frac{y}{2}}-(351i\sqrt{3}+189)e^{-\frac{16t}{3}-\frac{y}{2}}-32i\sqrt{3}+32}{(i\sqrt{3}-1)\left(18ie^{-\frac{16t}{3}-\frac{y}{2}}-27e^{-\frac{16t}{3}-\frac{y}{2}}-4\right)}.$$
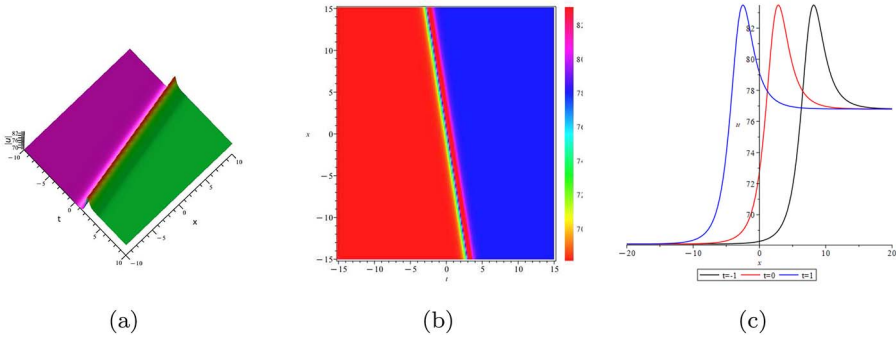
Figure 7. (a)–(c) describe the local structure, density, and intensity profiles with different times of one-soliton solution $|u|$. Parameters $r_{1,1} = 2$, $r_{2,1} = 1$, $\lambda = \frac{\pi}{6}$.

In accordance with the construction process of the RH problem, it can be known that there are singularities $\{0, \pm 1, \pm i\}$ of the spectral parameter z. Since we choose $z_0 \in \mathbb{C}^+$ here and $|z_0| = 1$, $z_0 = i$ is the singularity of solution $\hat{u}$. From the propagation of waves in figure 7, the bright soliton solution is the hump-kink interacted solution. Based on the intensity distribution figure captured at various time points, when the wave peaks and then drops a certain height, it continues to propagate but does not return to the same height as before.

Supposing $\lambda = \frac{\pi}{3}$, the solution can be obtained as follows

$$\hat{u}(x, t) = -2 \left(T_2^2 T_1^2 + \frac{T_3^2}{T_1^2} \right), \tag{3.84}$$

$$x = x(y, t) = y + 2 \ln T_1^2,$$

$$T_1^2 = \frac{1}{\varpi_1^2} \left((5520 - 696i + (1080 - 816i)\sqrt{3})e^{(-16t-y)\sqrt{3}} \right) +$$
$$\frac{1}{\varpi_1^2} \left(320 e^{-\frac{\sqrt{3}(16t+y)}{2}} \sqrt{3} + 64 - 96i + (64i - 32)\sqrt{3} \right),$$

$$T_2^2 = \frac{1}{\varpi_2^2} \left((-1056 - 1728i - (192 + 1248i)\sqrt{3})e^{(-16t-y)\sqrt{3}} + 896 - 1536i + 512\sqrt{3} \right) +$$
$$\frac{1}{\varpi_2^2} \left((-15360 + 28800i + (-9600 + 15360i)\sqrt{3})e^{-\frac{\sqrt{3}(16t+y)}{2}} - 896i\sqrt{3} \right),$$

$$T_3^2 = \frac{1}{\varpi_3^2} \left((1056 + 1728i - (192 + 1248i)\sqrt{3})e^{(-16t-y)\sqrt{3}} - 896 + 1536i + 512\sqrt{3} \right) +$$
$$\frac{1}{\varpi_3^2} \left((-7680 + 17280i + (5760 - 7680i)\sqrt{3})e^{-\frac{\sqrt{3}(16t+y)}{2}} - 896i\sqrt{3} \right),$$

$$\varpi_1^2 = 80(1 + i\sqrt{3})(2 - \sqrt{3})((3 + 6i\sqrt{3})e^{(-16t-y)\sqrt{3}}) + 4,$$
$$\varpi_2^2 = -40(3 + 2\sqrt{3})(i\sqrt{3} - 1)((3 + 6i)e^{-\frac{\sqrt{3}(16t+y)}{2}} + 4),$$

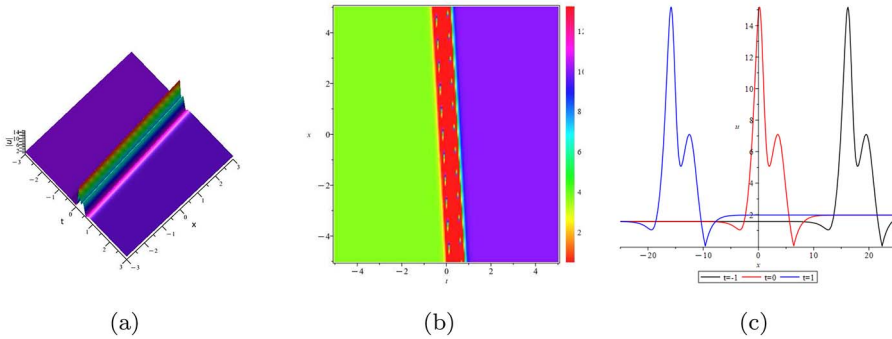Figure 8. (a)–(c) describe the local structure, density, and intensity profiles with different times of one-soliton solutions $|u|$. Parameters $r_{1,1} = 2$, $r_{2,1} = 1$, $\lambda = \frac{\pi}{3}$.

$$\varpi_3^2 = -320(2 - \sqrt{3})^2(i\sqrt{3} + 1)((6i\sqrt{3} - 3)e^{-\frac{\sqrt{3}(16t+y)}{2}} - 4).$$

From figure 8, it can be seen that the peak of soliton does not immediately propagate smoothly when it falls from the highest point but rather fluctuates and appears as a smaller peak.

3.3.2. Two-order pole solution

In an effort to construct the two-order pole solution of cCH equation under NZBCs, we consider the specific data as follows. This is for $N = 2$ case, and the expression of $u(x,t)$ defined in theorem 3.14.

Case 5: Assuming that $w_1 = z_0 = e^{i\lambda} \in \mathbb{C}^+$ be the two-order zero point of the scattering data $a(\mu)$, then $w_2 = w_1$ is also the two-order zero point of $a(\mu)$. The discrete spectrum is the set $\mathcal{X} = \{e^{i\lambda}, -e^{-i\lambda}, e^{-i\lambda}, -e^{i\lambda}\}$. For the purpose of displaying characteristics of soliton propagations more clearly, we select some appropriate parameters to construct the two-order pole solutions of cCH equation. The dynamic behaviours of solitons are shown in Figures (9)–(10).

Setting $\lambda = \frac{\pi}{6}$, $\{r_{1,1} = 1, r_{1,2} = 1, r_{2,1} = 1, r_{2,2} = 1\}$, $D = 1$, $\tilde{\alpha}_+ = 1$. Besides, $\Omega_{1,1} = \Omega_{1,3} = \Omega_{3,1} = \Omega_{3,3}$, $\Omega_{1,2} = \Omega_{1,4} = \Omega_{3,2} = \Omega_{3,4}$, $\Omega_{2,1} = \Omega_{2,3} = \Omega_{4,1} = \Omega_{4,3}$, $\Omega_{2,2} = \Omega_{2,4} = \Omega_{4,2} = \Omega_{4,4}$, $\tilde{\Omega}_{1,1} = \tilde{\Omega}_{1,3} = \tilde{\Omega}_{3,1} = \tilde{\Omega}_{3,3}$, $\tilde{\Omega}_{1,2} = \tilde{\Omega}_{1,4} = \tilde{\Omega}_{3,2} = \tilde{\Omega}_{3,4}$, $\tilde{\Omega}_{2,1} = \tilde{\Omega}_{2,3} = \tilde{\Omega}_{4,1} = \tilde{\Omega}_{4,3}$, $\tilde{\Omega}_{2,2} = \tilde{\Omega}_{2,4} = \tilde{\Omega}_{4,2} = \tilde{\Omega}_{4,4}$. In view of the construction process for simple high-order pole solutions under NZBCs and the parameter expressions of obtained solutions in theorem 3.14, we further obtain

$$\zeta_{1,0} = \zeta_{2,0} = e^{-\frac{2t}{3} - \frac{y}{2}},$$

$$\zeta_{1,1} = \zeta_{2,1} = \frac{8i(20t + 9y)e^{-\frac{2t}{3} - \frac{y}{2}}(i\sqrt{3} - 1)}{(i\sqrt{3} + 3)^3(\sqrt{3} + i)^2},$$

$$\Omega_{1,1} = \frac{5\sqrt{3}}{18}\left(\left(-\frac{21 + 27y}{20} + \frac{3i}{2} - 3t\right)\sqrt{3} + it + \frac{9i(y - 1)}{20} - \frac{9}{10}\right)e^{-\frac{2t}{3} - \frac{y}{2}},$$

$$\Omega_{1,2} = \frac{1440 e^{-\frac{2t}{3}-\frac{y}{2}}}{(i\sqrt{3}+3)^3(\sqrt{3}+i)^4}\left(\sqrt{3}\left(\frac{y-6}{20}+\frac{i}{10}+\frac{t}{9}\right)+it+\frac{9i(y+1)}{20}+\frac{1}{2}\right),$$

$$\Omega_{2,1} = -\frac{e^{-\frac{2t}{3}-\frac{y}{2}}(-2+i\sqrt{3})}{\sqrt{3}+i}, \quad \Omega_{2,2} = \frac{e^{-\frac{2t}{3}-\frac{y}{2}}(-3-2i\sqrt{3})}{(\sqrt{3}+i)^2},$$

$$\tilde{\Omega}_{1,1} = \frac{1}{24}\left((-6+3i+20it+9iy)\sqrt{3}+27+18i+20t+9y\right)e^{-\frac{2t}{3}-\frac{y}{2}},$$

$$\tilde{\Omega}_{1,2} = \frac{1120\left(\left(\frac{3}{10}+\frac{3i}{14}+\frac{t}{7}+\frac{9y}{140}\right)\sqrt{3}+it+\frac{9iy}{20}-\frac{27}{70}+\frac{9i}{10}\right)e^{-\frac{2t}{3}-\frac{y}{2}}}{(i\sqrt{3}-3)^3(i-\sqrt{3})^4},$$

$$\tilde{\Omega}_{2,1} = \frac{-i\sqrt{3}e^{-\frac{2t}{3}-\frac{y}{2}}}{i-\sqrt{3}}, \quad \tilde{\Omega}_{2,2} = \frac{(2i\sqrt{3}-1)e^{-\frac{2t}{3}-\frac{y}{2}}}{(i-\sqrt{3})^2},$$

$$\eta_{11}^1 = \eta_{21}^1 = \frac{-1280i\left(\left(-\frac{6}{5}+\frac{3i}{5}-t-\frac{9y}{20}\right)\sqrt{3}+it+\frac{9iy}{20}\right)e^{-\frac{2t}{3}-\frac{y}{2}}}{4(i\sqrt{3}-1)^2(i\sqrt{3}+3)^3(\sqrt{3}+i)^2},$$

$$\eta_{11}^2 = \eta_{21}^2 = \frac{(-1920i\sqrt{3}-7680it-3456iy+5760-2304i)e^{-\frac{2t}{3}-\frac{y}{2}}}{4(i\sqrt{3}+1)^2(i\sqrt{3}-3)^3(i-\sqrt{3})^2},$$

$$\eta_{12}^1 = \eta_{22}^1 = \frac{ie^{-\frac{2t}{3}-\frac{y}{2}}}{i\sqrt{3}-1}, \quad \eta_{12}^2 = \eta_{22}^2 = -\frac{3e^{-\frac{2t}{3}-\frac{y}{2}}}{2},$$

then substituting the above results into the expression of solution (Equation 3.82), we can obtain the two-order pole solution of cCH equation, Figure 9 shows the dynamic behaviour of soliton propagation, with three images presenting the local structure, density, and intensity profiles at different times of two-order pole solutions with the above parameters.

In addition, we select parameter $\lambda = \frac{5\pi}{6}$ and use the same process to obtain the dynamic propagation of the solution shown in Figure 10. From these two Figures 9–10, it can be seen that when the two waves collide at the centre, a larger peak is generated, and then they continue to propagate along the previous trajectory, but the energy decreases.

3.4. Multiple high-order pole solutions

The general circumstance that scattering data $a(\mu)$ has N high-order poles $z_1, z_2, \cdots, z_N$ will be considered, there $z_k \in \mathbb{C}^+$ for all $k = 1, 2, \ldots, N$, and their powers are $n_1, n_2, \ldots, n_N$, respectively. Let $w_1^k = z_k$ and $w_2^k = \frac{1}{z_k^*}$, $k = 1, 2, \ldots, N$, just like the case of one high-order pole, from the definition of $\tilde{M}(y,t,\mu)$, it can be seen that $\mu = w_j^k$ and $\mu = -w_j^{k*}$ $(j = 1, 2)$ are N-order pole points of $\tilde{M}_{11}$. Simultaneously, $\mu = w_j^{k*}$ and $\mu = -w_j^k$ $(j = 1, 2)$ are N-order pole points of $\tilde{M}_{12}$.
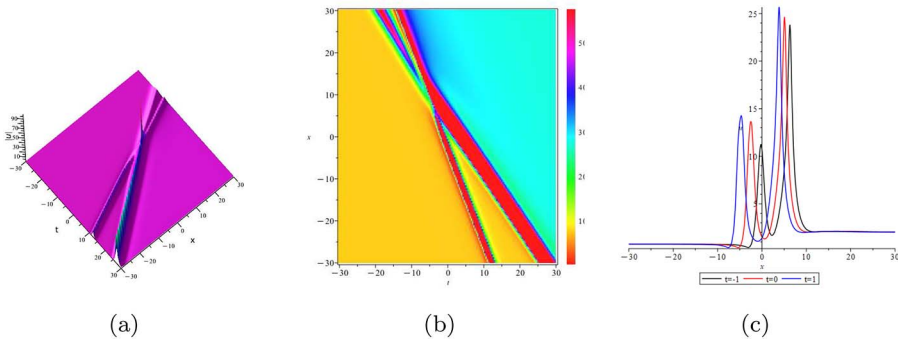
Figure 9. (a)–(c) describe the local structure, density, and intensity profiles with different times of the soliton solutions $|u|$ with one two-order pole. Parameters $r_{1,1} = 1$, $r_{1,2} = 1$, $r_{2,1} = 1$, $r_{2,2} = 1$, $\lambda = \frac{\pi}{6}$.
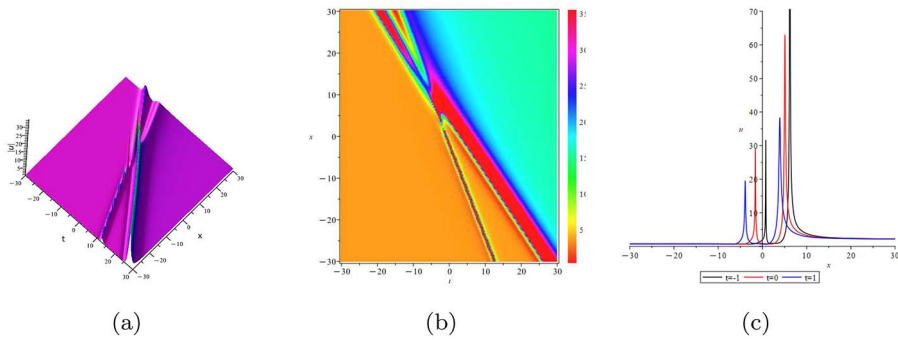

Figure 10. (a)–(c) describe the local structure, density, and intensity profiles with different times of the soliton solutions $|u|$ with one two-order pole. Parameters $r_{1,1} = 1$, $r_{1,2} = 1$, $r_{2,1} = 1$, $r_{2,2} = 1$, $\lambda = \frac{5\pi}{6}$.

Under the assumption that $\mu = w_j^k$ $(j = 1, 2)$ are n_k-order zero points of $a(\mu)$ for $k = 1, 2, \ldots, N$, by the Wronskian $a(\mu) = Wr(\check{\phi}_{-1}, \check{\phi}_{+2})$, there exist complex constants $b_{j,s}^k$ $(s = 1, 2, \ldots, n_N)$ satisfy the follows

$$\frac{\partial^m [\check{\phi}_+(y, t, w_j^k)]_2}{\partial \mu^m} = \sum_{l=0}^{m} \binom{m}{l} b_{j,m-l+1}^k \frac{\partial^l [\check{\phi}_-(y, t, w_j^k)]_1}{\partial \mu^l}, \qquad (3.85)$$

and

$$\frac{\partial^m [\check{\psi}_+(y, t, w_j^k)]_2}{\partial \mu^m} = \sum_{l=0}^{m} \binom{m}{l} b_{j,m-l+1}^k \frac{\partial^l [\check{\psi}_-(y, t, w_j^k)]_1 e^{2it\theta(w_j^k)}}{\partial \mu^l}, \qquad (3.86)$$

with $m = 0, 1, \ldots, n_{N-1}$. Utilizing the parallel approach in above, we define the $n_k \times n_k$ matrices $\Xi_{kd}^{j,p} = [\Xi_{kd}^{j,p}]_{s,q}$ and $\tilde{\Xi}_{kd}^{j,p} = [\tilde{\Xi}_{kd}^{j,p}]_{s,q}$ for $s, q = 1, 2, \ldots, n_k$ and

introduce the following notations for $j, p = 1, 2$

$$|\eta_j^{1k}\rangle = (\eta_{j1}^{1k}, \cdots, \eta_{jN}^{1k})^T, \quad |\eta_j^{2k}\rangle = (\eta_{j1}^{2k}, \cdots, \eta_{jN}^{2k})^T, \tag{3.87}$$

$$\eta_{js}^{1k} = \frac{i\tilde{\alpha}_+(y, t)}{4} \sum_{l=s}^{n_k} \sum_{m=0}^{l-s} \left(\frac{(-1)^m}{(w_j^k - 1)^{m+1}} - \frac{(-1)^m}{(w_j^k + 1)^{m+1}} \right) r_{j,l}^k \zeta_{j,l-s-m}^k,$$

$$\eta_{js}^{2k} = \sum_{l=s}^{n_k} \sum_{m=0}^{l-s} r_{j,l}^{k*} \zeta_{j,l-s-m}^{k*}$$

$$- \frac{i\tilde{\alpha}_+(y, t)}{4} D \sum_{l=s}^{n_k} \sum_{m=0}^{l-s} \left(\frac{(-1)^m}{(w_j^{k*} - 1)^{m+1}} + \frac{(-1)^m}{(w_j^{k*} + 1)^{m+1}} \right) r_{j,l}^{k*} \zeta_{j,l-s-m}^{k*},$$

$$r_{j,l}^k = \lim_{\mu \to w_j^k} \frac{b_{j,l}^k}{(n_k - l)!} \frac{\partial^{n_k - l}}{\partial (\mu - w_j^k)^{n_k - l}} \frac{(\mu - w_j^k)^{n_k}}{a(\mu)}, \tag{3.88}$$

$$|B_1\rangle = (|A_1^1\rangle, |A_1^2\rangle, \cdots, |A_1^N\rangle)^T, \quad |A_1^k\rangle = (|\eta_1^{1k}\rangle, |\eta_2^{1k}\rangle)^T,$$
$$|B_2\rangle = (|A_2^1\rangle, |A_2^2\rangle, \cdots, |A_2^N\rangle)^T, \quad |A_2^k\rangle = (-|\eta_1^{2k}\rangle, -|\eta_2^{2k}\rangle)^T, \tag{3.89}$$

$$\Xi_{k,d}^{j,p} = \sum_{l=s}^{n_k} \sum_{m=0}^{l-s} \binom{q+m-1}{m} r_{j,l}^k \zeta_{j,l-s-m}^k \left\{ \frac{(-1)^m}{(w_j^k - w_p^{k*})^{q+m}} + \frac{(-1)^{m+q+1}}{(w_j^k + w_p^k)^{q+m}} \right\}, \tag{3.90}$$

$$\tilde{\Xi}_{k,d}^{j,p} = \sum_{l=s}^{n_k} \sum_{m=0}^{l-s} \binom{q+m-1}{m} r_{j,l}^{k*} \zeta_{j,l-s-m}^{k*} \left\{ \frac{(-1)^m}{(w_j^{k*} - w_p^k)^{q+m}} + \frac{(-1)^{m+q}}{(w_j^{k*} + w_p^{k*})^{q+m}} \right\}, \tag{3.91}$$

$$\Xi = \begin{pmatrix} [\Xi_{11}^{j,p}] & \cdots & [\Xi_{1N}^{j,p}] \\ \vdots & \ddots & \vdots \\ [\Xi_{N1}^{j,p}] & \cdots & [\Xi_{NN}^{j,p}] \end{pmatrix}, \quad \tilde{\Xi} = \begin{pmatrix} [\tilde{\Xi}_{11}^{j,p}] & \cdots & [\tilde{\Xi}_{1N}^{j,p}] \\ \vdots & \ddots & \vdots \\ [\tilde{\Xi}_{N1}^{j,p}] & \cdots & [\tilde{\Xi}_{NN}^{j,p}] \end{pmatrix}, \tag{3.92}$$

$$I_\delta = \begin{pmatrix} I_{\delta_1} & & \\ & \ddots & \\ & & I_{\delta_N} \end{pmatrix}, \quad I_{\delta_k} = \begin{pmatrix} -I & \\ & -I \end{pmatrix}_{2n_k \times 2n_k} \quad k = 1, \ldots, N, \tag{3.93}$$

$$\langle \Lambda(\mu)| = (\langle \Lambda_1^1(\mu)|, \langle \Lambda_2^1(\mu)|, \ldots, \langle \Lambda_1^N(\mu)|, \langle \Lambda_2^N(\mu)|),$$
$$\langle \tilde{\Lambda}(\mu)| = (\langle \tilde{\Lambda}_1^1(\mu)|, \langle \tilde{\Lambda}_2^1(\mu)|, \ldots, \langle \tilde{\Lambda}_1^N(\mu)|, \langle \tilde{\Lambda}_2^N(\mu)|),$$
$$\langle \Lambda'(\mu)| = (\langle \Lambda_1^{1\prime}(\mu)|, \langle \Lambda_2^{1\prime}(\mu)|, \ldots, \langle \Lambda_1^{N\prime}(\mu)|, \langle \Lambda_2^{N\prime}(\mu)|),$$
$$\langle \tilde{\Lambda}'(\mu)| = (\langle \tilde{\Lambda}_1^{1\prime}(\mu)|, \langle \tilde{\Lambda}_2^{1\prime}(\mu)|, \ldots, \langle \tilde{\Lambda}_1^{N\prime}(\mu)|, \langle \tilde{\Lambda}_2^{N\prime}(\mu)|), \tag{3.94}$$

with

$$\langle \Lambda_j^k(\mu)| = (\Lambda_{j1}^k, \ldots, \Lambda_{jn_k}^k), \quad \Lambda_{js}^k = \frac{1}{(\mu - w_j^k)^s} + \frac{(-1)^s}{(\mu + w_j^{k*})^s},$$

$$\langle \tilde{\Lambda}_j^k(\mu)| = (\tilde{\Lambda}_{j1}^k, \ldots, \tilde{\Lambda}_{jn_k}^k), \quad \tilde{\Lambda}_{js}^k = \frac{1}{(\mu - w_j^{k*})^s} + \frac{(-1)^{s+1}}{(\mu + w_j^k)^s},$$

$$\langle \Lambda_j^{k\prime}(\mu)| = (\Lambda_{j1}^{k\prime}, \ldots, \Lambda_{jn_k}^{k\prime}), \quad \Lambda_{js}^{k\prime} = \frac{-s}{(\mu - w_j^k)^{s+1}} + \frac{s(-1)^{s+1}}{(\mu + w_j^{k*})^{s+1}},$$

$$\langle \tilde{\Lambda}_j^{k\prime}(\mu)| = (\tilde{\Lambda}_{j1}^{k\prime}, \ldots, \tilde{\Lambda}_{jn_k}^{k\prime}), \quad \tilde{\Lambda}_{js}^{k\prime} = \frac{-s}{(\mu - w_j^{k*})^{s+1}} + \frac{s(-1)^s}{(\mu + w_j^k)^{s+1}}.$$

Similar to theorem 3.14, we give the solution of cCH equation with multiple high-order poles as follows.

THEOREM 3.15 *The multiple high-order pole solutions of cCH equation under the nonzero condition can be obtained as follows*

$$\hat{u}(x,t) = -2(\tilde{M}'_{12}(i)\tilde{M}_{11}(i) + \tilde{M}'_{21}(i)\tilde{M}_{11}^{-1}(i)), \tag{3.95}$$

$$x = x(y,t) = y + 2\ln \tilde{M}_{11}(i),$$

with

$$\tilde{M}_{11}(i) = 1 - \frac{\tilde{\alpha}_+(y,t)D}{4} + \frac{\det(I_\delta + \Xi\tilde{\Xi} + \Xi|B_2\rangle\langle\Lambda(i)|) - \det(I_\delta + \Xi\tilde{\Xi} + |B_1\rangle\langle\Lambda(i)|)}{\det(I_\delta + \Xi\tilde{\Xi})},$$

$$\tilde{M}'_{12}(i) = \frac{\tilde{\alpha}_+(y,t)}{4} + \frac{\det(I_\delta + \tilde{\Xi}\Xi + |B_2\rangle\langle\tilde{\Lambda}'(i)|) - \det(I_\delta + \tilde{\Xi}\Xi + \tilde{\Xi}|B_1\rangle\langle\tilde{\Lambda}'(i)|)}{\det(I_\delta + \tilde{\Xi}\Xi)},$$

$$\tilde{M}'_{21}(i) = -\frac{\tilde{\alpha}_+(y,t)}{4}$$
$$+ \frac{\det(I_\delta + \Xi^*\tilde{\Xi}^* + |\Lambda_2^*\rangle\langle\tilde{\Lambda}'^*(-i)|) - \det(I_\delta + \Xi^*\tilde{\Xi}^* + \tilde{\Xi}^*|\Lambda_1^*\rangle\langle\tilde{\Lambda}'^*(-i)|)}{\det(I_\delta + \Xi^*\tilde{\Xi}^*)},$$

where D is shown in (Equation 3.21), $\tilde{\alpha}_+(y,t) \in \mathbb{R}$, the elements $|B_1\rangle$, $|B_2\rangle$, I_δ, Ξ, $\tilde{\Xi}$, $\langle\Lambda(i)|$, $\langle\tilde{\Lambda}(i)|$, $\langle\Lambda'(i)|$, and $\langle\tilde{\Lambda}'(i)|$ are defined in (Equation 3.89), (Equation 3.92), (Equation 3.93), and (Equation 3.94).

4. Conclusions

In summary, we have studied the multiple high-order pole solutions of the cCH equation with a linear dispersion term by the RH method with ZBCs and NZBCs. In comparison to the previous work presented in reference [43], we not only provide the multi-soliton solutions for this equation but also examine the multiple high-order pole solutions in the context of both boundary conditions.

Under ZBCs, in the direct scattering part, we make spectral analysis to the Lax pair. To address the issue of the multi-value problem arising from square roots, a uniformization variable μ is introduced, which leads to the Lax matrices U and V have singularities at the points $\mu = 0$, $\mu = \infty$, and the branch cut points $\mu = \pm i$ in the extended complex μ-plane. In order to control the behaviour of the eigenfunctions at these singular points, we introduce the suitable gauge transformations. This method has been developed by Boutet de Monvel and Shepelsky [6, 7]. In the

inverse scattering part, we construct the RH problem. According to the asymptotic behaviour of spectral parameter at point $\mu = i$, we successfully restore the solution of potential function u. Since the function $a(\mu)$ as a Wronskian of analytic Jost solutions, which vanishes to N-order at the N-order zero points, we obtain a linear expression for the Nth derivative of Jost solutions. Through solving an algebraic expression, we establish the solution of RH problem, which is the constituent elements of the solution of cCH equation. We select special parameters to provide specific examples of one-soliton solutions and two-order pole solutions and analysed the dynamic behaviours of solutions by analysing figures.

Under NZBCs, after making a transformation, the NZBCs are converted to ZBCs. Using a similar methodology as described earlier, we conduct spectral analysis on the new Lax pair and establish the RH problem. The differences between them are that the asymptotic behaviour at the new singularity point $\mu = \pm 1$ need to be controlled. There we generalize the processing method described in [5] and further using the RH method to study the high-order pole solutions of cCH equation with linear dispersion term νu_x. Then the solution of RH problem is used to construct the explicit and implicit expressions of the potential function u. Finally, we use the solution of RH problem to construct the high-order pole solutions of the cCH equation and analyse the propagation characteristics of solutions through some examples.

Acknowledgements

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