COMPOSITIO MATHEMATICA

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Compositio Math. **159** (2023), 1214–1249.

doi:10.1112/S0010437X23007182

FOUNDATION COMPOSITIO MATHEMATICA

LONDON MATHEMATICAL SOCIETY
EST. 1865

Compositio Math. 159 (2023) 1214–1249
doi:10.1112/S0010437X23007182

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Abstract

We give a new proof of Faltings's p-adic Eichler–Shimura decomposition of the modular curves via Bernstein–Gelfand–Gelfand (BGG) methods and the Hodge–Tate period map. The key property is the relation between the Tate module and the Faltings extension, which was used in the original proof. Then we construct overconvergent Eichler–Shimura maps for the modular curves providing 'the second half' of the overconvergent Eichler–Shimura map of Andreatta, Iovita and Stevens. We use higher Coleman theory on the modular curve developed by Boxer and Pilloni to show that the small-slope part of the Eichler–Shimura maps interpolates the classical p-adic Eichler–Shimura decompositions. Finally, we prove that overconvergent Eichler–Shimura maps are compatible with Poincaré and Serre pairings.

1. Introduction

Let p be a prime number, $\mathbb{A}_{\mathbb{Q}}^{\infty}$ the finite adèles of $\mathbb{Q}$, $\mathbb{A}_{\mathbb{Q}}^{\infty,p}$ the finite prime-to-p adèles, and $\mathbb{Z}_p$ the ring of p-adic integers. Let $\mathbb{C}_p$ be the p-adic completion of an algebraic closure of $\mathbb{Q}_p$, and $G_{\mathbb{Q}_p} = \mathrm{Gal}(\mathbb{C}_p/\mathbb{Q}_p)$ the absolute Galois group. From now on, we fix a neat compact open subgroup $K^p \subset \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty,p})$. Given an open compact subgroup $K_p \subset \mathrm{GL}_2(\mathbb{Q}_p)$, we denote by $Y_{K_p}^{\mathrm{alg}}$ the modular curve over $\operatorname{Spec} \mathbb{Q}_p$ of level $K^pK_p \subset \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty}) = \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty,p}) \times \mathrm{GL}_2(\mathbb{Q}_p)$, and by $X_{K_p}^{\mathrm{alg}}$ its compactification by adding cusps. Let Y_{K_p} and X_{K_p} be the rigid analytic varieties attached to the modular curves, seen as adic spaces over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ (cf. [Hub96]). Letting $D = X_{K_p} \backslash Y_{K_p}$ be the cusp divisor, we endow X_{K_p} with the log-structure defined by D.

Given a fine and saturated (fs) log adic space Z and $? \in \{\mathrm{an}, \text{ét}, \text{két}, \text{proét}, \text{prokét}\}$, we denote by $Z_?$ its analytic, étale, Kummer-étale, proétale and pro-Kummer-étale sites, respectively (see [Sch13] and [DLLZ23b]).

In [Fal87], Faltings described the Hodge–Tate decomposition of the étale cohomology (with coefficients) of the modular curve Y_{K_p}. More precisely, let E be the universal elliptic curve over Y_{K_p}. This admits an extension to a semi-abelian adic space E^{sm} over X_{K_p} (cf. [DR73]). Let $e : X_{K_p} \to E^{sm}$ be the unit section, $\omega_E = e^*\Omega^1_{E^{sm}/X}$ the modular sheaf and $T_pE = \varprojlim_n E[p^n]$ the Tate module over Y_{K_p}. We have the following theorem.

Theorem 1.0.1 (Faltings). *Let $k \geq 0$. There exists a Galois and Hecke equivariant isomorphism*

$$H^1_{\text{ét}}(Y_{K_p,\mathbb{C}_p}, \operatorname{Sym}^k T_pE) \otimes_{\mathbb{Q}_p} \mathbb{C}_p(1) = H^0_{\mathrm{an}}(X_{K_p,\mathbb{C}_p}, \omega_E^{k+2}) \oplus H^1_{\mathrm{an}}(X_{K_p,\mathbb{C}_p}, \omega_E^{-k})(k+1) \qquad (1)$$

called the Eichler–Shimura decomposition.

Received 26 October 2021, accepted in final form 2 February 2023, published online 22 May 2023.
2020 Mathematics Subject Classification 11F77 (primary), 11G18, 14G35 (secondary).
Keywords: Eichler–Shimura maps, p-adic modular symbols, modular curves, p-adic Hodge theory.

The first result of this paper is a new proof of Faltings's Eichler–Shimura decomposition using Bernstein–Gelfand–Gelfand (BGG) methods and the Hodge–Tate period map. Our proof is the proétale analogue of the BGG decomposition for the de Rham cohomology of Faltings and Chai [FC90, Ch. 5 and Theorem 5.5]. Let us develop the ideas behind it.

Let $X_\infty := \varprojlim_{K_p} X_{K_p}$ be Scholze's perfectoid modular curve and $\pi_{\mathrm{HT}} : X_\infty \to \mathbb{P}^1_{\mathbb{Q}_p}$ the Hodge–Tate period map [Sch15]. The morphism π_{HT} is $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant, where we see $\mathbb{P}^1_{\mathbb{Q}_p}$ as the left quotient of GL_2 by the upper triangular Borel $\mathbf{B}$. Let $\pi_{K_p} : X_\infty \to X_{K_p}$ be the natural map. We can see X_∞ as a pro-Kummer-étale K_p-torsor over X_{K_p}. We let $\widehat{\mathscr{O}}_{X_{K_p}}$ denote the p-adic completion of the structural sheaf over $X_{K_p,\text{prokét}}$. Let $\mathbb{Z}_p(1) = \varprojlim_n \mu_{p^n}$ be the Tate twist and $\widehat{\mathscr{O}}_{X_{K_p}}(i)$ the ith twist of $\widehat{\mathscr{O}}_{X_{K_p}}$. By [DLLZ23b, Theorem 4.6.1], T_pE admits a natural extension to the pro-Kummer-étale site of X_{K_p} which we denote in the same way. From now on, we fix the level K_p and write $Y = Y_{K_p}$ and $X = X_{K_p}$.

The map π_{HT} is defined from the Hodge–Tate exact sequence

$$0 \to \omega_E^{-1} \otimes_{\mathscr{O}_X} \widehat{\mathscr{O}}_X(1) \xrightarrow{\mathrm{HT}^\vee} T_pE \otimes_{\mathbb{Z}_p} \widehat{\mathscr{O}}_X \xrightarrow{\mathrm{HT}} \omega_E \otimes_{\mathscr{O}_X} \widehat{\mathscr{O}}_X \to 0, \tag{2}$$

which is the variation in families of the Hodge–Tate decomposition for elliptic curves (cf. [Tat67]). More precisely, let $\Psi : \mathbb{Z}_p^2 \xrightarrow{\sim} T_pE$ be the universal trivialization over X_∞; then (2) defines a line subbundle of $\mathscr{O}_{X_\infty}^{\oplus 2}$ which induces the map π_{HT}.

The $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariance of π_{HT} recovers (2) from a short exact sequence of GL_2-equivariant sheaves over $\mathbb{P}^1_{\mathbb{Q}_p}$. Indeed, the presentation $\mathbb{P}^1_{\mathbb{Q}_p} = \mathbf{B}\backslash\mathrm{GL}_2$ induces an equivalence between algebraic $\mathbf{B}$-representations and GL_2-equivariant vector bundles over $\mathbb{P}^1_{\mathbb{Q}_p}$. More explicitly, let V be a $\mathbf{B}$-representation; then one defines the vector bundle $\mathcal{V} = \mathrm{GL}_2 \times^{\mathbf{B}} V = \mathbf{B}\backslash(\mathrm{GL}_2 \times V)$, where in the right-hand side term $\mathrm{GL}_2 \times V$, the group $\mathbf{B}$ acts diagonally. Let $\mathscr{F}$ be a K_p-equivariant sheaf over $\mathbb{P}^1_{\mathbb{Q}_p}$; we shall identify $\pi_{\mathrm{HT}}^*(\mathscr{F})$ with the pro-Kummer-étale sheaf over X_{K_p} obtained by descent from the K_p-equivariant pullback over X_∞.

Let $\mathbf{T} \subset \mathbf{B}$ be the diagonal torus and let $\kappa = (k_1, k_2) \in X^*(\mathbf{T})$ be a character. We denote by $\mathbb{Q}_p(\kappa)$ the representation defined by κ and consider it as a $\mathbf{B}$-representation by letting the unipotent radical act trivially. We say that $\kappa = (k_1, k_2)$ is dominant if $k_1 \geq k_2$. Given a dominant character κ, we let $V_\kappa = \mathrm{Sym}^{k_1-k_2}\mathrm{St} \otimes \det^{k_2}$ be the irreducible representation of GL_2 of highest weight κ, and we denote by $V_{\kappa,\text{ét}}$ the associated pro-Kummer-étale sheaf obtained by descent from X_∞. Let $W = \{1, w_0\}$ be the Weyl group of GL_2. We denote by $\mathscr{L}(\kappa)$ the GL_2-equivariant sheaf over $\mathbb{P}^1_{\mathbb{Q}_p}$ given by $\mathbf{B}\backslash(\mathrm{GL}_2 \times w_0(\kappa))$. The standard representation St has a $\mathbf{B}$-filtration

$$0 \to \mathbb{Q}_p(1,0) \to \mathrm{St} \to \mathbb{Q}_p(0,1) \to 0. \tag{3}$$

By construction, the pullback of (3) by π_{HT} is equal to the Hodge–Tate exact sequence (2). In particular, $\pi_{\mathrm{HT}}^*(\mathrm{St} \otimes \mathscr{O}_{\mathbb{P}^1_{\mathbb{Q}_p}}) = T_pE \otimes \widehat{\mathscr{O}}_X$ and $\pi_{\mathrm{HT}}(\mathscr{L}(\kappa)) = \omega_E^{k_1-k_2} \otimes_{\mathscr{O}_X} \widehat{\mathscr{O}}_X(k_2)$. A natural question arises, namely, how to describe the pullbacks by π_{HT} of the GL_2-equivariant vector bundles over $\mathbb{P}^1_{\mathbb{Q}_p}$. We already know that the pullbacks of those vector bundles constructed from characters of $\mathbf{T}$ are related to modular sheaves, and it remains to understand the GL_2-equivariant sheaves constructed from non-semi-simple representations of $\mathbf{B}$.

Let $\mathscr{O}(\mathbf{B})$ be the ring of algebraic functions of $\mathbf{B}$ endowed with the right regular action; note that any algebraic representation of $\mathbf{B}$ occurs in $\mathscr{O}(\mathbf{B})$. The presentation $\mathbf{B} = \mathbf{T} \ltimes \mathbf{N}$ as a semidirect product gives an isomorphism $\mathscr{O}(\mathbf{B}) = \mathscr{O}(\mathbf{T}) \otimes \mathscr{O}(\mathbf{N})$, where $\mathbf{B}$ acts on the first factor by the right regular action of $\mathbf{T}$, and it acts on the second factor under the formula $(n, b) \mapsto t_b^{-1} n t_b n_b$, with $(n, b) \in \mathbf{N} \times \mathbf{B}$ and $b = (t_b, n_b) \in \mathbf{T} \ltimes \mathbf{N}$. We have the following theorem.

https://doi.org/10.1112/S0010437X23007182 Published online by Cambridge University Press

Theorem 1.0.2 (Theorem 4.1.2). *Let $\underline{\mathscr{O}}(\mathbf{N})$ be the* GL_2*-equivariant quasi-coherent sheaf over* $\mathbb{P}^1_{\mathbb{Q}_p}$ *associated to* $\mathscr{O}(\mathbf{N})$. *Let* $\mathscr{O}\mathbb{B}_{\mathrm{dR,log}}$ *be the geometric de Rham period sheaf of [Sch13] and [DLLZ23b], and let* $\mathscr{O}\mathbb{C}_{\log} = \mathrm{gr}^0\, \mathscr{O}\mathbb{B}_{\mathrm{dR,log}}$ *be the Hodge–Tate sheaf. We have a natural isomorphism of pro-Kummer-étale sheaves over* X,

$$\pi_{\mathrm{HT}}^*(\underline{\mathscr{O}}(\mathbf{N})) = \mathscr{O}\mathbb{C}_{\log}.$$

As a corollary one obtains the Eichler–Shimura decompositions.

Theorem 1.0.3 (Theorem 4.1.4). *Let* $\kappa = (k_1, k_2)$ *be a dominant character and set* $\alpha = (1, -1) \in X^*(\mathbf{T})$. *Let* $\mathrm{BGG}(\kappa)$ *be the dual BGG complex of weight* κ *(Proposition 2.4.3),*

$$0 \to V_\kappa \to V(\kappa) \to V(w_0(\kappa) - \alpha) \to 0.$$

We denote by $\underline{\mathrm{BGG}(\kappa)}$ *the associated* GL_2*-equivariant complex over* $\mathbb{P}^1_{\mathbb{Q}_p}$. *Then the pullback of* $\underline{\mathrm{BGG}(\kappa)}$ *by* π_{HT} *is the short exact sequence*

$$0 \to \mathrm{Sym}^{k_1-k_2}\, T_pE \otimes \widehat{\mathscr{O}}_X(k_2) \to \omega_E^{k_2-k_1} \otimes \mathscr{O}\mathbb{C}_{\log}(k_1) \to \omega_E^{k_1-k_2+2} \otimes \mathscr{O}\mathbb{C}_{\log}(k_2-1) \to 0.$$

Furthermore, let $\lambda : X_{\mathbb{C}_p,\text{prokét}} \to X_{\mathbb{C}_p,\mathrm{an}}$ *be the projection of sites. Then*

$$R\lambda_*(\mathrm{Sym}^k\, T_pE \otimes \widehat{\mathscr{O}}_X(1)) = \omega_E^{-k} \otimes \mathbb{C}_p(k+1)[0] \oplus \omega_E^{k+2}[-1].$$

Taking H^1*-cohomology in the analytic site of* $X_{\mathbb{C}_p}$, *we recover the Eichler–Shimura decomposition of Theorem 1.0.1.*

The proof of Theorem 1.0.2 follows from the isomorphism between Faltings extension $\mathrm{gr}^1\, \mathscr{O}\mathbb{B}^+_{\mathrm{dR,log}}$ and the sheaf $T_pE \otimes \widehat{\mathscr{O}}_X \otimes \omega_E$. This isomorphism was known to Faltings, and used in his original proof of Theorem 1.0.1. Our new proof provides a more explicit definition of the Eichler–Shimura maps in terms of cocycles and can be generalized to *any* Shimura variety. Moreover, using this method, one easily deduces the degeneration of the spectral sequence appearing in [Fal87], as well as its natural splitting using simple properties of the dual BGG resolution. It is worth mentioning that the isomorphism between the twist of the Tate module and the Faltings extension was used by Lue Pan in [Pan22] to compute the relative Sen operator of the modular curve.

The second goal of this paper is the interpolation of the Eichler–Shimura decomposition (1). The H^0-map of the overconvergent Eichler–Shimura maps was previously constructed by Andreatta, Iovita and Stevens in [AIS15]. The strategy followed in this paper is close to the construction of the Eichler–Shimura map for Shimura curves in [CHJ17]. Roughly speaking, we take pullbacks by π_{HT} of certain locally analytic sheaves over $\mathbb{P}^1_{\mathbb{Q}_p}$. In this way, we interpolate all the terms appearing in the Hodge–Tate exact sequence (2): we get overconvergent modular sheaves whose cohomology is the object of study in higher Coleman theory developed by Boxer and Pilloni [BP21, BP22]. The interpolation of the symmetric powers of the Tate module will be given by locally analytic principal series or locally analytic distributions as in [AS08]. Finally, the Hodge–Tate maps HT and $\mathrm{HT}^\vee$ can be put in families, obtaining the dlog map of [AIS15] as a particular case.

Let us sketch the main steps of the construction. Let $n \geq 1$ be an integer and let

$$\mathrm{Iw}_n := \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p^n\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}$$

be the Iwahori group modulo p^n. From now on, we will take $X = X_{\mathrm{Iw}_n}$. Let $\epsilon \geq \delta \geq n$ be rational numbers and (R, R^+) a uniform Tate algebra over $\mathbb{Q}_p$ which we may assume to be sheafy

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(i.e. such that the pre-sheaf of rational functions in $\mathrm{Spa}(R,R^+)$ is an actual sheaf). Let $T=\mathbf{T}(\mathbb{Z}_p)$ denote the $\mathbb{Z}_p$-points of the diagonal torus and $\chi=(\chi_1,\chi_2):T\to R^{+,\times}$ a δ-analytic character (cf. Lemma 2.2.5). We let $R\widehat{\otimes}\widehat{\mathscr{O}}_X$ be the p-adically complete tensor product of R and the completed sheaf $\widehat{\mathscr{O}}_X$. Given a character $\lambda:\mathbb{Z}_p^\times\to R^{+,\times}$, we denote by $R(\lambda)$ the $G_{\mathbb{Q}_p}$-module induced by the composition $G_{\mathbb{Q}_p}\xrightarrow{\chi_{\mathrm{cyc}}}\mathbb{Z}_p^\times\xrightarrow{\lambda}R^\times$. Finally, we write $\widehat{\mathscr{O}}_X(\lambda):=R(\lambda)\widehat{\otimes}\widehat{\mathscr{O}}_X$.

We begin with the construction of all the sheaves over $\mathbb{P}^1_{\mathbb{Q}_p}$: for $w\in W=\{1,w_0\}$ we define a family of overconvergent neighbourhoods $\{U_w(\epsilon)\,\mathrm{Iw}_n\}_{\epsilon\geq n}$ of $w\,\mathrm{Iw}_n$ in $\mathbb{P}^1_{\mathbb{Q}_p}$. The affinoid spaces $U_w(\epsilon)\,\mathrm{Iw}_n$ admit sections of the quotient map $\mathrm{GL}_2\to\mathbb{P}^1_{\mathbb{Q}_p}$. In particular, the natural $\mathbf{T}$-torsor $\mathbf{N}\backslash\mathrm{GL}_2\to\mathbb{P}^1_{\mathbb{Q}_p}$, where $\mathbf{N}$ is the unipotent radical of $\mathbf{B}$, has a trivialization over $U_w(\epsilon)\,\mathrm{Iw}_n$. We define a $R\widehat{\otimes}\mathscr{O}_{\mathbb{P}^1_{\mathbb{Q}_p}}$ line bundle $\mathscr{L}(\chi)$ in the analytic site of $U_w(\epsilon)\,\mathrm{Iw}_n$ in the same way we have defined the line bundles $\mathscr{L}(\kappa)$ for $\kappa\in X^*(\mathbf{T})$. Then we define the space of δ-analytic principal series of weight χ to be the R-Banach space

$$A^\delta_\chi=\Gamma(U_{w_0}(\delta)\,\mathrm{Iw}_n,\mathscr{L}(\chi)).$$

We define the δ-analytic distributions D^δ_χ to be the continuous dual of A^δ_χ endowed with the weak topology. The space A^δ_χ has a natural action of Iw_n, so that it defines a constant Iw_n-equivariant sheaf on $\mathbb{P}^1_{\mathbb{Q}_p}$. We let $A^\delta_{\chi,\text{ét}}$ and $D^\delta_{\chi,\text{ét}}$ denote the pro-Kummer-étale sheaves over X obtained by descent from the topological K_p-equivariant sheaves over X_∞.

In Proposition 2.4.4 we construct maps

$$R(\chi)\xrightarrow{\iota}A^\delta_\chi \text{ equivariant for the action of } \mathbf{B}(\mathbb{Z}_p)\cap\mathrm{Iw}_n,$$
$$A^\delta_\chi\xrightarrow{\mathrm{ev}_{w_0}}R(\chi) \text{ equivariant for the action of } w_0^{-1}\mathbf{B}(\mathbb{Z}_p)w_0\cap\mathrm{Iw}_n,$$

with ι being the highest weight vector, and ev_{w_0} the evaluation at w_0. We prove that these maps give rise morphisms of Iw_n-equivariant sheaves

$$\begin{aligned}\mathscr{L}(w_0(\chi))\to A^\delta_\chi\widehat{\otimes}_{\mathbb{Q}_p}\mathscr{O}_{\mathbb{P}^1_{\mathbb{Q}_p}} \text{ over } U_1(\epsilon)\,\mathrm{Iw}_n,\\ A^\delta_\chi\widehat{\otimes}_{\mathbb{Q}_p}\mathscr{O}_{\mathbb{P}^1_{\mathbb{Q}_p}}\to\mathscr{L}(\chi) \text{ over } U_{w_0}(\epsilon)\,\mathrm{Iw}_n\,.\end{aligned}\tag{4}$$

The next step is to translate all the previous constructions to the modular curve X. We start by defining the strict neighbourhoods of the w-ordinary locus $\{X_w(\epsilon)\}_{\epsilon>n}$; they are equal to $\pi_{\mathrm{Iw}_n}(\pi_{\mathrm{HT}}^{-1}(U_w(\epsilon)\,\mathrm{Iw}_n))$. The second object we descend to X are the overconvergent modular sheaves ω_E^χ; they are $R\widehat{\otimes}\mathscr{O}_X$-line bundles in the étale or analytic site of $X_w(\epsilon)$. We refer to [BP21] for the general construction of these sheaves. The dictionary provided by π_{HT} then gives (see Corollary 3.2.11)

$$\pi^*_{\mathrm{HT}}(\mathscr{L}(\chi))=\omega^\chi_E\otimes\widehat{\otimes}\widehat{\mathscr{O}}_X(\chi_2).\tag{5}$$

We continue with the pullback of the δ-analytic principal series and distributions, seen as Iw_n-equivariant sheaves over $\mathbb{P}^1_{\mathbb{Q}_p}$. By definition one has $\pi^*_{\mathrm{HT}}(A^\delta_\chi\widehat{\otimes}\mathscr{O}_{\mathbb{P}^1_{\mathbb{Q}_p}})=A^\delta_{\chi,\text{ét}}\widehat{\otimes}\widehat{\mathscr{O}}_X$ (and similarly for D^δ_χ). Finally, we pullback the maps (4) obtaining overconvergent Hodge–Tate maps of pro-Kummer-étale sheaves

$$\omega_E^{w_0(\chi)}\widehat{\otimes}\widehat{\mathscr{O}}_X(\chi_1)\xrightarrow{\mathrm{HT}^\vee}A^\delta_{\chi,\text{ét}}\widehat{\otimes}\widehat{\mathscr{O}}_X \text{ over } X_1(\epsilon),$$
$$A^\delta_{\chi,\text{ét}}\widehat{\otimes}\widehat{\mathscr{O}}_X\xrightarrow{\mathrm{HT}}\omega^\chi_E\widehat{\otimes}\widehat{\mathscr{O}}_X(\chi_2) \text{ over } X_{w_0}(\epsilon)$$

(and similarly for D^δ_χ). Taking pro-Kummer-étale cohomology, one obtains the following theorem.

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Theorem 1.0.4 (Theorem 4.2.2). *There are overconvergent Eichler–Shimura maps*

$$0 \to H^1_{1,c}(X_{\mathbb{C}_p}, \omega_E^{w_0(\chi)})_\epsilon(\chi_1) \xrightarrow{ES_A^\vee} H^1_{\text{prokét}}(X_{\mathbb{C}_p}, A^\delta_{\chi,\text{ét}} \widehat{\otimes} \widehat{\mathscr{O}}_X) \xrightarrow{ES_A} H^0_{w_0}(X_{\mathbb{C}_p}, \omega_E^{\chi+\alpha})_\epsilon(\chi_2 - 1) \to 0 \tag{6}$$

satisfying the following properties.

(i) *The composition $ES_A \circ ES_A^\vee$ is zero.*
(ii) *Assume that $\mathcal{V} = \operatorname{Spa}(R, R^+)$ is an affinoid subspace of the weight space $\mathcal{W}_T$ of $T = \mathbf{T}(\mathbb{Z}_p)$, and let $\kappa = (k_1, k_2) \in \mathcal{V}$ be a dominant weight of $\mathbf{T}$. Let $\alpha = (1, -1) \in X^*(\mathbf{T})$ and let $\chi = \chi_{\mathcal{V}}^{\text{un}}$ be the universal character of $\mathcal{V}$. Then we have the following commutative diagram.*

$$\begin{array}{ccccc}
H^1_{1,c}(X_{\mathbb{C}_p}, \omega_E^{w_0(\chi)})_\epsilon(\chi_1) & \xrightarrow{ES_A^\vee} & H^1_{\text{prokét}}(X_{\mathbb{C}_p}, A^\delta_{\chi,\text{ét}} \widehat{\otimes} \widehat{\mathscr{O}}_X) & \xrightarrow{ES_A} & H^0_{w_0}(X_{\mathbb{C}_p}, \omega_E^{\chi+\alpha})_\epsilon(\chi_2 - 1) \\
\downarrow & & \downarrow & & \downarrow \\
H^1_{1,c}(X_{\mathbb{C}_p}, \omega_E^{w_0(\kappa)})_\epsilon(k_1) & \longrightarrow & H^1_{\text{prokét}}(X_{\mathbb{C}_p}, A^\delta_{\kappa,\text{ét}} \widehat{\otimes} \widehat{\mathscr{O}}_X) & \longrightarrow & H^0_{w_0}(X_{\mathbb{C}_p}, \omega_E^{\kappa+\alpha})_\epsilon(k_2 - 1) \\
\downarrow{\scriptstyle \text{Cor}} & & \uparrow & & \uparrow{\scriptstyle \text{Res}} \\
H^1_{\text{an}}(X_{\mathbb{C}_p}, \omega_E^{w_0(\kappa)})(k_1) & \xrightarrow{ES^\vee} & H^1_{\text{prokét}}(X_{\mathbb{C}_p}, V_{\kappa,\text{ét}}) \otimes \mathbb{C}_p & \xrightarrow{ES} & H^0_{\text{an}}(X_{\mathbb{C}_p}, \omega_E^{\kappa+\alpha})(k_2 - 1)
\end{array}$$

(iii) *The maps of (ii) are Galois and U_p^t-equivariant with respect to the good normalizations of the U_p^t-operators. In particular, the above diagram restricts to the finite slope part with respect to the U_p^t-action.*
(iv) *Let $h < k_1 - k_2 + 1$. There exists an open affinoid $\mathcal{V}' \subset \mathcal{V}$ containing κ such that the $(\leq h)$-slope part of the restriction of (6) to $\mathcal{V}'$ is a short exact sequence of finite free $\mathbb{C}_p \widehat{\otimes}_{\mathbb{Q}_p} \mathscr{O}(\mathcal{V}')$-modules.*
(v) *Keep the hypothesis of (iv), and let χ be the universal character of $\mathcal{V}'$. Let $\widetilde{\chi} = \chi_1 - \chi_2 + 1 : \mathbb{Z}_p^\times \to R^{+,\times}$, and $b = d/dt|_{t=1}\widetilde{\chi}(t)$. Then we have a Galois equivariant split after inverting b*

$$H^1_{\text{prokét}}(X_{\mathbb{C}_p}, A^\delta_{\chi,\text{ét}} \widehat{\otimes} \widehat{\mathscr{O}}_X)_b^{\leq h} = [H^1_{1,c}(X_{\mathbb{C}_p}, \omega_E^{w_0(\chi)})_\epsilon^{\leq h}(\chi_1)]_b \oplus [H^0_{w_0}(X_{\mathbb{C}_p}, \omega_E^{\chi+\alpha})_\epsilon^{\leq h}(\chi_2 - 1)]_b.$$

Remark 1.0.5.

(i) The group $H^0_w(X_{\mathbb{C}_p}, -)_\epsilon$ is the overconvergent cohomology and $H^1_{w,c}(X_{\mathbb{C}_p}, -)_\epsilon$ the overconvergent cohomology with compact support around the w-ordinary locus of X (see [BP22] and Definition 3.2.12 below).
(ii) A similar statement holds for the distribution sheaves $D^\delta_{\chi,\text{ét}}$, in this case the overconvergent Eichler–Shimura map of [AIS15] is ES_D.
(iii) We also prove the theorem for the proétale cohomology with compact support of $A^\delta_{\chi,\text{ét}}$ and $D^\delta_{\chi,\text{ét}}$.
(iv) Note that if $\kappa = (k_1, k_2)$ with $k_1 + 1 \neq k_2$ (i.e. when the Hodge–Tate weights are not equal), one can choose $\mathcal{V}'$ small enough such that $b \neq 0$.

We finish with the compatibility of the oveconvergent Eichler–Shimura maps (6) with the Poincaré and Serre pairings. One can define a Poincaré pairing between the overconvergent proétale cohomologies

$$\langle -, - \rangle_P : H^1_{\text{proét},c}(Y_{\mathbb{C}_p}, D^\delta_{\chi,\text{ét}}(1)) \times H^1_{\text{proét}}(Y_{\mathbb{C}_p}, A^\delta_{\chi,\text{ét}}) \to \mathscr{O}(\mathcal{V}') \tag{7}$$

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where the left-hand side term is the proétale cohomology with compact support. On the other hand, one also has Serre pairings between overconvergent coherent cohomologies

$$\begin{aligned} \langle -,-\rangle_S &: H^1_{w,c}(X,\omega_E^{-\chi}(-D))_\epsilon \times H^0_w(X,\omega_E^{\chi+\alpha})_\epsilon \to \mathscr{O}(\mathcal{V}'), \\ \langle -,-\rangle_S &: H^1_{w,c}(X,\omega_E^{w_0(\chi)})_\epsilon \times H^0_w(X,\omega_E^{-w_0(\chi)+\alpha}(-D))_\epsilon \to \mathscr{O}(\mathcal{V}'). \end{aligned} \tag{8}$$

We have the following theorem.

THEOREM 1.0.6 (Theorem 4.2.3).

(i) *The Poincaré and Serre pairings (7) and (8) are compatible with the U_p-operators and the overconvergent Eichler–Shimura maps.*
(ii) *Let $\mathcal{W}_T$ be the weight space of $T = \mathbf{T}(\mathbb{Z}_p)$, let $\mathcal{V} \subset \mathcal{W}_T$ be an open affinoid and χ the universal character of $\mathcal{V}$. Let $\kappa = (k_1, k_2) \in \mathcal{V}$ be a dominant weight and fix $h < k_1 - k_2 + 1$. There exists an open affinoid $\mathcal{V}' \subset \mathcal{V}$ containing κ such that the $(\leq h)$-parts of the pairings (7) and (8) are perfect pairings of finite free $\mathbb{C}_p\widehat{\otimes}\mathscr{O}(\mathcal{V}')$-modules compatible with the Eichler–Shimura decomposition.*

The outline of the paper is as follows. In § 2 we develop the overconvergent theory over the flag variety. We define the affinoid subspaces $U_w(\epsilon)\,\mathrm{Iw}_n$ and the sheaves $\mathscr{L}(\chi)$. We construct the δ-analytic principal series A^δ_χ and the maps (4). We recall some facts of the BGG theory for irreducible representations of GL_2; in particular, we define the dual BGG complex $\mathrm{BGG}(\kappa)$.

Then in § 3, we translate all the previous constructions from $\mathbb{P}^1_{\mathbb{Q}_p}$ to the modular curves via π_{HT}. We define the strict neighbourhoods of the w-ordinary locus, the overconvergent modular sheaves and the overconvergent Hodge–Tate maps. We give the good normalizations of the Hecke operators and show that the Hodge–Tate maps are compatible with the normalized U_p-correspondence.

Finally, in § 4, we show how to obtain the classical Eichler–Shimura decomposition from the dual BGG complex, proving Theorems 1.0.3 and 1.0.1, in the process we also prove the theorem for the cohomology with compact support. Next, we construct the overconvergent Eichler–Shimura maps and obtain Theorem 1.0.4. Finally, we show the compatibility of Poincaré and Serre duality for the overconvergent Eichler–Shimura maps, obtaining Theorem 1.0.6.

Notation

Throughout this paper we fix a prime number p, we fix an algebraic closure of $\mathbb{Q}_p$ and denote by $\mathbb{C}_p$ its p-adic completion. We will work with adic spaces over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ which are either locally topologically of finite type over a non-archimedean extension K of $\mathbb{Q}_p$, or perfectoid spaces.

Let X be a log adic space over $\mathbb{Q}_p$; we will work with the proétale and pro-Kummer-étale site of X as introduced in [Sch13] and [DLLZ23a, DLLZ23b]. We denote by $X_?$, with $? \in \{\mathrm{an}, \text{ét}, \text{két}, \text{proét}, \text{prokét}\}$, the analytic, étale, Kummer-étale, proétale and pro-Kummer-étale sites of X, respectively. A space without an underlying log structure will be endowed with the trivial one. Fibre products are always fibre products of fs log adic spaces unless otherwise specified (cf. [DLLZ23b, Proposition 2.3.27]).

Finally, we will denote by $\mathscr{O}_X^{(+)}$ the uncompleted structural sheaves over $X_{\text{prokét}}$, and by $\widehat{\mathscr{O}}_X^{(+)}$ their p-adic completion, omitting the index X if the space is clear in the context. Let V be a topological $\mathbb{Z}_p$-module. By an abuse of notation we also denote by V the pro-Kummer-étale sheaf over $X_{\text{prokét}}$ whose points at an object U are equal to the space of continuous functions $\mathrm{Cont}(|U|, V)$, where $|U|$ is the underlying topological space attached to U.

2. Overconvergent sheaves over the flag variety

Let GL_2 be the algebraic group of invertible 2×2 matrices. Let $\mathbf{B}$ and $\mathbf{T}$ respectively be the upper triangular Borel and the diagonal torus of GL_2, and let $\mathbf{N} \subset \mathbf{B}$ be the unipotent radical consisting of upper triangular unipotent matrices. We also let $\overline{\mathbf{B}}$ and $\overline{\mathbf{N}}$ be the lower triangular Borel and its unipotent radical, respectively. Let $W = \{1, w_0\}$ be the Weyl group of GL_2. Given $n \geq 1$, we let $\mathrm{Iw}_n \subset \mathrm{GL}_2(\mathbb{Z}_p)$ denote the Iwahori subgroup of level p^n, that is, the subgroup of invertible matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that $c \equiv 0 \mod p^n$. We let $\mathrm{FL} = \mathbf{B}\backslash \mathrm{GL}_2$ be the flag variety and $\mathscr{F}\!\ell$ its analytification to an adic space over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. From now on, we see all the previous schemes as living over $\mathbb{Q}_p$.

The goal of this section is to introduce a family of Iw_n-stable overconvergent neighbourhoods of the Iw_n-orbit of $w \in W$ in $\mathscr{F}\!\ell$. Then we introduce some Iw_n-equivariant line bundles which play the role of the overconvergent modular sheaves over $\mathscr{F}\!\ell$. Finally, we construct some weight vector morphisms, which will be translated into the overconvergent Hodge–Tate maps over the modular curve.

For future reference we make the following convention.

Convention 2.0.7. Let $\mathbf{H}$ be an algebraic group scheme over $\mathrm{Spec}\,\mathbb{Z}_p$. We denote by $\mathcal{H}^0$ the rigid generic fibre of the p-adic completion of $\mathbf{H}$, and by $\mathcal{H}$ the analytification of the schematic generic fibre of $\mathbf{H}$ (see [Hub96]). Given a rational number $\delta > 0$, we let $\mathcal{H}(\delta) \subset \mathcal{H}^0 \subset \mathcal{H}$ denote the open subgroup whose (R, R^+)-points are

$$\mathcal{H}(\delta)(R, R^+) = \ker(\mathbf{H}(R^+) \to \mathbf{H}(R^+/p^\delta R^+)).$$

We call $\mathcal{H}(\delta)$ the δ-neighbourhood of the identity in $\mathcal{H}$.

It will be useful to introduce some particular open subgroups of $\mathcal{GL}_2$.

Definition 2.0.8. Let $\epsilon \geq \delta$ be positive rational numbers.

(i) We let

$$\mathcal{GL}_2(\epsilon, \delta) := \mathcal{N}(\delta) \times \mathcal{T}(\delta) \times \overline{\mathcal{N}}(\epsilon) \subset \mathcal{GL}_2\,.$$

(ii) Suppose that $\delta \geq n$. The δ-neighbourhood of Iw_n in $\mathcal{GL}_2$ is the open subgroup

$$\mathcal{I}\mathrm{w}_n(\delta) := \mathrm{Iw}_n\, \mathcal{GL}_2(\delta) = \mathcal{GL}_2(\delta)\, \mathrm{Iw}_n\,.$$

We refer to $\mathcal{I}\mathrm{w}_n(\delta)$ as an affinoid Iwahori subgroup of $\mathcal{GL}_2$.

(iii) We let T, B, N, etc. denote the $\mathbb{Z}_p$-points of $\mathbf{T}$, $\mathbf{B}$, $\mathbf{N}$, etc. Let $n \geq 1$. We define the following subgroups of T, N and $\overline{N}$

$$T_n = \begin{pmatrix} 1 + p^n\mathbb{Z}_p & 0 \\ 0 & 1 + p^n\mathbb{Z}_p \end{pmatrix}, \quad N_n = \begin{pmatrix} 1 & p^n\mathbb{Z}_p \\ 0 & 1 \end{pmatrix}, \quad \overline{N}_n = \begin{pmatrix} 1 & 0 \\ p^n\mathbb{Z}_p & 1 \end{pmatrix}.$$

2.1 GL_2-equivariant sheaves over the flag variety

In this subsection we fix notation for the representation theory of GL_2. Let $X^*(\mathbf{T})$ be the character group of the diagonal torus; we identify $X^*(\mathbf{T}) \cong \mathbb{Z}^2$ via the presentation

$$\mathbf{T} = \begin{pmatrix} \mathbb{G}_m & 0 \\ 0 & \mathbb{G}_m \end{pmatrix}.$$

A weight $\kappa \in X^*(\mathbf{T})$, written as $\kappa = (k_1, k_2)$, is said to be dominant if $k_1 \geq k_2$. We denote by $X^*(\mathbf{T})^+$ the cone of dominant weights. Given $\kappa \in X^*(\mathbf{T})^+$, we let V_κ denote the irreducible algebraic representation of GL_2 of highest weight κ. Letting St and det respectively be the standard and the determinant representations of GL_2, explicitly one has that $V_\kappa \cong \mathrm{Sym}^{k_1-k_2}\,\mathrm{St} \otimes \det^{k_2}$.

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Since $\mathrm{FL} = \mathbf{B}\backslash \mathrm{GL}_2$, there is an equivalence between the category of algebraic representations of $\mathbf{B}$ and the category of GL_2-equivariant vector bundles over FL. Explicitly, given W a representation of $\mathbf{B}$, one forms the GL_2-equivariant vector bundle

$$\mathcal{W} := \mathrm{GL}_2 \times^{\mathbf{B}} W := \mathbf{B}\backslash(\mathrm{GL}_2 \times W),$$

where in the right-hand-side term the group $\mathbf{B}$ acts diagonally.

DEFINITION 2.1.1. Let $\kappa \in X^*(\mathbf{T})$. We define $\mathscr{L}(\kappa)$ to be the GL_2-equivariant line bundle given by $\mathrm{GL}_w \times^{\mathbf{B}} w_0(\kappa)$.

Remark 2.1.2. (i) The line bundle $\mathcal{L}(\kappa)$ can be described in the following way. Let $\widetilde{\mathrm{FL}} = \mathbf{N}\backslash \mathrm{GL}_2$ be the natural $\mathbf{T}$-torsor over FL and $\pi : \widetilde{\mathrm{FL}} \to \mathrm{FL}$ the projection map. Then $\pi_* \mathscr{O}_{\widetilde{\mathrm{FL}}}$ is endowed with a left regular action of $\mathbf{T}$. One can construct the line bundle $\mathcal{L}(\kappa)$ as the following isotypic component:

$$\begin{aligned}\mathcal{L}(\kappa) &= \pi_* \mathscr{O}_{\widetilde{\mathrm{FL}}}[-w_0(\kappa)] \\ &= \{f \in \pi_* \mathscr{O}_{\widetilde{\mathrm{FL}}} : f(tx) = w_0(\kappa)(t) f(x), \ \text{for } t \in \mathbf{T}\}.\end{aligned}$$

The previous description shows that

$$\widetilde{\mathrm{FL}} = \underline{\mathrm{Isom}}(\mathscr{O}_{\mathrm{FL}}, \mathscr{L}(0,-1)) \times \underline{\mathrm{Isom}}(\mathscr{O}_{\mathrm{FL}}, \mathscr{L}(-1,0)).$$

(ii) The convention on the weight is made such that, if κ is dominant, then $\Gamma(\mathrm{FL}, \mathcal{L}(\kappa))$ is isomorphic to V_κ as a GL_2-representation.

2.2 Overconvergent line bundles over the flag variety

In order to define the overconvergent line bundles we first need to introduce some affinoid neighbourhoods of $w \in W$.

DEFINITION 2.2.1. Let $\epsilon > 0$ be a rational number, $w \in W$ and $w\,\mathcal{GL}_2(\epsilon)$ the ϵ-neighbourhood of w in $\mathcal{GL}_2$. We denote by $U_w(\epsilon)$ its image in $\mathscr{F}\ell$.

LEMMA 2.2.2.

(i) *The collection $\{U_w(\epsilon)\}_{\epsilon>0}$ is a basis of open affinoid neighbourhoods of $w \in \mathscr{F}\ell$. Moreover, we have a natural isomorphism*

$$\overline{\mathcal{N}}(\epsilon) w \xrightarrow{\sim} U_w(\epsilon).$$

(ii) *The Iwahori subgroups admit Iwahori decompositions*

$$\mathcal{I}\mathrm{w}_n(\epsilon) = (\overline{N}_n \overline{\mathcal{N}}(\epsilon)) \times (T\mathcal{T}(\epsilon)) \times (N\mathcal{N}(\epsilon)).$$

(iii) *Let $\epsilon \geq \delta \geq n \geq 1$. We have decompositions*

$$\mathcal{GL}_2(\epsilon, \delta)\,\mathrm{Iw}_n = (N\mathcal{N}(\delta)) \times (T\mathcal{T}(\delta)) \times (\overline{N}_n \overline{\mathcal{N}}(\epsilon)),$$

$$\mathcal{GL}_2(\epsilon, \delta) w_0\,\mathrm{Iw}_n = (N_n\mathcal{N}(\delta)) \times (T\mathcal{T}(\delta)) \times (\overline{N\mathcal{N}}(\epsilon)) w_0.$$

Proof. The collection $\{U_w(\epsilon)\}_{\epsilon>0}$ is a basis of neighbourhoods of w since $\mathscr{F}\ell$ is a locally spectral space and $\bigcap_{\epsilon>0} U_w(\epsilon) = \{w\}$. The isomorphism $U_w(\epsilon) \cong \overline{\mathcal{N}}(\epsilon) w$ is obvious. Next we prove (ii); part (iii) is proved in a similar way. It suffices to show the equality at (R, R^+)-points, with (R, R^+) a uniform affinoid $\mathbb{Q}_p$-algebra. By definition we have

$$\mathcal{I}\mathrm{w}_n(\epsilon) = \begin{pmatrix} \mathbb{Z}_p^\times(1 + p^\epsilon \mathbb{D}^1_{\mathbb{Q}_p}) & \mathbb{Z}_p + p^\epsilon \mathbb{D}^1_{\mathbb{Q}_p} \\ p^n \mathbb{Z}_p + p^\epsilon \mathbb{D}^1_{\mathbb{Q}_p} & \mathbb{Z}_p^\times(1 + p^\epsilon \mathbb{D}^1_{\mathbb{Q}_p}) \end{pmatrix},$$

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where $\mathbb{D}^1_{\mathbb{Q}_p} = \mathrm{Spa}(\mathbb{Q}_p\langle T\rangle, \mathbb{Z}_p\langle T\rangle)$ is the closed affinoid disc. Then

$$\mathcal{I}\mathrm{w}_n(\epsilon)(R,R^+) = \begin{pmatrix} \mathbb{Z}_p^\times(1+p^\epsilon R^+) & \mathbb{Z}_p + p^\epsilon R^+ \\ p^n\mathbb{Z}_p + p^\epsilon R^+ & \mathbb{Z}_p^\times(1+p^\epsilon R^+) \end{pmatrix}$$

Let $g \in \mathcal{I}\mathrm{w}_n(\epsilon)(R,R^+)$. Writing

$$g = \begin{pmatrix} 1 & x_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 & 0 \\ 0 & x_4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x_3 & 1 \end{pmatrix}$$

and solving the equations, one finds $x_3 \in p^n\mathbb{Z}_p + p^\epsilon R^+$, x_1 and $x_4 \in \mathbb{Z}_p^\times(1+p^\epsilon R^+)$, and $x_2 \in \mathbb{Z}_p + p^\epsilon R^+$ which gives (ii). □

Remark 2.2.3. Let us identify $\mathscr{F}\ell \cong \mathbb{P}^1_{\mathbb{Q}_p}$ by taking $[0:1] \in \mathbb{P}^1_{\mathbb{Q}_p}$ as the marked point, and where GL_2 acts by

$$[x:y]\begin{pmatrix} a & b \\ c & d \end{pmatrix} = [ax+cy : bx+dy].$$

Let $T = x/y$ be the canonical coordinate and $\epsilon = p^{-n}$. In the notation of [AI22, § 4.2] we have

$$U_1(\epsilon) = U^{(n)}_{0,0} = \{[x:y] \in \mathbb{P}^1_{\mathbb{Q}_p} : |T/p^n| \leq 1\},$$
$$U_{w_0}(\epsilon) = U^{(n)}_{\infty} = \{[x:y] \in \mathbb{P}^1_{\mathbb{Q}_p} : |1/(p^nT)| \leq 1\}.$$

Lemma 2.2.4 describes the dynamics of the element $\varpi = \mathrm{diag}(1,p)$ over $\mathscr{F}\ell$. This action has only two fixed points represented by the elements of W, and expands or shrinks neighbourhoods of 1 and w_0, respectively.

Lemma 2.2.4. *Let $\varpi = \mathrm{diag}(1,p)$. The following assertions hold.*

(i) $U_1(\epsilon)\varpi = U_1(\epsilon-1)$ *and* $U_{w_0}(\epsilon)\varpi = U_{w_0}(\epsilon+1)$.
(ii) *Let $\epsilon \geq n \geq 1$. Then* $U_1(\epsilon)\,\mathrm{Iw}_n\,\varpi = U_1(\epsilon-1)\,\mathrm{Iw}_{n-1}$ *and* $U_{w_0}(\epsilon)\,\mathrm{Iw}_n\,\varpi = U_{w_0}(\epsilon+1)N_1$.

Proof. The proof follows from Lemma 2.2.2 and the computation

$$\begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} = \begin{pmatrix} a & pb \\ p^{-1}c & d \end{pmatrix}.$$ □

Let Γ be a finite $\mathbb{Z}_p$-module. Abstractly, Γ is isomorphic to $\mathbb{Z}_p^s \bigoplus \Gamma_{\mathrm{tor}}$ where $s \in \mathbb{N}$ and $\Gamma_{\mathrm{tor}} \subset \Gamma$ is the torsion subgroup; we call such an isomorphism a chart of Γ. Let $\mathcal{V} = \mathrm{Spa}(R,R^+)$ be an affinoid adic space with R an uniform Tate $\mathbb{Q}_p$-algebra, and $\chi : \Gamma \to R^{+,\times}$ a continuous character.

Lemma 2.2.5 [Urb11, Lemma 3.4.6]. *Let $\psi : \Gamma \cong \mathbb{Z}_p^s \times \Gamma_{\mathrm{tor}}$ be a chart. There exists $\delta > 0$ such that χ extends to a character*

$$\chi : (\mathbb{Z}_p + p^\delta \mathbb{D}^1_{\mathbb{Q}_p})^s \times \Gamma_{\mathrm{tor}} \times \mathcal{V} \to \mathbb{G}_m. \tag{9}$$

We say that χ is a δ-analytic character of Γ with respect to the chart ψ.

Remark 2.2.6. In the following we will take $\Gamma = T$, and we say that χ is δ-analytic if it extends to a character of $T\mathcal{T}(\delta)$. Let $\mathfrak{W}_T = \mathrm{Spf}\,\mathbb{Z}_p[[T]]$ be the weight space of T and $\mathcal{W}_T$ its rigid generic fibre, in practice we will take $\mathcal{V} = \mathrm{Spa}(R,R^+) \subset \mathcal{W}_T$ an affinoid subspace and $\chi = \chi_{\mathrm{univ}}$ the universal character over $\mathcal{V}$.

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DEFINITION 2.2.7. Let $\epsilon \geq \delta \geq n \geq 1$, $w \in W$, and $\widetilde{\mathscr{F}\ell} = \mathcal{N}\backslash\mathcal{GL}_2$ the natural $\mathcal{T}$-torsor over $\mathscr{F}\ell$.

(i) We define the following neighbourhood of w in $\widetilde{\mathscr{F}\ell}$:

$$\widetilde{U}_w(\epsilon, \delta) = \mathcal{N}(\delta)\backslash\mathcal{GL}_2(\epsilon, \delta)w.$$

Let $\mathrm{pr} : \widetilde{U}_w(\epsilon, \delta)\,\mathrm{Iw}_n \to U_w(\epsilon)\,\mathrm{Iw}_n$ denote the natural projection, Lemma 2.2.2(iii) implies that pr is a trivial $T\mathcal{T}(\delta)$-torsor.

(ii) Let $\chi : T \to R^\times$ be a δ-analytic character. We define the $\mathscr{O}_{\mathscr{F}\ell}\widehat{\otimes}R$-line bundle

$$\mathcal{L}(\chi) = \mathrm{pr}_* \mathscr{O}_{\widetilde{U}_w(\epsilon,\delta)}\widehat{\otimes}R[-w_0(\chi)].$$

In other words, $\mathcal{L}(\chi)$ is the line bundle whose sections over $U \subset U_w(\epsilon)\,\mathrm{Iw}_n$ are

$$\mathcal{L}(\chi)(U) = \{f \in \mathscr{O}_{\mathscr{F}\ell}(\mathrm{pr}^{-1}(U))\widehat{\otimes}R : f(tx) = w_0(\chi)(t)f(x), \text{ for } t \in T\mathcal{T}(\delta)\}.$$

Remark 2.2.8. Let $\varpi = \mathrm{diag}(1, p)$, $c = \mathrm{diag}(p, p)$, and $\Lambda = \langle \varpi, c\rangle \subset \mathbf{T}(\mathbb{Q}_p)$. The natural map $\widetilde{\mathscr{F}\ell} \to \Lambda\backslash\widetilde{\mathscr{F}\ell}$ identifies $\widetilde{U}_w(\epsilon, \delta)$ with an open affinoid of the quotient, namely, the Λ orbit of $\widetilde{U}_w(\epsilon, \delta)$ in $\widetilde{\mathscr{F}\ell}$ is the disjoint union

$$\bigsqcup_{\gamma \in \Lambda} \widetilde{U}_w(\epsilon, \delta)\gamma.$$

By construction, the sheaves $\mathcal{L}(\chi)$ are independent of δ and ϵ. They fit in a U_p-correspondence as follows. Let $\varpi = \mathrm{diag}(1, p)$ and consider the double coset $\mathrm{Iw}_n\,\varpi\,\mathrm{Iw}_n$. Then one has that

$$\mathrm{Iw}_n\,\varpi\,\mathrm{Iw}_n = \bigsqcup_{a=0}^{p-1} \mathrm{Iw}_n \begin{pmatrix} 1 & -a \\ 0 & p \end{pmatrix}.$$

Let us denote $U_{p,a} = \left(\begin{smallmatrix} 1 & -a \\ 0 & p \end{smallmatrix}\right)$.

DEFINITION 2.2.9. We define the U_p-correspondence of $\widetilde{\mathscr{F}\ell}$ (respectively, the normalized U_p-correspondence of $\Lambda\backslash\widetilde{\mathscr{F}\ell}$) to be the diagram

$$\begin{array}{ccc} & \bigsqcup_{a=0}^{p-1}\widetilde{\mathscr{F}\ell} & \\ {\scriptstyle p_1}\swarrow & & \searrow{\scriptstyle p_2} \\ \widetilde{\mathscr{F}\ell} & & \widetilde{\mathscr{F}\ell} \end{array} \tag{10}$$

(respectively, for $\Lambda\backslash\widetilde{\mathscr{F}\ell}$), where $\widetilde{\mathscr{F}\ell}_a = \widetilde{\mathscr{F}\ell}$, $p_1|_{\widetilde{\mathscr{F}\ell}_a} = \mathrm{id}_{\mathscr{F}\ell}$, and $p_2|_{\mathscr{F}\ell_a} = R_{U_{p,a}^{-1}}$ is right multiplication by $U_{p,a}^{-1}$.

Remark 2.2.10. The correspondence (10) is equivariant for the natural action of Iw_n, namely, the group acts by right multiplication on the bottom spaces, and it acts on the disjoint union $\bigsqcup_{a=0}^{p-1}\widetilde{\mathscr{F}\ell}$ as follows. Let $\gamma \in \mathrm{Iw}_n$ and $a, a' \in \{0, \ldots, p-1\}$ such that

$$U_{p,a}\gamma \in \mathrm{Iw}_n\, U_{p,a'}.$$

Then, given $x \in \mathscr{F}\ell_a$, we define $x \cdot \gamma$ to be $x\gamma \in \mathscr{F}\ell_{a'}$. This action satisfies the following properties.

(i) The map p_1 is Iw_n-equivariant.

(ii) The map p_2 preserves Iw_n-orbits, that is, the composition $\bigsqcup_{a=0}^{p-1}\widetilde{\mathscr{F}\ell} \xrightarrow{p_2} \widetilde{\mathscr{F}\ell}/\,\mathrm{Iw}_n$ onto the quotient stack[1] factors through $(\bigsqcup_{a=0}^{p-1}\widetilde{\mathscr{F}\ell})/\,\mathrm{Iw}_n$.

[1] By considering the quotient as a v-stack (see [Sch17]).

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The previous points show that we have a correspondence of stacks

$$\begin{array}{ccc} & (\bigsqcup_{a=0}^{p-1} \widetilde{\mathscr{F}\ell})/\operatorname{Iw}_n & \\ & \swarrow^{p_1} \quad \searrow^{p_2} & \\ \widetilde{\mathscr{F}\ell}/\operatorname{Iw}_n & & \widetilde{\mathscr{F}\ell}/\operatorname{Iw}_n \end{array} \tag{11}$$

In § 3 we will relate this diagram to the U_p-correspondence of modular curves.

The following lemma describes the dynamics of the correspondence on neighbourhoods of 1 and w_0.

Lemma 2.2.11. *The following assertions hold.*

- (i) $p_2(p_1^{-1}(\widetilde{U}_{w_0}(\epsilon,\delta)\operatorname{Iw}_n)) \supset \widetilde{U}_{w_0}(\epsilon-1,\delta)\operatorname{Iw}_n$.
- (ii) $p_2(p_1^{-1}(\widetilde{U}_1(\epsilon,\delta)\operatorname{Iw}_n)) \subset \widetilde{U}_1(\epsilon+1,\delta)\operatorname{Iw}_n$.
- (iii) $p_1(p_2^{-1}(\widetilde{U}_{w_0}(\epsilon,\delta)\operatorname{Iw}_n)) \subset \widetilde{U}_{w_0}(\epsilon+1,\delta)\operatorname{Iw}_n$.
- (iv) $p_1(p_2^{-1}(\widetilde{U}_1(\epsilon,\delta)\operatorname{Iw}_n)) \supset \widetilde{U}_1(\epsilon-1,\delta)\operatorname{Iw}_n$.

Proof. These assertions follow from the definition of the correspondence and Lemma 2.2.4. □

Definition 2.2.12.

- (i) Let $\kappa \in X^*(\mathbf{T})$. We define the $U_{p,\kappa}$- and $U_{p,\kappa}^t$-correspondences of $\mathscr{L}(\kappa)$

$$U_{p,\kappa}: p_2^*\mathscr{L}(\kappa) \to p_1^*\mathscr{L}(\chi) \quad \text{and} \quad U_{p,\kappa}^t: p_1^*\mathscr{L}(\kappa) \to p_2^*\mathscr{L}(\chi),$$

  to be the maps constructed by taking $(-w_0(\kappa))$-isotypic components of the structural sheaves of diagram (10).
- (ii) Let χ be a δ-analytic character of T. We define the normalized U_p- and U_p^t-correspondences of $\mathscr{L}(\chi)$,

$$U_p: p_2^*\mathscr{L}(\chi) \to p_1^*\mathscr{L}(\chi) \quad \text{and} \quad U_p^t: p_1^*\mathscr{L}(\chi) \to p_2^*\mathscr{L}(\chi),$$

  to be the maps constructed by taking $(-w_0(\chi))$-isotypic components of the structural sheaves of the diagrams

$$\begin{array}{ccc} & p_2^{-1}(\widetilde{U}_{w_0}(\epsilon+1,\delta)\operatorname{Iw}_n) & \\ & \swarrow^{p_1} \quad \searrow^{p_2} & \\ \widetilde{U}_{w_0}(\epsilon+1,\delta)\operatorname{Iw}_n & & \widetilde{U}_{w_0}(\epsilon,\delta)\operatorname{Iw}_n \end{array}$$

$$\begin{array}{ccc} & p_1^{-1}(\widetilde{U}_1(\epsilon-1,\delta)\operatorname{Iw}_n) & \\ & \swarrow^{p_1} \quad \searrow^{p_2} & \\ \widetilde{U}_1(\epsilon-1,\delta)\operatorname{Iw}_n & & \widetilde{U}_1(\epsilon,\delta)\operatorname{Iw}_n \end{array}$$

  obtained from the normalized diagram (10) and Lemma 2.2.11.

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Remark 2.2.13. Let κ be a classical weight. The relation between the classical and the normalized U_p-operators for $\mathscr{L}(\kappa)$ is given by the formulas

$$U_p = \frac{1}{w_0(\kappa)(\varpi^{-1})} U_{p,\kappa} \quad \text{and} \quad U_p^t = \frac{1}{w_0(\kappa)(\varpi)} U_{p,\kappa}^t \quad \text{over } U_1(\epsilon)\,\mathrm{Iw}_n,$$
$$U_p = \frac{1}{\kappa(\varpi^{-1})} U_{p,\kappa} \quad \text{and} \quad U_p^t = \frac{1}{\kappa(\varpi)} U_{p,\kappa}^t \quad \text{over } U_{w_0}(\epsilon)\,\mathrm{Iw}_n\,.$$

2.3 Analytic principal series and distributions

In this subsection we introduce some non-archimedean spaces interpolating the finite-dimensional representations of GL_2. Let $\mathcal{V} = \mathrm{Spa}(R, R^+)$ be a uniform affinoid adic space over $\mathbb{Q}_p$ and $\chi : T \to R^{+,\times}$ a δ-analytic character.

DEFINITION 2.3.1. Let $\delta \geq n$.

(i) We define the δ-analytic principal series of weight χ to be the Iw_n-module

$$A_\chi^\delta = \Gamma(U_{w_0}(\delta)\,\mathrm{Iw}_n, \mathscr{L}(\chi)),$$

seen as a Banach space over R. Equivalently, we have that

$$A_\chi^\delta = \{f : w_0\,\mathcal{I}\mathrm{w}_n(\epsilon,\delta) \to \mathbb{A}^1_R | f(bw_0x) = w_0(\chi)(b) f(w_0x), \text{ for } b \in \mathcal{B} \cap w_0\,\mathcal{I}\mathrm{w}_n(\epsilon,\delta) w_0^{-1}\}.$$

(ii) We define the δ-analytic distributions of weight χ to be the continuous weak dual

$$D_\chi^\delta = \mathrm{Hom}^0_R(A_\chi^\delta, R).$$

Remark 2.3.2. It will come in handy to define lattices in the principal series and distributions, namely, let $\mathscr{L}^+(\chi) = f_*\mathscr{O}^+_{\tilde{U}_w(\epsilon,\delta)\,\mathrm{Iw}_n}[-w_0(\chi)]$, $A_\chi^{\delta,+} = \Gamma(U_{w_0}(\epsilon)\,\mathrm{Iw}_n, \mathscr{L}^+(\chi))$, and $D_\chi^{\delta,+} = \mathrm{Hom}_{R^+}(A_\chi^{\delta,+}, R^+)$. The space $N\mathcal{N}(\delta)$ is a disjoint union of closed discs; in particular, $\mathscr{O}^+(N\mathcal{N}(\delta))$ is an orthonormalizable $\mathbb{Z}_p$-algebra. Let $\{e_i\}_{i\in I}$ be an orthonormalizable basis of $\mathscr{O}^+(N\mathcal{N}(\delta))$. Using the Iwahori decomposition, one has isomorphisms of R^+-modules

$$A_\chi^{\delta,+} \cong \widehat{\bigoplus_{i\in I}} R^+ e_i \quad \text{and} \quad D_\chi^{\delta,+} \cong \prod_{i\in I} R^+ e_i^\vee.$$

Remark 2.3.3. It is easy to compare the δ-analytic principal series and distributions defined above with those used in [AIS15]. Let $\chi : T \to R^{+,\times}$ be a δ-analytic character written as $\chi = (\chi_1, \chi_2)$. Consider the set $\mathbb{Z}_p^\times \times \mathbb{Z}_p$ endowed with right multiplication by Iw_n and left multiplication by $\mathbb{Z}_p^\times$. We let $A^{\delta,+}_{\chi_1-\chi_2}$ be the space of functions $f : \mathbb{Z}_p^\times \times \mathbb{Z}_p \to R^{+,\times}$ satisfying the following conditions.

(i) $f|_{1\times\mathbb{Z}_p}$ extends to an analytic function of $\mathbb{Z}_p + p^\delta \mathbb{D}^1_{\mathbb{Q}_p}$.
(ii) $f(tx) = (\chi_1 - \chi_2)(t) f(x)$ for $t \in \mathbb{Z}_p^\times$ and $x \in \mathbb{Z}_p^\times \times \mathbb{Z}_p$.

Note that $\mathbb{Z}_p^\times \times \mathbb{Z}_p$ endowed with the action of Iw_n and $\mathbb{Z}_p^\times$ is isomorphic to the quotient

$$\mathbb{Z}_p^\times \times \mathbb{Z}_p = \begin{pmatrix} 1 & 0 \\ p^n\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix} \Big\backslash \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p^n\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ p^n\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix} \Big\backslash \mathrm{Iw}_n\,.$$

Thus, we have an isomorphism

$$A_\chi^{\delta,+} = A^{\delta,+}_{\chi_1-\chi_2} \otimes (\det)^{\chi_2}.$$

Remark 2.3.4. Let $\delta' > \delta \geq n$. The inclusion $U_{w_0}(\delta')\,\mathrm{Iw}_n \subset U_{w_0}(\delta)\,\mathrm{Iw}_n$ induces maps $A_\chi^{\delta,+} \to A_\chi^{\delta',+}$ and $D_\chi^{\delta',+} \to D_\chi^{\delta,+}$. Furthermore, let $\widehat{A}_\chi^{\delta,+}$ be the completion of $A_\chi^{\delta,+}$ in $A_\chi^{\delta',+}$. Since the inclusion of affinoids above is strict, one has that

$$\widehat{A}_{\chi}^{\delta,+}=\prod_{i\in I}R^+e_i,$$

endowed with the product topology. Let $D_{\chi}^{\delta,b,+}$ be the continuous R^+-dual of $\widehat{A}_{\chi}^{\delta,+}$ endowed with the p-adic topology. Then the arrow $D_{\chi}^{\delta',+}\to D_{\chi}^{\delta,+}$ factors through $D_{\chi}^{\delta,b,+}$. Moreover, we have that

$$D_{\chi}^{\delta,b,+}\cong\widehat{\bigoplus_{i\in I}}R^+e_i^{\vee}.$$

The previous discussion shows that the directed (respectively, inverse) systems $\{A_{\chi}^{\delta,+}\}_{\delta\geq n}$ and $\{\widehat{A}_{\chi}^{\delta,+}\}_{\delta\geq n}$ (respectively, $\{D_{\chi}^{\delta,+}\}_{\delta\geq n}$ and $\{D_{\chi}^{\delta,b,+}\}_{\delta\geq n}$) are isomorphic as systems of topological R^+-modules.

For future reference we will prove a devisage of A_{χ}^{δ} and D_{χ}^{δ} in terms of finite Iw_n-modules. We need the following lemma.

Lemma 2.3.5. *Let $(F,\mathcal{O}_F)$ be a non-archimedean field. Let $\mathcal{H}=\mathrm{Spa}(A,A^+)$ be an affinoid adic analytic group over F, and $Z=\mathrm{Spa}(R,R^+)$ be an affinoid adic space topologically of finite type over $\mathrm{Spa}(F,\mathcal{O}_F)$. Let $\Theta:\mathcal{H}\times Z\to Z$ be an action of $\mathcal{H}$ over Z. Then for all $N>0$ there exists a neighbourhood $1\in U\subset\mathcal{H}$ such that for all $g\in U$, $z\in Z$ and $f\in\mathscr{O}^+(Z)$, we have $|f(z)-f(gz)|\leq|p|^N$.*

Proof. As $\mathscr{O}^+(Z)=R^+$ is topologically of finite type over $\mathcal{O}_F$, it suffices to prove the proposition for a single $f\in R^+$. Let $\Theta^*:R^+\to(A^+\widehat{\otimes}_{\mathcal{O}_K}R^+)^+$ be the pullback of the multiplication map. Let $V\subset\mathcal{H}\times Z$ be the open affinoid subspace defined by the equation

$$|1\otimes f-\Theta^*(f)|\leq|p|^N.$$

As V contains $1\times Z$ and this is a quasi-compact closed subset of $\mathcal{H}\times Z$, there exists $1\in U_f\subset U$ such that $U_f\times Z\subset V$. Therefore, for all $g\in U_f$ and $z\in Z$ we have $|f(z)-f(gz)|\leq|p|^N$. □

Corollary 2.3.6. *Let $s\geq 1$. There exists an open subgroup $H\subset\mathrm{Iw}_n$, depending on s, which acts trivially on $A_{\chi}^{\delta,+}/p^s$. In particular, we can write $A_{\chi}^{\delta,+}/p^s$ as a colimit of finite $R/p^s[\mathrm{Iw}_n]$-modules (dually, we can write $D_{\chi}^{\delta,+}/p^s$ as a projective limit of finite $R/p^s[\mathrm{Iw}_n]$-modules).*

Proof. The affinoid group $\mathcal{I}\mathrm{w}_n(\delta)$ acts on the space $\widetilde{U}_{w_0}(\delta,\delta)$ by right multiplication. The corollary follows by Lemma 2.3.5 since $A_{\chi}^{\delta,+}$ is by definition an isotypic component of the global functions of $\widetilde{U}_{w_0}(\delta,\delta)$. Dually, consider the map $D_{\chi}^{\delta,+}\to D_{\chi}^{\delta-1,+}$ and define $\mathrm{Fil}^s D_{\chi}^{\delta,+}=D_{\chi}^{\delta,+}\cap p^sD_{\chi}^{\delta-1,+}$. Then the quotients $D_{\chi}^{\delta,+}/\mathrm{Fil}^s D_{\chi}^{\delta,+}$ are finite R^+/p^s-modules and the weak topology of $D_{\chi}^{\delta,+}$ is the same as the inverse limit topology (see [AIS15, Proposition 3.10])

$$D_{\chi}^{\delta,+}=\varprojlim_s D_{\chi}^{\delta,+}/\mathrm{Fil}^s D_{\chi}^{\delta,s}.$$

The corollary follows. □

Definition 2.3.7. Let V be a Banach space over $\mathbb{Q}_p$ and $V^+\subset V$ a lattice. We define the completed tensor products

$$A_{\chi}^{\delta}\widehat{\otimes}V=\Big(\varprojlim_s A_{\chi}^{\delta,+}/p^s\otimes V^+/p^s\Big)\left[\frac{1}{p}\right]\quad\text{and}\quad D_{\chi}^{\delta}\widehat{\otimes}V=\Big(\varprojlim_s D_{\chi}^{\delta,+}/\mathrm{Fil}^s D_{\chi}^{\delta,+}\otimes V^+/p^s\Big)\left[\frac{1}{p}\right].$$

https://doi.org/10.1112/S0010437X23007182 Published online by Cambridge University Press

Next, let us explain how A^δ_χ and D^δ_χ interpolate the finite-dimensional representations of GL_2.

PROPOSITION 2.3.8. *Let $\kappa \in X^*(\mathbf{T})$ be a dominant weight and $\delta \geq n$. There is a natural Iw_n-equivariant inclusion $V_\kappa \to A^\delta_\kappa$. Dually, there is a natural Iw_n-equivariant surjective map $D^\delta_\kappa \to V^\vee_\kappa = V_{-w_0(\kappa)}$.*

Proof. The second map is just the dual of the first, and the arrow $V_\kappa \to A^\delta_\kappa$ arises from the global sections functor applied to the inclusion $U_{w_0}(\delta)\,\mathrm{Iw}_n \subset \mathscr{F}\!\ell$ and to the line bundle $\mathscr{L}(\kappa)$. □

Finally, let us briefly discuss the U_p-correspondence of the principal series and distributions. We let $A^\delta_\chi \widehat{\otimes} \mathscr{O}_{\mathscr{F}\!\ell}$ and $D^\delta_\chi \widehat{\otimes} \mathscr{O}_{\mathscr{F}\!\ell}$ be the constant Iw_n-equivariant quasi-coherent sheaves over $\mathscr{F}\!\ell$ induced by A^δ_χ and D^δ_χ. We have isomorphisms of $\mathscr{O}_{\mathscr{F}\!\ell}$-sheaves

$$A^\delta_\chi \widehat{\otimes} \mathscr{O}_{\mathscr{F}\!\ell} = \widehat{\bigoplus_{i\in I}} (R \widehat{\otimes} \mathscr{O}_{\mathscr{F}\!\ell}) e_i \quad \text{and} \quad D^\delta_\chi \widehat{\otimes} \mathscr{O}_{\mathscr{F}\!\ell} = \prod_{i \in I} (R^+ \widehat{\otimes} \mathscr{O}^+_{\mathscr{F}\!\ell}) e_i^\vee \left[\frac{1}{p}\right].$$

By Remark 2.2.8 one can endow A^δ_χ with an action of ϖ commuting with Iw_n (dually, one can endow D^δ_χ with an action of ϖ^{-1}). Moreover, the multiplication by ϖ on A^δ_χ (respectively, D^δ_χ) factors through $A^{\delta-1}_\chi$ (respectively, $D^{\delta+1}_\chi$) in accordance with Lemma 2.2.4. Using this action and diagram (10), one defines maps

$$U_p^t : p_1^*(A^\delta_\chi \widehat{\otimes} \mathscr{O}_{\mathscr{F}\!\ell}) \to p_2^*(A^{\delta-1}_\chi \widehat{\otimes} \mathscr{O}_{\mathscr{F}\!\ell}) \quad \text{and} \quad U_p : p_2^*(D^\delta_\chi \widehat{\otimes} \mathscr{O}_{\mathscr{F}\!\ell}) \to p_1^*(D^{\delta+1}_\chi \widehat{\otimes} \mathscr{O}_{\mathscr{F}\!\ell}) \tag{12}$$

(see [AI22, §§ 3.5.2 and 4.5] for more details). We highlight that the previous U_p-correspondence is compatible with the maps of Proposition 2.3.8, after normalizing the action of ϖ and c on V_κ (see Remark 2.2.13 and the first remark of [AS08, § 3.11]).

2.4 The dual BGG complex and the weight vector maps

Let $W = \{1, w_0\}$ be the Weyl group of GL_2 and $\mathbf{B}w_0\mathbf{N} \subset \mathrm{GL}_2$ the big cell. We have a commutative diagram of torsors as follows.

$$\begin{array}{ccc} \mathbf{B}w_0\mathbf{N} & \longrightarrow & \mathrm{GL}_2 \\ \downarrow & & \downarrow \\ \mathbf{B}\backslash\mathbf{B}w_0\mathbf{N} & \longrightarrow & \mathrm{FL} \end{array} \tag{13}$$

Let $\kappa \in X^*(\mathbf{T})$ be a character and $\mathscr{L}(\kappa)$ the associated GL_2-equivariant line bundle over FL (see Definition 2.1.1). Recall that, if κ is dominant, the global sections of $\mathscr{L}(\kappa)$ are isomorphic to V_κ.

DEFINITION 2.4.1. We define the $(\mathfrak{g}, \mathbf{B})$-representation $V(\kappa) := \Gamma(\mathbf{B}\backslash\mathbf{B}w_0\mathbf{N}, \mathscr{L}(\kappa))$, where the action of $(\mathfrak{g}, \mathbf{B})$ is induced by the right regular action on the big cell.

As a $\mathbf{B}$-module, $V(\kappa)$ is a twist of the algebra of regular functions of $\mathbf{N}$. Indeed, there is an isomorphism of affine schemes $\mathbf{B}\backslash\mathbf{B}w_0\mathbf{N} \cong \mathbf{N}$, and one has

$$V(\kappa) \cong \kappa \otimes V(1) \cong \kappa \otimes \mathscr{O}(\mathbf{N}), \tag{14}$$

where the action of $\mathbf{B} = \mathbf{T} \ltimes \mathbf{N}$ on $\mathscr{O}(\mathbf{N})$ is induced from the map $(n, b) \mapsto t_b^{-1} n t_b n_b$ for $(n, b) \in \mathbf{N} \times \mathbf{B}$ and $b = (t_b, n_b) \in \mathbf{T} \ltimes \mathbf{N}$.

Remark 2.4.2. The $(\mathfrak{g}, \mathbf{B})$-module $V(\kappa)$ is in fact the admissible dual of the Verma module of highest weight $-w_0(\kappa)$ (see § 3.10 of [AS08]).

Let κ be a dominant weight. Taking the global sections of $\mathscr{L}(\kappa)$ in the bottom arrow of diagram (13), one obtains a map

$$V_\kappa \to V(\kappa).$$

Writing $\kappa = (k_1, k_2) \in \mathbb{Z}^2$, and $\mathbb{G}_a \cong \mathbf{N}$ via $X \mapsto \left(\begin{smallmatrix} 1 & X \\ 0 & 1 \end{smallmatrix}\right)$, the map of V_κ in $V(\kappa)$ is identified in (14) with the inclusion $\mathbb{Q}_p[X]_{k_1-k_2} \subset \mathbb{Q}_p[X] \cong \mathscr{O}(\mathbf{N})$ of polynomials of degree at most $k_1 - k_2$. We have the following proposition.

Proposition 2.4.3. *Let $\alpha = (1, -1) \in \mathbb{Z}^2 \cong X^*(\mathbf{T})$ and let $\kappa = (k_1, k_2)$ be a dominant weight. There is a short exact sequence of $(\mathfrak{g}, \mathbf{B})$-representations*

$$\mathrm{BGG}(\kappa) : \left[0 \to V_\kappa \to V(\kappa) \to V(w_0(\kappa) - \alpha) \to 0\right]$$

called the dual BGG complex of weight κ. As such it is identified with the short exact sequence

$$0 \to \kappa \otimes \mathbb{Q}_p[X]_{\leq k_1-k_2} \to \kappa \otimes \mathbb{Q}_p[X] \xrightarrow{(\frac{d}{dX})^{k_1-k_2+1}} (w_0(\kappa) - \alpha) \otimes \mathbb{Q}_p[X] \to 0, \tag{15}$$

where $\mathbb{Q}_p[X] = \mathscr{O}(\mathbf{N})$.

Proof. We have a weight decomposition of $V(\kappa)$ with respect to $\mathbf{T}$,

$$V(\kappa) = \bigoplus_{n \geq 0} (\kappa - n\alpha)\mathbb{Q}_p,$$

where $(\kappa - n\alpha)\mathbb{Q}_p$ is identified with $\kappa \otimes \mathbb{Q}_p X^n$ under the isomorphism (14). As V_κ is the irreducible representation of highest weight κ, it has a weight decomposition $V_\kappa \cong \bigoplus_{0 \leq n \leq k_1-k_2} (\kappa - n\alpha)\mathbb{Q}_p$. This shows that $V_\kappa \subset V(\kappa)$ is identified with the inclusion $\kappa \otimes \mathbb{Q}_p[X]_{\leq k_1-k_2} \subset \kappa \otimes \mathbb{Q}_p[X]$. As $\kappa \otimes X^{k_1-k_2+1}$ has weight $(w_0(\kappa) - \alpha)$, the isomorphism of $\mathrm{BGG}(\kappa)$ with (15) as $\mathbf{B}$-representations is clear. □

The representation V_κ has a $\mathbf{B}$-filtration whose highest and lowest weight vectors are $\mathbb{Q}_p(\kappa) \to V_\kappa$ and $V_\kappa \to \mathbb{Q}_p(w_0(\kappa))$, respectively. Taking the associated GL_2-equivariant vector bundles over FL, one finds morphisms $\Psi^\vee_{-w_0(\kappa)} : \mathscr{L}(w_0(\kappa)) \to \mathscr{O}_{\mathrm{FL}} \otimes V_\kappa$ and $\Psi_\kappa : \mathscr{O}_{\mathscr{F}\!\ell} \otimes V_\kappa \to \mathscr{L}(\kappa)$. In Propositions 2.4.4 and 2.4.5 we interpolate these maps on neighbourhoods of $w \in W$.

Proposition 2.4.4. *Let $\epsilon \geq \delta \geq n$. Let (R, R^+) be a uniform Tate $\mathbb{Q}_p$-algebra and $\chi : T = \mathbf{T}(\mathbb{Z}_p) \to R^{\times,+}$ a δ-analytic character.*

(i) *There is a $\mathcal{B} \cap \mathcal{I}\mathrm{w}_n(\delta)$-equivariant map $\iota : R(\chi) \to A^\delta_\chi$ (the highest weight vector map). It induces a morphism of Iw_n-equivariant sheaves over $U_1(\epsilon)\,\mathrm{Iw}_n$,*

$$\Psi^{A,\vee}_{-w_0(\chi)} : \mathscr{L}(w_0(\chi)) \to A^\delta_\chi \widehat{\otimes} \mathscr{O}_{U_1(\epsilon)\,\mathrm{Iw}_n}.$$

Dually, we have equivariant maps $D^\delta_\chi \to R(-\chi)$ and $\Psi^D_{-w_0(\chi)} : D^\delta_\chi \widehat{\otimes} \mathscr{O}_{U_1(\epsilon)\,\mathrm{Iw}_n} \to \mathscr{L}(-w_0(\chi))$.

(ii) *There is a $\overline{\mathcal{B}} \cap \mathcal{I}\mathrm{w}_n(\delta)$-equivariant map $\mathrm{ev}_{w_0} : A^\delta_\chi \to R(\chi)$ (the lowest weight vector quotient). Moreover, it induces a morphism of Iw_n-equivariant sheaves over $U_{w_0}(\epsilon)\,\mathrm{Iw}_n$,*

$$\Psi^A_\chi : A^\delta_\chi \widehat{\otimes} \mathscr{O}_{U_{w_0}(\epsilon)\,\mathrm{Iw}_n} \to \mathscr{L}(\chi).$$

Dually, we have equivariant maps $R(-\chi) \to D^\delta_\chi$ and $\Psi^{D,\vee}_\chi : \mathscr{L}(-\chi) \to D^\delta_\chi \widehat{\otimes} \mathscr{O}_{U_{w_0}(\epsilon)\,\mathrm{Iw}_n}$.

Moreover, the morphisms of sheaves above are compatible with the U_p-correspondence (10).

Proof. It is enough to prove the statements for the principal series. Recall that $A^\delta_\chi = \mathscr{O}_{\widetilde{U}_{w_0}(\epsilon,\delta)}[-w_0(\chi)]$, where one takes isotypic components with respect to the left regular action of $T\mathcal{T}(\delta)$.

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(i) By Lemma 2.2.2(iii) we have an isomorphism of R-modules $A^\delta_\chi \cong \mathscr{O}(\overline{N\mathcal{N}}(\delta)w_0)$; this shows that the $N\mathcal{N}(\delta)$-invariants of A^δ_χ are isomorphic to the $\mathcal{B}\cap\mathcal{I}\mathrm{w}_n(\delta)$-module $R(\chi)$, and thus provides the first arrow. The map $\Psi^{A,\vee}_{-w_0(\chi)} : \mathscr{L}(w_0(\chi)) \to A^\delta_\chi \widehat{\otimes}\mathscr{O}_{U_1(\epsilon)\,\mathrm{Iw}_n}$ is constructed from the arrow $R(\chi) \to A^\delta_\chi$ via the presentation $U_{w_0}(\epsilon)\,\mathrm{Iw}_n = \mathcal{B}\cap\mathcal{I}\mathrm{w}_n(\delta)\backslash\mathcal{GL}_2(\epsilon,\delta)\,\mathrm{Iw}_n$. Indeed, one has that

$$\mathscr{L}(w_0(\chi)) = \mathcal{GL}_2(\epsilon,\delta)w_0\,\mathrm{Iw}_n \times^{\mathcal{B}} R(\chi) \text{ and } A^\delta_\chi\widehat{\otimes}\mathscr{O}_{U_1(\epsilon)\,\mathrm{Iw}_n} = \mathcal{GL}_2(\epsilon,\delta)w_0\,\mathrm{Iw}_n \times^{\mathcal{B}} A^\delta_\chi.$$

(ii) By Lemma 2.2.2 we have the presentation

$$\widetilde{U}_{w_0}(\epsilon,\delta)\,\mathrm{Iw}_n = N_n\mathcal{N}(\delta)\backslash(N_n\mathcal{N}(\delta)\times T\mathcal{T}(\delta)\times\overline{N\mathcal{N}}(\epsilon)w_0).$$

Then a straightforward computation shows that the evaluation map $\mathrm{ev}_{w_0} : A^\delta_\chi \to R(\chi)$ is $\overline{\mathcal{B}}\cap\mathcal{I}\mathrm{w}_n(\delta)$-equivariant. The arrow $\Psi^A_\chi : A^\delta_\chi\widehat{\otimes}\mathscr{O}_{U_{w_0}(\epsilon)\,\mathrm{Iw}_n} \to \mathscr{L}(\chi)$ is just the natural map induced by the global sections since $A^\delta_\chi = \Gamma(U_{w_0}(\epsilon)\,\mathrm{Iw}_n, \mathscr{L}(\chi))$. Notice that it factors through the co-invariants of A^δ_χ by the action of $\overline{N}_n\overline{\mathcal{N}}(\delta)$.

By definition of the U_p-correspondence, it is enough to show that the maps $\Psi^{A,\vee}_{-w_0(\chi)}$ and Ψ^A_χ are equivariant for the action of $\varpi = \mathrm{diag}(1,p)$; this is clear since the multiplication by ϖ on $\mathscr{L}(\chi)$ and A^δ_χ is induced by the right multiplication of ϖ on $\Lambda\backslash\widetilde{\mathscr{F}\ell}$ (see Remark 2.2.8 and Definition 2.2.12). □

Proposition 2.4.5. *Let $\kappa\in X^*(\mathbf{T})$ be a dominant character. There are commutative diagrams of Iw_n-equivariant sheaves*

$$\begin{array}{ccc} A^\delta_\kappa\widehat{\otimes}\mathscr{O}_{\mathscr{F}\ell} & \xrightarrow{\Psi^A_\kappa} & \mathscr{L}(\kappa) \\ \uparrow & \nearrow_{\Psi_\kappa} & \\ V_\kappa\otimes\mathscr{O}_{\mathscr{F}\ell} & & \end{array} \qquad \begin{array}{ccc} \mathscr{L}(w_0(\kappa)) & \xrightarrow{\Psi^{A,\vee}_{-w_0(\kappa)}} & A^\delta_\kappa\widehat{\otimes}\mathscr{O}_{\mathscr{F}\ell} \\ & \searrow^{\Psi^\vee_{-w_0(\kappa)}} & \uparrow \\ & & V_\kappa\otimes\mathscr{O}_{\mathscr{F}\ell} \end{array}$$

A dual statement holds for D^δ_κ. Moreover, these diagrams are compatible with the (normalized) U_p-correspondence (Definition 2.2.9).

Proof. The commutativity of the diagrams is obvious from the definition of the maps Ψ^A_κ and $\Psi^{A,\vee}_{-w_0(\kappa)}$ of Proposition 2.4.4, and the fact that $\Psi^\vee_{-w_0(\kappa)}$ and Ψ_κ are induced by the invariants and co-invariants for the action of $\mathbf{N}$ on V_κ, respectively. The compatibility with the (normalized) U_p-correspondence follows from the fact that the (normalized) action of ϖ on any of the sheaves involved is induced by the right multiplication on $\Lambda\backslash\widetilde{\mathscr{F}\ell}$ (see Remark 2.2.8 and Definition 2.2.12). □

Remark 2.4.6. In [AI22, §§ 4.7 and 4.8] the authors define some filtrations attached to the sheaves of modular symbols over the modular curves; it turns out that these can be constructed directly from the flag variety. Indeed, Andreatta and Iovita use the formalism of vector bundles with marked sections to define the filtrations (see [AI21, Corollary 2.6]), and the data of a vector bundle with marked section lives over affinoid neighbourhoods of $w\in W$. For example, let St be the standard representation of GL_2 and $\mathrm{St}^+\subset\mathrm{St}$ its natural lattice, and let us denote by e_1, e_2 the canonical basis of St^+. Over $U_1(\epsilon)$ the natural map $\mathscr{L}^+(0,1)\to\mathrm{St}^+\otimes\mathscr{O}_{U_1(\epsilon)}$ induces an isomorphism $\mathscr{L}^+(0,1)/p^\epsilon = e_1\otimes\mathscr{O}^+_{U_1(\epsilon)}/p^\epsilon$. Therefore, we have the data of a vector bundle with marked sections $(\mathrm{St}^+\otimes\mathscr{O}_{U_1(\epsilon)}, \mathscr{L}(0,1), e_1)$. The previous discussion shows in particular that the map of [AI22, Proposition 4.15 ii.] is Ψ^D_χ.

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Following the work of Pan [Pan22], there is a better way to study the filtrations above. Let $\mathfrak{gl}_2^0 = \mathfrak{gl}_2 \otimes \mathscr{O}_{\mathscr{F}\ell}$ be the constant Lie algebroid over $\mathscr{F}\ell$, and let $\mathfrak{n}^0 \subset \mathfrak{gl}_2^0$ be the GL_2-equivariant line bundle whose fibre at a point $x \in \mathscr{F}\ell$ is $\mathfrak{n}^0(x) = \mathrm{Lie}\, x\mathbf{N}x^{-1}$, that is, the Lie algebra of the unipotent group fixing x. Then $\mathfrak{n}^0$ is an ideal of $\mathfrak{gl}_2^0$, and the natural action of $\mathfrak{n}^0$ on $A_\chi^\delta \widehat{\otimes} \mathscr{O}_{\mathscr{F}\ell}$ induced by derivations of $\mathfrak{gl}_2^0$ defines the filtrations of Andreatta and Iovita.

3. Overconvergent sheaves over the modular curve

Let $\mathbb{A}_{\mathbb{Q}}^\infty$ and $\mathbb{A}_{\mathbb{Q}_p}^{\infty,p}$ be the rings of finite and finite prime-to-p adèles of $\mathbb{Q}$, respectively. From now on, we fix a neat compact open subgroup $K^p \subset \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty,p})$. Letting $n \geq 0$, we denote by $\Gamma(p^n)$, $\Gamma_1(p^n)$ and $\Gamma_0(p^n)$ the principal congruence subgroups

$$\Gamma(p^n) = \{g \in \mathrm{GL}_2(\mathbb{Z}_p) : g \equiv 1 \bmod p^n\},$$

$$\Gamma_1(p^n) = \left\{g \in \mathrm{GL}_2(\mathbb{Z}_p) : g \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod p^n\right\},$$

$$\Gamma_0(p^n) = \left\{g \in \mathrm{GL}_2(\mathbb{Z}_p) : g \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod p^n\right\}.$$

Let $K_p \subset \mathrm{GL}_2(\mathbb{Q}_p)$ be a compact open subgroup. We denote by $Y_{K_p}^{\mathrm{alg}}$ and $X_{K_p}^{\mathrm{alg}}$ the affine and compact modular curves of level K^pK_p over $\mathrm{Spec}\, \mathbb{Q}_p$ (cf. [DR73]). We let $Y^{\mathrm{alg}}(p^n)$, $Y_1^{\mathrm{alg}}(p^n)$ and $Y_0^{\mathrm{alg}}(p^n)$ be the modular curves of level $K^p\Gamma(p^n)$, $K^p\Gamma_1(p^n)$ and $K^p\Gamma_0(p^n)$, respectively (and similarly for the compact modular curves). We let Y_{K_p} and X_{K_p} denote their p-adic analytification to adic spaces over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ (see [Hub96]). We endow X_{K_p} with the log structure defined by the cusp divisor $D = X_{K_p} \backslash Y_{K_p}$.

Let $E^{\mathrm{alg}}/Y_{K_p}^{\mathrm{alg}}$ be the universal elliptic curve and $E^{\mathrm{alg,sm}}/X_{K_p}^{\mathrm{alg}}$ its extension to a semi-abelian scheme. Let $e : X_{K_p}^{\mathrm{alg}} \to E^{\mathrm{alg,sm}}$ be the unit section and $\omega_E = e^*\Omega^1_{E^{\mathrm{alg,sm}}/X^{\mathrm{alg}}}$ the modular sheaf. Given an integer $k \in \mathbb{Z}$, we denote by $\omega_E^k = \omega_E^{\otimes k}$ the modular sheaf of weight k. We define the modular torsor to be the $\mathcal{T}$-torsor over X_{K_p} defined by

$$\mathcal{T}_{\mathrm{mod}} = \underline{\mathrm{Isom}}(\mathscr{O}_X, \omega_E) \times \underline{\mathrm{Isom}}(\mathscr{O}_X, \omega_E^{-1}). \tag{16}$$

Let $E[p^n]/Y_{K_p}$ be the étale local system of p^n-torsion points of the universal elliptic curve. By [DLLZ23b, Theorem 4.6.1], the sheaf $E[p^n]$ has a natural extension to a Kummer-étale local system over X_{K_p}, which by an abuse of notation we also write as $E[p^n]$. We let $T_pE = \varprojlim_n E[p^n]$ be the Tate module of E, seen as a pro-Kummer-étale local system over X_{K_p}. Given a dominant weight $\kappa \in X^*(\mathbf{T})^+$, we let $V_{\kappa,\text{ét}}$ denote the pro-Kummer-étale local system over X_{K_p} attached to the K_p-representation V_κ. We let $R\Gamma_{\text{proét}}(Y_{\mathbb{C}_p}, V_{\kappa,\text{ét}})$ and $R\Gamma_{\text{proét},c}(Y_{\mathbb{C}_p}, V_{\kappa,\text{ét}})$ respectively be the étale cohomology and the étale cohomology with compact support of $V_{\kappa,\text{ét}}$.

3.1 The Hodge–Tate period map

Let $\mathbb{Q}_p^{\mathrm{cyc}}$ be the p-adic completion of the p-adic cyclotomic field $\mathbb{Q}_p(\mu_{p^\infty})$. Scholze proved in [Sch15] that the inverse limit $X(p^\infty) = \varprojlim_n X(p^n)$ has a natural interpretation as a perfectoid space. Furthermore, he constructed a Hodge–Tate period map $\pi_{\mathrm{HT}} : X(p^\infty) \to \mathbb{P}^1_{\mathbb{Q}_p}$ parametrizing the Hodge–Tate filtration of elliptic curves at geometric points; let us briefly recall how it is constructed. The Hodge–Tate exact sequence is the following short exact sequence of pro-Kummer-étale sheaves over X_{K_p}:

$$0 \to \omega_E^{-1} \otimes_{\mathscr{O}} \widehat{\mathscr{O}}(1) \to T_pE \otimes_{\mathbb{Z}_p} \widehat{\mathscr{O}} \to \omega_E \otimes_{\mathscr{O}} \widehat{\mathscr{O}} \to 0. \tag{17}$$

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On the other hand, the perfectoid space $X(p^\infty)$ parametrizes trivializations of T_pE. If $\psi : \mathbb{Z}_p^2 \cong T_pE$ denotes the universal trivialization of the Tate module, the pullback of (17) by ψ gives rise a line subbundle of $\mathscr{O}_{X(p^\infty)}^{\oplus 2}$ which defines the map $\pi_{\mathrm{HT}} : X(p^\infty) \to \mathscr{F}\!\ell = \mathbb{P}^1_{\mathbb{Q}_p}$. Let us summarize the previous discussion and some properties of π_{HT} in the following theorem.

Theorem 3.1.1 [Sch15, Theorem III.3.18]. *There exists a perfectoid space $X(p^\infty)$ over $\mathbb{Q}_p^{\mathrm{cyc}}$ satisfying the tilde limit property [SW13, Definition 2.4.1]*

$$X(p^\infty) \sim \varprojlim_n X(p^n).$$

Moreover, let $[x : y] \in \mathscr{F}\!\ell = \mathbb{P}^1_{\mathbb{Q}_p}$ denote the projective coordinates of the projective space. There is a $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant Hodge–Tate period map

$$\pi_{\mathrm{HT}} : X(p^\infty) \to \mathscr{F}\!\ell$$

such that for any open rational subset U of $U_1 = \{[x : y]||x/y| \leq 1\}$ or $U_\infty = \{[x : y]||y/x| \leq 1\}$ of $\mathscr{F}\!\ell$, the inverse image $\pi_{\mathrm{HT}}^{-1}(U) \subset X(p^\infty)$ is an affinoid perfectoid subspace, and there exist $n \gg 0$ and an open affinoid $V_n \subset X(p^n)$ whose inverse image to $X(p^\infty)$ is equal to $\pi_{\mathrm{HT}}^{-1}(U)$.

Another feature of the Hodge–Tate period map is that it encodes the modular sheaves in terms of GL_2-equivariant line bundles over $\mathscr{F}\!\ell$. More precisely, letting St be the standard representation, we have an exact sequence of $\mathbf{B}$-modules

$$0 \to \mathbb{Q}_p(1,0) \to \mathrm{St} \to \mathbb{Q}_p(0,1) \to 0.$$

Taking the associated GL_2-equivariant vector bundles over $\mathscr{F}\!\ell$, we have a short exact sequence

$$0 \to \mathscr{L}(0,1) \to \mathrm{St} \otimes \mathscr{O}_{\mathscr{F}\!\ell} \to \mathscr{L}(1,0) \to 0. \tag{18}$$

Now the map π_{HT} induces a $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant morphism of ringed sites

$$\pi_{\mathrm{HT}} : (X(p^\infty)_{\mathrm{prok\acute{e}t}}, \widehat{\mathscr{O}}_X) \to (\mathscr{F}\!\ell, \mathscr{O}_{\mathscr{F}\!\ell}).$$

In particular, we can take pullbacks of GL_2-equivariant $\mathscr{O}_{\mathscr{F}\!\ell}$-vector bundles over $\mathscr{F}\!\ell$ to $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant $\widehat{\mathscr{O}}_X$-vector bundles over $X(p^\infty)_{\mathrm{prok\acute{e}t}}$.

Convention 3.1.2. Given a K_p-equivariant sheaf $\mathscr{F}$ over $\mathscr{F}\!\ell$, we will identify $\pi_{\mathrm{HT}}^*(\mathscr{F})$ with the pro-Kummer-étale sheaf over X_{K_p} defined by descent from the K_p-equivariant sheaf over $X(p^\infty)_{\mathrm{prok\acute{e}t}}$.

Remark 3.1.3. Essentially by definition, the pullback of (18) by π_{HT} is the Hodge–Tate exact sequence (17). Our convention differs from that of [AI22, § 4.3], where the pullback of the standard representation by π_{HT} is identified with the dual of T_pE.

Let $\kappa \in X^*(\mathbf{T})^+$ be a dominant weight and V_κ the irreducible representation of GL_2 of highest weight κ. Treating V_κ as a constant $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant sheaf over $\mathscr{F}\!\ell$, one has that $\pi_{\mathrm{HT}}^*(V_\kappa) = V_{\kappa,\mathrm{\acute{e}t}}$ as pro-Kummer-étale sheaves. This implies that the pullback by π_{HT} of the GL_2-equivariant vector bundle $V_\kappa \otimes \mathscr{O}_{\mathscr{F}\!\ell}$ is equal to $V_{\kappa,\mathrm{\acute{e}t}} \otimes \widehat{\mathscr{O}}_X$. Concerning the GL_2-equivariant line bundles over $\mathscr{F}\!\ell$, one has the following result (cf. [CS17, Proposition 2.3.9]).

Proposition 3.1.4. *Let $\kappa = (k_1, k_2) \in X^*(\mathbf{T})$ be an algebraic weight and $\mathscr{L}(\kappa)$ the GL_2-equivariant line bundle over $\mathscr{F}\!\ell$ of weight κ. There is a natural isomorphism of pro-Kummer-étale*

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sheaves over X_{K_p},

$$\pi_{\mathrm{HT}}^*(\mathscr{L}(\kappa)) = \omega_E^{k_1-k_2} \otimes_{\mathscr{O}} \widehat{\mathscr{O}}(k_2).$$

Equivalently, let $\widetilde{\mathscr{F}\ell} = \mathcal{N}\backslash\mathcal{GL}_2$ and $\pi_{K_p} : X(p^\infty) \to X_{K_p}$. We have an isomorphism of $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant $\mathcal{T}$-torsors over $X(p^\infty)$,

$$\pi_{\mathrm{HT}}^*(\widetilde{\mathscr{F}\ell}) = \pi_{K_p}^*(\mathcal{T}_{\mathrm{mod}})(-1,0),$$

where $\mathcal{T}_{\mathrm{mod}}(-1,0)$ is a Tate twist in the first component of the modular torsor (16).

Proof. We only need to show the isomorphism of torsors; this follows from Remark 2.1.2(i) and the definition of the modular torsor. In fact, since the pullback of (18) is the Hodge–Tate exact sequence, one has that $\pi_{\mathrm{HT}}^*(\mathscr{L}(1,0)) = \omega_E^1 \otimes \widehat{\mathscr{O}}_X$ and $\pi_{\mathrm{HT}}^*(\mathscr{L}(0,1)) = \omega_E^{-1} \otimes \widehat{\mathscr{O}}(1)$, and the proposition follows since $\mathscr{L}(\kappa) = \mathscr{L}(1,0)^{\otimes k_1} \otimes \mathscr{L}(0,1)^{\otimes k_2}$. □

3.2 Overconvergent modular forms

Throughout the rest of this section we fix $n \geq 1$ and write $X_\infty = X(p^\infty)$ and $X = X_0(p^n)$. Let $\pi_{\mathrm{Iw}_n} : X_\infty \to X$ be the natural projection and $\pi_{\mathrm{HT}} : X_\infty \to \mathscr{F}\ell \cong \mathbb{P}^1_{\mathbb{Q}_p}$ the Hodge–Tate period map. Let $\overline{X}_\infty^{\mathrm{ord}} = \pi_{\mathrm{HT}}^{-1}(\mathscr{F}\ell(\mathbb{Q}_p))$ be the closure of the ordinary locus at infinite level [Sch15, Lemma III.3.6.], and $\overline{X}^{\mathrm{ord}} = \pi_{\mathrm{Iw}_n}(\overline{X}_\infty^{\mathrm{ord}})$ the closure of the ordinary locus of X.

Let $C_n^{\mathrm{can}} \subset E[p^n]$ be the canonical subgroup over $\overline{X}^{\mathrm{ord}}$ and $w \in W = \{1, w_0\}$. We let $\overline{X}_w^{\mathrm{ord}} \subset \overline{X}^{\mathrm{ord}}$ denote the w-ordinary locus, that is, the ordinary locus where C_n^{can} has relative position w with respect to the universal subgroup $H_n \subset E[p^n]$. In other words, $\overline{X}_1^{\mathrm{ord}}$ is the ordinary locus where $C_n^{\mathrm{can}} = H_n$ and $\overline{X}_{w_0}^{\mathrm{ord}}$ the locus where $C_n^{\mathrm{can}} \cap H_n = 0$. We can also write $\overline{X}_w^{\mathrm{ord}} = \pi_{\mathrm{Iw}_n}(\pi_{\mathrm{HT}}^{-1}(w\,\mathrm{Iw}_n))$.

Let $\epsilon \geq n \geq 1$ and $w \in W = \{1, w_0\}$. In § 2.2 we defined affinoid neighbourhoods $\{U_w(\epsilon)\,\mathrm{Iw}_n\}_{\epsilon \geq n}$ of $w \in \mathscr{F}\ell$. By Theorem 3.1.1 their pullback to X_∞ is an affinoid perfectoid arising from some finite level modular curve X_{K_p}. Furthermore, as $\pi_{\mathrm{HT}}^{-1}(U_w(\epsilon)\,\mathrm{Iw}_n)$ is Iw_n-stable, we can take $K_p = \mathrm{Iw}_n$. The previous discussion leads to the following definition.

Definition 3.2.1. There exists a unique open affinoid subspace $X_w(\epsilon) \subset X$ such that $\pi_{\mathrm{Iw}_n}^{-1}(X_w(\epsilon)) = \pi_{\mathrm{HT}}^{-1}(U_w(\epsilon)\,\mathrm{Iw}_n)$.

Remark 3.2.2. The following properties are deduced from Lemma 2.2.2.

(i) $X_w(\epsilon') \subset X_w(\epsilon)$ is a strict immersion for $\epsilon' > \epsilon$.

(ii) $\{X_w(\epsilon)\}_{\epsilon \geq n}$ is a basis of strict neighbourhoods of $\overline{X}_w^{\mathrm{ord}}$, namely, $\bigcap_{\epsilon \geq n} X_w(\epsilon) = \overline{X}_w^{\mathrm{ord}}$.

The affine modular curve $Y \subset X$ parametrizes triples (E, H_n, ψ_N) where E is an elliptic curve E, ψ_N is some prime-to-p level structure, and $H_n \subset E[p^n]$ is a cyclic subgroup of order p^n. Let $\varpi = \mathrm{diag}(1,p)$. In the following we study the dynamics of the U_p-correspondence (cf. [BP22, § 5.3]).

Definition 3.2.3. The U_p-correspondence of X is the finite flat correspondence C fitting into the diagram

$$\begin{array}{ccc} & C & \\ {\scriptstyle p_1}\swarrow & & \searrow{\scriptstyle p_2} \\ X & & X \end{array} \tag{19}$$

and parametrizing tuples (E, H_n, ψ_N, H') where (E, H_n, ψ_N) defines a point in X, and $H' \subset E[p]$ is a cyclic subgroup of order p such that $H_n \cap H' = 0$. We define $p_1(E, H_n, \psi_N, H') = (E, H_n, \psi_N)$

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and $p_2(E, H_n, \psi_N, H') = (E/H', \overline{H}_n, \overline{\psi}_N)$, where $\overline{\psi}_N$ and $\overline{H}_n$ are the images of ψ_N and H_n in the quotient E/H'. Let $\pi : p_1^*E \to p_2^*E$ be the universal isogeny over C and $\pi^\vee : p_2^*E \to p_1^*E$ its dual. For a subspace $Z \subset X$ let us denote $U_p(Z) = p_1(p_2^{-1}(Z))$ and $U_p^t(Z) = p_2(p_1^{-1}(Z))$.

The following lemma uses the same strategy as [AI22, § 4.5]. Notice, however, that the convention on π_{HT} differs (see Remark 3.1.3).

LEMMA 3.2.4. *Let $\epsilon \geq n$. The following assertions hold.*

(i) $U_p^t(X_1(\epsilon)) \subset X_1(\epsilon + 1)$ *and* $U_p(X_1(\epsilon)) \supset X_1(\epsilon - 1)$ *if* $\epsilon \geq n + 1$.
(ii) $U_p(X_{w_0}(\epsilon)) \subset X_{w_0}(\epsilon + 1)$ *and* $U_p^t(X_{w_0}(\epsilon)) \supset X_{w_0}(\epsilon - 1)$ *if* $\epsilon \geq n + 1$.

Proof. The perfectoid modular curve X_∞ parametrizes triples $(E, \psi_N, (e_1, e_2))$ where E is an elliptic curve, ψ_N a prime-to-p level structure, and (e_1, e_2) a basis of T_pE. Let $C_\infty = X_\infty \times_{X,p_1} C$. The perfectoid curve C_∞ parametrizes $(E, \psi_N, (e_1, e_2), H')$, where $(E, \psi_N, (e_1, e_2)) \in X_\infty$ and $H' \subset E[p^n]$ is a cyclic subgroup of order p such that $\langle \overline{e}_1 \rangle \cap H' = 0 \mod p$. Write $C_\infty = \bigsqcup_{a \in \mathbb{F}_p} C_{\infty,a}$, with $C_{\infty,a}$ the locus where $H' = \langle \overline{e}_2 + a\overline{e}_1 \rangle$. Note that the map $p_1 : C_{\infty,a} \to X_\infty$ is an isomorphism for all a. We have a diagram

$$\begin{array}{ccc} & \bigsqcup_{a=0}^{p-1} C_{\infty,a} & \\ \swarrow^{p_1} & & \searrow^{p_2} \\ X_\infty & & X_\infty \end{array}$$

with $p_1(E, \psi_N, (e_1, e_2), H') = (E, \psi_N, (e_1, e_2))$ and $p_2(E, \psi_N, (e_1, e_2), H') = (E/H', \overline{\psi}_N, (\pi(e_1), \tilde{e}_2))$, such that the restriction of p_2 to $C_{\infty,a}$ is given by $\tilde{e}_2 = (1/p)(\pi(e_2) + a\pi(e_1))$ for $0 \leq a < p$ lifting a. Let $U_{p,a} := \left(\begin{smallmatrix} 1 & -a \\ 0 & p \end{smallmatrix}\right)$. Composing with the Hodge–Tate period map $\pi_{\mathrm{HT}} : X_\infty \to \mathscr{F}\ell$, we have a commutative diagram of correspondences (see diagram (10))

$$\begin{array}{ccccc} & & \bigsqcup_a C_{\infty,a} & & \\ & \swarrow^{p_1} & \downarrow & \searrow^{p_2} & \\ X_\infty & & & & X_\infty \\ \downarrow{\scriptstyle \pi_{\mathrm{HT}}} & & \bigsqcup_{a=0}^{p-1} \mathscr{F}\ell_a & & \downarrow{\scriptstyle \pi_{\mathrm{HT}}} \\ & \swarrow^{p_1} & & \searrow^{p_2} & \\ \mathscr{F}\ell & & & & \mathscr{F}\ell \end{array} \tag{20}$$

where the map $p_{2,a} : \mathscr{F}\ell_a \to \mathscr{F}\ell$ is the right multiplication by $U_{p,a}^{-1}$. Indeed, if e_1, e_2 is the canonical basis of the standard representation of GL_2 we have that $p_{2,a}^*(e_1) = e_1$ and $p_{2,a}^*(e_2) = (1/p)(e_2 + ae_1)$. The lemma follows by Lemma 2.2.11 and the definition of the affinoids $X_w(\epsilon)$. □

Remark 3.2.5. Taking Iw_n-quotients in (20), one obtains the morphism of correspondences

$$\begin{array}{ccccc} & & C & & \\ & \swarrow^{p_1} & \downarrow & \searrow^{p_2} & \\ X & & & & X \\ \downarrow{\scriptstyle \pi_{\mathrm{HT}}} & & (\bigsqcup_{a=0}^{p-1} \mathscr{F}\ell_a)/\mathrm{Iw}_n & & \downarrow{\scriptstyle \pi_{\mathrm{HT}}} \\ & \swarrow^{p_1} & & \searrow^{p_2} & \\ \mathscr{F}\ell/\mathrm{Iw}_n & & & & \mathscr{F}\ell/\mathrm{Iw}_n \end{array}$$

where the bottom correspondence is that of (11). Therefore, the U_p-correspondence of X is simply the pullback of the U_p-correspondence of the stack $\mathscr{F}\ell/\mathrm{Iw}_n$.

Let $\mathcal{T}^0 \subset \mathcal{T}$ be the affinoid bounded torus given by the generic fibre of the p-adic completion of $\mathbf{T}$. Let $\mathcal{T}_{\mathrm{mod,ét}}$ be the base change of $\mathcal{T}_{\mathrm{mod}}$ to a $\mathcal{T}$-torsor over the étale site of X. In order to construct overconvergent modular sheaves we have to find refinements of the torsor $\mathcal{T}_{\mathrm{mod}}$. It turns out that the torsor $\mathcal{T}_{\mathrm{mod}}$ admits an integral reduction to an étale torsor.

Theorem 3.2.6 [BP22, § 4.6]. *Let $\widetilde{\mathscr{F}\ell}^0 = \mathcal{N}^0\backslash\mathcal{GL}_2^0$ be the natural $\mathcal{GL}_2^0$-equivariant $\mathcal{T}^0$-torsor over $\mathscr{F}\ell$. There exists an étale $\mathcal{T}^0$-torsor $\mathcal{T}^0_{\mathrm{mod,ét}}$ over X such that*

$$\mathcal{T}_{\mathrm{mod,ét}} = \mathcal{T}\times^{\mathcal{T}^0}\mathcal{T}^0_{\mathrm{mod,ét}} \quad \textit{and} \quad \pi_{\mathrm{HT}}^*(\widetilde{\mathscr{F}\ell}^0) = \pi^*_{\mathrm{Iw}_n}(\mathcal{T}^0_{\mathrm{mod,ét}})(-1,0).$$

Remark 3.2.7. The existence of the integral torsor holds in greater generality for any Shimura variety. The theorem follows from the fact that $\pi_{\mathrm{HT}}^*(\widetilde{\mathscr{F}\ell}^0)$ is a $\mathrm{GL}(\mathbb{Z}_p)$-equivariant open subspace of the twisted torsor $\pi_{\mathrm{Iw}_n}(\mathcal{T}_{\mathrm{mod}})(-1,0)$ over X_∞, and it follows that this open subspace descends to some finite level providing, locally étale on X, an integral trivialization of $\mathcal{T}_{\mathrm{mod}}$.

The following definition is justified by the proof of Theorem 3.2.6 (see [BP21, Proposition 4.6.12] or [BP22, Proposition 5.15]).

Definition 3.2.8. Let $\epsilon\geq\delta\geq n\geq 1$ and $w\in W=\{1,w_0\}$. Consider the Iw_n-equivariant $T\mathcal{T}(\delta)$-torsor of Definition 2.2.7,

$$\widetilde{U}_w(\epsilon,\delta)\,\mathrm{Iw}_n \to U_w(\epsilon)\,\mathrm{Iw}_n\,.$$

The restriction of $\mathcal{T}^0_{\mathrm{mod,ét}}$ to $X_w(\epsilon)$ admits a reduction to an étale $T\mathcal{T}(\delta)$-torsor $\mathcal{T}_{\mathrm{mod}}(\delta)$ determined by the equality

$$\pi^*_{\mathrm{HT}}(\widetilde{U}_w(\epsilon,\delta)\,\mathrm{Iw}_n) = \pi^*_{\mathrm{Iw}_n}(\mathcal{T}_{\mathrm{mod}}(\delta))(-1,0) \tag{21}$$

as open subspaces of $\pi^*_{\mathrm{HT}}(\widetilde{\mathscr{F}\ell}) = \pi^*_{\mathrm{Iw}_n}(\mathcal{T})(-1,0)$.

Definition 3.2.9. Let (R,R^+) be a uniform affinoid Tate $\mathbb{Q}_p$-algebra, and $\chi: T=\mathbf{T}(\mathbb{Z}_p)\to R^{+,\times}$ a δ-analytic character. Let $\mathscr{O}_{\mathcal{T}_{\mathrm{mod}}(\delta)}$ be the algebra of regular functions of $\mathcal{T}_{\mathrm{mod}}(\delta)$, seen as an étale Banach $\mathscr{O}_X$-algebra over $X_w(\epsilon)$. The sheaf of overconvergent modular forms of weight χ is given by

$$\begin{aligned}\omega_E^\chi &= \mathscr{O}_{\mathcal{T}_{\mathrm{mod}}(\delta)}\widehat{\otimes}R[-w_0(\chi)]\\ &= \{f\in\mathscr{O}_{\mathcal{T}_{\mathrm{mod}}(\delta)}\widehat{\otimes}R : f(tx)=w_0(\chi)(t)f(x) \text{ for } t\in T\mathcal{T}(\delta)\}.\end{aligned}$$

Remark 3.2.10. In [BP21, Proposition 4.6.15] it is shown that the torsor $\mathcal{T}_{\mathrm{mod}}(\delta)$ is trivial in a finite étale covering of $X_w(\epsilon)$. This implies that the étale sheaf ω_E^χ is locally in the étale topology an orthonormalizable $\mathscr{O}_X$-sheaf, and equal to the pullback to the étale site of an $\mathscr{O}_{X_w(\epsilon)}\widehat{\otimes}R$-line bundle over the analytic site of $X_w(\epsilon)$.

From Definition 3.2.8 we deduce the following overconvergent analogue of Proposition 3.1.4.

Corollary 3.2.11. *Let (R,R^+) and δ be as in Definition 3.2.9, and write $\chi=(\chi_1,\chi_2)$. Let $\chi_{\mathrm{cyc}}: G_{\mathbb{Q}_p}\to\mathbb{Z}_p^\times$ be the cyclotomic character and $\chi_2\circ\chi_{\mathrm{cyc}}: G_{\mathbb{Q}_p}\to R^{+,\times}$ its composition with χ_2. We set $\widehat{\mathscr{O}}_X(\chi_2) := R(\chi_2\circ\chi_{\mathrm{cyc}})\widehat{\otimes}\widehat{\mathscr{O}}_X$. There is a Galois equivariant isomorphism of pro-Kummer-étale sheaves over $X_w(\epsilon)$,*

$$\pi^*_{\mathrm{HT}}(\mathscr{L}(\chi)) = \omega_E^\chi\widehat{\otimes}_{R\widehat{\otimes}\mathscr{O}_X}\widehat{\mathscr{O}}_X(\chi_2).$$

We can finally define the overconvergent modular forms and the overconvergent cohomology classes appearing in higher Coleman theory. We refer to [Urb11] for the notion of perfect Banach complexes and compact operators of perfect Banach complexes. See [Sta20, Tag 0A39] for the definition of cohomology with supports in a closed subspace.

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Definition 3.2.12. Let $w \in W = \{1, w_0\}$ and let $\mathscr{F}$ be a sheaf over $X_w(\epsilon)_{\mathrm{an}}$. Denote $X_w(>\epsilon) := \bigcup_{\epsilon' > \epsilon} X_w(\epsilon)$. We define the cohomology complexes

$$R\Gamma_w(X, \mathscr{F})_\epsilon := R\Gamma_{\mathrm{an}}(X_w(\epsilon), \mathscr{F}) \quad \text{and} \quad R\Gamma_{w,c}(X, \mathscr{F})_\epsilon := R\Gamma_{\mathrm{an}, \overline{X_w(>\epsilon+1)}}(X_w(\epsilon), \mathscr{F}).$$

Set $H^0_w(X, \mathscr{F}) := H^0(R\Gamma_w(X, \mathscr{F})_\epsilon)$ and $H^1_{w,c}(X, \mathscr{F}) = H^1(R\Gamma_{w,c}(X, \mathscr{F})_\epsilon)$. When $\mathscr{F} = \omega_E^\chi$, we call $H^0_w(X, \omega_E^\chi)_\epsilon$ and $H^1_{w,c}(X, \omega_W^\chi)_\epsilon$ the space of overconvergent modular forms and the overconvergent cohomology with compact support of weight χ, respectively.

3.2.1 *Hecke operators.* We end this subsection with the definition of the U_p-operators for the overconvergent modular forms. First, let us recall the definition for the classical modular sheaves. Let $X \xleftarrow{p_1} C \xrightarrow{p_2} X$ be the U_p-correspondence. We let $\pi : p_1^* E \to p_2^* E$ be the universal isogeny over C and $\pi^\vee : p_2^* E \to p_1^* E$ its dual. We denote by $\pi^* : p_2^* \omega_E \to p_1^* \omega_E$ and $\pi_* : p_1^* \omega_E^{-1} \to p_2^* \omega_E^{-1}$ the pullback and pushforward maps of π (respectively, for $\pi^\vee$). For a quasi-coherent sheaf $\mathscr{F}$ over X we let $\mathrm{Tr}_{p_i} : p_{i,*} p_i^* \mathscr{F} \to \mathscr{F}$ be the trace map of p_i. Let $\kappa = (k_1, k_2) \in X^*(\mathbf{T})$ be a character of $\mathbf{T}$, and recall that we have made the convention $\omega_E^\kappa = \omega_E^{k_1} \otimes \omega_E^{-k_2} = \omega_E^{k_1 - k_2}$.

Definition 3.2.13. The Hecke operator $U_{p,\kappa}$ acting over $R\Gamma_{\mathrm{an}}(X, \omega_E^\kappa)$ is the composition

$$R\Gamma_{\mathrm{an}}(X, \omega_E^\kappa) \xrightarrow{p_2^*} R\Gamma_{\mathrm{an}}(C, p_2^* \omega_E^\kappa) \xrightarrow{(\pi^{\vee,*,-1})^{\otimes k_1} \otimes (\pi_*^{-1})^{\otimes k_2}} R\Gamma_{\mathrm{an}}(C, p_1^* \omega_E^\kappa) \xrightarrow{\mathrm{Tr}_{p_1}} R\Gamma_{\mathrm{an}}(X, \omega_E^\kappa).$$

We define the $U^t_{p,\kappa}$ operator by shifting the roles of p_1 and p_2, and by composing with the map $(\pi^{\vee,*})^{\otimes k_1} \otimes (\pi_*)^{\otimes k_2}$.

Remark 3.2.14. The $U_{p,\kappa}$ above is equal to the operator $p^{-k_1} U^{\mathrm{naive}}_{p, k_1 - k_2}$ of [BP22]. Indeed, $(\pi^{\vee,*,-1})^{\otimes k_1} = p^{-k_1} (\pi^*)^{\otimes k_1}$ and $(\pi_*^{-1})^{\otimes k_2} = (\pi^*)^{\otimes -k_2}$. In other words, $U^{\mathrm{naive}}_{p,k} = U_{p,(0,-k)}$.

Let us explain the normalization of the Hecke operator $U_{p,\kappa}$ of the previous definition. It turns out that it is induced from the $U_{p,\kappa}$-correspondence of the sheaf $\mathscr{L}(\kappa)$ over $\mathscr{F}\!\ell$ (see Definition 2.2.12(i)). Recall that the Hodge–Tate exact sequence

$$0 \to \omega_E^{-1} \otimes \widehat{\mathscr{O}}_X(1) \to T_p E \otimes \widehat{\mathscr{O}} \to \omega_E \otimes \widehat{\mathscr{O}} \to 0$$

is the pullback by π_{HT} of the exact sequence of GL_2-equivariant vector bundles

$$0 \to \mathscr{L}(0,1) \to \mathrm{St} \otimes \mathscr{O}_{\mathscr{F}\!\ell} \to \mathscr{L}(1,0) \to 0.$$

By the proof of Lemma 3.2.4, the U_p-correspondence of X at level X_∞ commutes with the U_p-correspondence of $\mathscr{F}\!\ell$ defined in (10). Then the natural U_p-correspondence of $\pi_{\mathrm{HT}}^* \mathrm{St} \otimes \mathscr{O}_{\mathscr{F}\!\ell} = T_p E \otimes \mathscr{O}_{X_\infty}$ induced by the $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant structure of St is compatible with the natural U_p-correspondence of $\pi_{\mathrm{HT}}^*(\mathcal{L}(0,1)) = \omega_E^{-1} \otimes \mathscr{O}_{X_\infty}(1)$ and $\pi_{\mathrm{HT}}^*(\mathcal{L}(1,0)) = \omega_E \otimes \mathscr{O}_{X_\infty}$. But the correspondence of $T_p E$ is just the natural isogeny $\pi : p_1^*(T_p E) \to p_2^*(T_p E)$ and we have the following commutative diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & p_1^*(\omega_E^{-1}) \otimes \mathscr{O}_{X_\infty}(1) & \longrightarrow & p_1^*(T_p E) \otimes \mathscr{O}_{X_\infty} & \longrightarrow & p_1^*(\omega_E) \otimes \mathscr{O}_{X_\infty} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi^{\vee,*}} & & \\ 0 & \longrightarrow & p_2^*(\omega_E^{-1}) \otimes \mathscr{O}_{X_\infty}(1) & \longrightarrow & p_2^*(T_p E) \otimes \mathscr{O}_{X_\infty} & \longrightarrow & p_2^*(\omega_E) \otimes \mathscr{O}_{X_\infty} & \longrightarrow & 0 \end{array}$$

Then the natural U_p correspondences of $\omega_E \otimes \mathscr{O}_{X_\infty}$ and $\omega_E^{-1} \otimes \mathscr{O}_{X_\infty}(1)$, defined by $\mathscr{L}(1,0)$ and $\mathscr{L}(0,1)$ respectively, are given by the maps

$$\pi^{\vee,*} : p_1^* \omega_E \to p_2^* \omega_E \quad \text{and} \quad \pi_* : p_1^* \omega_E^{-1} \otimes \mathscr{O}_{X_\infty}(1) \to p_2^* \omega_E^{-1} \otimes \mathscr{O}_{X_\infty}(1).$$

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With the previous explanation in mind, we can now define the U_p-correspondence for the overconvergent modular forms. We refer to [BP22, Proposition 5.8] for the details.

Definition 3.2.15. Consider the U_p-correspondence (19) and its restriction to the overconvergent neighbourhoods of $\overline{X}_w^{\mathrm{ord}}$ (see Lemma 3.2.4):

$$\begin{array}{ccccccc} & p_2^{-1}(X_{w_0}(\epsilon+1)) & & & & p_1^{-1}(X_1(\epsilon-1)) & \\ {}^{p_1}\swarrow & & \searrow^{p_2} & & {}^{p_1}\swarrow & & \searrow^{p_2} \\ X_{w_0}(\epsilon+1) & & X_{w_0}(\epsilon) & & X_1(\epsilon-1) & & X_1(\epsilon) \end{array}$$

(i) We define the maps

$$U_p : p_2^*\omega_E^\chi \to p_1^*\omega_E^\chi \quad \text{and} \quad U_p^t : p_1^*\omega_E^\chi \to p_2^*\omega_E^\chi$$

to be the unique maps whose base changes to $\widehat{\mathscr{O}}_X$-modules coincide with the pullback by π_{HT} of the U_p-correspondence of Definition 2.2.12(ii).

(ii) The U_p operator on $R\Gamma_1(X,\omega_E^\chi)_\epsilon$ and $R\Gamma_{w_0,c}(X,\omega_E^\chi)_\epsilon$ is the one induced by the map U_p above.

(iii) The U_p^t operator on $R\Gamma_{w_0}(X,\omega_E^\chi)_\epsilon$ and $R\Gamma_{1,c}(X,\omega_E^\chi)_\epsilon$ is the one induced by the map U_p^t above.

In order to state the classicality result we need to normalize the U_p-operators.

Definition 3.2.16. Let (R,R^+) be a uniform Tate $\mathbb{Q}_p$-algebra and $\chi : T \to R^{+,\times}$ be a δ-analytic character. Let $\kappa \in X^*(\mathbf{T})$. We define normalizations of U_p and U_p^t:

$$U_p^{\mathrm{good}} = \begin{cases} \frac{1}{p}U_p & \text{over } R\Gamma_1(X,\omega_E^\chi)_\epsilon, \\ U_p & \text{over } R\Gamma_{w_0,c}(X,\omega_E^\chi)_\epsilon, \\ p^{-\min\{1-k_1,-k_2\}}U_{p,\kappa} & \text{over } R\Gamma_{\mathrm{an}}(X,\omega^k), \end{cases}$$

$$U_p^{t,\mathrm{good}} = \begin{cases} \frac{1}{p}U_p^t & \text{over } R\Gamma_{w_0}(X,\omega_E^\chi)_\epsilon, \\ U_p^t & \text{over } R\Gamma_{1,c}(X,\omega_E^\chi)_\epsilon, \\ p^{-\min\{1-k_1,-k_2\}}U_{p,\kappa}^t & \text{over } R\Gamma_{\mathrm{an}}(X,\omega_E^\kappa). \end{cases}$$

We shall need the following classicality theorem for overconvergent cohomologies.

Theorem 3.2.17 [BP22, Theorem 5.13]. *Let $\kappa=(k_1,k_2)\in X^*(\mathbf{T})$ be an algebraic weight.*

(i) *The U_p^{good} operator has non-negative slopes on $H_1^0(X,\omega_E^\kappa)_\epsilon$ and $H^1_{w_0,c}(X,\omega_E^\kappa)_\epsilon$.*

(ii) *The $U_p^{t,\mathrm{good}}$ operator has non-negative slopes on $H^0_{w_0}(X,\omega_E^\kappa)_\epsilon$ and $H^1_{1,c}(X,\omega_E^\kappa)_\epsilon$.*

Furthermore, we have isomorphisms of small-slope cohomologies

$$H_1^0(X,\omega_E^\kappa)_\epsilon^{U_p^{\mathrm{good}}<k_1-k_2-1} = H^0_{\mathrm{an}}(X,\omega_E^\kappa)^{U_p^{\mathrm{good}}<k_1-k_2-1},$$

$$H^1_{w_0,c}(X,\omega_E^\kappa)_\epsilon^{U_p^{\mathrm{good}}<1+k_2-k_1} = H^1_{\mathrm{an}}(X,\omega_E^\kappa)^{U_p^{\mathrm{good}}<1+k_2-k_1},$$

$$H^0_{w_0}(X,\omega_E^\kappa)_\epsilon^{U_p^{t,\mathrm{good}}<k_1-k_2-1} = H^0_{\mathrm{an}}(X,\omega_E^\kappa)^{U_p^{t,\mathrm{good}}<k_1-k_2-1},$$

$$H^1_{1,c}(X,\omega_E^\kappa)_\epsilon^{U_p^{t,\mathrm{good}}<1+k_2-k_1} = H^1_{\mathrm{an}}(X,\omega_E^\kappa)^{U_p^{t,\mathrm{good}}<1+k_2-k_1}.$$

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3.3 Overconvergent modular symbols

Let (R, R^+) be a uniform affinoid Tate algebra over $\mathbb{Q}_p$, and $\chi : T \to R^{+,\times}$ a δ-analytic character. Let A_χ^δ and $D_\chi^\delta = \mathrm{Hom}_R^0(A_\chi^\delta, R)$ be the principal series and distributions of Definition 2.3.1. These are topological $\mathbb{Q}_p$-vector spaces, A_χ^δ being a Banach space and D_χ^δ endowed with the weak topology. Consider the constant Iw_n-equivariant quasi-coherent sheaves $A_\chi^\delta \widehat{\otimes} \mathscr{O}_{\mathscr{F}\ell}$ and $D_\chi^\delta \widehat{\otimes} \mathscr{O}_{\mathscr{F}\ell}$ over $\mathscr{F}\ell$, where the completed tensor products are as in Definition 2.3.7. Their pullbacks by π_{HT} are identified with pro-Kummer-étale sheaves $A_{\chi,\text{ét}}^\delta \widehat{\otimes} \widehat{\mathscr{O}}_X$ and $D_{\chi,\text{ét}}^\delta \widehat{\otimes} \widehat{\mathscr{O}}_X$, where $A_{\chi,\text{ét}}^\delta$ and $D_{\chi,\text{ét}}^\delta$ are the sheaves over $X_{\text{prokét}}$ obtained by descent from the Iw_n-equivariant constant sheaves over X_∞ induced by the corresponding topological Iw_n-modules.

Before introducing the spaces of overconvergent modular symbols, let us define the proétale cohomology with compact support.

Definition 3.3.1. Let $\mathscr{F}$ be a proétale sheaf over $Y = Y_0(p^n)$, and let $j_{\text{prokét}} : Y_{\text{proét}} \to X_{\text{prokét}}$ be the natural morphism of sites. The proétale cohomology with compact support of $\mathscr{F}$ is the complex

$$R\Gamma_{\text{proét},c}(Y_{\mathbb{C}_p}, \mathscr{F}) = R\Gamma_{\text{prokét}}(X_{\mathbb{C}_p}, j_{\text{prokét},!}\mathscr{F}).$$

Remark 3.3.2. Let $j_{\text{proét}} : Y_{\text{proét}} \to X_{\text{proét}}$ be the natural morphism of sites and let $\mathbb{L}$ be a proétale $\mathbb{Z}_p$-local system over Y. The proétale cohomology with compact support of $\mathbb{L}$ is usually defined as $R\Gamma_{\text{proét},c}(Y_{\mathbb{C}_p}, \mathbb{L}) := R\Gamma_{\text{proét}}(X_{\mathbb{C}_p}, j_{\text{proét},!}\mathbb{L})$. On the other hand, [DLLZ23b, Lemma 4.4.27] implies that this cohomology can be computed in the pro-Kummer-étale site, that is, that we have a quasi-isomorphism

$$R\Gamma_{\text{proét}}(X_{\mathbb{C}_p}, j_{\text{proét},!}\mathbb{L}) = R\Gamma_{\text{prokét}}(X_{\mathbb{C}_p}, j_{\text{prokét},!}\mathbb{L}).$$

In other words, if $\mathscr{F} = \mathbb{L}$ is a proétale local system over Y. The cohomology with compact support of Definition 3.3.1 coincides with the classical one.

Definition 3.3.3. We define the overconvergent modular symbols as the cohomology complexes

$$R\Gamma_{\text{proét}}(Y_{\mathbb{C}_p}, A_{\chi,\text{ét}}^\delta) \quad \text{and} \quad R\Gamma_{\text{proét}}(Y_{\mathbb{C}_p}, D_{\chi,\text{ét}}^\delta).$$

We also define the overconvergent modular symbols with compact support in the obvious way.

Remark 3.3.4. By purity on p-torsion local systems [DLLZ23b, Theorem 6.4.1] and the devisage of $A_\chi^{\delta,+}$ and $D_\chi^{\delta,+}$ of Corollary 2.3.6, one has quasi-isomorphisms

$$R\Gamma_{\text{proét}}(Y_{\mathbb{C}_p}, A_{\chi,\text{ét}}^\delta) = R\Gamma_{\text{prokét}}(X_{\mathbb{C}_p}, A_{\chi,\text{ét}}^\delta),$$
$$R\Gamma_{\text{proét}}(Y_{\mathbb{C}_p}, D_{\chi,\text{ét}}^\delta) = R\Gamma_{\text{prokét}}(X_{\mathbb{C}_p}, D_{\chi,\text{ét}}^\delta).$$

The primitive comparison theorem also applies for the modular symbols as follows.

Lemma 3.3.5. *Let $\iota : D \subset X$ be the cusp divisor endowed with the log structure induced by X. Let $\iota : D_{\text{prokét}} \to X_{\text{prokét}}$ be the natural morphism, and let $\widehat{\mathscr{I}}^+ = \ker(\widehat{\mathscr{O}}_X^+ \to \iota_* \widehat{\mathscr{O}}_D^+)$ be the ideal defining the cusps. Then $(j_{\text{prokét},!} A_{\chi,\text{ét}}^{\delta,+}) \widehat{\otimes} \widehat{\mathscr{O}}_X^+ = A_{\chi,\text{ét}}^{\delta,+} \widehat{\otimes} \widehat{\mathscr{I}}^+$. Furthermore, we have almost quasi-isomorphisms*

$$\begin{aligned} R\Gamma_{\text{prokét}}(X_{\mathbb{C}_p}, A_{\chi,\text{ét}}^{\delta,+}) \widehat{\otimes} \mathcal{O}_{\mathbb{C}_p} &=^a R\Gamma_{\text{prokét}}(X_{\mathbb{C}_p}, A_{\chi,\text{ét}}^{\delta,+} \widehat{\otimes} \widehat{\mathscr{O}}_X^+), \\ R\Gamma_{\text{prokét}}(X_{\mathbb{C}_p}, j_{\text{prokét},!} A_{\chi,\text{ét}}^{\delta,+}) \widehat{\otimes} \mathcal{O}_{\mathbb{C}_p} &=^a R\Gamma_{\text{prokét}}(X_{\mathbb{C}_p}, A_{\chi,\text{ét}}^{\delta,+} \widehat{\otimes} \widehat{\mathscr{I}}^+). \end{aligned} \tag{22}$$

An analogous statement holds for the sheaf $D_{\chi,\text{ét}}^\delta$.

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Proof. We only give the proof for $A^{\delta,+}_{\chi,\text{ét}}$, the other being identical. By Corollary 2.3.6 we can write $A^{\delta,+}_{\chi,\text{ét}} = R\varprojlim_i(\varinjlim_j \mathbb{L}_{i,j})$, where $\mathbb{L}_{i,j}$ are finite Kummer-étale local systems over X. Then, by the primitive comparison theorem for finite local systems, one has that

$$\begin{aligned} R\Gamma_{\text{prokét}}(X_{\mathbb{C}_p}, A^{\delta,+}_{\chi,\text{ét}})\widehat{\otimes}\mathcal{O}_{\mathbb{C}_p} &= R\varprojlim_i \operatorname{hocolim}_j(R\Gamma_{\text{két}}(X_{\mathbb{C}_p}, \mathbb{L}_{i,j}) \otimes \mathcal{O}_{\mathbb{C}_p}) \\ &=^a R\varprojlim_i \operatorname{hocolim}_j(R\Gamma_{\text{két}}(X_{\mathbb{C}_p}, \mathbb{L}_{i,j} \otimes \mathscr{O}^+_X)) \\ &= R\Gamma_{\text{proket}}(X_{\mathbb{C}_p}, A^{\delta,+}_{\chi,\text{ét}}\widehat{\otimes}\widehat{\mathscr{O}}^+_X). \end{aligned}$$

To prove the case of cohomology with compact support we need the following observation. Consider the exact sequence of Kummer-étale sheaves over X,

$$0 \to j_{\text{két},!}\mathbb{Z}/p^s \to \mathbb{Z}/p^s \to \iota_*\mathbb{Z}/p^s \to 0.$$

Tensoring with $\mathscr{O}^+_X$ and taking projective limits on s, one obtains the short exact sequence

$$0 \to \widehat{\mathscr{I}}^+ \to \widehat{\mathscr{O}}^+_X \to \iota_*\widehat{\mathscr{O}}_D \to 0.$$

Therefore, tensoring with $A^{\delta,+}_{\chi,\text{ét}}$, one gets a short exact sequence

$$0 \to A^{\delta,+}_{\chi,\text{ét}}\widehat{\otimes}\widehat{\mathscr{I}}^+ \to A^{\delta,+}_{\chi,\text{ét}}\widehat{\otimes}\widehat{\mathscr{O}}^+_X \to A^{\delta,+}_{\chi,\text{ét}}\widehat{\otimes}\iota_*\widehat{\mathscr{O}}^+_D \to 0. \tag{23}$$

It is easy to see that (23) is the (completed) $\widehat{\mathscr{O}}^+_X$-scalar extension of the short exact sequence

$$0 \to j_{\text{proket},!}A^{\delta,+}_{\chi} \to A^{\delta,+}_{\chi,\text{ét}} \to \iota_*\iota^*A^{\delta,+}_{\chi,\text{ét}} \to 0.$$

In particular, from the previous two short exact sequences one deduces that $j_{\text{proket},!}A^{\delta,+}_{\chi,\text{ét}}\widehat{\otimes}\widehat{\mathscr{O}}^+_X = A^{\delta,+}_{\chi,\text{ét}}\widehat{\otimes}\widehat{\mathscr{I}}^+$. Moreover, the second almost equality of (22) holds after taking pro-Kummer-étale cohomology of the triangle (23), by applying the primitive comparison theorem for both X and D (see [LLZ23, Theorem 2.2.1]). $\square$

Next, we define the U_p-operators for overconvergent modular symbols. These are obtained by pulling back the maps of (12).

Definition 3.3.6. Consider the U_p-correspondence C of X. The U^t_p and U_p-correspondences of $A^{\delta}_{\chi,\text{ét}}$ and $D^{\delta}_{\chi,\text{ét}}$ are the morphisms

$$U^t_p : p_1^*(A^{\delta}_{\chi,\text{ét}}) \to U_p : p_2^*(A^{\delta}_{\chi,\text{ét}}) \quad \text{and} \quad U_p : p_2^*(D^{\delta}_{\chi,\text{ét}}) \to p_1^*(D^{\delta}_{\chi,\text{ét}})$$

defined by the pullback of (12) by π_{HT}.

We shall need the following classicality result (see [AS08, Theorem 6.4.1] and [AIS15, Theorem 3.16]).

Theorem 3.3.7. *Let $\kappa = (k_1, k_2) \in X^*(\mathbf{T})^+$ be a dominant weight. The maps $D^{\delta}_{\kappa} \to V_{-w_0(\kappa)}$ and $V_\kappa \to A^{\delta}_{\kappa}$ induce isomorphisms of the $(< k_1 - k_2 + 1)$-slope part for the action of the (normalized) U_p-operators*

$$\begin{aligned} H^1_{\text{proket}}(X_{\mathbb{C}_p}, D^{\delta}_{\kappa,\text{ét}})^{U_p<k_1-k_2+1} &\xrightarrow{\sim} H^1_{\text{proket}}(X_{\mathbb{C}_p}, V_{-w_0(\kappa),\text{ét}})^{U_p<k_1-k_2+1}, \\ H^1_{\text{proket}}(X_{\mathbb{C}_p}, V_{\kappa,\text{ét}})^{U^t_p<k_1-k_2+1} &\xrightarrow{\sim} H^1_{\text{proket}}(X_{\mathbb{C}_p}, A^{\delta}_{\kappa,\text{ét}})^{U^t_p<k_1-k_2+1}. \end{aligned}$$

A similar result holds for the cohomology with compact support.

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Remark 3.3.8. In [AS08], Theorem 3.3.7 is proved only for the cohomology of distributions. However, the same strategy works for the principal series and the cohomology with compact support. In particular, the bounds of the classicality theorem are motivated by [AS08, Theorem 3.11.1].

3.4 Overconvergent Hodge–Tate maps

We end this section with the definition of overconvergent Hodge–Tate maps interpolating the morphisms $\mathrm{HT}^k : \mathrm{Sym}^k T_pE \otimes \widehat{\mathscr{O}}_X \to \omega_E^k \otimes_{\mathscr{O}_X} \widehat{\mathscr{O}}_X$ and $\mathrm{HT}^{k,\vee} : \omega_E^{-k} \otimes_{\mathscr{O}_X} \widehat{\mathscr{O}}_X(k) \to \mathrm{Sym}^k T_pE \otimes \widehat{\mathscr{O}}_X$.

Definition 3.4.1. Let $\epsilon \geq \delta \geq n$, (R, R^+) a uniform affinoid Tate $\mathbb{Q}_p$-algebra and $\chi = (\chi_1, \chi_2) : T = \mathbf{T}(\mathbb{Z}_p) \to R^{+,\times}$ a δ-analytic character.

(i) We define the map of pro-Kummer-étale sheaves over $X_1(\epsilon)$,

$$\mathrm{HT}^{A,\vee}_{-w_0(\chi)} : \omega_E^{w_0(\chi)} \otimes_{R\widehat{\otimes}\mathscr{O}_X} \widehat{\mathscr{O}}_X(\chi_1) \to A^{\delta}_{\chi,\text{ét}} \widehat{\otimes} \widehat{\mathscr{O}}_X,$$

as the pullback of the highest weight vector map $\Psi^{A,\vee}_{-w_0(\chi)} : \mathscr{L}(w_0(\chi)) \to A^{\delta}_{\chi} \widehat{\otimes} \mathscr{O}_{\mathscr{F}\ell}$ over $U_1(\epsilon)\,\mathrm{Iw}_n \subset \mathscr{F}\ell$ by π_{HT} (cf. Proposition 2.4.4). We let $\mathrm{HT}^{D}_{-w_0(\chi)} : D^{\delta}_{\chi,\text{ét}} \widehat{\otimes} \widehat{\mathscr{O}}_X \to \omega_E^{-w_0(\chi)} \otimes \widehat{\mathscr{O}}_X(-\chi_1)$ be the dual of $\mathrm{HT}^{A,\vee}_{-w_0(\chi)}$.

(ii) We define the map of pro-Kummer-étale sheaves over $X_{w_0}(\epsilon)$,

$$\mathrm{HT}^{A}_{\chi} : A^{\delta}_{\chi,\text{ét}} \widehat{\otimes} \widehat{\mathscr{O}}_X \to \omega_E^{\chi} \otimes \widehat{\mathscr{O}}_X(\chi_2),$$

as the pullback by π_{HT} of the lowest weight vector map $A^{\delta}_{\chi} \widehat{\otimes} \mathscr{O}_{\mathscr{F}\ell} \to \mathscr{L}(\chi)$ over $U_{w_0}(\epsilon)\,\mathrm{Iw}_n \subset \mathscr{F}\ell$. We let $\mathrm{HT}^{D,\vee}_{\chi} : \omega_E^{-\chi} \otimes \widehat{\mathscr{O}}_X(-\chi_2) \to D^{\delta}_{\chi,\text{ét}} \widehat{\otimes} \widehat{\mathscr{O}}_X$ be the dual of HT^{A}_{χ}.

Lemma 3.4.2. *The overconvergent Hodge–Tate maps of Definition 3.4.1 are compatible with the U_p-correspondences of Definitions 3.2.15 and 3.3.6. Moreover, they are compatible with the normalized U_p-correspondences of the sheaves $V_{\kappa,\text{ét}}$ for $\kappa \in X^*(\mathbf{T})^+$ (see Remark 2.2.13).*

Proof. The lemma is an immediate consequence of the definitions and Proposition 2.4.5 (see Remarks 2.2.10 and 3.2.5). □

4. p-adic Eichler–Shimura maps

Throughout this section we fix a neat compact open subgroup $K^p \subset \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty,p})$, and, given $K_p \subset \mathrm{GL}_2(\mathbb{Q}_p)$, we let $Y = Y_{K_p}$ and $X = X_{K_p}$ denote the affine and compact modular curves of level K_p, respectively. We let $D = X\backslash Y$ be the cusp divisor. Let $f : E^{sm} \to X$ be the semi-abelian scheme extending the universal elliptic curve over Y, and $\overline{E}$ be its relative compactification to a log smooth adic space over X (cf. [DR73]). We denote by $\mathrm{DR}_X(\overline{E})$ the relative log de Rham complex of $\overline{E}$ over X, and by $\mathscr{H}^1_{\mathrm{dR}} := R^1 f_{\mathrm{an},*}(\mathrm{DR}_X(\overline{E}))$ the first relative de Rham cohomology group. The sheaf $\mathscr{H}^1_{\mathrm{dR}}$ is endowed with a log connection

$$\nabla : \mathscr{H}^1_{\mathrm{dR}} \to \mathscr{H}^1_{\mathrm{dR}} \otimes_{\mathscr{O}_X} \Omega^1_X(\log)$$

and a Hodge filtration $0 \to \omega_E \to \mathscr{H}^1_{dR} \to \omega_E^{-1} \to 0$ with $\mathrm{Fil}^0\, \mathscr{H}^1_{\mathrm{dR}} = \mathscr{H}^1_{\mathrm{dR}}$, $\mathrm{Fil}^1\, \mathscr{H}^1_{\mathrm{dR}} = \omega_E$ and $\mathrm{Fil}^2\, \mathscr{H}^1_{\mathrm{dR}} = 0$, satisfying Griffiths transversality. This last section is dedicated to the construction of the Eichler–Shimura decomposition for the étale cohomology of the modular curves. We first provide a new proof of Faltings's Eichler–Shimura decomposition of the cohomology of the local systems $V_{\kappa,\text{ét}}$ (cf. [Fal87]). Our method uses the Hodge–Tate period map and the dual BGG resolution of Proposition 2.4.3. Next, we use the overconvergent Hodge–Tate maps of Definition 3.4.1

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to define overconvergent Eichler–Shimura maps. We shall recover the results of [AIS15] as well as a new map from the H^1-cohomology with compact support of overconvergent modular forms to overconvergent modular symbols. Finally, we show that the overconvergent Eichler–Shimura maps are compatible with the Poincaré and Serre duality pairings, and that, for small slope, we have a perfect pairing.

4.1 A proétale Eichler–Shimura decomposition

Let $\kappa=(k_1,k_2)\in X^*(\mathbf{T})^+$ be a dominant weight, V_κ the irreducible representation of highest weight κ, and $V_{\kappa,\text{ét}}$ the pro-Kummer-étale local system over X defined by V_κ. Let $\alpha=(1,-1)\in X^*(\mathbf{T})$. Let us recall a theorem of Faltings.

Theorem 4.1.1 [Fal87, Theorem 6]. *There are Hecke and Galois equivariant isomorphisms*

$$H^1_{\text{proét}}(Y_{\mathbb{C}_p},V_{\kappa,\text{ét}})\otimes_{\mathbb{Z}_p}\mathbb{C}_p=H^1_{\text{an}}(X_{\mathbb{C}_p},\omega_E^{w_0(\kappa)})(k_1)\oplus H^0_{\text{an}}(X_{\mathbb{C}_p},\omega_E^{\kappa+\alpha})(k_2-1),$$

$$H^1_{\text{proét},c}(Y_{\mathbb{C}_p},V_{\kappa,\text{ét}})\otimes_{\mathbb{Z}_p}\mathbb{C}_p=H^1_{\text{an}}(X_{\mathbb{C}_p},\omega_E^{w_0(\kappa)}(-D))(k_1)\oplus H^0_{\text{an}}(X_{\mathbb{C}_p},\omega_E^{\kappa+\alpha}(-D))(k_2-1).$$

Let $\mathbb{B}_{\text{dR}}^+$ be the de Rham period sheaf of $X_{\text{prokét}}$, $\theta:\mathbb{B}^+_{\text{dR}}\to\widehat{\mathscr{O}}_X$ the Fontaine map, and $\xi\in\ker\theta$ a local generator of the kernel; we set $\mathbb{B}_{\text{dR}}:=\mathbb{B}^+_{\text{dR}}[1/\xi]$. Let $\mathscr{O}\mathbb{B}^+_{\text{dR},\log}$ be the geometric de Rham period sheaf and $\mathscr{O}\mathbb{B}_{\text{dR},\log}=\mathscr{O}\mathbb{B}^+_{\text{dR},\log}[1/\xi]$, we denote by $\mathscr{O}\mathbb{C}_{\log}$ the sheaf $\operatorname{gr}^0(\mathscr{O}\mathbb{B}_{\text{dR},\log})$. We refer to [Sch13] and [DLLZ23a] for the definition of the period sheaves.

The main ingredient of our proof of Theorem 4.1.1 is an explicit relation between the Faltings extension $\operatorname{gr}^1\mathscr{O}\mathbb{B}^+_{\text{dR},\log}$ and the Tate module T_pE. This arises naturally in the study of pullbacks of GL_2-equivariant vector bundles of $\mathscr{F}\!\ell$ by π_{HT}. Recall that $\mathrm{FL}=\mathbf{B}\backslash\mathrm{GL}_2$, so that we have an equivalence of categories between GL_2-equivariant vector bundles over $\mathscr{F}\!\ell$ and algebraic representations of $\mathbf{B}$. Let $\mathscr{O}(\mathbf{B})$ be the ring of algebraic functions of $\mathbf{B}$ endowed with the right regular action; note that any finite representation of $\mathbf{B}$ occurs in $\mathscr{O}(\mathbf{B})$. Writing $\mathbf{B}=\mathbf{T}\ltimes\mathbf{N}$ as a product of the diagonal torus and its unipotent radical, one has that $\mathscr{O}(\mathbf{B})=\mathscr{O}(\mathbf{T})\otimes\mathscr{O}(\mathbf{N})$. The action of $\mathbf{B}$ on $\mathscr{O}(\mathbf{T})$ factors through $\mathbf{T}$, so that this ring can be decomposed in terms of characters of the torus. By Proposition 3.1.4 we already know what the pullback by π_{HT} of the quasi-coherent sheaf associated to $\mathscr{O}(\mathbf{T})$ is; it admits an explicit description in terms of modular sheaves. On the other hand, the action of $\mathbf{B}$ on $\mathscr{O}(\mathbf{N})$ is determined by the right action $(n,b)\mapsto t_b^{-1}nt_bn_b$, where $n\in\mathbf{N}$ and $b=(t_b,n_b)\in\mathbf{B}=\mathbf{T}\ltimes\mathbf{N}$. Let $\underline{\mathscr{O}}(\mathbf{N})$ be the GL_2-equivariant quasi-coherent sheaf over $\mathscr{F}\!\ell$ attached to $\mathscr{O}(\mathbf{N})$.

Theorem 4.1.2 ([Fal87, Theorem 5] and [Pan22, Theorem 4.2.2]). *There is a natural isomorphism of pro-Kummer-étale sheaves over X,*

$$\pi^*_{\text{HT}}(\underline{\mathscr{O}}(\mathbf{N}))=\mathscr{O}\mathbb{C}_{\log}.$$

Furthermore, let $\mathscr{O}(\mathbf{N})^{\leq 1}\subset\mathscr{O}(\mathbf{N})$ be the subrepresentation consisting on polynomials of degree at most 1. We have an isomorphism as $\mathbf{B}$-representations $\mathscr{O}(\mathbf{N})^{\leq 1}=\mathrm{St}\otimes\mathbb{Q}_p(-1,0)$; in particular, $\pi^*_{\text{HT}}(\underline{\mathscr{O}}(\mathbf{N})^{\leq 1})=T_pE\otimes\widehat{\mathscr{O}}_X(-1)\otimes\omega_E$. Moreover, there is an isomorphism of extensions

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widehat{\mathscr{O}}_X(1) & \xrightarrow{\mathrm{HT}^\vee} & T_pE\otimes\widehat{\mathscr{O}}_E\otimes\omega_E & \xrightarrow{\mathrm{HT}} & \omega_E^2\otimes\widehat{\mathscr{O}}_X & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle\mathrm{id}} & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle -KS} & & \\
0 & \longrightarrow & \widehat{\mathscr{O}}_X(1) & \longrightarrow & \operatorname{gr}^1\mathscr{O}\mathbb{B}^+_{\text{dR},\log} & \longrightarrow & \Omega^1_X(\log)\otimes\widehat{\mathscr{O}}_X & \longrightarrow & 0
\end{array}$$

where KS is the Kodaira–Spencer isomorphism.

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Proof. It is enough to show the second statement, namely, that if $\pi_{\mathrm{HT}}^*(\underline{\mathscr{O}}(\mathbf{N})^{\leq 1}) = T_pE \otimes \widehat{\mathscr{O}}(-1) \otimes \omega_E = \mathrm{gr}^1\,\mathscr{O}\mathbb{B}^+_{\mathrm{dR,log}}(-1)$, one has

$$\mathscr{O}\mathbb{C}_{\log} = \varinjlim_k \mathrm{Sym}^k(\mathrm{gr}^1\,\mathscr{O}\mathbb{B}^+_{\mathrm{dR,log}}(-1)) = \varinjlim_k \pi_{\mathrm{HT}}^*(\mathrm{Sym}^k\,\underline{\mathscr{O}}(\mathbf{N})^{\leq 1}) = \pi_{\mathrm{HT}}^*(\underline{\mathscr{O}}(\mathbf{N})).$$

Let $\mathscr{F}$ be a sheaf endowed with an integral log connection ∇; we denote by $\mathrm{DR}(\mathscr{F},\nabla)$ the log de Rham complex of $\mathscr{F}$. Let $f:\overline{E}\to X$ be the compactification of the elliptic curve as a log smooth adic space over X. We have a quasi-isomorphism of complexes over $\overline{E}_{\mathrm{prok\acute{e}t}}$,

$$T_p\mathbb{G}_m \otimes_{\widehat{\mathbb{Z}}_p} \mathbb{B}_{\mathrm{dR},\overline{E}} \simeq T_p\mathbb{G}_m \otimes_{\widehat{\mathbb{Z}}_p} \mathrm{DR}(\mathscr{O}\mathbb{B}_{\mathrm{dR,log},\overline{E}}, d) = \mathrm{DR}(\mathscr{O}\mathbb{B}_{\mathrm{dR,log},\overline{E}}, d)(1).$$

Taking $R^1 f_{\mathrm{prok\acute{e}t},*}$ one obtains by [DLLZ23a, Theorem 3.2.7 (5)] or [Sch13, Theorem 8.8]

$$T_pE\otimes \mathbb{B}_{\mathrm{dR},X} \simeq T_pE \otimes_{\widehat{\mathbb{Z}}_p} \mathrm{DR}(\mathscr{O}\mathbb{B}_{\mathrm{dR,log},X}, d) \cong \mathrm{DR}(\mathscr{H}^1_{dR}\otimes \mathscr{O}\mathbb{B}_{\mathrm{dR,log},X},\nabla)(1). \tag{24}$$

Let $\mathbb{M} := T_pE(-1)\otimes \mathbb{B}^+_{\mathrm{dR},X} = (T_pE(-1)\otimes \mathscr{O}\mathbb{B}^+_{\mathrm{dR,log},X})^{\nabla=0}$ and $\mathbb{M}_0 = (\mathscr{H}^1_{\mathrm{dR}}\otimes \mathscr{O}\mathbb{B}^+_{\mathrm{dR,log},X})^{\nabla=0}$. Both $\mathbb{M}_0$ and $\mathbb{M}$ are $\mathbb{B}^+_{\mathrm{dR},X}$-lattices of $T_pE(-1)\otimes\mathbb{B}_{\mathrm{dR},X}$. The Hodge filtration of $\mathscr{H}^1_{\mathrm{dR}}$ is concentrated in degrees 0 and 1, and is equal to

$$0\to\omega_E\to\mathscr{H}^1_{\mathrm{dR}}\to\omega_E^{-1}\to 0.$$

This implies that $\xi\mathbb{M}\subset\mathbb{M}_0\subset\mathbb{M}$, and that $(\mathrm{Fil}^1(\mathscr{H}^1_{\mathrm{dR}}\otimes\mathscr{O}\mathbb{B}^+_{\mathrm{dR,log},X}))^{\nabla=0}=\xi\mathbb{M}$. Then Proposition 7.9 of [Sch13] implies that

$$\mathbb{M}_0/\xi\mathbb{M} = \mathrm{gr}^0\,\mathscr{H}^1_{\mathrm{dR}}\otimes\widehat{\mathscr{O}}_X = \omega_E^{-1}\otimes\widehat{\mathscr{O}}_X,$$
$$\mathbb{M}/\mathbb{M}_0 = \mathrm{gr}^1\,\mathscr{H}^1_{\mathrm{dR}}\otimes\widehat{\mathscr{O}}_X(-1) = \omega_E\otimes\widehat{\mathscr{O}}_X(-1).$$

In particular,

$$0\to\xi\mathbb{M}_0/\xi^2\mathbb{M}\to\xi\mathbb{M}/\xi^2\mathbb{M}\to\xi\mathbb{M}/\xi\mathbb{M}_0\to 0$$

is just the Hodge–Tate exact sequence of $T_pE\otimes\widehat{\mathscr{O}}_X$ (note the multiplication by ξ induced by the Tate twist in (24)), and

$$0\to\xi\mathbb{M}/\xi\mathbb{M}_0\to\mathbb{M}_0/\xi\mathbb{M}_0\to\mathbb{M}_0/\xi\mathbb{M}\to 0$$

is the Hodge exact sequence of $\mathscr{H}^1_{dR}\otimes\widehat{\mathscr{O}}_X$.

Consider the map of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{M} & \longrightarrow & \mathbb{M}\otimes\mathscr{O}\mathbb{B}^+_{\mathrm{dR,log},X} & \xrightarrow{d} & \mathbb{M}\otimes\mathscr{O}\mathbb{B}^+_{\mathrm{dR,log},X}\otimes\Omega^1_X(\log) & \longrightarrow & 0\\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathbb{M}_0 & \longrightarrow & \mathbb{M}_0\otimes\mathscr{O}\mathbb{B}^+_{\mathrm{dR,log},X} & \xrightarrow{d} & \mathbb{M}_0\otimes\mathscr{O}\mathbb{B}^+_{\mathrm{dR,log},X}\otimes\Omega^1_X(\log) & \longrightarrow & 0
\end{array} \tag{25}$$

and let $\tilde\theta:\mathscr{O}\mathbb{B}^+_{\mathrm{dR,log},X}\to\widehat{\mathscr{O}}_X$ be Fontaine's map.

Taking the first graded piece in the upper short exact sequence, one finds

$$0\to\xi\mathbb{M}/\xi^2\mathbb{M}\to\frac{\mathbb{M}\otimes(\ker\tilde\theta)}{\mathbb{M}\otimes(\ker\tilde\theta)^2}\xrightarrow{\overline{\nabla}}\frac{\mathbb{M}\otimes\mathscr{O}\mathbb{B}^+_{\mathrm{dR,log},X}}{\mathbb{M}\otimes(\ker\tilde\theta)}\otimes\Omega^1_X(\log)\to 0.$$

Since $\xi\mathbb{M}\subset\mathbb{M}_0$, taking the intersection with the image of the lower short exact sequence in (25), one obtains a short exact sequence

$$0\to\frac{\xi\mathbb{M}}{\xi^2\mathbb{M}}\to\frac{\mathbb{M}_0\otimes(\ker\tilde\theta)+\xi\mathbb{M}\otimes\mathscr{O}\mathbb{B}^+_{\mathrm{dR,log},X}}{\mathbb{M}_0\otimes(\ker\tilde\theta)^2+\xi\mathbb{M}\otimes(\ker\tilde\theta)}\xrightarrow{\overline{\nabla}}\frac{\mathbb{M}_0\otimes\mathscr{O}\mathbb{B}^+_{\mathrm{dR,log},X}}{\mathbb{M}_0\otimes(\ker\tilde\theta)+\xi\mathbb{M}\otimes\mathscr{O}\mathbb{B}^+_{\mathrm{dR,log},X}}\otimes\Omega^1_X(\log)\to 0. \tag{26}$$

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The quotient term of the short exact sequence (26) is equal to $\mathbb{M}_0/\xi\mathbb{M}\otimes\Omega^1_X(\log)=\omega_E^{-1}\otimes\Omega^1_X(\log)\otimes\widehat{\mathscr{O}}_X$. The middle term of the short exact sequence is equal to

$$\mathrm{gr}^1(\mathscr{H}^1_{\mathrm{dR}}\otimes\mathscr{O}\mathbb{B}^+_{\mathrm{dR},\log,X})=\omega_E\otimes\widehat{\mathscr{O}}_X\oplus\omega_E^{-1}\otimes\mathrm{gr}^1\mathscr{O}\mathbb{B}^+_{\mathrm{dR},\log,X}.$$

Note that the restriction of $\overline{\nabla}$ to $\omega_E\otimes\widehat{\mathscr{O}}_X$ is the Kodaira–Spencer map by definition. Indeed, if $\nabla:\mathscr{H}^1_{dR}\to\mathscr{H}^1_{dR}\otimes\Omega^1_X(\log)$ is the connection, taking the first graded piece we get the map

$$KS:\omega_E\to\omega_E^{-1}\otimes\Omega^1_X(\log).$$

Therefore, we have constructed a short exact sequence

$$0\to T_pE\otimes\widehat{\mathscr{O}}_X\xrightarrow{\mathrm{HT}\oplus\alpha}\omega_E\otimes\widehat{\mathscr{O}}_X\oplus\omega_E^{-1}\otimes\mathrm{gr}^1\mathscr{O}\mathbb{B}^+_{\mathrm{dR},\log,X}\xrightarrow{KS\oplus\overline{\nabla}}\omega_E^{-1}\otimes\Omega^1_X(\log)\otimes\widehat{\mathscr{O}}_X\to 0.$$

Thus, as KS is an isomorphism so is α, and we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \omega_E^{-1}\otimes\widehat{\mathscr{O}}_X(1) & \xrightarrow{\mathrm{HT}^\vee} & T_pE\otimes\widehat{\mathscr{O}}_X & \xrightarrow{\mathrm{HT}} & \omega_E\otimes\widehat{\mathscr{O}}_X & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle -KS} & & \\
0 & \longrightarrow & \omega_E^{-1}\otimes\widehat{\mathscr{O}}_X(1) & \longrightarrow & \omega_E^{-1}\otimes\mathrm{gr}^1\mathscr{O}\mathbb{B}^+_{\mathrm{dR},\log,X} & \xrightarrow{\overline{\nabla}} & \omega_E^{-1}\otimes\Omega^1_X(\log)\otimes\widehat{\mathscr{O}}_X & \longrightarrow & 0
\end{array}$$

which finishes the proof. □

Remark 4.1.3. The previous proposition is the key tool necessary to compute the relative Sen operator for the modular curve in Pan's locally analytic vectors (cf. [Pan22]).

We deduce the Eichler–Shimura decompositions for the local systems $V_{\kappa,\text{ét}}$.

Theorem 4.1.4. *Let $\alpha=(1,-1)$. Let $\kappa=(k_1,k_2)\in X^*(\mathbf{T})^+$ be a dominant weight and* $\mathrm{BGG}(\kappa)$ *the BGG complex of Proposition 2.4.3:*

$$\mathrm{BGG}(\kappa):[0\to V_{\kappa,\text{ét}}\to V(\kappa)\to V(w_0(\kappa)-\alpha)\to 0].$$

Let $\underline{\mathrm{BGG}(\kappa)}$ be the GL_2-equivariant complex of sheaves defined by $\mathrm{BGG}(\kappa)$. We have a quasi-isomorphism of complexes over $X_{\text{proét}}$,

$$\pi^*_{\mathrm{HT}}(\underline{\mathrm{BGG}(\kappa)})=[0\to V_\kappa\otimes\widehat{\mathscr{O}}_X\to\omega_E^{w_0(\kappa)}\otimes\mathscr{O}\mathbb{C}_{\log}(k_1)\to\omega_E^{\kappa+\alpha}\otimes\mathscr{O}\mathbb{C}_{\log}(k_2-1)\to 0].$$

Moreover, let $\lambda:X_{\mathbb{C}_p,\text{proét}}\to X_{\mathbb{C}_p,\mathrm{an}}$ be the projection of sites. Let $\iota:D_{\text{proét}}\to X_{\text{proét}}$ be the natural morphism, and $\widehat{\mathscr{I}}=\ker(\widehat{\mathscr{O}}_X\to\iota_*\widehat{\mathscr{O}}_D)$. We have

$$R\lambda_*(V_{\kappa,\text{ét}}\otimes\widehat{\mathscr{O}}_X)=\omega_E^{w_0(\kappa)}\otimes\mathbb{C}_p(k_1)[0]\oplus\omega_E^{\kappa+\alpha}\otimes\mathbb{C}_p(k_2-1)[-1],$$

$$R\lambda_*(V_{\kappa,\text{ét}}\otimes\widehat{\mathscr{I}}_X)=\omega_E^{w_0(\kappa)}(-D)\otimes\mathbb{C}(k_1)[0]\oplus\omega_E^{\kappa+\alpha}(-D)\otimes\mathbb{C}(k_2-1)[-1].$$

Then, taking the H^1-cohomology over $X_{\mathbb{C}_p,\mathrm{an}}$, one obtains Theorem 4.1.1.

Proof. Note that $V(\kappa)=\kappa\otimes V(0)$ as a $\mathbf{B}$-module, thus the first part of the theorem follows from Theorem 4.1.2 and Proposition 3.1.4. On the other hand, by [DLLZ23a, Lemma 3.3.2] we know that $R\lambda_*\mathscr{O}\mathbb{C}_{\log}=\mathscr{O}_{X_{\mathbb{C}_p}}$ and $R\lambda_*(\mathscr{O}\mathbb{C}_{\log}\otimes\iota_*\widehat{\mathscr{O}}_D)=\iota_*\mathscr{O}_D$, in particular that $R\lambda_*(\mathscr{O}\mathbb{C}_{\log}\otimes\widehat{\mathscr{I}})=\mathscr{O}(-D)$. Therefore,

$$\begin{aligned}R\lambda_*(V_{\kappa,\text{ét}}\otimes\widehat{\mathscr{O}}_X)&=[\omega_E^{w_0(\kappa)}\otimes\mathbb{C}_p(k_1)\to\omega_E^{\kappa+\alpha}\otimes\mathbb{C}_p(k_2-1)],\\ R\lambda_*(V_{\kappa,\text{ét}}\otimes\widehat{\mathscr{I}})&=[\omega_E^{w_0(\kappa)}(-D)\otimes\mathbb{C}_p(k_1)\to\omega_E^{\kappa+\alpha}(-D)\otimes\mathbb{C}_p(k_2-1)].\end{aligned}\tag{27}$$

But the arrows of (27) are 0 since the sheaf $\omega_E^{w_0(\kappa)}\otimes\mathbb{C}_p(k_1)$ already factors through $V_{\kappa,\text{ét}}\otimes\widehat{\mathscr{O}}_X$ via $\mathrm{HT}^\vee_{-w_0(\kappa)}:\omega_E^{w_0(\kappa)}\otimes\widehat{\mathscr{O}}_X(k_1)\to V_{\kappa,\text{ét}}\otimes\widehat{\mathscr{O}}_X$. The theorem follows. □

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Remark 4.1.5. In [LLZ23], Lan, Liu and Zhu gave another proof of the Eichler–Shimura decomposition for arbitrary Shimura varieties (also called BGG decomposition; see [LLZ23, Theorem 6.2.3]). Their proof uses the p-adic Riemann–Hilbert correspondence and the BGG decomposition in terms of Verma modules; they then apply Faltings's strategy to construct a complex of $\mathcal{D}$-modules quasi-isomorphic to the de Rham complex of the corresponding vector bundle with connection. In our situation, we use the dual BGG decomposition and the associated GL_2-equivariant sheaves over $\mathscr{F}\!\ell$ instead. Our key ingredient to compute the proétale cohomology of $V_\kappa \otimes \widehat{\mathscr{O}}_X$ is Theorem 4.1.2, which serves as a dictionary between vector bundles over $\mathscr{F}\!\ell$ and $\widehat{\mathscr{O}}_X$-vector bundles over X.

We finish this section with the compatibility of the Eichler–Shimura decomposition with Poincaré and Serre duality.

Proposition 4.1.6 [LLZ23, Theorem 6.2.3]. *Let* $\mathrm{Tr}_P : H^2_{\text{proét},c}(Y_{\mathbb{C}_p}, \mathbb{Q}_p(1)) \to \mathbb{Q}_p$ *and* $\mathrm{Tr}_S : H^1_{\text{an}}(X_{\mathbb{C}_p}, \Omega^1_X) \to \mathbb{C}_p$ *respectively be the Poincaré and Serre traces. Then the Poincaré pairing*

$$H^1_{\text{proét}}(Y_{\mathbb{C}_p}, V_{\kappa,\text{ét}})(1) \times H^1_{\text{proét},c}(Y_{\mathbb{C}_p}, V_{-w_0(\kappa),\text{ét}}) \xrightarrow{\cup} H^2_{\text{proét},c}(Y_{\mathbb{C}_p}, \mathbb{Q}_p(1)) \xrightarrow{\mathrm{Tr}_P} \mathbb{Q}_p$$

and the Serre pairing

$$H^1_{\text{an}}(X_{\mathbb{C}_p}, \omega_E^{-\kappa}) \times H^0_{\text{an}}(X_{\mathbb{C}_p}, \omega_E^{\kappa+\alpha}(-D)) \xrightarrow{KS\circ\cup} H^1_{\text{an}}(X_{\mathbb{C}_p}, \Omega^1_X) \xrightarrow{\mathrm{Tr}_S} \mathbb{C}_p$$

(respectively, for $\omega_E^{w_0(\kappa)}(-D)$ *and* $\omega_E^{-w_0(\kappa)}$*) are compatible with the Eichler–Shimura decomposition.*

4.2 The overconvergent Eichler–Shimura maps

Let $n \geq 1$ be a fixed integer. In this subsection we will work with $Y = Y_0(p^n)$ and $X = X_0(p^n)$, the modular curves of level $K^p \,\mathrm{Iw}_n$. Let $\epsilon \geq \delta \geq n$ be rational numbers, (R, R^+) a uniform affinoid Tate $\mathbb{Q}_p$-algebra and $\chi = (\chi_1, \chi_2) : T \to R^{+,\times}$ a δ-analytic character. Let $w \in W = \{1, w_0\}$ be an element in the Weyl group of GL_2 and $X_w(\epsilon)$ the ϵ-neighbourhood of the w-ordinary locus (cf. Definition 3.2.1). Let ω_E^χ be the sheaf of overconvergent modular forms of weight χ over $X_w(\epsilon)$ (cf. Definition 3.2.9), and let $A^\delta_{\chi,\text{ét}}$ and $D^\delta_{\chi,\text{ét}}$ be the pro-Kummer-étale sheaves of δ-analytic principal series and distributions over X (cf. § 3.3). We can finally state the main theorem of this section, but first we need the following lemma.

Lemma 4.2.1. *Let* $\alpha = (1, -1) \in X^*(\mathbf{T})$. *The overconvergent Hodge–Tate morphisms of Definition 3.4.1 give rise to Galois and* U_p^t*-equivariant maps of cohomology groups (see Definition 3.2.16 for the good normalizations)*

$$\begin{aligned} H^1_{\text{prokét}}(X_{\mathbb{C}_p}, A^\delta_{\chi,\text{ét}} \widehat{\otimes} \widehat{\mathscr{O}}_X) &\xrightarrow{ES_A} H^0_{w_0}(X_{\mathbb{C}_p}, \omega_E^{\chi+\alpha})_\epsilon(\chi_2 - 1), \\ H^1_{1,c}(X_{\mathbb{C}_p}, \omega_E^{w_0(\chi)})_\epsilon(\chi_1) &\xrightarrow{ES_A^\vee} H^1_{\text{prokét}}(X_{\mathbb{C}_p}, A^\delta_{\chi,\text{ét}} \widehat{\otimes} \widehat{\mathscr{O}}_X). \end{aligned} \tag{28}$$

Dually, we have Galois and U_p*-equivariant maps of cohomology groups*

$$\begin{aligned} H^1_{w_0,c}(X_{\mathbb{C}_p}, \omega_E^{-\chi})_\epsilon(-\chi_2) &\xrightarrow{ES_D^\vee} H^1_{\text{prokét}}(X_{\mathbb{C}_p}, D^\delta_{\chi,\text{ét}} \widehat{\otimes} \widehat{\mathscr{O}}_X), \\ H^1_{\text{prokét}}(X_{\mathbb{C}_p}, D^\delta_{\chi,\text{ét}} \widehat{\otimes} \widehat{\mathscr{O}}_X) &\xrightarrow{ES_D} H^0_1(X_{\mathbb{C}_p}, \omega_E^{-w_0(\chi)+\alpha})_\epsilon(-\chi_1 - 1). \end{aligned}$$

An analogous statement holds by exchanging $A^\delta_{\chi,\text{ét}}$ *(respectively,* $D^\delta_{\chi,\text{ét}}$*) with* $j_{\text{prokét},!}A^\delta_{\chi,et}$ *(respectively,* $j_{\text{prokét},!}D^\delta_{\chi,\text{ét}}$*), and* ω_E^χ *with* $\omega_E^\chi(-D)$.

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Proof. Let $\lambda : X_{\mathbb{C}_p,\text{prokét}} \to X_{\mathbb{C}_p,\text{an}}$ be the natural projection. First, let us show that

$$\begin{aligned} R\lambda_*(\omega_E^{\chi}\widehat{\otimes}\widehat{\mathscr{O}}_X) &= \omega_E^{\chi}[0] \oplus \omega_E^{\chi+\alpha} \otimes \mathbb{C}_p(-1)[-1], \\ R\lambda_*(\omega_E^{\chi}\widehat{\otimes}\widehat{\mathscr{I}}) &= \omega_E^{\chi}(-D)[0] \oplus \omega_E^{\chi+\alpha}(-D) \otimes \mathbb{C}_p(-1)[-1]. \end{aligned} \tag{29}$$

By Remark 3.2.10, the sheaf $\omega_E^{\chi,+}$ is an orthonormalizable $\mathscr{O}^+_{X,\text{ét}}\widehat{\otimes}R^+$ sheaf locally for the étale topology of X. Let $\nu : X_{\mathbb{C}_p,\text{prokét}} \to X_{\mathbb{C}_p,\text{két}}$ be the natural projection of sites. Then, locally étale, we can write $\omega_E^{\chi,+}\widehat{\otimes}\widehat{\mathscr{O}}_X = \widehat{\bigoplus_i}\widehat{\mathscr{O}}^+_X\widehat{\otimes}R^+e_i$. We get that

$$\begin{aligned} R\nu_*(\omega_E^{\chi}\widehat{\otimes}\widehat{\mathscr{O}}_X) &= R\nu_*(\omega_E^{\chi,+}\widehat{\otimes}\widehat{\mathscr{O}}^+_X)\left[\frac{1}{p}\right] \\ &= R\varprojlim_s R\nu_*\Big(\bigoplus_i(\mathscr{O}^+_X/p^s \otimes R^+)e_i\Big)\left[\frac{1}{p}\right] \\ &= R\varprojlim_s \bigoplus_i(R\nu_*\widehat{\mathscr{O}}^+_X/p^s \otimes R^+)e_i\left[\frac{1}{p}\right] \\ &= \omega_E^{\chi}\widehat{\otimes}_{R\widehat{\otimes}\widehat{\mathscr{O}}_X} R\nu_*(\widehat{\mathscr{O}}_X\widehat{\otimes}R). \end{aligned}$$

Since R is a $\mathbb{Q}_p$-Banach space, it has a orthonormalizable basis over $\mathbb{Q}_p$, and the same reasoning as before shows that $\omega_E^{\chi}\widehat{\otimes}_{R\widehat{\otimes}\widehat{\mathscr{O}}_X} R\nu_*(\widehat{\mathscr{O}}_X\widehat{\otimes}R) = \omega_E^{\chi}\widehat{\otimes}_{\widehat{\mathscr{O}}_X} R\nu_*\widehat{\mathscr{O}}_X$. On the other hand, by Theorem 4.1.4 we know that $R\nu_*\widehat{\mathscr{O}}_X = \mathscr{O}_{X,\text{két}}[0] \oplus \omega_E^{\alpha}(-1)[-1]$. Lemma 5.5 of [Sch13] and Lemma 6.17 of [DLLZ23b] imply that the integral structure obtained by $R\nu_*(\widehat{\mathscr{O}}^+_X)$ defines the same topology as that given by $\mathscr{O}^+_{X,\text{két}}[0] \oplus \omega_E^{\alpha,+}(-1)[-1]$ (in fact, the lemmas cited show that both complexes differ just by bounded torsion when evaluated at affinoids). Therefore,

$$R\nu_*(\omega_E^{\chi}\widehat{\otimes}\widehat{\mathscr{O}}_X) = \omega_E^{\chi}[0] \oplus \omega_E^{\chi+\alpha}(-1)[0]$$

over the Kummer-étale site of $X_w(\epsilon)$. Finally, let $\mu : X_{\mathbb{C}_p,\text{két}} \to X_{\mathbb{C}_p,\text{an}}$ be the projection map. In order to descend to the analytic site we recall that ω_E^{χ} is a projective Banach sheaf over $X_w(\epsilon)$ (cf. [BP22, § 5.5.2]). Thus, it is a direct summand of an orthonormalizable Banach sheaf $\widehat{\bigoplus_i}\mathscr{O}_X$ over $X_w(\epsilon)$. But we know that the Kummer-étale cohomology of $\mathscr{O}^+_X$ in affinoids admitting a Kummer-étale map to a torus $\mathbb{T} = \mathrm{Spa}(\mathbb{Q}_p\langle T^{\pm 1}\rangle, \mathbb{Z}_p\langle T^{\pm 1}\rangle)$ or a disc $\mathbb{D} = \mathrm{Spa}(\mathbb{Q}_p\langle U\rangle, \mathbb{Z}_p\langle U\rangle)$ has bounded torsion (by computing the cohomology via the pullback of the perfectoid torus or disc, and using Lemma 5.5 of [Sch13] or Lemma 6.1.7 of [DLLZ23b] again). An argument similar to that before using derived limits shows that $R\mu_*(\widehat{\bigoplus_i}\mathscr{O}_{X,\text{két}}) = \widehat{\bigoplus_i}\mathscr{O}_{X,\text{an}}$, whence $R\mu_*\omega_E^{\chi} = \omega_E^{\chi}$. Finally, to prove the second equality of (29), it is enough to show the analogous property for $\omega_E^{\chi}\widehat{\otimes}\iota_*\widehat{\mathscr{O}}_D$, which follows from the previous argument applied to the log adic space defined by the cusps (notice that even if D is a disjoint union of points, the log structure is not trivial!).

Next, we construct the overconvergent Eichler–Shimura maps; we only explain the procedure for the sheaf $A^{\delta}_{\chi,\text{ét}}$ and the pro-Kummer-étale cohomology; the case of $D^{\delta}_{\chi,\text{ét}}$ or the cohomology with compact support follows the same steps. Consider the overconvergent Hodge–Tate maps of Definition 3.4.1,

$$\begin{aligned} \mathrm{HT}^{A,\vee}_{-w_0(\chi)} &: \omega_E^{w_0(\chi)}\widehat{\otimes}\widehat{\mathscr{O}}_X(\chi_1) \to A^{\delta}_{\chi,\text{ét}}\widehat{\otimes}\widehat{\mathscr{O}}_X \text{ over } X_1(\epsilon), \\ \mathrm{HT}^{A}_{\chi} &: A^{\delta}_{\chi,\text{ét}}\widehat{\otimes}\widehat{\mathscr{O}}_X \to \omega_E^{\chi}\widehat{\otimes}\widehat{\mathscr{O}}_X(\chi_2) \text{ over } X_{w_0}(\epsilon). \end{aligned}$$

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Taking the projection from the pro-Kummer-étale site to the analytic site, one gets maps

$$\omega_E^{w_0(\chi)} \widehat{\otimes} R\lambda_* \widehat{\mathscr{O}}_X(\chi_1) \to R\lambda_*(A^\delta_{\chi,\text{ét}} \widehat{\otimes} \widehat{\mathscr{O}}_X) \text{ over } X_1(\epsilon),$$
$$R\lambda_*(A^\delta_{\chi,\text{ét}} \widehat{\otimes} \widehat{\mathscr{O}}_X) \to \omega_E^{\chi} \widehat{\otimes} R\lambda_* \widehat{\mathscr{O}}_X(\chi_2) \text{ over } X_{w_0}(\epsilon).$$

Taking the overconvergent cohomologies of Definition 3.2.12 and using (29), we obtain maps

$$\begin{aligned} R\Gamma_{1,c}(X_{\mathbb{C}_p}, \omega_E^{w_0(\chi)})_\epsilon(\chi_1) &\to R\Gamma_{1,c}(X_{\mathbb{C}_p}, R\lambda_*(A^\delta_{\chi,\text{ét}} \widehat{\otimes} \widehat{\mathscr{O}}_X))_\epsilon, \\ R\Gamma_{w_0}(X_{\mathbb{C}_p}, R\lambda_*(A^\delta_{\chi,\text{ét}} \widehat{\otimes} \widehat{\mathscr{O}}_X))_\epsilon &\to R\Gamma_{w_0}(X_{\mathbb{C}_p}, \omega_E^{\chi+\alpha}))_\epsilon(\chi_2-1)[-1]. \end{aligned} \tag{30}$$

On the other hand, we have restriction and corestriction maps

$$R\Gamma_{1,c}(X_{\mathbb{C}_p}, R\lambda_*(A^\delta_{\chi,\text{ét}} \widehat{\otimes} \widehat{\mathscr{O}}_X))_\epsilon \xrightarrow{\text{Cor}} R\Gamma_{\text{prokét}}(X_{\mathbb{C}_p}, A^\delta_{\chi,\text{ét}} \widehat{\otimes} \widehat{\mathscr{O}}_X) \xrightarrow{\text{Res}} R\Gamma_{w_0}(X_{\mathbb{C}_p}, R\lambda_*(A^\delta_{\chi,\text{ét}} \widehat{\otimes} \widehat{\mathscr{O}}_X))_\epsilon. \tag{31}$$

Taking H^1-cohomology in (30), and composing with the morphisms of (31), one obtains the maps in (28). The Galois equivariance is clear as the Hodge–Tate maps are defined over $X_w(\epsilon) \subset X$. The compatibility with respect to the good normalization of the U_p-operators follows from Lemma 3.4.2 and the fact that $U_{p,\alpha}^{\text{good}} = U_{p,\alpha}$ for the correspondence associated to ω_E^α (see Definition 3.2.16). $\square$

THEOREM 4.2.2. *Let $\epsilon \geq \delta \geq n$, (R, R^+) be a uniform affinoid Tate $\mathbb{Q}_p$-algebra and $\chi : T = \mathbf{T}(\mathbb{Z}_p) \to R^{+,\times}$ be a δ-analytic character. The following assertions hold.*

(i) *The composition of the Eichler–Shimura maps $ES_A \circ ES_A^\vee$ is zero. Consider the following sequence:*

$$\begin{aligned} 0 \to H^1_{1,c}(X_{\mathbb{C}_p}, \omega_E^{w_0(\chi)})_\epsilon(\chi_1) &\xrightarrow{ES_A^\vee} H^1_{\text{prokét}}(X_{\mathbb{C}_p}, A^\delta_{\chi,\text{ét}} \widehat{\otimes} \widehat{\mathscr{O}}_X) \\ &\xrightarrow{ES_A} H^0_{w_0}(X_{\mathbb{C}_p}, \omega_E^{\chi+\alpha})_\epsilon(\chi_2 - 1) \to 0. \end{aligned} \tag{32}$$

(ii) *Assume that $\mathcal{V} = \operatorname{Spa}(R, R^+)$ is an affinoid subspace of the weight space $\mathcal{W}_T$ of T, and let $\kappa = (k_1, k_2) \in \mathcal{V}$ be a dominant weight of $\mathbf{T}$. Let $\alpha = (1, -1) \in X^*(\mathbf{T})$ and let $\chi = \chi_{\mathcal{V}}^{\text{un}}$ be the universal character of $\mathcal{V}$. The following diagram commutes.*

$$\begin{array}{ccccc} H^1_{1,c}(X_{\mathbb{C}_p}, \omega_E^{w_0(\chi)})_\epsilon(\chi_1) & \xrightarrow{ES_A^\vee} & H^1_{\text{prokét}}(X_{\mathbb{C}_p}, A^\delta_{\chi,\text{ét}} \widehat{\otimes} \widehat{\mathscr{O}}_X) & \xrightarrow{ES_A} & H^0_{w_0}(X_{\mathbb{C}_p}, \omega_E^{\chi+\alpha})_\epsilon(\chi_2 - 1) \\ \downarrow & & \downarrow & & \downarrow \\ H^1_{1,c}(X_{\mathbb{C}_p}, \omega_E^{w_0(\kappa)})_\epsilon(k_1) & \longrightarrow & H^1_{\text{prokét}}(X_{\mathbb{C}_p}, A^\delta_{\kappa,\text{ét}} \widehat{\otimes} \widehat{\mathscr{O}}_X) & \longrightarrow & H^0_{w_0}(X_{\mathbb{C}_p}, \omega_E^{\kappa+\alpha})_\epsilon(k_2 - 1) \\ \downarrow{\scriptstyle \text{Cor}} & & \uparrow & & \uparrow{\scriptstyle \text{Res}} \\ H^1_{\text{an}}(X_{\mathbb{C}_p}, \omega_E^{w_0(\kappa)})(k_1) & \xrightarrow{ES^\vee} & H^1_{\text{prokét}}(X_{\mathbb{C}_p}, V_{\kappa,\text{ét}}) \otimes \mathbb{C}_p & \xrightarrow{ES} & H^0_{\text{an}}(X_{\mathbb{C}_p}, \omega_E^{\kappa+\alpha})(k_2 - 1) \end{array}$$

(iii) *The maps of (ii) are Galois and U_p^t-equivariant with respect to the good normalizations of the Hecke operator (Definition 3.2.16). In particular, the diagram above restricts to the finite slope part with respect to the action of U_p^t.*

(iv) *Let $h < k_1 - k_2 + 1$. There exists an open affinoid $\mathcal{V}' \subset \mathcal{V}$ containing κ such that the $(\leq h)$-slope part of the restriction of (32) to $\mathcal{V}'$ is a short exact sequence of finite free $\mathbb{C}_p \widehat{\otimes}_{\mathbb{Q}_p} \mathscr{O}(\mathcal{V}')$-modules.*

(v) *Retain the hypothesis of (iv), and let* χ *be the universal character of* $\mathcal{V}'$. *Let* $\widetilde{\chi} = \chi_1 - \chi_2 + 1 : \mathbb{Z}_p^\times \to R^{+,\times}$ *and* $b = d/dt|_{t=1}\widetilde{\chi}(t)$. *Then we have a Galois equivariant split after inverting* b*:*

$$H^1_{\text{prok\'et}}(X_{\mathbb{C}_p}, \mathcal{A}^\delta_\chi \widehat{\otimes} \widehat{\mathscr{O}}_X)^{\leq h}_b = [H^1_{1,c}(X_{\mathbb{C}_p}, \omega_E^{w_0(\chi)})^{\leq h}_\epsilon(\chi_1)]_b \oplus [H^0_{w_0}(X_{\mathbb{C}_p}, \omega_E^{\chi+\alpha})^{\leq h}_\epsilon(\chi_2 - 1)]_b.$$

Analogous statements hold for the cohomology with compact support and for the sheaf $D^\delta_{\chi,\text{ét}}$.

Proof. Part (i) follows from the fact that the composition of the restriction and corestriction maps (31) is zero.

Parts (ii) and (iii) follow from Lemma 4.2.1, and the compatibility of the formation of $A^\delta_{\chi,\text{ét}}$ and ω_E^χ with the character χ. The commutation of the lower diagram is a direct consequence of the constructions and Corollary 2.4.5.

For part (iv) we follow the same arguments as [AIS15]. The finite-slope theory (cf. [Urb11, Buz07]) implies that there is an affinoid open subspace $\mathcal{V}' \subset \mathcal{V}$ containing κ such that the $(\leq h)$-part of the sequence (32) restricted to $\mathcal{V}'$ is a sequence of finite free $\mathbb{C}_p\widehat{\otimes}_{\mathbb{Q}_p}\mathscr{O}(\mathcal{V}')$-modules. Moreover, by classicality (Theorems 3.2.17 and 3.3.7) and the classical Eichler–Shimura decomposition (Theorem 4.1.1), we can take $\mathcal{V}'$ such that the $(\leq h)$-slope of the sequence (32) is short and exact.

Finally, we briefly sketch the argument for part (v). Let $\mathcal{V}'$ be as in (iv), let $R = \mathscr{O}(\mathcal{V}')$, and consider the short exact sequence of the $(\leq h)$-slope of (32). Taking basis of the finite free $\mathbb{C}_p\widehat{\otimes}R$-modules of (32) and tensoring with the Tate twist $R(1-\chi_2)$, we are left to prove that the localization by b of $H^1(G_{\mathbb{Q}_p}, \mathbb{C}_p\widehat{\otimes}R(\chi_1 - \chi_2 + 1))$ vanishes. By almost étale descent one has

$$H^1(G_{\mathbb{Q}_p}, \mathbb{C}_p\widehat{\otimes}R(\chi_1 - \chi_2 + 1)) = H^1(\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{cyc}}/\mathbb{Q}_p), \mathbb{Q}_p^{\mathrm{cyc}}\widehat{\otimes}_{\mathbb{Q}_p}R(\chi_1 - \chi_2 + 1)). \tag{33}$$

We identify $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{cyc}}/\mathbb{Q}_p)$ with $\mathbb{Z}_p^\times$ via χ_{cyc}. By Sen theory, to show that (33) is of b-torsion it is enough to prove that $H^1(\mathrm{Lie}\,\mathbb{Z}_p^\times, R(\chi_1 - \chi_2 + 1))_b = 0$, but this is clear as $H^1(\mathrm{Lie}\,\mathbb{Z}_p^\times, R(\chi_1 - \chi_2 + 1)) \cong R/bR$. □

4.2.1 *Previous works in the literature.* Here we briefly discuss some previous works and their connection with Theorem 4.2.2. Our main result is the complement of the work of Andreatta, Iovita and Stevens; they constructed the map ES_D from $H^1_{\text{proét}}(Y_{\mathbb{C}_p}, D^\delta_{\chi,\text{ét}})\widehat{\otimes}\mathbb{C}_p$ to the space of overconvergent modular forms of weight $-w_0(\chi)$ (cf. [AIS15, Theorem 6.1]). Our theorem constructs a new map from the overconvergent H^1-cohomology with supports of higher Coleman theory [BP22], to the space of modular symbols defined by the distributions. In addition, we have discussed the dual picture with the principal series, and with the proétale cohomology with compact support instead.

On the other hand, the first work on the subject which uses the perfectoid modular curve to construct the ES_D map goes back to Chojecki, Hansen and Johansson. Additionally, they constructed the map for Shimura curves, and they translated a theorem analogous to Theorem 4.2.2 in terms of the eigencurve (see [CHJ17, Theorem 5.14]).

The work of Sean Howe [How21] studies natural pairings between some local cohomologies attached to the flag variety $\mathscr{F}\ell = \mathbb{P}^1_{\mathbb{Q}_p}$ and overconvergent modular forms; these take values in the locally analytic vectors of the completed cohomology of the modular curve. The local cohomologies are the cohomology with supports $\varprojlim_\epsilon H^1_{\overline{U_w(\epsilon)}}(\mathbb{P}^1_{\mathbb{Q}_p}, \mathscr{L}(\chi))$ or the overconvergent cohomology $\varinjlim_\epsilon H^1(U_w(\epsilon), \mathscr{L}(\chi))$ (see [How21, Lemma 4.3.1 and Remark 1.2.12]; in the notation of [How21] one has $0 = [0:1]$, which is represented by $1 \in \mathrm{GL}_2$). It is expected that these pairings provide a more geometric interpretation of the $ES_A^\vee$ map of Theorem 4.2.2, namely, they are

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closely related to the description the of completed cohomology of Lue Pan that we briefly explain next.

In his recent work [Pan22], Lue Pan gives an exhausting description of the χ-isotypic part of the locally analytic vectors of the completed cohomology for the action of the Borel algebra Lie$\mathbf{B}$ (see [Pan22, Theorem 1.0.1]). His method uses a new tool in p-adic Hodge theory which is the geometric Sen operator; then, using the dictionary between representation theory over $\mathscr{F}\ell$ and proétale sheaves over the modular curve provided by π_{HT}, he shows that the completed cohomology can be decomposed in terms of overconvergent modular forms. The intersection between locally analytic vectors of completed cohomology and Theorem 4.2.2 lies in the fact that the cohomology group $H^1_{\text{proét}}(Y_{\mathbb{C}_p}, A^{\delta}_{\chi,\text{ét}})$ is a subspace of the locally analytic completed cohomology, consisting on those δ-analytic cohomology classes admitting an action of Iw_n, such that $\mathbf{B}(\mathbb{Z}_p) \cap \mathrm{Iw}_n$ acts via $-\chi$. Finally, the maps $ES_A^{\vee}$ and ES_A are instances of the spaces $M_{\mu,1}$ and $M_{\mu,w}$ appearing in [Pan22, Theorem 5.4.2].

4.2.2 *The pairings.* We end this section with the compatibility of the overconvergent Eichler–Shimura maps and Poincaré and Serre duality. Let $\epsilon \geq \delta \geq n$, and let (R, R^+) and χ be as in previous sections. By definition there is a natural pairing between the δ-principal series and distributions

$$A^{\delta}_{\chi} \times D^{\delta}_{\chi} \to R.$$

It is easy to see that it induces a Poincaré pairing

$$\langle -,- \rangle_P : H^1_{\text{proét},c}(Y_{\mathbb{C}_p}, D^{\delta}_{\chi,\text{ét}}(1)) \times H^1_{\text{proét}}(Y_{\mathbb{C}_p}, A^{\delta}_{\chi,\text{ét}}) \to H^2_{\text{proét},c}(Y_{\mathbb{C}_p}, R(1)) \xrightarrow{\mathrm{Tr}_P} R,$$

where the first arrow is a Yoneda pairing, and the last arrow is induced by the Poincaré trace $H^1_{\text{proét},c}(Y_{\mathbb{C}_p}, \mathbb{Z}_p(1)) \xrightarrow{\mathrm{Tr}_P} \mathbb{Z}_p$.

On the other hand, in [BP22] the authors have defined overconvergent Serre pairings in families

$$\langle -,- \rangle_S : H^1_{w,c}(X_{\mathbb{C}_p}, \omega_E^{-\chi}(-D))_{\epsilon} \times H^0_w(X_{\mathbb{C}_p}, \omega_E^{\chi+\alpha})_{\epsilon} \to \mathbb{C}_p \widehat{\otimes} R$$

compatible with the classical Serre pairings. The pairings are constructed by taking the Yoneda's product

$$\cup : H^1_{w,c}(X_{\mathbb{C}_p}, \omega_E^{-\chi}(-D))_{\epsilon} \times H^0_w(X_{\mathbb{C}_p}, \omega_E^{\chi+\alpha})_{\epsilon} \xrightarrow{\cup} H^1_{w,c}(X_{\mathbb{C}_p}, \omega_E^{\alpha}(-D)\widehat{\otimes} R)_{\epsilon} \xrightarrow{KS} H^1_{w,c}(X_{\mathbb{C}_p}\Omega^1_X \widehat{\otimes} R)_{\epsilon}$$

and composing with the Serre trace map of X,

$$H^1_{w,c}(X_{\mathbb{C}_p}, \Omega^1_X \widehat{\otimes} R)_{\epsilon} \xrightarrow{\mathrm{Cor}} H^1_{\mathrm{an}}(X_{\mathbb{C}_p}, \Omega^1_X \widehat{\otimes} R) \xrightarrow{\mathrm{Tr}_S} \mathbb{C}_p \widehat{\otimes} R.$$

THEOREM 4.2.3. *Retain the notation of Theorem 4.2.2. The following assertions hold.*

(i) *The Poincaré and Serre pairings of overconvergent cohomologies are compatible with the good normalizations of the U_p-operators (Definition 3.2.16). Moreover, they are compatible with the classical Eichler–Shimura maps of Theorem 4.1.1.*

(ii) *Let $\mathcal{W}_T$ be the weight space of $T = \mathbf{T}(\mathbb{Z}_p)$, let $\mathcal{V} \subset \mathcal{W}_T$ be an open affinoid, and let $\chi = \chi^{\mathrm{un}}_{\mathcal{V}}$ be the universal character of $\mathcal{V}$. Let $\kappa = (k_1, k_2) \in \mathcal{V}$ be a classical weight and fix $h < k_1 - k_2 + 1$. There exists an open affinoid $\mathcal{V}' \subset \mathcal{V}$ containing κ such that we have perfect pairings of finite free $\mathbb{C}_p \widehat{\otimes} \mathscr{O}(\mathcal{V}')$-modules*

$$\langle -,- \rangle_P : H^1_{\text{proét},c}(Y_{\mathbb{C}_p}, D^{\delta}_{\chi,\text{ét}}(1))^{\leq h} \times H^1_{\text{proét}}(Y_{\mathbb{C}_p}, A^{\delta}_{\chi,\text{ét}})^{\leq h} \to \mathscr{O}(\mathcal{V}')$$

https://doi.org/10.1112/S0010437X23007182 Published online by Cambridge University Press

and

$$\langle -,- \rangle_S : H^1_{w,c}(X_{\mathbb{C}_p}, \omega_E^{-\chi}(-D))^{\leq h}_{\epsilon} \times H^0_w(X_{\mathbb{C}_p}, \omega_E^{\chi+\alpha})^{\leq h}_{\epsilon} \to \mathbb{C}_p \widehat{\otimes} \mathscr{O}(\mathcal{V}'),$$
$$\langle -,- \rangle_S : H^1_{w,c}(X_{\mathbb{C}_p}, \omega_E^{w_0(\chi)})^{\leq h}_{\epsilon} \times H^0_w(X_{\mathbb{C}_p}, \omega_E^{-w_0(\chi)+\alpha}(-D))^{\leq h}_{\epsilon} \to \mathbb{C}_p \widehat{\otimes} \mathscr{O}(\mathcal{V}'),$$

compatible with the overconvergent Eichler–Shimura maps.

Proof. The Hecke operators are compatible with the pairings by their definition via finite flat correspondences (see Definitions 3.2.15 and 3.3.6).

In the following we forget the Galois action. Let $\lambda : X_{\mathbb{C}_p,\text{prokét}} \to X_{\mathbb{C}_p,\text{an}}$ be the projection of sites. We have the following commutative diagram of Yoneda's products.

$$\begin{array}{ccccc}
H^1_{1,c}(X_{\mathbb{C}_p}, R\lambda_*(\omega_E^{w_0(\chi)} \widehat{\otimes} \widehat{\mathscr{O}}_X))_{\epsilon} & \times & H^1_1(X_{\mathbb{C}_p}, R\lambda_*(\omega_E^{-w_0(\kappa)} \widehat{\otimes} \widehat{\mathscr{I}}))_{\epsilon} & \longrightarrow & H^2_{1,c}(X_{\mathbb{C}_p}, R\lambda_*(R\widehat{\otimes}\widehat{\mathscr{I}}))_{\epsilon} \\
\downarrow & & \uparrow & & \downarrow \text{Cor} \\
H^1_{\text{proként}}(X_{\mathbb{C}_p}, A^{\delta}_{\chi,\text{ét}} \widehat{\otimes} \widehat{\mathscr{O}}_X) & \times & H^1_{\text{proként}}(X_{\mathbb{C}_p}, D^{\delta}_{\chi,\text{ét}} \widehat{\otimes} \widehat{\mathscr{I}}) & \longrightarrow & H^2_{\text{proként}}(X_{\mathbb{C}_p}, R\widehat{\otimes}\widehat{\mathscr{I}}) \\
\downarrow & & \uparrow & & \uparrow \text{Cor} \\
H^1_{w_0}(X_{\mathbb{C}_p}, R\lambda_*(\omega_E^{\chi} \widehat{\otimes} \widehat{\mathscr{O}}_X))_{\epsilon} & \times & H^1_{w_0,c}(X_{\mathbb{C}_p}, R\lambda_*(\omega_E^{-\chi} \widehat{\otimes} \widehat{\mathscr{I}}))_{\epsilon} & \longrightarrow & H^2_{w_0,c}(X_{\mathbb{C}_p}, R\lambda_*(R\widehat{\otimes}\widehat{\mathscr{I}}))_{\epsilon}
\end{array}$$

On the other hand, we also have compatible pairings provided by the Faltings extension (cf. [Sch13, Corollary 6.14])

$$\begin{array}{ccccc}
H^1_{w,c}(X_{\mathbb{C}_p}, \omega_E^{-\chi}(-D))_{\epsilon} & \times & H^0_w(X_{\mathbb{C}_p}, \omega_E^{\chi+\alpha})_{\epsilon} & \xrightarrow{\text{Cor}\circ\cup} & H^1_{\text{an}}(X_{\mathbb{C}_p}, R\widehat{\otimes}\Omega^1_X) \\
\downarrow & & \uparrow & & \downarrow FE \\
H^1_{w_0,c}(X_{\mathbb{C}_p}, R\lambda_*(\omega_E^{-\chi} \widehat{\otimes} \widehat{\mathscr{I}}))_{\epsilon} & \times & H^1_{w_0}(X_{\mathbb{C}_p}, R\lambda_*(\omega_E^{\chi} \widehat{\otimes} \widehat{\mathscr{O}}_X))_{\epsilon} & \xrightarrow{\text{Cor}\circ\cup} & H^2_{\text{proként}}(X_{\mathbb{C}_p}, R\widehat{\otimes}\widehat{\mathscr{I}})
\end{array}$$

The compatibility of Poincaré and Serre traces [LLZ23, Theorem 4.4.1(4)] implies part (i). Part (ii) follows along the same lines of the proof of Theorem 4.2.2 using the fact that the pairings are perfect for the classical Eichler–Shimura decomposition. $\square$

Acknowledgements

None of this work would have been possible without the support of my advisor V. Pilloni, and I want to express my deep gratitude for the many hours he dedicated to explaining to me the theory of overconvergent modular forms. I want to thank Professors A. Iovita and F. Andreatta for very fruitful exchanges during the spring of 2020, and the opportunity to give a talk on this subject in the workshop on higher Coleman theory in December of the same year. I want to thank G. Boxer, D. Junger, J. R. Jacinto and J.-F. Wu for corrections to and comments on an earlier version of this paper. Finally, I want to thank the anonymous referees for the many suggestions and corrections that improved the presentation of this paper. This work was written while the author was a PhD student at the ENS de Lyon.

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