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THE SYMMETRIES OF GENUS ONE HANDLEBODIES

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The symmetries of manifolds are a focal point of study in low-dimensional topology and yet, outside of some totally asymmetrical 3- and 4-manifolds, there are very few cases in which a complete classification has been attained. In this work we provide such a classification for symmetries of the orientable and nonorientable 3-dimensional handlebodies of genus one. Our classification includes a description, up to isomorphism, of all of the finite groups which can arise as symmetries on these manifolds, as well as an enumeration of the different ways in which they can arise. To be specific, we will classify the equivalence, weak equivalence and strong equivalence classes of (effective) finite group actions on the genus one handlebodies. This work continues the study of finite group actions on handlebodies which was begun in [MMZ] and [KM]. In the latter paper, prime order cyclic actions on orientable handlebodies with genus $g > 1$ were explicitly classified, however the arguments do not directly extend to the case of the solid torus ($g = 1$). Indeed, the handlebodies of genus one are clearly exceptional in that they have circle actions, and, in particular, they have infinitely many nonisomorphic finite symmetries. This contrasts with orientable handlebodies with genus larger than one which admit only finitely many symmetries up to equivalence [KM].

The genus one handlebodies are the solid torus $V_1 = D^2 \times S^1$ (orientable) and the solid Klein bottle $\tilde{V}_1 = D^2 \widetilde{\times} S^1$ (nonorientable). To define our basic terms, suppose V is a genus one handlebody and let G be a finite group. Then a *G-action* on V is an injective homomorphism $\phi: G \longrightarrow \mathrm{Diff}(V)$. Two G-actions ϕ and ψ are *weakly equivalent* if their images are conjugate in $\mathrm{Diff}(V)$. They are *equivalent* if there is $h \in \mathrm{Diff}(V)$ so that $\psi(g) = h\phi(g)h^{-1}$ for all $g \in G$, and they are *strongly equivalent* if the diffeomorphism h may be chosen to be homotopic to the identity. In each case, if ϕ and ψ are orientation preserving G-actions then we further require h to be orientation preserving (this is merely a convenience for stating our results, see Section 2). We also say that the two G-actions have the same *quotient type* if their orbifold quotients V/ϕ and V/ψ are homeomorphic. Thus a quotient type is a homeomorphism class of orbifolds. The quotient type $\mathcal{Q}$ is said to be *G-admissible* if there is a G-action on V whose quotient orbifold is of type $\mathcal{Q}$. Observe that strongly equivalent actions are equivalent, equivalent actions are weakly equivalent, and weakly equivalent actions have the same quotient type. In the classification of the various equivalence classes of actions the pivotal underlying question is: to what extent are G-actions on V with the same quotient type weakly equivalent?

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Our approach to the classification is as follows: (1) determine all possible quotient types which can arise as quotients of actions on V; (2) for a fixed quotient type $\mathcal{Q}$, find all finite groups G for which $\mathcal{Q}$ is G-admissible; (3) for fixed G and fixed $\mathcal{Q}$, describe the various equivalence classes of G-actions. The equivariant loop theorem and the Smith conjecture (as generalized in [MY2]) are used in carrying out step (1), and the descriptions of "handlebody orbifolds" in terms of graphs of groups from [MMZ] (as expanded in Section 3 below) are crucial for step (2). The basic approach to step (3) is given in Section 1. The main result of that section uses orbifold covering space theory to reduce the classification problem into group theoretic problems involving the orbifold fundamental group of the quotient type and its group of "realizable" automorphisms. (An automorphism is realizable if it is induced by an orbifold mapping class.)

In Section 2 we give the classification of the orientation preserving actions on the solid torus V_1 using the results from Section 1. The quotient types (which are orientable) occur in two families $\{(A0,k)\}$ and $\{(B0,k)\}$ where k is a positive integer. An analysis of these families shows that a finite group G which acts on V_1 preserving orientation has one of the forms $\mathbf{Z_m} \times \mathbf{Z}_\ell$ or $\mathrm{Dih}(\mathbf{Z_m} \times \mathbf{Z}_\ell)$ (where it is assumed that m divides ℓ), and the corresponding quotient type is respectively either $(A0,k)$ or $(B0,k)$ where k is an integer which is divisible by m and divides ℓ. In both cases G contains an element of order k which has a circle of fixed points. In theorems 2.3 and 2.7 we will show that the number of weak equivalence classes of G-actions with respective quotient type either $(A0,k)$ or $(B0,k)$ equals $\phi((\frac{k}{m},\frac{\ell}{k}))$ (the Euler phi function applied to the greatest common divisor of $\frac{k}{m}$ with $\frac{\ell}{k}$). Thus, for example, if $\ell \neq 2$ then $\mathbf{Z}_\ell \times \mathbf{Z}_\ell$ has the form $\mathbf{Z_m} \times \mathbf{Z}_\ell$ and there is exactly one $\mathbf{Z}_\ell \times \mathbf{Z}_\ell$ action on V_1 up to weak equivalence (here $\ell = m = k$). In the case of orientation preserving cyclic group actions of order $\ell \neq 2$, the number of weak equivalence classes is obtained by summing $\phi((k,\frac{\ell}{k}))$ over all divisors k of ℓ. (The cyclic group of order two can be expressed in both forms $\mathbf{Z_m} \times \mathbf{Z}_\ell$ and $\mathrm{Dih}(\mathbf{Z_m} \times \mathbf{Z}_\ell)$, and there are two $\mathbf{Z_2}$-admissible quotient types $(A0,2)$ and $(B0,1)$ which must be considered. Similar problems arise for all groups G with exponent 2, so these are almost always exceptional cases in the various classification theorems which we obtain.) The result concerning cyclic group actions was previously proved by P. Kim [Ki] in two special cases: (1) where the $\mathbf{Z}_\ell$-actions are free (which is equivalent to $k = 1$); and (2) where the $\mathbf{Z}_\ell$-actions have a fixed point (which is equivalent to $k = \ell$)—in each of these cases there is exactly one weak equivalence class. In [Ki] it was also shown that there are three weak equivalence classes of orientation preserving $\mathbf{Z_4}$-actions on V_1, and this follows from our classification as well. For another illustration consider the $\mathbf{Z_{25}}$-actions on V_1. Note that all such actions must preserve orientation and that $\mathbf{Z_{25}}$ has the form $\mathbf{Z_m} \times \mathbf{Z}_\ell$ where $m = 1$ and $\ell = 25$. By the results described above, there are 6 weak equivalence classes of such actions: one is free ($k = 1$), one has fixed point ($k = 25$), and the remaining four have the quotient type $(A0,5)$ ($k = 5$). The latter four $\mathbf{Z_{25}}$-actions are respectively generated by the composition of a longitudinal rotation through $\frac{2\pi}{5}$ with a meridional rotation thru $\frac{2n\pi}{25}$ where $1 \leq n \leq 4$. As this example suggests, representatives for each of the weak equivalence and equivalence classes of G-actions can be described in terms of meridional and longitudinal components of rotation (see remark (4) following theorem 2.3).

In Section 3 we consider the quotient types of orientation reversing actions on orientable handlebodies of genus $g \geq 1$, and of actions on nonorientable handlebodies. We classify these quotient types in terms of graphs of groups as was previously done in [MMZ] for orientation preserving actions. In the special case $g = 1$ the quotient types occur in eleven families denoted $\{(A1,k)\},\ldots,\{(A3,k)\},\{(B1,k)\},\ldots,\{(B8,k)\}$. In Section 4 we analyze these nonorientable families to classify the finite group actions on $\tilde{V}_1$. The results show that the only finite groups G which can arise as symmetries on the solid Klein bottle are, up to isomorphism, the subgroups of $\mathbf{Z_2} \times \mathbf{D_s}$, $s \geq 1$. Furthermore, it is shown that two such G-actions are weakly equivalent if and only if they have the same quotient type (Theorem 4.2).

The finite groups G which act reversing orientation on V_1 involve the same eleven quotient types but are more complicated to describe up to isomorphism. Here there are twenty two forms for G. These forms are described in theorem 5.1 and it is also shown that G-actions with the same quotient type are weakly equivalent except that there are four families of quotient types each of which have exactly 2 weak equivalence classes. In this setting, as well as those of Section 2 and Section 4, a description of the equivalence and strong equivalence classes of G-actions can be readily obtained from the weak equivalence classes with some additional information involving Aut(G). This is carried out in Section 2 and in Section 4, but in Section 5, where the groups G are more complicated, we carry out the necessary computations only when $|G|$ has order congruent to 2 mod 4 (corollary 5.2) and when G is abelian (corollary 5.3).

For the special case of involutions ($G = \mathbf{Z_2}$) it is easy to observe that the notions of strong equivalence and weak equivalence coincide. (On genus one handlebodies our results even imply that involutions are strongly equivalent if they have the same quotient type.) We find a total of nine weak equivalence classes of $\mathbf{Z_2}$-actions on V_1, three are orientation preserving and six are not (these six are described in corollary 5.3). Of the nine classes of involutions, seven preserve the product fibering of $V_1 \approx D^2 \times S^1$, and these were enumerated in [KT] (see also [To]). The remaining two classes preserve the (2, 1) fibering on V_1, one has a Mobius band as fixed point set and the other has a disk and an isolated point as its fixed point set. There are five weak equivalence classes of involutions on the solid Klein bottle (Theorem 4.2), and they are also fiber preserving. More generally, our results readily imply that each finite group action on a genus one handlebody preserves a fibering. This is proved by observing that each of the (thirteen) possible quotient types that we find admits an orbifold Seifert fibering, and this lifts to an equivariant fibering on the handlebody. In addition this result follows as a corollary of the main theorem of [MS]. The fibered classification of the thirteen families of quotient types are listed as the "local types" of Seifert fiberings in [BS].

The other groups of exponent two which can act on the solid torus are $\mathbf{Z_2} \times \mathbf{Z_2}$, $\mathbf{Z_2} \times \mathbf{Z_2} \times \mathbf{Z_2}$, and $\mathbf{Z_2} \times \mathbf{Z_2} \times \mathbf{Z_2} \times \mathbf{Z_2}$. There are eighteen weak equivalence classes of $\mathbf{Z_2} \times \mathbf{Z_2}$-actions, three of which are orientation preserving. There are nine weak equivalence classes of $\mathbf{Z_2} \times \mathbf{Z_2} \times \mathbf{Z_2}$-actions, one of which is orientation preserving. And there is one weak equivalence class of $\mathbf{Z_2} \times \mathbf{Z_2} \times \mathbf{Z_2} \times \mathbf{Z_2}$-action; it reverses orientation.

From a different perspective, the results of this paper can also be interpreted as giving group theoretic information about the groups $\mathit{Diff}(V_1)$ and $\mathrm{Diff}(\tilde{V}_1)$. In this framework, our classification enumerates the conjugacy classes of finite subgroups, and of finite group imbeddings into these groups. There are infinitely many isomorphism types of these subgroups and imbeddings, but each isomorphism type contains only finitely many conjugacy classes.

Throughout this work we refer to the following specific groups: the cyclic group $\mathbf{Z_k}$ of order k; the group of units $\mathbf{Z}_\mathbf{k}^\star$ in the ring of integers mod k (note that both $\mathbf{Z_1}$ and $\mathbf{Z}_\mathbf{1}^\star$ are groups consisting of one element); the dihedral group $\mathbf{D_k}$ of order $2k$; the generalized dihedral group $\mathrm{Dih}(A)$ over an abelian group A (which is defined to be the semidirect product $A \circ \mathbf{Z_2}$ whose action is given by inverting each element of A); the symmetric and alternating groups $\mathbf{S_k}$ and $\mathbf{A_k}$ on k letters. In addition an important role is played by the Euler phi function which is defined by $\phi(k) = |\mathbf{Z}_\mathbf{k}^\star|$ (for positive integer k). This function is characterized by two properties:

(i) for p is prime and $k > 0$ then $\phi(p^k) = p^k(1 - \frac{1}{p})$

(ii) if m and n are relatively prime then $\phi(mn) = \phi(m)\phi(n)$.

1. **Algebraic characterizations of equivalence.** In this section we give algebraic formulations of the various concepts of equivalence of finite group actions in the context where the finite group G and the quotient type $\mathcal{Q}$ are fixed. The main theorem is directly based on results from [KM] and we will refer there for some of the details of the proof. This theorem holds for actions on many other spaces besides the solid torus and Klein bottle (such as handlebodies of arbitrary genus), except for the description of the strong equivalence classes which is given in part (c) of the theorem. The latter relies on the special fact that each equivalence class of a G-action on V_1 or $\tilde{V}_1$ is contained in a homotopy class. (If two G-actions on V_1 are equivalent then the corresponding abstract kernels $G \longrightarrow \mathrm{Out}(\pi_1(V_1))$ differ by conjugation in $\mathrm{Out}(\pi_1(V_1))$; as this is abelian, the two equivalent G-actions on V_1 must be homotopic.) In a more general setting part (c) should be rephrased so as to pertain only to the strong equivalence classes contained in the intersection of an equivalence class with a homotopy class; this is the viewpoint of Section 3 of [KM].

Let G be a fixed finite group. We consider all G-actions having a common quotient type $\mathcal{Q}$ which is represented by the orbifold $V(\mathcal{Q})$. The set of all weak equivalence classes of G-actions on V_1 with quotient type $\mathcal{Q}$ is denoted by $\mathcal{WE}^{\mathcal{Q}}(G)$. Similarly, $\mathcal{E}^{\mathcal{Q}}(G)$ denotes the set of equivalence classes of G-actions with quotient type $\mathcal{Q}$ and $\mathcal{SE}^{\mathcal{Q}}(G)$ denotes the set of strong equivalence classes of actions with quotient type $\mathcal{Q}$.

Let $\Pi(\mathcal{Q})$ be the orbifold fundamental group of $V(\mathcal{Q})$ and let $\Pi^+(\mathcal{Q})$ be its orientation subgroup. The *set of G-kernels in* $\Pi(\mathcal{Q})$ is

$$\mathcal{K}^{\mathcal{Q}}(G) = \{N \mid N \triangleleft \Pi(\mathcal{Q}), N < \Pi^+(\mathcal{Q}), N \cong \mathbf{Z}, \text{ and } \Pi(\mathcal{Q})/N \cong G\} .$$

For each $N \in \mathcal{K}^{\mathcal{Q}}(G)$, we choose once and for all an exact sequence

$$(i_N, \lambda_N) : 1 \longrightarrow \mathbf{Z} \xrightarrow{i_N} \Pi(\mathcal{Q}) \xrightarrow{\lambda_N} G \longrightarrow 1$$

with $i_N(\mathbf{Z}) = N$. There is a left action of the group $\mathrm{Aut}(\Pi(\mathcal{Q}))$ on the set $\mathcal{K}^{\mathcal{Q}}(G) \times \mathrm{Aut}(G) \times \mathrm{Aut}(\mathbf{Z})$ given by:

$$\alpha \cdot (N, \gamma, \beta) = \left(\alpha(N), \left(\lambda_{\alpha(N)}\alpha\lambda_N^{-1}\right)\gamma, (i_{\alpha(N)}^{-1}\alpha\, i_N)\beta\right)$$

for $\alpha \in \mathrm{Aut}(\Pi(\mathcal{Q}))$ and $(N, \gamma, \beta) \in \mathcal{K}^{\mathcal{Q}}(G) \times \mathrm{Aut}(G) \times \mathrm{Aut}(\mathbf{Z})$. This action restricts to give actions on $\mathcal{K}^{\mathcal{Q}}(G)$ and on $\mathcal{K}^{\mathcal{Q}}(G) \times \mathrm{Aut}(G)$. Let $\mathrm{Aut}_R(\Pi(\mathcal{Q}))$ be the subgroup of $\mathrm{Aut}(\Pi(\mathcal{Q}))$ consisting of automorphisms which are induced by (basepoint preserving) homeomorphisms of $V(\mathcal{Q})$.

THEOREM 1.1.

(a) *The set* $\mathcal{WE}^{\mathcal{Q}}(G)$ *of weak equivalence classes of G-actions on the solid torus with quotient type* $\mathcal{Q}$ *is in 1–1 correspondence with the orbit space of the action of* $\mathrm{Aut}_R(\Pi(\mathcal{Q}))$ *on* $\mathcal{K}^{\mathcal{Q}}(G)$.

(b) *The set* $\mathcal{E}^{\mathcal{Q}}(G)$ *of equivalence classes of G-actions on the solid torus with quotient type* $\mathcal{Q}$ *is in 1–1 correspondence with the orbit space of the action of* $\mathrm{Aut}_R(\Pi(\mathcal{Q}))$ *on* $\mathcal{K}^{\mathcal{Q}}(G) \times \mathrm{Aut}(G)$.

(c) *The set* $\mathcal{SE}^{\mathcal{Q}}(G)$ *of strong equivalence classes of G-actions on the solid torus with quotient type* $\mathcal{Q}$ *is in 1–1 correspondence with the orbit space of the action of* $\mathrm{Aut}_R(\Pi(\mathcal{Q}))$ *on* $\mathcal{K}^{\mathcal{Q}}(G) \times \mathrm{Aut}(G) \times \mathrm{Aut}(\mathbf{Z})$.

PROOF. Let $\Lambda^{\mathcal{Q}}(G)$ be the set of epimorphisms from $\Pi(\mathcal{Q})$ to G whose kernels are infinite cyclic and contained in $\Pi^+(\mathcal{Q})$. There is a left action of $\mathrm{Aut}_R(\Pi(\mathcal{Q}))$ on $\Lambda^{\mathcal{Q}}(G)$ given by $\alpha \cdot \lambda = \lambda\alpha^{-1}$. By theorem 1.3 of [KM], $\mathcal{E}^{\mathcal{Q}}(G)$ is in 1–1 correspondence with the orbit space of $\Lambda^{\mathcal{Q}}(G)$ under this action. The function from $\Lambda^{\mathcal{Q}}(G)$ to $\mathcal{K}^{\mathcal{Q}}(G) \times \mathrm{Aut}(G)$ defined by $\lambda \longmapsto \left(N, \lambda_N\lambda^{-1}\right)$ where $N = \ker\lambda$ is a bijection which is equivariant with respect to the $\mathrm{Aut}_R(\Pi(\mathcal{Q}))$ actions. Thus the orbit spaces are in 1–1 correspondence and (b) follows.

To prove (a) we must first describe the 1–1 correspondence given in theorem 1.3 of [KM]. Let ϕ be a G-action on V_1 with quotient type $\mathcal{Q}$. This determines (in nonunique fashion) a covering map $\nu: V_1 \longrightarrow V(\mathcal{Q})$ whose group of covering transformations is $\phi(G)$. Let $\rho: \Pi(\mathcal{Q}) \longrightarrow \mathrm{Diff}(V_1)$ be the covering transformation homomorphism for ν and define $\lambda_\phi \in \Lambda^{\mathcal{Q}}(G)$ by $\lambda_\phi = \phi^{-1}\rho$. The aforementioned bijection from $\mathcal{E}^{\mathcal{Q}}(G)$ to the orbit space of $\Lambda^{\mathcal{Q}}(G)$ is given by sending the equivalence class of ϕ to the orbit of λ_ϕ [KM]. (Note: The fact that this is surjective relies on the knowledge that a covering space of $V(\mathcal{Q})$ with infinite cyclic fundamental group must be a solid torus.) It follows that the bijection of part (b) takes the equivalence class of ϕ to the orbit of $(N, \lambda_N\lambda_\phi^{-1})$ where $N = \ker\lambda_\phi$.

Now suppose that ψ and ϕ are weakly equivalent G-actions with quotient type $\mathcal{Q}$. Then ψ is equivalent to $\phi\gamma$ for some $\gamma \in \mathrm{Aut}(G)$. But then λ_ψ and $\lambda_{\phi\gamma} = \gamma^{-1}\lambda_\phi$ are in the same orbit of $\Lambda^{Q}(G)$—that is, $\lambda_\psi = \gamma^{-1}\lambda_\phi\alpha^{-1}$ for some $\alpha \in \mathrm{Aut}_R(\Pi(Q))$. It follows that $\ker\lambda_\psi = \ker(\gamma^{-1}\lambda_\phi\alpha^{-1}) = \ker(\lambda_\phi\alpha^{-1}) = \alpha(\ker\lambda_\phi)$. Conversely, if $\ker\lambda_\psi = \alpha(\ker\lambda_\phi)$ for some $\alpha \in \mathrm{Aut}_R(\Pi(Q))$ then ψ and ϕ are weakly equivalent. Now consider the following diagram whose vertical arrows are natural projections:

$$\begin{array}{ccc} \mathcal{E}^{Q}(G) & \longrightarrow & \text{Orbit space of } \mathcal{K}^{Q}(G)\times\mathrm{Aut}(G) \\ \downarrow & & \downarrow \\ \mathcal{WE}^{Q}(G) & \longrightarrow & \text{Orbit space of } \mathcal{K}^{Q}(G) \end{array}$$

The diagonal map from $\mathcal{E}^{Q}(G)$ to the orbit space of $\mathcal{K}^{Q}(G)$ is given by taking the equivalence class of ϕ to the orbit of λ_ϕ. The above remarks thus imply that the diagram induces a function from $\mathcal{WE}^{Q}(G)$ to the orbit space of $\mathcal{K}^{Q}(G)$ and that this is a bijection. This proves (a).

For part (c) we consider the set $\mathrm{Ext}^{Q}(G)$ of exact sequences of the form

$$(i,\lambda)\colon 1 \longrightarrow \mathbf{Z} \xrightarrow{\ i\ } \Pi(Q) \xrightarrow{\ \lambda\ } G \longrightarrow 1$$

with $i(\mathbf{Z}) \subset \Pi^{+}(Q)$. There is a left $\mathrm{Aut}_R(\Pi(Q))$ action on this set defined by $\alpha\cdot(i,\lambda) = (\alpha i, \lambda\alpha^{-1})$. By lemma 3.2 of [KM] the set $\mathcal{SE}^{Q}(G)$ is in 1–1 correspondence with the orbit space of $\mathrm{Ext}^{Q}(G)$ under this action. The function from $\mathrm{Ext}^{Q}(G)$ to $\mathcal{K}^{Q}(G)\times\mathrm{Aut}(G)\times\mathrm{Aut}(\mathbf{Z})$ given by $(i,\lambda)\mapsto(N,\lambda_N\lambda^{-1}, i_N^{-1}i)$ where $N = \ker\lambda$ induces a bijection on the $\mathrm{Aut}_R(\Pi(Q))$ orbit spaces and (c) follows. ∎

REMARK. There are forgetful surjections from $\mathcal{SE}^{Q}(G)$ to $\mathcal{E}^{Q}(G)$ and from $\mathcal{E}^{Q}(G)$ to $\mathcal{WE}^{Q}(G)$. Under the correspondences of theorem 1.1, these are readily seen to induce the standard surjections on orbit spaces coming from the projections
$\mathcal{K}^{Q}(G)\times\mathrm{Aut}(G)\times\mathrm{Aut}(\mathbf{Z}) \longrightarrow \mathcal{K}^{Q}(G)\times\mathrm{Aut}(G) \longrightarrow \mathcal{K}^{Q}(G)$. ∎

For $N \in \mathcal{K}^{Q}(G)$, let $\mathrm{Aut}_R^N(\Pi(Q))$ be the stabilizer of N in $\mathrm{Aut}_R(\Pi(Q))$ and let $\hat\lambda_N\colon \mathrm{Aut}_R^N(\Pi(Q)) \longrightarrow \mathrm{Aut}(G)$ be the homomorphism defined by $\hat\lambda_N(\alpha) = \lambda_N\alpha\lambda_N^{-1}$. Furthermore, let $\mathrm{Aut}_R^N(G)$ be the image of $\hat\lambda_N$.

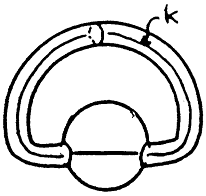
COROLLARY 1.2. *Let $\{N_i\}_{i=1}^{t}$ be a set of representatives for $\mathcal{K}^{Q}(G)/\mathrm{Aut}_R(\Pi(Q))$. Then*

$$|\mathcal{E}^{Q}(G)| = \sum_{i=1}^{t}[\mathrm{Aut}(G) : \mathrm{Aut}_R^{N_i}(G)] \text{ and}$$

$$|\mathcal{SE}^{Q}(G)| = \sum_{i=1}^{t} j_i[\mathrm{Aut}(G) : \mathrm{Aut}_R^{N_i}(G)]$$

where $j_i = 1$ if there is $\alpha \in \ker(\hat\lambda_{N_i})$ which inverts N_i and $j_i = 2$ otherwise.

PROOF. Consider the orbit of $\mathcal{K}^{Q}(G)$ represented by $N = N_i$. Let (N,γ) and (N,γ') be representatives for two $\mathrm{Aut}_R(\Pi(Q))$ orbits in $\mathcal{K}^{Q}(G)\times\mathrm{Aut}(G)$. These orbits will coincide if and only if there is $\alpha \in \mathrm{Aut}_R(\Pi(Q))$ such that $(\alpha(N), (\lambda_{\alpha(N)}\alpha\lambda_N^{-1})\gamma) = (N,\gamma')$. This is equivalent to the existence of $\alpha \in \mathrm{Aut}_R^N(\Pi(Q))$ such that $\gamma'\gamma^{-1} = \hat\lambda_N(\alpha)$; and so it is equivalent to γ and γ' being in the same $\mathrm{Aut}_R^N(G)$ coset of $\mathrm{Aut}(G)$. Summing over all i this yields the first part of the corollary. Now suppose that there is



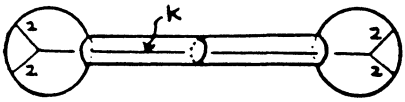


FIGURE 1

$\alpha \in \ker(\hat{\lambda}_N)$ which inverts N, then for each $(N,\gamma,\beta) \in \mathcal{K}^{\mathcal{Q}}(G) \times \mathrm{Aut}(G) \times \mathrm{Aut}(\mathbf{Z})$ we have $\alpha \cdot (N,\gamma,\beta) = (N,\gamma,-\beta)$. Otherwise, (N,γ,β) and $(N,\gamma,-\beta)$ are in different orbits. ∎

We now consider the G-actions on the solid Klein bottle $\tilde{V}_1$. Denote the sets of weak equivalence classes, equivalence classes, and strong equivalence classes of G-actions on $\tilde{V}_1$ with quotient type $\mathcal{Q}$ by $\widetilde{\mathcal{WE}}^{\mathcal{Q}}(G)$, $\widetilde{\mathcal{E}}^{\mathcal{Q}}(G)$, and $\widetilde{\mathcal{SE}}^{\mathcal{Q}}(G)$. The *set of G-kernels for* $\tilde{V}_1$ *in* $\Pi(\mathcal{Q})$ is

$$\widetilde{\mathcal{K}}^{\mathcal{Q}}(G) = \{N \mid N \lhd \Pi(\mathcal{Q}), N \not\subseteq \Pi^+(\mathcal{Q}), N \cong \mathbf{Z}, \text{ and } \Pi(\mathcal{Q})/N \cong G\} .$$

A direct analogue of the proof of theorem 1.1 yields the following.

THEOREM 1.3.
(a) The set $\widetilde{\mathcal{WE}}^{\mathcal{Q}}(G)$ *of weak equivalence classes of G-actions on the solid Klein bottle with quotient type* $\mathcal{Q}$ *is in 1–1 correspondence with the orbit space of the action of* $\mathrm{Aut}_R(\Pi(\mathcal{Q}))$ *on* $\widetilde{\mathcal{K}}^{\mathcal{Q}}(G)$.
(b) The set $\widetilde{\mathcal{E}}^{\mathcal{Q}}(G)$ *of equivalence classes of G-actions on the solid Klein bottle with quotient type* $\mathcal{Q}$ *is in 1–1 correspondence with the orbit space of the action of* $\mathrm{Aut}_R(\Pi(\mathcal{Q}))$ *on* $\widetilde{\mathcal{K}}^{\mathcal{Q}}(G) \times \mathrm{Aut}(G)$.
(c) The set $\widetilde{\mathcal{SE}}^{\mathcal{Q}}(G)$ *of strong equivalence classes of G-actions on the solid Klein bottle with quotient type* $\mathcal{Q}$ *is in 1–1 correspondence with the orbit space of the action of* $\mathrm{Aut}_R(\Pi(\mathcal{Q}))$ *on* $\widetilde{\mathcal{K}}^{\mathcal{Q}}(G) \times \mathrm{Aut}(G) \times \mathrm{Aut}(\mathbf{Z})$.

The analogue of corollary 1.2 also holds for actions on $\tilde{V}_1$.

2. **Orientation preserving actions on the solid torus.** For orientation preserving actions on V_1 the appropriate quotient types $\mathcal{Q}$ are classified into two types which we denote by $(A0,k)$ and $(B0,k)$, $k \geq 1$. These are represented by the handlebody orbifolds $V(A0,k)$ and $V(B0,k)$ shown in Figure 1.

That these are the only possible quotient types follows from Proposition 7.1 of [MMZ] and Proposition 1.1 of [KM]. (This is discussed in detail in Section 3 below.) We fix presentations for the corresponding orbifold fundamental groups $\Pi(\mathcal{Q})$:

$$\Pi(A0,k) = \langle a,b \mid a^k = [a,b] = 1\rangle \cong \mathbf{Z_k} \times \mathbf{Z}$$

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$$\Pi(B0,k) = \langle a,b,c \mid a^k = c^2 = [a,b] = 1,\ a^c = a^{-1},\ b^c = b^{-1} \rangle \cong \mathrm{Dih}(\mathbf{Z_k} \times \mathbf{Z})$$

The finite groups which act preserving orientation are quotients of these groups by infinite cyclic normal subgroups. Such a quotient is either a finite abelian group whose rank is at most 2 when $\mathcal{Q} = (A0,k)$, or a dihedral group over a finite abelian group whose rank is at most 2 when $\mathcal{Q} = (B0,k)$. It follows that any finite group G which acts preserving orientation on V_1 is isomorphic to either $\mathbf{Z_m} \times \mathbf{Z}_\ell$ or $\mathrm{Dih}(\mathbf{Z_m} \times \mathbf{Z}_\ell)$. Without loss of generality we always assume that m divides ℓ. We shall investigate the $\mathbf{Z_m} \times \mathbf{Z}_\ell$-actions with quotient type $(A0,k)$ first (culminating in Theorem 2.3) before considering the case of $\mathrm{Dih}(\mathbf{Z_m} \times \mathbf{Z}_\ell)$-actions with quotient type $(B0,k)$ (Theorem 2.7).

LEMMA 2.1. *The quotient type $\mathcal{Q} = (A0,k)$ is $\mathbf{Z_m} \times \mathbf{Z}_\ell$-admissible if and only if m divides k and k divides ℓ. The quotient type $\mathcal{Q} = (B0,k)$ is $\mathbf{Z_m} \times \mathbf{Z}_\ell$-admissible if and only if $\ell \leq 2$ (so that $\mathbf{Z_m} \times \mathbf{Z}_\ell = 1, \mathbf{Z_2}$, or $\mathbf{Z_2} \times \mathbf{Z_2}$ and $k = 1$ or 2).*

PROOF. When $m|k|\ell$ it is easy to construct a finite-injective epimorphism from $\Pi(A0,k)$ to $\mathbf{Z_m} \times \mathbf{Z}_\ell$ (send a to $(1, \frac{\ell}{k})$ and b to $(0,1)$). Conversely, suppose that $\Pi(A0,k)$ is $\mathbf{Z_m} \times \mathbf{Z}_\ell$ admissible and let $\lambda : \Pi(A0,k) \longrightarrow \mathbf{Z_m} \times \mathbf{Z}_\ell$ be a finite-injective epimorphism. Since $\lambda(a)$ is an element of order k, $k$must divide the exponent of $\mathbf{Z_m} \times \mathbf{Z}_\ell$ which equals ℓ. Now suppose that m does not divide k. Then for some prime p there is a power p^M which divides m but not k. Let $\bar{\lambda}$ be the composition of λ with the projection of $\mathbf{Z_m} \times \mathbf{Z}_\ell$ onto $(\mathbf{Z_m} \times \mathbf{Z}_\ell) \otimes \mathbf{Z_p} = \mathbf{Z_p} \times \mathbf{Z_p}$. Then $a \in \ker(\bar{\lambda})$ so that $\bar{\lambda}(\Pi(A0,k))$ is generated by $\bar{\lambda}(b)$. This contradicts the fact that $\bar{\lambda}$ is surjective, and the proof of the first assertion of the lemma follows. Now suppose that $\mathbf{Z_m} \times \mathbf{Z}_\ell$ is $\Pi(B0,k) \cong \mathrm{Dih}(\mathbf{Z_k} \times \mathbf{Z})$ admissible. Then $\mathbf{Z_m} \times \mathbf{Z}_\ell$ contains a $\mathbf{D_k}$-subgroup and so k is at most 2 as $\mathbf{D_k}$ must be abelian. If $k = 1$ then $\Pi(B0,k) \cong \mathbf{D}_\infty$ and the abelian quotients are $\mathbf{D_1} \cong \mathbf{Z_2}$ and $\mathbf{D_2} \cong \mathbf{Z_2} \times \mathbf{Z_2}$. If $k = 2$ then $\Pi(B0,k) \cong \mathbf{Z_2} \times \mathbf{D}_\infty$ and the only abelian quotient of rank ≤ 2 is $\mathbf{Z_2} \times \mathbf{Z_2}$. ∎

We now focus on a fixed k with $m|k|\ell$. Let $p_1, \ldots, p_s$ be the distinct prime divisors of ℓ and let

$$I = \{ i \mid p_i \text{ does not divide } \ell / k \}.$$

For any integer J dividing ℓ, let J_i be the power of p_i in the prime decomposition of J, so that $J = \prod_{i=1}^{s} p_i^{J_i}$. Furthermore let $J' = \prod_{\{i \in I\}} p_i^{J_i}$ and let J'' be given by $J = J'J''$ (in other words $J'' = \prod_{\{i \notin I\}} p_i^{J_i}$). With this notation we have $m = \prod_{i=1}^{s} p_i^{m_i}$, $k = \prod_{i=1}^{s} p_i^{k_i}$ and $\ell = \prod_{i=1}^{s} p_i^{\ell_i}$ where $m_i \leq k_i \leq \ell_i$ for each i. Also observe that $I = \{ i | k_i = \ell_i \}$ and that $k' = \ell'$.

PROPOSITION 2.2. *Suppose m divides k and k divides ℓ. Then the set $\mathcal{K}^{(A0,k)}(\mathbf{Z_m} \times \mathbf{Z}_\ell)$ of $\mathbf{Z_m} \times \mathbf{Z}_\ell$ kernels in $\Pi(A0,k)$ is in 1–1 correspondence with $\mathbf{Z}_{\mathbf{k'/m'}} \times \mathbf{Z}^\star_{\mathbf{k''/m''}}$. Under the correspondence the pair (x,y) is identified with the infinite cyclic group $N(x,y)$ generated by $a^{xm'k''+ym''k'} b^{m\ell/k}$.*

PROOF. Suppose that $\lambda : \Pi(A0,k) \longrightarrow \mathbf{Z_m} \times \mathbf{Z}_\ell$ is an epimorphism whose kernel is $N = \langle a^r b^t \rangle \in \mathcal{K}^{(A0,k)}(\mathbf{Z_m} \times \mathbf{Z}_\ell)$. The group $\Pi(A0,k)/N \cdot \langle a \rangle = \langle a,b | a^k = [a,b] = 1, a^r = b^{-t}, a = 1 \rangle$ is cyclic with order t and it is also isomorphic to $\mathbf{Z_m} \times \mathbf{Z}_\ell / \langle \lambda(a) \rangle$

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which has order $m\ell/k$ (since $\lambda(a)$ has order k) . It follows that $t = m\ell/k$. We will identify a^r by first determining its order q, which equals the order of $\lambda(a^r)$. Let J be the order of $\lambda(b)$ in $\mathbf{Z_m} \times \mathbf{Z}_\ell$. Since $\lambda(b)^t = \lambda(a^r)^{-1}$ it follows that $J = qt = qm\ell/k = qm\ell''/k''$. Since it is the exponent of $\mathbf{Z_m} \times \mathbf{Z}_\ell = \langle \lambda(a), \lambda(b) \rangle$, ℓ equals the least common multiple of the orders of $\lambda(a)$ and $\lambda(b)$. Thus

$$\ell = \operatorname{lcm}\{k, J\} = \operatorname{lcm}\{k'k'', J'J''\} = \ell' \operatorname{lcm}\{k'', J''\}$$

as $k' = \ell'$ and J' divides ℓ' . Therefore

$$\prod_{\{i \notin I\}} p_i^{\ell_i} = \ell'' = \operatorname{lcm}\{k'', J''\} = \prod_{\{i \notin I\}} p_i^{\max\{k_i, J_i\}}$$

and $\ell_i = \max\{k_i, J_i\}$ for $i \notin I$. But k_i is less than ℓ_i for $i \notin I$ so we conclude that $J_i = \ell_i$ and that $J'' = \ell''$. Thus $\ell'' = \frac{J}{J'} = \frac{qm\ell''}{k''J'}$ which determines the order of a^r as

$$q = \frac{k''J'}{m} = \frac{J'}{m'}\frac{k''}{m''}.$$

In particular, $q = q'q''$ where $q' = J'/m'$ which divides $\ell'/m' = k'/m'$, and $q'' = k''/m''$. If we write

$$\langle a \rangle = \langle a^{k''} \rangle \times \langle a^{k'} \rangle \cong \mathbf{Z}_{\mathbf{k}'} \times \mathbf{Z}_{\mathbf{k}''}$$

then it follows immediately that $a^r \in \langle a^{m'k''} \rangle \times \langle a^{m''k'} \rangle^\star$. This shows that $a^r = a^{xm'k''+ym''k'}$ for some $x \in \mathbf{Z}_{\mathbf{k}'/\mathbf{m}'}$ and $y \in \mathbf{Z}^\star_{\mathbf{k}''/\mathbf{m}''}$, and $N = N(x,y)$ (where the latter is as described in the statement of the proposition).

The above yields a function from $\mathcal{K}^{(A0,k)}(\mathbf{Z_m} \times \mathbf{Z}_\ell)$ to $\mathbf{Z}_{\mathbf{k}'/\mathbf{m}'} \times \mathbf{Z}^\star_{\mathbf{k}''/\mathbf{m}''}$ by taking N to (x,y) . To complete the proof we need to show that if $(x,y) \in \mathbf{Z}_{\mathbf{k}'/\mathbf{m}'} \times \mathbf{Z}^\star_{\mathbf{k}''/\mathbf{m}''}$ then $N(x,y) \in \mathcal{K}^{(A0,k)}(\mathbf{Z_m} \times \mathbf{Z}_\ell)$. Thus it must be shown that if $G = \Pi(A0,k)/N(x,y)$ then $G \cong \mathbf{Z_m} \times \mathbf{Z}_\ell$. Let λ be the projection from $\Pi(A0,k)$ to G . Since $\lambda(b)^{m\ell/k} = \lambda(a^{xm'k''+ym''k'}) \in \langle \lambda(a) \rangle$ it follows that $G/\langle \lambda(a) \rangle \cong \mathbf{Z}_{\mathbf{m}\ell/\mathbf{k}}$ and hence $|G| = k(m\ell/k) = m\ell$. Furthermore $\lambda(b)$ has order $m\ell/k$ times the order of $(x,y) \in \mathbf{Z}_{\mathbf{k}'/\mathbf{m}'} \times \mathbf{Z}^\star_{\mathbf{k}''/\mathbf{m}''}$; if the order of x is L' then the order of $\lambda(b)$ equals

$$\frac{m\ell}{k} L' \frac{k''}{m''} = \frac{m}{m''} L' \frac{\ell k''}{k} = \frac{m'm''}{m''} L' \frac{\ell'\ell''k''}{k'k''} = m'L'\ell''.$$

Hence $G = \langle \lambda(a), \lambda(b) \rangle$ has exponent equal to

$$\operatorname{lcm}\{k, m'L'\ell''\} = \operatorname{lcm}\{\ell'k'', m'L'\ell''\} = \ell$$

since $m'L'$ divides ℓ' , and k'' divides ℓ'' . As G is an abelian group with order $m\ell$, rank at most two, and exponent ℓ , it follows that $G \cong \mathbf{Z_m} \times \mathbf{Z}_\ell$. ∎

We now describe $\operatorname{Aut}(\Pi(A0,k))$ and its action on $\mathcal{K}^{(A0,k)}(\mathbf{Z_m} \times \mathbf{Z}_\ell)$. Each $\alpha \in \operatorname{Aut}(\Pi(A0,k))$ is determined by $u = u_\alpha \in \mathbf{Z}^\star_{\mathbf{k}}, v = v_\alpha \in \mathbf{Z}_{\mathbf{k}'}, w = w_\alpha \in \mathbf{Z}_{\mathbf{k}''}$, and $\epsilon = \epsilon_\alpha \in \{-1, 1\}$ according to the following:

$$\alpha(a) = a^u$$
$$\alpha(b) = a^{vk''}a^{wk'}b^\epsilon \quad .$$

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For $N(x,y) \in \mathcal{K}^{(A0,k)}(\mathbf{Z_m} \times \mathbf{Z}_\ell)$ (where $x \in \mathbf{Z}_{\mathbf{k'}/\mathbf{m'}}$ and $y \in \mathbf{Z}^\star_{\mathbf{k''}/\mathbf{m''}}$) we have

$$\begin{aligned}\alpha \cdot N(x,y) &= \alpha(\langle a^{xm'k''} a^{ym''k'} b^{m\ell/k}\rangle)\\ &= \langle a^{(ux+vm''\ell/k)m'k''} a^{(uy+wm'\ell/k)m''k'} b^{\epsilon m\ell/k}\rangle \\ &= N(\epsilon(ux+vm''\ell/k), \epsilon(uy+wm'\ell/k)) \quad .\end{aligned}$$

Also, by Proposition 2.1 of [KM], α can be realized by an orientation preserving homeomorphism if and only if $u_\alpha = \epsilon_\alpha (\text{mod } k)$.

THEOREM 2.3. *Suppose m divides k and k divides ℓ. The number of weak equivalence, equivalence and strong equivalence classes of $\mathbf{Z_m} \times \mathbf{Z}_\ell$-actions on the solid torus with quotient type $(A0,k)$ are respectively given as follows.*

(a) $|\mathcal{WE}^{(A0,k)}(\mathbf{Z_m} \times \mathbf{Z}_\ell)| = \phi\left(\left(\frac{k}{m}, \frac{\ell}{k}\right)\right)$.

(b) $|\mathcal{E}^{(A0,k)}(\mathbf{Z_m} \times \mathbf{Z}_\ell)| = \frac{1}{\delta_0} m\phi(m)\phi(\ell)\prod_{\{i|m_i<k_i<\ell_i\}}\left(1-\frac{1}{p_i}\right)\prod_{\{i|m_i=\ell_i\}}\left(1+\frac{1}{p_i}\right)$

where $\delta_0 = \begin{cases} 1 & \text{if } k \le 2 \text{ and } \frac{m\ell}{k} \le 2 \\ 2 & \text{otherwise} \end{cases}$.

(c)

$$|\mathcal{SE}^{(A0,k)}(\mathbf{Z_m} \times \mathbf{Z}_\ell)| = m\phi(m)\phi(\ell)\prod_{\{i|m_i<k_i<\ell_i\}}\left(1-\frac{1}{p_i}\right)\prod_{\{i|m_i=\ell_i\}}\left(1+\frac{1}{p_i}\right).$$

(Here ϕ denotes the Euler phi function, (a,b) denotes the greatest common divisor of integers a and b, and $\prod p_i^{m_i}$, $\prod p_i^{k_i}$, and $\prod p_i^{\ell_i}$ are the respective decompositions of m, k and ℓ into prime powers.)

PROOF. (a) By Theorem 1.1, we need to count the $\text{Aut}_R(\Pi(A0,k))$ orbits in the set of kernels $\mathcal{K}^{(A0,k)}(\mathbf{Z_m} \times \mathbf{Z}_\ell)$. To this end we will first calculate the orders of the stabilizers under this action.

Let $N = N(x,y) \in \mathcal{K}^{(A0,k)}(\mathbf{Z_m} \times \mathbf{Z}_\ell)$ and let $\alpha \in \text{Aut}_R(\Pi(A0,k))$. Using the description of the action given above, it is apparent that $\alpha \cdot N = N$ if and only if the following two equations hold:

$$\left\{\begin{array}{ll} x = \epsilon_\alpha(u_\alpha x + v_\alpha m''\ell/k) & (\text{mod } k'/m') \\ y = \epsilon_\alpha(u_\alpha y + w_\alpha m'\ell/k) & (\text{mod } k''/m'') \end{array}\right\}.$$

Using $\epsilon_\alpha u_\alpha = 1$ these directly reduce to

$$\left\{\begin{array}{ll} \epsilon_\alpha v_\alpha m''\ell/k = 0 & (\text{mod } k'/m') \\ \epsilon_\alpha w_\alpha m'\ell/k = 0 & (\text{mod } k''/m'') \end{array}\right\}$$

or, since $\left(\frac{\ell}{k}, \frac{k'}{m'}\right) = \left(\frac{\ell''}{k''}, \frac{k'}{m'}\right) = 1$ and $(m'', \frac{k'}{m'}) = (m', \frac{k''}{m''}) = 1$,

$$\left\{\begin{array}{rl} v_\alpha = 0 & (\text{mod } k'/m') \\ (\ell''/k'')w_\alpha = 0 & (\text{mod } k''/m'') \end{array}\right\}.$$

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Thus α is in the stabilizer of N if and only if $v_\alpha \in \mathbf{Z}_{\mathbf{m}'} \leq \mathbf{Z}_{\mathbf{k}'}$ and $w_\alpha \in \mathbf{Z}_{\mathbf{J}''} \leq \mathbf{Z}_{\mathbf{k}''}$ where $J'' = (\frac{m''\ell''}{k''}, k'')$. The number of such α's is $2m'J''$.

Since the order of the stabilizer of N is $2m'J''$ which depends only on m , k and ℓ (and not on N) , each orbit in $\mathcal{K}^{(A0,k)}(\mathbf{Z_m} \times \mathbf{Z}_\ell)$ contains the same number of elements $(|\operatorname{Aut}_R(\Pi(A0,k))| / 2m'J'')$. Therefore using Proposition 2.2 and noting that $|\operatorname{Aut}_R(\Pi(A0,k))| = 2k$ we have

$$\begin{aligned}
&|\mathcal{K}^{(A0,k)}(\mathbf{Z_m} \times \mathbf{Z}_\ell) / \operatorname{Aut}_R(\Pi(A0,k))| = |\mathcal{K}^{(A0,k)}(\mathbf{Z_m} \times \mathbf{Z}_\ell)| / |\text{orbit}| \\
&= \frac{k'}{m'}\phi\left(\frac{k''}{m''}\right)\left(2m'J''/|\operatorname{Aut}_R(\Pi(A0,k))|\right) = \frac{J''}{k''}\phi\left(\frac{k''}{m''}\right) \\
&= \frac{J''}{k''}\left(\frac{k''}{m''}\prod_{\{i\notin I | k_i > m_i\}}\left(1-\frac{1}{p_i}\right)\right) = \left(\frac{k''}{m''}, \frac{\ell''}{k''}\right)\prod_{\{i\notin I | k_i > m_i\}}\left(1-\frac{1}{p_i}\right) \\
&= \phi\left(\left(\frac{k''}{m''}, \frac{\ell''}{k''}\right)\right) = \phi\left(\left(\frac{k}{m}, \frac{\ell}{k}\right)\right)
\end{aligned}$$

This completes the proof of part (a) .

(b) Now we must count the $\operatorname{Aut}_R(\Pi(A0,k))$ orbits in $\mathcal{K}^{(A0,k)}(\mathbf{Z_m} \times \mathbf{Z}_\ell) \times \operatorname{Aut}(\mathbf{Z_m} \times \mathbf{Z}_\ell)$. We first consider the stabilizer of an arbitrary $(N,\gamma) \in \mathcal{K}^{(A0,k)}(\mathbf{Z_m} \times \mathbf{Z}_\ell) \times \operatorname{Aut}(\mathbf{Z_m} \times \mathbf{Z}_\ell)$. If α is a realizable automorphism in this stabilizer then $\alpha \cdot N = N$ (so that α satisfies the conditions discussed in the proof of (a)) and $\lambda_N \alpha \lambda_N^{-1}\gamma = \gamma$, or , equivalently, $\hat{\lambda}_N(\alpha) = 1$. ($\hat{\lambda}_N$ was defined before Corollary 1.2.) Therefore

$$\begin{aligned}
&\hat{\lambda}_N(\alpha)(\lambda_N(a)) = \lambda_N(\alpha(a)) = \lambda_N(a)^{u_\alpha}, \quad \text{and} \\
&\hat{\lambda}_N(\alpha)(\lambda_N(b)) = \lambda_N(\alpha(b)) = \lambda_N\left(a^{v_\alpha k''}a^{w_\alpha k'}\right)\lambda_N(b)^{\epsilon_\alpha}.
\end{aligned}$$

It follows that $\hat{\lambda}_N(\alpha) = 1$ if and only if $u_\alpha = 1$ and $\lambda_N\left(a^{v_\alpha k''}a^{w_\alpha k'}\right)\lambda_N(b)^{\epsilon_\alpha} = \lambda_N(b)$. If $\epsilon_\alpha = 1$ then $a^{v_\alpha k''}a^{w_\alpha k'} = 1$ and $\alpha = 1$. As a consequence, the order of the stabilizer of (N,γ) is 1 or 2, and it equals 2 if and only if there is an $\alpha \in \operatorname{Aut}_R(\Pi(A0,k))$ with $\alpha \cdot (N,\gamma) = (N,\gamma)$ and $\epsilon_\alpha = -1$. If α satisfies these conditions then $\lambda_N(b)^2 = \lambda_N\left(a^{v_\alpha k''}a^{w_\alpha k'}\right)$. This implies that $\langle \lambda_N(a)\rangle \cong \mathbf{Z_k}$ is a subgroup of index ≤ 2 in $\mathbf{Z_m} \times \mathbf{Z}_\ell$ so that $\frac{m\ell}{k} \leq 2$. Furthermore, $k \leq 2$ since $u_\alpha = 1$, $\epsilon_\alpha = -1$, and $u_\alpha = \epsilon_\alpha (\operatorname{mod} k)$. On the other hand, if $\frac{m\ell}{k} \leq 2$ and $k \leq 2$ it is readily seen that there is an α with $\alpha \cdot (N,\gamma) = (N,\gamma)$ and $\epsilon_\alpha = -1$. This shows that the stabilizer of (N,γ) has order $\frac{2}{\delta_0}$ and each orbit contains $|\operatorname{Aut}_R(\Pi(A0,k))| / \delta_0 = k\delta_0$ elements.

In order to count the number of orbits in $\mathcal{K}^{(A0,k)}(\mathbf{Z_m} \times \mathbf{Z}_\ell) \times \operatorname{Aut}(\mathbf{Z_m} \times \mathbf{Z}_\ell)$ we need to know the order of $\operatorname{Aut}(\mathbf{Z_m} \times \mathbf{Z}_\ell)$. It is evident that $\operatorname{Aut}(\mathbf{Z_m} \times \mathbf{Z}_\ell) \cong \prod_{i=1}^{s} \operatorname{Aut}\left(\mathbf{Z}_{\mathbf{p_i^{m_i}}} \times \mathbf{Z}_{\mathbf{p_i^{\ell_i}}}\right)$. If $m_i = \ell_i$ then

$$|\operatorname{Aut}\left(\mathbf{Z}_{\mathbf{p_i^{m_i}}} \times \mathbf{Z}_{\mathbf{p_i^{\ell_i}}}\right)| = p_i^{2\ell_i}\phi\left(p_i^{\ell_i}\right)^2\left(1+\frac{1}{p_i}\right);$$

this is determined by the standard procedure of counting the oriented bases for $\mathbf{Z}_{\mathbf{p_i^{m_i}}} \times \mathbf{Z}_{\mathbf{p_i^{\ell_i}}}$ (the number of elements of order $p_i^{\ell_i}$ is $p_i^{\ell_i}\phi\left(p_i^{\ell_i}\right)\left(1+\frac{1}{p_i}\right)$). Similarly by counting the number of generating sets in $\mathbf{Z}_{p_i^{m_i}} \times \mathbf{Z}_{p_i^{\ell_i}}$ which contain one element of order $p_i^{\ell_i}$ and one element of order $p_i^{m_i}$, we see that, if $m_i < \ell_i$ then

$$| \operatorname{Aut}\left(\mathbf{Z}_{p_i^{m_i}} \times \mathbf{Z}_{p_i^{\ell_i}}\right) | = p_i^{2m_i}\phi\left(p_i^{m_i}\right)\phi\left(p_i^{\ell_i}\right)$$

(the number of elements of order $p_i^{\ell_i}$ is $\phi\left(p_i^{\ell_i}\right)p_i^{m_i}$). Putting this together we have

$$\begin{aligned} | \operatorname{Aut}(\mathbf{Z_m} \times \mathbf{Z}_\ell)| &= \prod_{\{i|m_i<\ell_i\}} p_i^{2m_i}\phi\left(p_i^{m_i}\right)\phi\left(p_i^{\ell_i}\right) \cdot \prod_{\{i|m_i=\ell_i\}} p_i^{2\ell_i}\phi\left(p_i^{\ell_i}\right)^2\left(1+\frac{1}{p_i}\right) \\ &= \prod_{i=1}^{s} p_i^{2m_i}\phi\left(p_i^{m_i}\right)\phi\left(p_i^{\ell_i}\right) \cdot \prod_{\{i|m_i=\ell_i\}}\left(1+\frac{1}{p_i}\right) \\ &= m^2\phi(m)\phi(\ell)\prod_{\{i|m_i=\ell_i\}}\left(1+\frac{1}{p_i}\right) \quad . \end{aligned}$$

Now the number of orbits in $\mathcal{K}^{(A0,k)}(\mathbf{Z_m} \times \mathbf{Z}_\ell) \times \operatorname{Aut}(\mathbf{Z_m} \times \mathbf{Z}_\ell)$ is

$$\begin{aligned} &| \mathcal{K}^{(A0,k)}(\mathbf{Z_m} \times \mathbf{Z}_\ell)| \cdot | \operatorname{Aut}(\mathbf{Z_m} \times \mathbf{Z}_\ell)| / (k\delta_0) \\ &\quad= \frac{k'}{m'}\phi\left(\frac{k''}{m''}\right) m^2\phi(m)\phi(\ell)\prod_{\{i|m_i=\ell_i\}}\left(1+\frac{1}{p_i}\right) / (k\delta_0) \\ &\quad= \frac{1}{\delta_0} m\phi(m)\phi(\ell)\frac{m''}{k''}\phi\left(\frac{k''}{m''}\right)\prod_{\{i|m_i=\ell_i\}}\left(1+\frac{1}{p_i}\right) \\ &\quad= \frac{1}{\delta_0} m\phi(m)\phi(\ell)\prod_{\{i\notin I|k_i>m_i\}}\left(1-\frac{1}{p_i}\right)\prod_{\{i|m_i=\ell_i\}}\left(1+\frac{1}{p_i}\right). \end{aligned}$$

(c) To prove (c) we must count the $\operatorname{Aut}_R(\Pi(A0,k))$ orbits in

$$\mathcal{K}^{(A0,k)}(\mathbf{Z_m} \times \mathbf{Z}_\ell) \times \operatorname{Aut}(\mathbf{Z_m} \times \mathbf{Z}_\ell) \times \operatorname{Aut}(\mathbf{Z}).$$

If $\alpha \in \operatorname{Aut}_R(\Pi(A0,k))$ stabilizes the element (N,γ,β) of this set then $\beta\, i_N^{-1}\alpha\, i_N = \beta$ which implies that $i_N^{-1}\alpha\, i_N = 1$. Therefore

$$i_N^{-1}\left(a^{xm'k''}a^{ym''k'}b^{m\ell/k}\right) = i_N^{-1}\alpha\left(a^{xm'k''}a^{ym''k'}b^{m\ell/k}\right) = i_N^{-1}\left(a^{*}b^{\epsilon_\alpha m\ell/k}\right)$$

and $\epsilon_\alpha = 1$. Referring back to the proof of (b) we see that $\alpha = 1$; which implies that all stabilizers are trivial. It follows that the number of orbits is

$$| \mathcal{K}^{(A0,k)}(\mathbf{Z_m} \times \mathbf{Z}_\ell)| \cdot | \operatorname{Aut}(\mathbf{Z_m} \times \mathbf{Z}_\ell)| \cdot | \operatorname{Aut}(\mathbf{Z})| / | \operatorname{Aut}_R(\Pi(A0,k))|$$

$$= | \mathcal{K}^{(A0,k)}(\mathbf{Z_m} \times \mathbf{Z}_\ell)| \cdot | \operatorname{Aut}(\mathbf{Z_m} \times \mathbf{Z}_\ell)| / k.$$

The result now follows using the final computation from the proof of (b) . ∎

REMARKS. (1) From the proof of the theorem it follows that each weak equivalence class in $\mathcal{WE}^{(A0,k)}(\mathbf{Z_m} \times \mathbf{Z}_\ell)$ contains the same number of equivalence classes. This number is

$$\frac{|\operatorname{Aut}(\mathbf{Z_m} \times \mathbf{Z}_\ell)|}{\delta_0 m' J''}.$$

(2) From part (a) of the theorem we deduce the following: If ℓ / m is squarefree, or, more generally, if the only square integers n^2 dividing ℓ / m are 1 or 4, then two orientation preserving $\mathbf{Z_m} \times \mathbf{Z}_\ell$ actions on V_1 are weakly equivalent if and only if they have the same quotient type. Conversely, if ℓ / m is divisible by a square integer other than 1 or 4 then there are orientation preserving $\mathbf{Z_m} \times \mathbf{Z}_\ell$ actions on V_1 which have the same quotient type but are not weakly equivalent.
(3) As consequences of Theorem 2.3 we have :

$$|\mathcal{WE}^{(A0,k)}(\mathbf{Z_m} \times \mathbf{Z}_\ell)| = \prod_{i=1}^{s} \phi\left(\left(p_i^{k_i - m_i}, p_i^{\ell_i - k_i}\right)\right) = \prod_{i=1}^{s} |\mathcal{WE}^{(A0,p_i^{k_i})}(\mathbf{Z}_{\mathbf{p_i^{m_i}}} \times \mathbf{Z}_{\mathbf{p_i^{\ell_i}}})|$$

and

$$|\mathcal{E}^{(A0,k)}(\mathbf{Z_m} \times \mathbf{Z}_\ell)| = \frac{1}{2}\prod_{i=1}^{s} 2|\mathcal{E}^{(A0,p_i^{k_i})}(\mathbf{Z}_{\mathbf{p_i^{m_i}}} \times \mathbf{Z}_{\mathbf{p_i^{\ell_i}}})|.$$

(4) If m, ℓ and k are powers of a common prime p then a set of representatives for the weak equivalence classes of $\mathbf{Z_m} \times \mathbf{Z}_\ell$ actions on V_1 with quotient type $(A0, k)$ can be described using the constructions in the proof of theorem 2.3 as follows:

Let $j = \min\{\frac{k}{m}, \frac{\ell}{k}\}$, and observe that $\mathbf{Z_j}$ has $\phi\left(\left(\frac{k}{m}, \frac{\ell}{k}\right)\right)$ units. For $y \in \mathbf{Z_j}^\star$ choose $z \in \mathbf{Z}_{\frac{\mathbf{k}}{\mathbf{m}}}{}^\star$ so that $yz = -1 (\operatorname{mod} \frac{k}{m})$. Let ϕ_y be the $\mathbf{Z_m} \times \mathbf{Z}_\ell$ action defined by taking $\phi_y(1, 0)$ to be the meridional rotation of V_1 through angle $\frac{2\pi}{k}(1 + yz)$ followed by the longitudinal rotation through angle $-\frac{2\pi z}{m}$, and taking $\phi_y(0, 1)$ to be the meridional rotation through $-\frac{2\pi y}{\ell}$ followed by longitudinal rotation through angle $\frac{2\pi k}{m\ell}$. The weak equivalence classes in $\mathcal{WE}^{(A0,k)}(\mathbf{Z_m} \times \mathbf{Z}_\ell)$ are in 1–1 correspondence with the actions $\{\phi_y\}$.
In light of remark (3) this result can be used to describe representative actions for all of the weak equivalence classes of $\mathbf{Z_m} \times \mathbf{Z}_\ell$ actions on V_1 with quotient type $(A0, k)$ even when m, ℓ and k are not powers of a common prime p.
(5) In studying orientation preserving actions one may also define equivalence, weak equivalence, and strong equivalence in terms of actions being conjugate by *homeomorphisms which may reverse orientation*. Denote the corresponding sets of equivalence classes by $\mathcal{E}_\pm$, $\mathcal{WE}_\pm$ and $\mathcal{SE}_\pm$. The proof of Theorem 2.3 is easily modified to obtain the following classification (the main change is that $\operatorname{Aut}_R(\Pi(A0, k))$ is slightly larger now—it contains the automorphism $a \longmapsto a^{-1}, b \longmapsto b$). If j is a positive integer we define $\delta(j)$ to equal 1 if $j \leq 2$ and to equal 2 otherwise.

COROLLARY 2.4.
(a) $|\mathcal{WE}_\pm^{(A0,k)}(\mathbf{Z_m} \times \mathbf{Z}_\ell)| = \frac{D}{(D,2)}$ *where* $D = \phi\left(\left(\frac{k}{m}, \frac{\ell}{k}\right)\right)$.
(b) $|\mathcal{E}_\pm^{(A0,k)}(\mathbf{Z_m} \times \mathbf{Z}_\ell)| = \frac{\delta_0}{\delta(k)\delta(\frac{m\ell}{k})}|\mathcal{E}^{(A0,k)}(\mathbf{Z_m} \times \mathbf{Z}_\ell)|$

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(c) $|\mathcal{SE}_{\pm}^{(A0,k)}(\mathbf{Z_m}\times\mathbf{Z}_\ell)| = \frac{1}{\delta(k)}|\mathcal{SE}^{(A0,k)}(\mathbf{Z_m}\times\mathbf{Z}_\ell)|$. ∎

We now consider the actions of Dih ($\mathbf{Z_m}\times\mathbf{Z}_\ell$) on V_1 (where m divides ℓ as previously). The associated quotient types are $(B0,k), k \geq 1$. Observe that $\Pi(B0,k)$ contains $\Pi(A0,k)$ as a subgroup of index two; it is the subgroup generated by a and b (using the presentations given at the beginning of this section). This subgroup is characteristic since it is the centralizer of any nontrivial element whose order is not two. Every element of $\Pi(B0,k)-\Pi(A0,k)$ has order two and conjugates each element of $\Pi(A0,k)$ to its inverse. Thus any infinite cyclic subgroup of $\Pi(B0,k)$ is contained in $\Pi(A0,k)$, and it follows that $\mathcal{K}^{(B0,k)}(\mathrm{Dih}\,(\mathbf{Z_m}\times\mathbf{Z}_\ell))$ equals $\mathcal{K}^{(A0,k)}(\mathbf{Z_m}\times\mathbf{Z}_\ell)$.

LEMMA 2.5. *The quotient type* $(B0,k)$ *is* Dih ($\mathbf{Z_m}\times\mathbf{Z}_\ell$) *admissible if and only if mdivides k, and kdivides* ℓ.

PROOF. We will show that $(B0,k)$ is Dih ($\mathbf{Z_m}\times\mathbf{Z}_\ell$) admissible if and only if $(A0,k)$is $\mathbf{Z_m}\times\mathbf{Z}_\ell$-admissible . The result then follows from Lemma 2.1 . Suppose there is a finite-injective epimorphism $\lambda\colon\Pi(B0,k)\longrightarrow \mathrm{Dih}\,(\mathbf{Z_m}\times\mathbf{Z}_\ell)$. Under λ each element of $\Pi(B0,k)-\Pi(A0,k)$ gets mapped to an element of order two in Dih ($\mathbf{Z_m}\times\mathbf{Z}_\ell$)which inverts every element of $\lambda(\Pi(A0,k))$. It readily follows that $\lambda(\Pi(A0,k))\cong\mathbf{Z_m}\times\mathbf{Z}_\ell$. Thus the restriction of λ to $\Pi(A0,k)$ is a finite-injective epimorphism to $\mathbf{Z_m}\times\mathbf{Z}_\ell$. Conversely, any finite-injective epimorphism $\bar{\lambda}$ from $\Pi(A0,k)$ to $\mathbf{Z_m}\times\mathbf{Z}_\ell$ extends to a finite-injective epimorphism λ from $\Pi(B0,k)$ to Dih ($\mathbf{Z_m}\times\mathbf{Z}_\ell$) by defining $\lambda(c)$ to be any element of Dih ($\mathbf{Z_m}\times\mathbf{Z}_\ell$) $-$ ($\mathbf{Z_m}\times\mathbf{Z}_\ell$). ∎

LEMMA 2.6. *An automorphism* α *of* $\Pi(B0,k)$ *is in* $\mathrm{Aut}_R(\Pi(B0,k))$ *if and only if its restriction* $\mathrm{res}(\alpha)$ *to* $\Pi(A0,k)$ *is in* $\mathrm{Aut}_R(\Pi(A0,k))$.

PROOF. Suppose that $\alpha\in\mathrm{Aut}_R(\Pi(B0,k))$. Then $\alpha=h_\#$ for some homeomorphism h of $V(B0,k)$. There is a covering map $V(A0,k)\longrightarrow V(B0,k)$ corresponding to the characteristic $\Pi(A0,k)$ subgroup of $\Pi(B0,k)$. Lift h to a homeomorphism $\tilde{h}$ on $V(A0,k)$. Then $\tilde{h}_\# = \mathrm{res}(\alpha)$ and thus $\mathrm{res}(\alpha)\in\mathrm{Aut}_R(\Pi(A0,k))$.

To prove the converse we will first show that the automorphisms ρ,τ and $\mu(x), x\in\Pi(B0,k)$, are in $\mathrm{Aut}_R(\Pi(B0,k))$ where $\mu(x)$ is conjugation by x and ρ and τ are given by

$$\left\{\begin{array}{l}\rho(a)=a^{-1}\\ \rho(b)=b^{-1}\\ \rho(c)=b^{-1}c\end{array}\right\}\left\{\begin{array}{l}\tau(a)=a\\ \tau(b)=ab\\ \tau(c)=ac\end{array}\right\}.$$

That $\mu(x)$ is in $\mathrm{Aut}_R(\Pi(B0,k))$ follows by sliding the basepoint of $V(B0,k)$ around a loop in the nonsingular set of $V(B0,k)$ corresponding to x. The automorphism ρ is realized by a 180° rotation interchanging the two 0-handles of $V(B0,k)$ (this orbifold is depicted in Figure 1) and τ is obtained by spinning the right-hand 0-handle by 180° around the $\mathbf{Z_k}$ arc of the singular set.

Now suppose that $\text{res}(\alpha) \in \text{Aut}_R(\Pi(A0,k))$. By the remark before Theorem 2.3, $\text{Aut}_R(\Pi(A0,k))$ is generated by the restrictions of the automorphisms ρ and τ. Therefore there is $\alpha' \in \text{Aut}_R(\Pi(B0,k))$ such that $\text{res}(\alpha') = \text{res}(\alpha)$, and $\alpha'\alpha^{-1}$ restricts to the identity on $\Pi(A0,k)$. If $\alpha'\alpha^{-1}(c) = a^q b^r c$ then a straightforward computation shows that $\alpha'\alpha^{-1} = [\rho,\tau]^{-q}(\mu(c)\rho)^r \in \text{Aut}_R(\Pi(B0,k))$. Since $\alpha' \in \text{Aut}_R(\Pi(B0,k))$ this completes the proof of the Lemma. ∎

THEOREM 2.7. *Suppose m divides k and k divides ℓ. The number of weak equivalence, equivalence and strong equivalence classes of* $\text{Dih}(\mathbf{Z_m} \times \mathbf{Z}_\ell)$*-actions on the solid torus with quotient type $(B0,k)$ can be determined from the following.*

(a) $|\mathcal{WE}^{(B0,k)}(\text{Dih}(\mathbf{Z_m} \times \mathbf{Z}_\ell))| = |\mathcal{WE}^{(A0,k)}(\mathbf{Z_m} \times \mathbf{Z}_\ell)|$.

(b) $|\mathcal{E}^{(B0,k)}(\text{Dih}(\mathbf{Z_m} \times \mathbf{Z}_\ell))| = |\mathcal{E}^{(A0,k)}(\mathbf{Z_m} \times \mathbf{Z}_\ell)|$ *if* $\ell \neq 2$.

(c) $|\mathcal{SE}^{(B0,k)}(\text{Dih}(\mathbf{Z_m} \times \mathbf{Z}_\ell))| = |\mathcal{SE}^{(A0,k)}(\mathbf{Z_m} \times \mathbf{Z}_\ell)|$ *if* $\ell \neq 2$.

(d) *For* $\ell = 2$ *we have*

$$|\mathcal{E}^{(B0,1)}(\mathbf{Z_2} \times \mathbf{Z_2})| = |\mathcal{SE}^{(B0,1)}(\mathbf{Z_2} \times \mathbf{Z_2})| = 3$$
$$|\mathcal{E}^{(B0,2)}(\mathbf{Z_2} \times \mathbf{Z_2})| = |\mathcal{SE}^{(B0,2)}(\mathbf{Z_2} \times \mathbf{Z_2})| = 3$$
$$|\mathcal{E}^{(B0,2)}(\mathbf{Z_2} \times \mathbf{Z_2} \times \mathbf{Z_2})| = |\mathcal{SE}^{(B0,2)}(\mathbf{Z_2} \times \mathbf{Z_2} \times \mathbf{Z_2})| = 21.$$

PROOF. Let

$$\mathcal{K} = \mathcal{K}^{(A0,k)}(\mathbf{Z_m} \times \mathbf{Z}_\ell) = \mathcal{K}^{(B0,k)}(\text{Dih}(\mathbf{Z_m} \times \mathbf{Z}_\ell)).$$

The groups $\text{Aut}_R(\Pi(A0,k))$ and $\text{Aut}_R(\Pi(B0,k))$ act on $\mathcal{K}$ with orbit spaces in 1–1 correspondence with $\mathcal{WE}^{(A0,k)}(\mathbf{Z_m} \times \mathbf{Z}_\ell)$ and $\mathcal{WE}^{(B0,k)}(\text{Dih}(\mathbf{Z_m} \times \mathbf{Z}_\ell))$ respectively . Let N_1 and N_2 be elements of $\mathcal{K}$. If $\alpha \cdot N_2 = N_1$ for some $\alpha \in \text{Aut}_R(\Pi(B0,k))$ then $\text{res}(\alpha)\cdot N_2 = N_1$ and $\text{res}(\alpha) \in \text{Aut}_R(\Pi(A0,k))$. On the other hand, if $\bar{\alpha} \in \text{Aut}_R(\Pi(A0,k))$ and $\bar{\alpha} \cdot N_2 = N_1$ then there is an automorphism α of $\Pi(B0,k)$ with $\text{res}(\alpha) = \bar{\alpha}$ (define α to equal $\bar{\alpha}$ on $\Pi(A0,k)$ and to map c to c). By Lemma 2.6, $\alpha \in \text{Aut}_R(\Pi(B0,k))$. Since $\alpha \cdot N_2 = N_1$ we have shown that N_1 and N_2 are in the same $\text{Aut}_R(\Pi(A0,k))$-orbit if and only if they are in the same $\text{Aut}_R(\Pi(B0,k))$-orbit. This proves (a).

Consider $\mathbf{Z_m} \times \mathbf{Z}_\ell$ as the $(\mathbf{Z_m} \times \mathbf{Z}_\ell)\circ\{1\}$ subgroup of $\text{Dih}(\mathbf{Z_m} \times \mathbf{Z}_\ell) = (\mathbf{Z_m} \times \mathbf{Z}_\ell)\circ \mathbf{Z_2}$. If $\ell \neq 2$ then $\mathbf{Z_m} \times \mathbf{Z}_\ell$ is the centralizer of any element of order ℓ. It follows that, when $\ell \neq 2$, $\mathbf{Z_m} \times \mathbf{Z}_\ell$ is a characteristic subgroup of $\text{Dih}(\mathbf{Z_m} \times \mathbf{Z}_\ell)$ so there is a restriction homomorphism res: $\text{Aut}(\text{Dih}(\mathbf{Z_m} \times \mathbf{Z}_\ell)) \longrightarrow \text{Aut}(\mathbf{Z_m} \times \mathbf{Z}_\ell)$. For each $N \in \mathcal{K}$ we choose finite-injective epimorphisms $\bar{\lambda}_N \colon \Pi(A0,k) \longrightarrow \mathbf{Z_m} \times \mathbf{Z}_\ell$ and $\lambda_N \colon \Pi(B0,k) \longrightarrow \text{Dih}(\mathbf{Z_m} \times \mathbf{Z}_\ell)$ so that $\lambda_N\ |_{\Pi(A0,k)} = \bar{\lambda}_N$. For each N , we also choose an isomorphism $i_N \colon \mathbf{Z} \longrightarrow N \leq \Pi(A0,k) \leq \Pi(B0,k)$.

To prove (b) recall that $\mathcal{E}^{(B0,k)}(\text{Dih}(\mathbf{Z_m} \times \mathbf{Z}_\ell))$ (respectively $\mathcal{E}^{(A0,k)}(\mathbf{Z_m} \times \mathbf{Z}_\ell)$) is in 1–1 correspondence with the orbit space of $\mathcal{K} \times \text{Aut}(\text{Dih}(\mathbf{Z_m} \times \mathbf{Z}_\ell))$ (respectively $\mathcal{K} \times \text{Aut}(\mathbf{Z_m} \times \mathbf{Z}_\ell)$) under the action of $\text{Aut}_R(\Pi(B0,k))$ (respectively $\text{Aut}_R(\Pi(A0,k))$). Define a surjection

$$r \colon \mathcal{K} \times \text{Aut}(\text{Dih}(\mathbf{Z_m} \times \mathbf{Z}_\ell)) \longrightarrow \mathcal{K} \times \text{Aut}(\mathbf{Z_m} \times \mathbf{Z}_\ell)$$

by $r(N,\gamma) = (N, \mathrm{res}(\gamma))$. Then, for $\alpha \in \mathrm{Aut}_R(\Pi(B0,k))$,

$$\begin{aligned} r(\alpha\cdot(N,\gamma)) &= r\left(\alpha(N), \lambda_{\alpha(N)}\alpha\lambda_N^{-1}\gamma\right) \\ &= \left(\mathrm{res}(\alpha)(N), \bar{\lambda}_{\alpha(N)}\,\mathrm{res}(\alpha)\bar{\lambda}_N^{-1}\,\mathrm{res}(\gamma)\right) = \mathrm{res}(\alpha)\cdot r(N,\gamma)\end{aligned}$$

and so r induces a surjection on the orbit spaces. If $r(N_2,\gamma_2)$ is in the same orbit as $r(N_1,\gamma_1)$ then there is $\bar{\alpha} \in \mathrm{Aut}_R(\Pi(A0,k))$ such that $r(N_2,\gamma_2) = \bar{\alpha}\cdot r(N_1,\gamma_1)$. This implies that $(N_2, \mathrm{res}(\gamma_2)) = \left(\bar{\alpha}(N_1), \bar{\lambda}_{\bar{\alpha}(N)}\bar{\alpha}\bar{\lambda}_N^{-1}\,\mathrm{res}(\gamma_1)\right)$. Now define $\alpha \in \mathrm{Aut}(\Pi(B0,K))$ to agree with $\bar{\alpha}$ on $\Pi(A0,k)$ and to satisfy $\alpha(c) \in \lambda_{\alpha(N_1)}^{-1}\gamma_2\gamma_1^{-1}\lambda_{N_1}(c)$. By Lemma 2.6, $\alpha \in \mathrm{Aut}_R(\Pi(B0,k))$. We have

$$\alpha\cdot(N_1,\gamma_1) = \left(\bar{\alpha}(N_1), \lambda_{\alpha(N_1)}\alpha\lambda_{N_1}^{-1}\gamma_1\right) = \left(N_2, \lambda_{\alpha(N_1)}\alpha\lambda_{N_1}^{-1}\gamma_1\right).$$

Now

$$\mathrm{res}\left(\lambda_{\alpha(N_1)}\alpha\lambda_{N_1}^{-1}\gamma_1\right) = \bar{\lambda}_{\alpha(N_1)}\bar{\alpha}\bar{\lambda}_{N_1}^{-1}\,\mathrm{res}(\gamma_1) = \mathrm{res}(\gamma_2)$$

and

$$\left(\lambda_{\alpha(N_1)}\alpha\lambda_{N_1}^{-1}\gamma_1\right)\left[\gamma_1^{-1}\lambda_{N_1}(c)\right] = \lambda_{\alpha(N_1)}\alpha(c) = \gamma_2\left[\gamma_1^{-1}\lambda_{N_1}(c)\right].$$

Since $\mathrm{Dih}(\mathbf{Z_m}\times\mathbf{Z}_\ell)$ is generated by $\mathbf{Z_m}\times\mathbf{Z}_\ell$ together with $\gamma_1^{-1}\lambda_{N_1}(c)$ it follows that $\lambda_{\alpha(N_1)}\alpha\lambda_{N_1}^{-1}\gamma_1 = \gamma_2$ and consequently $\alpha\cdot(N_1,\gamma_1) = (N_2,\gamma_2)$. This shows that r induces a bijection on orbit spaces, completing the proof of (b).

To prove (c) we recall that $\mathcal{SE}^{(B0,k)}(\mathrm{Dih}(\mathbf{Z_m}\times\mathbf{Z}_\ell))$ (respectively $\mathcal{SE}^{(A0,k)}(\mathbf{Z_m}\times\mathbf{Z}_\ell)$) is in 1–1 correspondence with the orbit space of $\mathcal{K}\times\mathrm{Aut}(\mathrm{Dih}(\mathbf{Z_m}\times\mathbf{Z}_\ell))\times\mathrm{Aut}(\mathbf{Z})$ (respectively $\mathcal{K}\times\mathrm{Aut}(\mathbf{Z_m}\times\mathbf{Z}_\ell)\times\mathrm{Aut}(\mathbf{Z})$) under the action of $\mathrm{Aut}_R(\Pi(B0,k))$ (respectively $\mathrm{Aut}_R(\Pi(A0,k))$). Let $\hat{r}$ be the surjection from $\mathcal{K}\times\mathrm{Aut}(\mathrm{Dih}(\mathbf{Z_m}\times\mathbf{Z}_\ell))\times\mathrm{Aut}(\mathbf{Z})$ to $\mathcal{K}\times\mathrm{Aut}(\mathbf{Z_m}\times\mathbf{Z}_\ell)\times\mathrm{Aut}(\mathbf{Z})$ given by $\hat{r} = r\times id$. If $\alpha\in\mathrm{Aut}_R(\Pi(B0,k))$ then it is easily checked that $\hat{r}(\alpha\cdot(N,\gamma,\beta)) = \mathrm{res}(\alpha)\cdot(\hat{r}(N,\gamma,\beta))$. Thus $\hat{r}$ induces a surjection on orbit spaces. Suppose that $\bar{\alpha}\cdot\hat{r}(N_1,\gamma_1,\beta_1) = \hat{r}(N_2,\gamma_2,\beta_2)$ for some $\bar{\alpha}\in\mathrm{Aut}_R(\Pi(A0,k))$. Then $\bar{\alpha}\cdot r(N_1,\gamma_1) = r(N_2,\gamma_2)$ and $\bar{\alpha}\cdot\beta_1 = \beta_2$. It was shown in the previous paragraph that there is $\alpha\in\mathrm{Aut}_R(\Pi(B0,k))$ with $\mathrm{res}(\alpha) = \bar{\alpha}$ and $\alpha\cdot(N_1,\gamma_1) = (N_2,\gamma_2)$. It follows that

$$\begin{aligned}\alpha\cdot(N_1,\gamma_1,\beta_1) &= (N_2,\gamma_2, i_{\alpha(N_1)}^{-1}\alpha i_{N_1}\beta_1) = (N_2,\gamma_2, i_{\alpha(N_1)}^{-1}\bar{\alpha} i_{N_1}\beta_1) \\ &= (N_2,\gamma_2,\bar{\alpha}\cdot\beta_1) = (N_2,\gamma_2,\beta_2).\end{aligned}$$

This shows that $\hat{r}$ induces a bijection on the orbit spaces and completes the proof of (c).

We now consider the cases where $\ell = 2$. There are three possibilities $(m,k,\ell) = (1,1,2), (1,2,2)$, or $(2,2,2)$. In each case, using (a) and Theorem 2.3(a), we see that there is one weak equivalence class of action. When $m = 1$, $\mathrm{Dih}(\mathbf{Z_m}\times\mathbf{Z}_\ell)\cong\mathbf{Z_2}\times\mathbf{Z_2}$ and, after choosing an oriented basis $\mathcal{B}$, we may identify $\mathrm{Aut}(\mathrm{Dih}(\mathbf{Z_m}\times\mathbf{Z}_\ell))$ with $SL(2,\mathbf{Z_2})$ which has order 6 . Similarly when $m = 2$, $\mathrm{Dih}(\mathbf{Z_m}\times\mathbf{Z}_\ell)\cong\mathbf{Z_2}\times\mathbf{Z_2}\times\mathbf{Z_2}$ and we may identify $\mathrm{Aut}(\mathrm{Dih}(\mathbf{Z_m}\times\mathbf{Z}_\ell))$ with $SL(3,\mathbf{Z_2})$, a group of order 168 .

CASE 1: $(m,k,\ell) = (1,1,2)$. Then $\Pi(B0,k) = \mathrm{Dih}(\mathbf{Z}) = \langle b,c\rangle$ and $N = \langle b^2\rangle \in \mathcal{K}$. By Lemma 2.6 and the remarks preceding Theorem 2.3, an automorphism α is in $\mathrm{Aut}_R^N(\Pi(B0,k))$ if and only if $\alpha(b) = b^{\pm1}$. Choosing $\mathcal{B} = \{\lambda_N(b), \lambda_N(c)\}$ we see that the image of $\hat{\lambda}_N$ is the subgroup of upper triangular matrices which has index 3 in $SL(2,\mathbf{Z_2})$. If $\alpha(b) = b^{-1}$ and $\alpha(c) = c$ then $\alpha \in \ker(\hat{\lambda}_N)$ and α inverts each element of N. This completes the proof of this case using Corollary 1.2.

CASE 2: $(m,k,\ell) = (1,2,2)$. Then $\Pi(B0,k) = \mathrm{Dih}(\mathbf{Z_2}\times\mathbf{Z}) = \langle a,b,c\rangle$ and $N = \langle b\rangle \in \mathcal{K}$. As in the previous case, we determine that $\alpha \in \mathrm{Aut}_R^N(\Pi(B0,k))$ if and only if $\alpha(a) = a$ and $\alpha(b) = b^{\pm 1}$. Choosing $\mathcal{B} = \{\lambda_N(a), \lambda_N(c)\}$, again we see that *image*$(\hat{\lambda}_N)$ is the subgroup of upper triangular matrices in $SL(2,\mathbf{Z_2})$—which has index 3. Since any element of $\mathrm{Aut}_R^N(\Pi(B0,k))$ which inverts b and fixes c is in the kernel of $\hat{\lambda}_N$ and inverts the elements of N, the proof of this case is complete by Corollary 1.2.

CASE 3: $(m,k,\ell) = (2,2,2)$. Then $\Pi(B0,k) = \mathrm{Dih}(\mathbf{Z_2}\times\mathbf{Z}) = \langle a,b,c\rangle$ and $N = \langle b^2\rangle \in \mathcal{K}$. In this case $\alpha \in \mathrm{Aut}_R^N(\Pi(B0,k))$ if and only if $\alpha(a) = a$ and $\alpha(b) = a^* b^{\pm 1}$. Choosing $\mathcal{B} = \{\lambda_N(a), \lambda_N(b), \lambda_N(c)\}$, the image of $\hat{\lambda}_N$ is the upper triangular subgroup in $SL(3,\mathbf{Z_2})$ which has index $\frac{168}{8} = 21$. The automorphism α with $\alpha(a) = a, \alpha(b) = b^{-1}$ and $\alpha(c) = c$ is in the kernel of $\hat{\lambda}_N$ and inverts each element of N. ∎

REMARK. Noting that Lemma 2.6 holds when Aut_R is replaced by the group of automorphisms that are realized by homeomorphisms which may reverse orientation, the above proof extends to prove that Theorem 2.7 is also true if $\mathcal{WE}$, $\mathcal{E}$ and $\mathcal{SE}$ are respectively replaced with $\mathcal{WE}_\pm$, $\mathcal{E}_\pm$ and $\mathcal{SE}_\pm$. Analogues of the other remarks following Theorem 2.3 clearly hold for $\mathrm{Dih}(\mathbf{Z_m}\times\mathbf{Z}_\ell)$ actions as well. ∎

The results of this section now lead to a complete description of the orientation preserving actions on the solid torus V_1 (not necessarily with fixed quotient type) in terms of the different forms of equivalence of actions. This is indicated in the next corollary where $\mathcal{E}(G)$ (resp. $\mathcal{WE}(G)$) denotes the set of equivalence (resp. weak equivalence) classes of orientation preserving G-actions on V_1. Recall that the only groups G which can act preserving orientation on V_1 have the form $\mathbf{Z_m}\times\mathbf{Z}_\ell$ or $\mathrm{Dih}(\mathbf{Z_m}\times\mathbf{Z}_\ell)$ where m divides ℓ. For convenience we assume in the corollary that $\ell \neq 2$ but the actions of $\mathbf{Z_2}$, $\mathbf{Z_2}\times\mathbf{Z_2}$, and $\mathbf{Z_2}\times\mathbf{Z_2}\times\mathbf{Z_2}$ can also be determined from Theorems 2.3 and 2.7.

COROLLARY 2.8. *Suppose that* $\ell = \prod_{i=1}^s p_i^{\ell_i}$, *and* $m = \prod_{i=1}^s p_i^{m_i}$ *where* $m_i \le \ell_i$ *and* $\ell \neq 2$. *Then*

(a) $|\mathcal{WE}(\mathbf{Z_m}\times\mathbf{Z}_\ell)| = |\mathcal{WE}(\mathrm{Dih}(\mathbf{Z_m}\times\mathbf{Z}_\ell))| = \prod_{i=1}^s A_i$

$$\text{where } A_i = \begin{cases} 1 & \text{if } m_i = \ell_i \\ 2p_i^{\frac{\ell_i - m_i - 1}{2}} & \text{if } \ell_i - m_i \text{ is odd} \\ (p_i+1)p_i^{\frac{\ell_i - m_i - 2}{2}} & \text{if } \ell_i - m_i > 0 \text{ is even} \end{cases}$$

(b) $|\mathcal{E}(\mathbf{Z_m}\times\mathbf{Z}_\ell)| = |\mathcal{E}(\mathrm{Dih}(\mathbf{Z_m}\times\mathbf{Z}_\ell))|$

$$= \tfrac{1}{2} m\phi(m)\phi(\ell)\prod_{i=1}^s 2\left(2 + (\ell_i - m_i - 1)\left(1 - \tfrac{1}{p_i}\right)\right).$$

PROOF. First consider $|\mathcal{WE}(\mathbf{Z_{p^M}}\times\mathbf{Z_{p^L}})|$ where p is prime, $p^L \neq 2$ and $M \le L$. We have

$$|\mathcal{WE}(\mathbf{Z_{p^M}}\times\mathbf{Z_{p^L}})| = \sum_{K=M}^{L} |\mathcal{WE}^{(A0,p^K)}(\mathbf{Z_{p^M}}\times\mathbf{Z_{p^L}})| = \sum_{K=M}^{L} \phi\left((p^{K-M}, p^{L-K})\right)$$

If $L = M$ this equals 1 , otherwise, putting $J(K) = min\{K - M, L - K\}$, it equals

$$2 + \sum_{K=M+1}^{L-1} p^{J(K)}\left(1 - \frac{1}{p}\right) = 2 + (p-1)\sum_{K=M+1}^{L-1} p^{J(K)-1}$$
$$= 2 + (p-1)\begin{cases} 2(1 + p + \cdots + p^{\frac{L-M-3}{2}}) & \text{if } L - M \text{ is odd} \\ 2(1 + p + \cdots + p^{\frac{L-M-2}{2}}) - p^{\frac{L-M-2}{2}} & \text{if } L - M \text{ is even} \end{cases}$$
$$= \begin{cases} 2p^{\frac{L-M-1}{2}} & \text{if } L - M \text{ is odd} \\ (p+1)p^{\frac{L-M-2}{2}} & \text{if } L - M \text{ is even (and } L \neq M) \end{cases}$$

By applying Remark (3) which follows Theorem 2.3, we have

$$|\mathcal{WE}(\mathbf{Z_m} \times \mathbf{Z}_\ell)| = \prod_{i=1}^{s} |\mathcal{WE}(\mathbf{Z}_{\mathbf{p_i}^{\mathbf{m_i}}} \times \mathbf{Z}_{\mathbf{p_i}^{\ell_i}})|$$

and this proves (a). Similar computations for equivalence classes give (b).

3. **Nonorientable handlebody orbifolds.** In this section we classify the quotient types of the finite group actions on genus one handlebodies. We will find thirteen families of these quotient types which we denote by $(A0, k), \ldots, (A3, k), (B0, k), \ldots, (B8, k)$ $(k \geq 1)$. The families $(A0, k)$ and $(B0, k)$ were discussed in the previous section as they are quotients of orientation preserving actions on V_1. The remaining eleven families are nonorientable, they arise from orientation reversing actions on V_1 and from actions on $\tilde{V}_1$. In order to obtain the classification of these "handlebody orbifolds with Euler characteristic zero" we will first characterize the nonorientable handlebody orbifolds with arbitrary Euler characteristic in terms of graphs of groups. This extends the description of orientable handlebody orbifolds which was given in [MMZ].

An *orbifold 0-handle* is a 3-orbifold which is the quotient of a linear action on the 3-ball. Up to homeomorphism there are 14 different orbifold 0-handles (see for example [Du], or Table 2 of [CM]). They are depicted in Figure 2 together with their fundamental groups.

In this figure, the first five orbifolds are orientable, the eighth orbifold has underlying space a cone over the projective plane, and each of the remaining orbifolds has mirrors. An *orbifold 1-handle* is the product $I \times O^2$ of an interval with a 2-orbifold O^2 which is the quotient of a linear action on the 2-ball. There are two forms of orbifold 1-handles as shown in Figure 3; the first is orientable and the second has mirrors. A *handlebody orbifold* is a connected orbifold which is obtained by gluing orbifold 1-handles to orbifold 0-handles via attaching maps which are inclusions from $\partial I \times O^2$ to the boundary of the 0-handles. Each handlebody orbifold has an associated graph of groups whose underlying graph has one vertex for each orbifold 0-handle and one edge for each orbifold 1-handle. The vertex and edge groups are the fundamental groups of the respective 0- and 1-handles, and the edge-to-vertex monomorphisms are induced by the attaching maps. By Van Kampen's theorem the fundamental group of the handlebody orbifold is

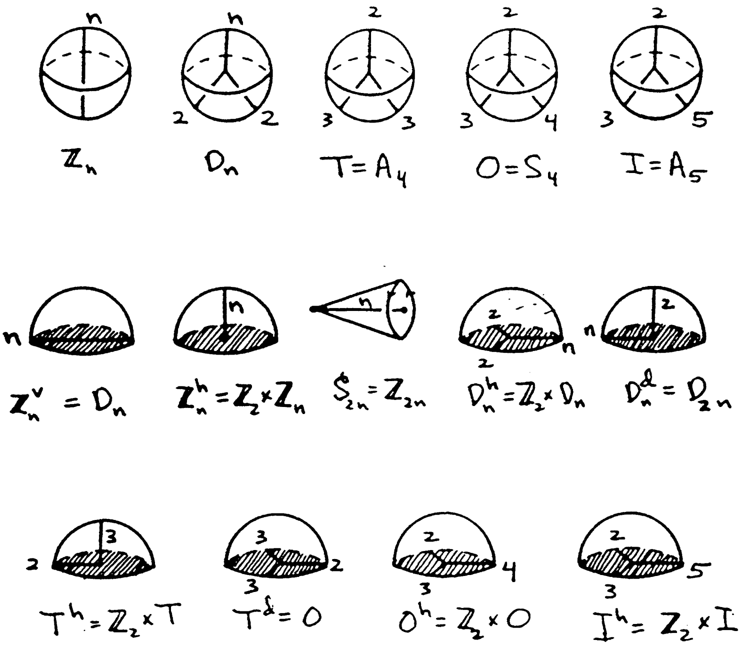

FIGURE 2

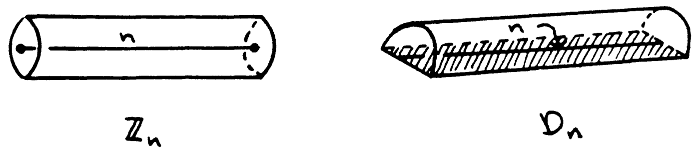

FIGURE 3

isomorphic to the fundamental group of the associated graph of groups. We will now describe "normalized conditions" on a graph of groups which essentially characterize the graphs of groups that can arise from this construction.

Let $(\Gamma, \mathcal{G})$ be a finite graph of groups. The graph Γ consists of vertices v and (oriented) edges e where $\bar{e}$ denotes the reverse edge of e, and the initial and terminal vertices of e are $\delta_0(e)$ and $\delta_1(e) = \delta_0(\bar{e})$ respectively. The data $\mathcal{G}$ consists of collections of vertex groups $\{ G_v \mid v \text{ is a vertex of } \Gamma \}$; of edge groups $\{ G_e \mid e \text{ is an edge of } \Gamma \}$, subject to the condition that $G_e = G_{\bar{e}}$; and, of edge-to-vertex monomorphisms $\{ f_e \colon G_e \longrightarrow G_{\delta_0(e)} \mid e \text{ is an edge of } \Gamma \}$. The *Euler characteristic* of $(\Gamma, \mathcal{G})$ is the rational number $\chi(\Gamma, \mathcal{G}) = \sum \frac{1}{|G_v|} - \sum \frac{1}{|G_e|}$ where the sums are taken over all vertices v and over all unoriented edges e. An edge e in Γ is called a *trivial edge* of $(\Gamma, \mathcal{G})$ provided that $\delta_0(e) \neq \delta_1(e)$ and f_e is an isomorphism. We say that $(\Gamma, \mathcal{G})$ is in *standard form* if it has no trivial edges and if it satisfies the property that whenever e_1 and e_2 are edges with $\delta_0(e_1) = v = \delta_0(e_2)$ and $f_{e_1}(G_{e_1})$ is conjugate in G_v to a subgroup of $f_{e_2}(G_{e_2})$ then $f_{e_1}(G_{e_1}) \subseteq f_{e_2}(G_{e_2})$. We may assume that a graph of groups is in standard form without any loss of generality since each trivial edge may be collapsed and each edge-to-vertex monomorphism may be composed with an inner automorphism to fulfill the second property; neither of these modifications changes the fundamental group (cf. Lemma 1.1 of [MMZ]). A loop $e_1 \cdots e_n$ in Γ is a *closed $\mathcal{G}$-loop* if $f_{\bar{e}_i}(G_{e_i}) = f_{e_{i+1}}(G_{e_{i+1}})$ for $i = 1, \ldots, n-1$ and $f_{\bar{e}_n}(G_{e_n}) = f_{e_1}(G_{e_1})$. Each such closed $\mathcal{G}$-loop determines an *associated automorphism* on $f_{e_1}(G_{e_1})$ given by $(f_{\bar{e}_n} f_{e_n}^{-1}) \cdots (f_{\bar{e}_1} f_{e_1}^{-1})$. We say that a second graph of groups $(\Gamma, \mathcal{G}^+)$ with the same underlying graph Γ is a *compatible subgraph* of $(\Gamma, \mathcal{G})$ provided that for each vertex v and edge e we have: $G_v^+ \subseteq G_v$, $G_e^+ = f_e^{-1}(G^+_{\delta_0(e)})$, and $f_e^+ = f_e|_{G_e^+}$.

A finite graph of groups $(\Gamma, \mathcal{G})$ is said to satisfy the *normalized conditions* if it is in standard form and contains a compatible subgraph $(\Gamma, \mathcal{G}^+)$ which has the properties (1) through (4) listed below. The subgraph $(\Gamma, \mathcal{G}^+)$ is called a *compatible orientation* for $(\Gamma, \mathcal{G})$.

(1) VERTEX GROUPS: For each vertex v the pair (G_v, G_v^+) is isomorphic to one of the fourteen pairs in Table 1. (The groups G_v in this list are those which act on the 2-sphere and the G_v^+ are the orientation preserving subgroups.)

G_v	$\mathbf{Z_n}$	$\mathbf{D_n}$	$\mathbf{T} = \mathbf{A_4}$	$\mathbf{O} = \mathbf{S_4}$	$\mathbf{I} = \mathbf{A_5}$
G_v^+	$\mathbf{Z_n}$	$\mathbf{D_n}$	$\mathbf{T}$	$\mathbf{O}$	$\mathbf{I}$

G_v	$\mathbf{Z_n^v} = \mathbf{D_n}$	$\mathbf{Z_n^h} = \mathbf{Z_2} \times \mathbf{Z_n}$	$\mathcal{S}_{\mathbf{2n}} = \mathbf{Z_{2n}}$	$\mathbf{D_n^h} = \mathbf{Z_2} \times \mathbf{D_n}$	$\mathbf{D_n^d} = \mathbf{D_{2n}}$
G_v^+	$\mathbf{Z_n}$	$\mathbf{Z_n} = \mathbf{1} \times \mathbf{Z_n}$	$\mathbf{Z_n}$	$\mathbf{D_n} = \mathbf{1} \times \mathbf{D_n}$	$\mathbf{D_n}$

G_v	$\mathbf{T^h} = \mathbf{Z_2} \times \mathbf{T}$	$\mathbf{T^d} = \mathbf{O}$	$\mathbf{O^h} = \mathbf{Z_2} \times \mathbf{O}$	$\mathbf{I^h} = \mathbf{Z_2} \times \mathbf{I}$	
G_v^+	$\mathbf{T} = \mathbf{1} \times \mathbf{T}$	$\mathbf{T}$	$\mathbf{O} = \mathbf{1} \times \mathbf{O}$	$\mathbf{I} = 1 \times \mathbf{I}$	

TABLE 1.

(2) EDGE GROUPS: For each edge e the pair (G_e, G_e^+) is isomorphic to either $(\mathbf{Z_n}, \mathbf{Z_n})$ or $(\mathbf{D_n}, \mathbf{Z_n})$. We will distinguish between $G_e = \mathbf{Z_2}$ and $G_e = \mathbf{D_1}$ as the first implicitly has $G_e^+ = \mathbf{Z_2}$ while the second has $G_e^+ = 1$.

(3) INCIDENCE RESTRICTIONS: The configuration of all edges e which are incident to a fixed vertex v and have $G_e^+ \neq 1$ is a subconfiguration of one of the configurations shown in Figure 4. In each case (where $G_e^+ \neq 1$) $f_e(G_e^+)$ is a maximal cyclic subgroup of G_v^+. If e' is a second edge incident to v for which $G_{e'}^+ \neq 1$ and $G_{e'} \cong G_e$ then $f_e(G_e) \neq f_{e'}(G_{e'})$ if and only if G_v equals $\mathbf{D_n}$ or $\mathbf{D_n^h}$ for some even integer n (and G_e is respectively either $\mathbf{Z_2}$ or $\mathbf{D_2}$). If v is a vertex with vertex group $\mathcal{S}_{\mathbf{2n}}$ then there is no incident edge e with $G_e = \mathbf{D_1}$; otherwise there are no

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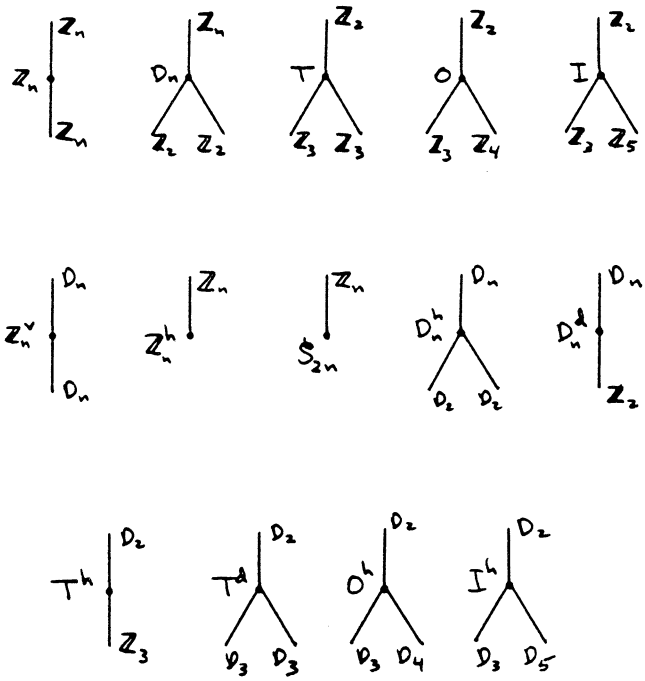

FIGURE 4

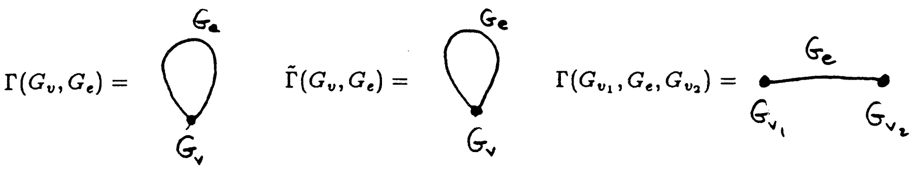

FIGURE 5

restrictions on the number of edges whose edge group is 1 or $\mathbf{D_1}$. (However note that by compatibility of orientations there are no $\mathbf{D_1}$ edges incident to any vertex v with $G_v^+ = G_v$.)

(4) CLOSED LOOP RESTRICTIONS: The associated automorphism for each closed $\mathcal{G}$-loop in $(\Gamma, \mathcal{G})$ is either the identity or an inversion automorphism. An *inversion automorphism* is defined on $\mathbf{Z_n}$ to invert each element, and, defined on $\mathbf{D_n}$ to interchange a pair of order two elements which generate $\mathbf{D_n}$.

We use the notation given in Figure 5 below for certain graphs of groups satisfying the normalized conditions. Each of the first two graphs has a closed $\mathcal{G}$-loop; its associated automorphism is the identity in the first case and an inversion in the second case.

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We now consider the graph of groups associated with a handlebody orbifold. This graph of groups may not be in standard form, so we modify it to a graph in standard form using the procedure mentioned above. The resulting graph of groups can be seen to satisfy the normalized conditions provided that we choose $\mathcal{G}^+$ to consist of the orientation subgroups of the fundamental groups of the 0- and 1-handles. To illustrate, the orbifold 1-handles incident to an orbifold 0-handle are limited by the incidence restrictions given in the normalized condition (3) as can be seen by comparison with the pictures of the orbifold 0-handles in Figure 2. As another illustration, if a 0-handle has group $\mathcal{S}_{2\mathbf{n}}$ then it has no mirror in its boundary and there can be no incident edge with group $\mathbf{D_1}$ and this is in agreement with condition (3). The remaining properties (1)–(4) can be readily checked. There is a converse to this construction given by:

THEOREM 3.1. *If* $(\Gamma, \mathcal{G})$ *is a finite graph of groups satisfying the normalized conditions then there is a handlebody orbifold* $V(\Gamma, \mathcal{G})$ *with* $\pi_1^{\text{orb}}(V(\Gamma, \mathcal{G})) \cong \pi_1(\Gamma, \mathcal{G})$ *and* $\chi_{\text{orb}}(V(\Gamma, \mathcal{G})) = \chi(\Gamma, \mathcal{G})$.

PROOF. If $(\Gamma, \mathcal{G})$ satisfies the normalized conditions then we may construct a handlebody orbifold $V(\Gamma, \mathcal{G})$ as follows: for each vertex v choose a 0-handle whose fundamental group and orientation subgroup is isomorphic to the pair (G_v, G_v^+); for each edge e choose a 1-handle whose fundamental group and orientation subgroup is isomorphic to (G_e, G_e^+); now attach these 0- and 1-handles according to the incidence relations in the graph Γ, do this in such a way that a loop in $V(\Gamma, \mathcal{G})$ coming from a closed $\mathcal{G}$-loop in Γ is orientable if and only if its associated automorphism is the identity automorphism. The associated graph of groups $(\Gamma, \mathcal{G}')$ of the orbifold $V(\Gamma, \mathcal{G})$ which we have constructed is not the graph $(\Gamma, \mathcal{G})$ as the edge-to-vertex monomorphisms need not be the same. However the proof of Proposition 3.2 of [MMZ] adapts to the present setting to imply that $\pi_1(\Gamma, \mathcal{G}) \cong \pi_1(\Gamma', \mathcal{G}')$. The statement about Euler characteristics follows using the observations that $\chi_{\text{orb}}(V(\Gamma, \mathcal{G})) = \chi(\Gamma, \mathcal{G}')$ and that $\chi(\Gamma, \mathcal{G}') = \chi(\Gamma, \mathcal{G})$. ∎

It should be cautioned that, in contrast to the orientable setting of [MMZ], the handlebody orbifold $V(\Gamma, \mathcal{G})$ constructed in the previous theorem may not be uniquely determined up to orbifold homeomorphism; however the indeterminacies are easy to categorize. For example a graph of groups consisting of one vertex whose vertex group is cyclic of order 6 can be realized as a handlebody orbifold in three distinct ways since this group is isomorphic to each of $\mathbf{Z_6}$, $\mathbf{Z_3^h}$, and $\mathcal{S}_6$. (The latter two have the same compatible orientations.) There are similar isomorphisms between other vertex groups listed in the normalized condition (1) which lead to more indeterminacies, but notice that the notation of labelling allows one to distinguish these. Thus for example $\mathbf{Z_6}$, $\mathbf{Z_3^h}$ and $\mathcal{S}_6$ are distinct *labelled* graphs of groups, and in this way we distinguish their corresponding orbifolds. However even the procedure of labelling does not eliminate all indeterminacy in general. For example Figure 6 shows nonhomeomorphic handlebody orbifolds each

FIGURE 6

of which corresponds to the graph of groups $\Gamma(\mathbf{Z}_5^h, \mathbf{D}_1, \mathbf{T}^d)$. At any rate, we will see that labelling does suffice to distinguish the orbifolds in which we are interested, those whose associated graph of groups has vanishing Euler characteristic.

THEOREM 3.2. *If a finite group G acts on a (possibly nonorientable) handlebody V then the orbifold quotient is homeomorphic to a handlebody orbifold $V(\Gamma, \mathcal{G})$ for some graph of groups $(\Gamma, \mathcal{G})$ satisfying the normalized conditions. If V has genus g then $1-g = |G|\chi(\Gamma, \mathcal{G})$.*

PROOF. By the equivariant Dehn's Lemma [MY1] there is an equivariant handle decomposition of the handlebody V into 0- and 1-handles such that the action restricts to a product action on the 1-handles. The stabilizer of each 0-handle is equivalent to a linear action on the 3-ball—this follows by the generalized Smith conjecture of [Th] if any nontrivial element has fixed point set with dimension larger than 0 and by [Li] when the fixed point set is 0-dimensional (cf. [MS] for related discussions and other references). Therefore the images of the 0- and 1-handles of V in the orbifold quotient of the action are orbifold 0- and 1-handles and the quotient is a handlebody orbifold. As remarked prior to Theorem 3.1 the modified graph of groups $(\Gamma, \mathcal{G})$ associated with this orbifold satisfies the normalized conditions and, from the description given in the proof of Theorem 3.1, it is homeomorphic to $V(\Gamma, \mathcal{G})$. If V has genus g, then the multiplicativity of Euler characteristic via coverings gives

$$1 - g = \chi(V) = |G|\chi(V(\Gamma, \mathcal{G})) = |G|\chi(\Gamma, \mathcal{G}). \qquad \blacksquare$$

THEOREM 3.3. *A complete list of the quotient types of finite group actions on V_1 and $\tilde{V}_1$ is given in Table 2. Each quotient type $\mathcal{Q}$ corresponds to a graph of groups $\Gamma(\mathcal{Q})$ satisfying the normalized conditions, and this in turn determines a handlebody orbifold $V(\mathcal{Q}) = V(\Gamma(\mathcal{Q}))$ which represents the quotient type $\mathcal{Q}$. The orbifold fundamental group of $V(\mathcal{Q})$ is denoted by $\Pi(\mathcal{Q})$ and its orientation subgroup is denoted by $\Pi^+(\mathcal{Q})$.*

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Q	$\Gamma(Q)$	$\Pi(Q)$	$\Pi^+(Q)$
$(A0,k)$	$\Gamma(\mathbf{Z_k},\mathbf{Z_k})$	$\mathbf{Z_k}\times\mathbf{Z}$	orientable
$(A1,k)$	$\tilde{\Gamma}(\mathbf{Z_k},\mathbf{Z_k})$	$\langle a,t \mid a^k=1, a^t=a^{-1}\rangle = \mathbf{Z_k}\circ\mathbf{Z}$	$\langle a,t^2\rangle$
$(A2,k)$	$\Gamma(\mathbf{Z_k^v},\mathbf{D_k})$	$\langle a,b,t \mid a^k=b^2=1, a^b=a^{-1}, \{a,b\}\leftrightarrow t\rangle = \mathbf{D_k}\times\mathbf{Z}$	$\langle a,t\rangle$
$(A3,k)$	$\tilde{\Gamma}(\mathbf{Z_k^v},\mathbf{D_k})$	$\langle b,c,t \mid b^2=c^2=(bc)^k=1, b^t=c, c^t=b\rangle = \mathbf{D_k}\circ\mathbf{Z}$	$\langle bc,bt\rangle$
$(B0,k)$	$\Gamma(\mathbf{D_k},\mathbf{Z_k},\mathbf{D_k})$	$\mathrm{Dih}(\mathbf{Z_k}\times\mathbf{Z})$	orientable
$(B1,k)$	$\Gamma(\mathbf{D_k},\mathbf{Z_k},\mathbf{Z_k^h})$	$\langle a,b,c \mid a^k=b^2=c^2=1, a^b=a^{-1}, a\leftrightarrow c\rangle = \mathbf{Z_k}\circ\mathbf{D_\infty}$	$\langle a,b,(bc)^2\rangle$
$(B2,k)$	$\Gamma(\mathbf{D_k},\mathbf{Z_k},\mathcal{S}_{\mathbf{2k}})$	$\langle a,b \mid a^{2k}=b^2=1, (a^2)^b=a^{-2}\rangle = (\mathbf{Z_k}\circ\mathbf{Z})\circ\mathbf{Z_2}$	$\langle a^2,b,(ab)^2\rangle$
$(B3,k)$	$\Gamma(\mathbf{Z_k^h},\mathbf{Z_k},\mathbf{Z_k^h})$	$\langle a,b,c \mid a^2=b^k=c^2=1, b\leftrightarrow\{a,c\}\rangle = \mathbf{Z_k}\times\mathbf{D_\infty}$	$\langle b,ac\rangle$
$(B4,k)$	$\Gamma(\mathbf{Z_k^h},\mathbf{Z_k},\mathcal{S}_{\mathbf{2k}})$	$\langle a,b \mid a^2=b^{2k}=1, a\leftrightarrow b^2\rangle = \mathbf{D_\infty}\circ\mathbf{Z_{2k}}$	$\langle ab,b^2\rangle$
$(B5,k)$	$\Gamma(\mathcal{S}_{\mathbf{2k}},\mathbf{Z_k},\mathcal{S}_{\mathbf{2k}})$	$\langle a,b \mid a^{2k}=b^{2k}=1, a^2=b^2\rangle = \mathbf{Z}\circ\mathbf{Z_{2k}}$	$\langle a^2,ab^{-1}\rangle$
$(B6,k)$	$\Gamma(\mathbf{D_k^h},\mathbf{D_k},\mathbf{D_k^h})$	$\langle a,b,c,d \mid a^2=b^2=c^2=(bc)^k=d^2=1, \{b,c\}\leftrightarrow\{a,d\}\rangle = \mathbf{D_k}\times\mathbf{D_\infty}$	$\langle ab,bc,ad\rangle$
$(B7,k)$	$\Gamma(\mathbf{D_k^h},\mathbf{D_k},\mathbf{D_k^d})$	$\langle a,b,c \mid a^2=b^{2k}=c^2=1, b^c=b^{-1}, a\leftrightarrow\{b^2,c\}\rangle = \mathbf{D_\infty}\circ\mathbf{D_{2k}}$	$\langle b^2,ac,bc\rangle$
$(B8,k)$	$\Gamma(\mathbf{D_k^d},\mathbf{D_k},\mathbf{D_k^d})$	$\langle a,b,c \mid a^{2k}=b^2=c^{2k}=1, a^b=a^{-1}, c^b=c^{-1}, a^2=c^2\rangle = \mathbf{Z}\circ\mathbf{D_{2k}}$	$\langle a^2,ab,ac^{-1}\rangle$

TABLE 2. The quotient types of actions on genus one handlebodies.

(notation: $x^y = yxy^{-1}$, and, we write $x\leftrightarrow y$ if and only if x and y commute.)

PROOF. Suppose that G acts on V_1 or $\tilde{V}_1$. By Theorem 3.2, the orbifold quotient is homeomorphic to $V(\Gamma,\mathcal{G})$ for some graph of groups $(\Gamma,\mathcal{G})$ satisfing the normalized conditions and having Euler characteristic 0. (In this setting the use of Thurston's theorem in Theorem 3.2 is not necessary. The orientation preserving subgroup of G contains an abelian subgroup of index at most 2 (Theorem 8.2 of [MMZ]). Thus there are no icosahedral point stabilizers in the G-action and [MY2] applies.) It is readily seen that the Euler characteristic of any graph of groups in standard form with two or more unoriented edges is negative. Therefore, $(\Gamma,\mathcal{G})$ must have exactly one unoriented edge. If this unoriented edge is not a loop then $(\Gamma,\mathcal{G})$ (having vanishing Euler characteristic) must be of the form $\Gamma(G_{v_1},G_e,G_{v_2})$ where $f_e(G_e)$ and $f_{\bar{e}}(G_e)$ have index two in their respective vertex groups. Thus by the normalized condition (3) each vertex-edge group pair must be one of: $(\mathbf{D_k},\mathbf{Z_k})$, $(\mathbf{Z_k^h},\mathbf{Z_k})$, $(\mathcal{S}_{\mathbf{2k}},\mathbf{Z_k})$, $(\mathbf{D_k^h},\mathbf{D_k})$, or$(\mathbf{D_k^d},\mathbf{D_k})$; and $(\Gamma,\mathcal{G})$ is one of the graphs $\Gamma(B0,k),\dots,\Gamma(B8,k)$ in Table 2. If the unoriented edge is a loop then we conclude that $(\Gamma,\mathcal{G})$ must have the form of $\Gamma(G_v,G_e)$ or $\tilde{\Gamma}(G_v,G_e)$ where f_e is an isomorphism. Again consulting normalized condition (3) we see that $(\Gamma,\mathcal{G})$ is one of $\Gamma(A0,k),\dots,\Gamma(A3,k)$. This shows that Table 2 contains all possible quotient types for actions on V_1 or $\tilde{V}_1$.

By Theorem 3.1, each of the graphs of groups $\Gamma(Q)$ listed in the table gives rise to a handlebody orbifold $V(Q)$ whose fundamental group is isomorphic to $\pi_1(\Gamma(Q))$. As each of these groups has a torsion free subgroup of finite index, the corresponding orbifold covering provides a finite group action on V_1 or $\tilde{V}_1$ whose quotient type is Q. Note that $\Gamma(G_v, G_e)$ and $\tilde{\Gamma}(G_v, G_e)$ have fundamental group of the form $G_v *_{G_e}$ while the fundamental group of $\Gamma(G_{v_1}, G_e, G_{v_2})$ is of the form $G_{v_1} *_{G_e} G_{v_2}$. Using this it is straightforward to obtain the presentations in the table for the groups $\Pi(Q) \cong \pi_1(\Gamma(Q))$. ∎

4. **Actions on the solid Klein bottle.** In this section we describe the finite group actions on the solid Klein bottle $\tilde{V}_1$. A special feature is that only finitely many quotient types occur for these actions—for each possible type Q it turns out that k must equal 1 or 2 . In order to enumerate the various equivalence classes of actions using the results of § 1 we first need to understand the groups of realizable automorphisms $\mathrm{Aut}_R(\Pi(Q))$. To describe these we use the presentations for the groups $\Pi(Q)$ given in Table 2. In the following lemma the automorphisms ϕ_1 and ϕ_2 fix all generators which are not indicated and the inner automorphism determined by x is denoted by $\mu(x)$ where $\mu(x)(y) = y^x = xyx^{-1}$.

LEMMA 4.1. *For* $Q = (A1,k),\ldots,(A3,k),(B1,k),\ldots,(B8,k)$ *the group* $\mathrm{Aut}_R(\Pi(Q))$ *of automorphisms of* $\Pi(Q)$ *induced by basepoint-preserving homeomorphisms of* $V(Q)$ *is given as follows :*

(1) $\mathrm{Aut}_R(\Pi(A1,k)) = \langle \phi_1, \phi_2, \mu(t)\rangle$ where $\phi_1\colon t \mapsto t^{-1}, \phi_2\colon t \mapsto at$

(2) $\mathrm{Aut}_R(\Pi(A2,k)) = \langle \phi_1, \phi_2\rangle$ where $\phi_1\colon t \mapsto t^{-1}, \phi_2\colon \left\{ \begin{smallmatrix} a\mapsto a^{-1} \\ b\mapsto ab \end{smallmatrix}\right.$

(3) $\mathrm{Aut}_R(\Pi(A3,k)) = \langle \phi_1, \mu(t)\rangle$ where $\phi_1\colon t \mapsto t^{-1}$

(4) $\mathrm{Aut}_R(\Pi(B1,k)) = \langle \phi_1, \phi_2, \mu(a), \mu(b)\rangle$ where $\phi_1\colon a \mapsto a^{-1}, \phi_2\colon b \mapsto ab$

(5) $\mathrm{Aut}_R(\Pi(B2,k)) = \langle \phi_1, \phi_2, \mu(a), \mu(b)\rangle$ where $\phi_1\colon a \mapsto a^{-1}, \phi_2\colon b \mapsto a^2b$

(6) $\mathrm{Aut}_R(\Pi(B3,k)) = \langle \phi_1, \phi_2\rangle$ where $\phi_1\colon b \mapsto b^{-1}, \phi_2\colon \left\{ \begin{smallmatrix} a\mapsto c \\ c\mapsto a \end{smallmatrix}\right.$

(7) $\mathrm{Aut}_R(\Pi(B4,k)) = \langle \phi_1, \mu(b)\rangle$ where $\phi_1\colon b \mapsto b^{-1}$

(8) $\mathrm{Aut}_R(\Pi(B5,k)) = \langle \phi_1, \phi_2, \mu(a), \mu(b)\rangle$ where $\phi_1\colon \left\{ \begin{smallmatrix} a\mapsto a^{-1} \\ b\mapsto b^{-1} \end{smallmatrix}\right., \phi_2\colon \left\{ \begin{smallmatrix} a\mapsto b \\ b\mapsto a \end{smallmatrix}\right.$

(9) $\mathrm{Aut}_R(\Pi(B6,k)) = \langle \phi_1, \phi_2\rangle$ where $\phi_1\colon \left\{ \begin{smallmatrix} a\mapsto d \\ d\mapsto a \end{smallmatrix}\right., \phi_2\colon \left\{ \begin{smallmatrix} b\mapsto c \\ c\mapsto b \end{smallmatrix}\right.$

(10) $\mathrm{Aut}_R(\Pi(B7,k)) = \langle \phi_1, \mu(bc)\rangle$ where $\phi_1\colon \left\{ \begin{smallmatrix} b\mapsto b^{-1} \\ c\mapsto b^2c \end{smallmatrix}\right.$

(11) $\mathrm{Aut}_R(\Pi(B8,k)) = \langle \phi_1, \phi_2, \mu(ab), \mu(ac^{-1})\rangle$ where $\phi_1\colon c \mapsto aca^{-1}, \phi_2\colon \left\{ \begin{smallmatrix} a\mapsto c \\ c\mapsto a \end{smallmatrix}\right.$.

PROOF. Let $S(Q)$ be the boundary of $V(Q)$. Each homeomorphism of $V(Q)$ induces a homeomorphism of $S(Q)$ and, since inclusion induces a surjection $\pi_1^{\mathrm{orb}}(S(Q)) \longrightarrow \Pi(Q)$, the determination of $\mathrm{Aut}_R(\Pi(Q))$ can be made by studying the realizable auto-

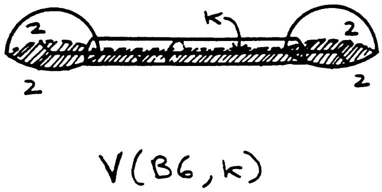

FIGURE 7

morphisms of $\pi_1^{orb}(S(Q))$ that extend to $\Pi(Q)$. Writing $Q = (*, k)$, the (Euclidean) 2-orbifolds $S(Q)$ which arise from the eleven types are: a Klein bottle for $* = A1, B5$; a mirrored annulus for $* = A2, B3$; a mirrored Mobius band for $* = A3, B4$; a projective plane with two singular points of order 2 for $* = B2$; a mirrored disk with two cone points of order 2 for $* = B1, B8$; a mirrored disk with two corners of order 2 and one cone point of order 2 for $* = B7$; and, a mirrored disk with four corners of order 2 for $* = B6$. In each case the homeomorphisms of $S(Q)$ are easy to classify up to basepoint preserving isotopy, and, except when $* = B2$, $B6$ or $B7$, each isotopy class extends to $V(Q)$. To illustrate the procedure for obtaining specific generators for $\mathrm{Aut}_R(\Pi(Q))$ consider the case $* = B6$.

On $S(B6, k)$ there is a $\mathbf{D_4}$ group of homeomorphisms generated by an order 4 rotation and a reflection through the axis L pictured above, and each homeomorphism of $S(B6, k)$ can be isotoped to one of these eight elements. The order 4 rotation, however, cannot extend to $V(B6, k)$ and thus the extendible homeomorphisms constitute the $\mathbf{D_2}$ subgroup of $\mathbf{D_4}$ which is generated by reflections through L and a second axis perpendicular to L. Writing

$$\pi_1^{orb}(S(B6,k)) = \langle A, B, C, D \mid A^2 = B^2 = C^2 = D^2 = 1, \{B, C\} \leftrightarrow \{A, D\} \rangle$$

we observe that one reflection induces the automorphism $A \mapsto A, B \mapsto C, C \mapsto B, D \mapsto D$ and the second induces the automorphism $A \mapsto D, B \mapsto B, C \mapsto C, D \mapsto A$. The inclusion induced homomorphism $\pi_1^{orb}(S(B6,k)) \longrightarrow \pi(B6,k)$ is given by $A \mapsto a, B \mapsto b, C \mapsto c, D \mapsto d$ and this verifies case (9) of the Lemma. We remark that in this case $S(B6, k)$ contains no essential simple closed curve, and $\mathrm{Aut}_R(B6, k)$ does not contain any nontrivial inner automorphisms. In general a basepoint slide around a simple closed curve in $S(Q)$ will induce an inner automorphism by the corresponding element of the fundamental group. (This accounts for the inner automorphisms arising in some other cases.) ∎

THEOREM 4.2. *If G is a finite group acting on the solid Klein bottle $\tilde{V}_1$ then G is isomorphic to a subgroup of $\mathbf{Z_2} \times \mathbf{D_s}$ for some s. The quotient types as well as the equivalence, weak equivalence, and strong equivalence classes of these actions are enumerated in Table 3. In this table s always represents an integer larger than two.*

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G	G-admissible types $\mathcal{Q}$	$\lvert\widetilde{\mathcal{WE}}^{\mathcal{Q}}(G)\rvert$	$\lvert\widetilde{\mathcal{E}}^{\mathcal{Q}}(G)\rvert$	$\lvert\widetilde{\mathcal{SE}}^{\mathcal{Q}}(G)\rvert$
$\mathbf{Z_2}$	$(A1,2); (A2,1); (A3,1);$ $(B1,1); (B2,1)$	$1;1;1;$ $1;1$	$1;1;1;$ $1;1$	$1;1;1;$ $1;1$
$\mathbf{Z_2} \times \mathbf{Z_2}$	$(A2,2); (B1,2); (B6,1);$ $(B7,1); (B8,1)$	$1;1;1;$ $1;1$	$6;6;3;$ $6;3$	$6;6;6;$ $6;3$
$\mathbf{Z_2} \times \mathbf{Z_2} \times \mathbf{Z_2}$	$(B6,2)$	1	168	168
$\mathbf{Z_s}$, s odd	$(A1,1)$	1	$\frac{1}{2}\phi(s)$	$\phi(s)$
$\mathbf{Z_s}$, $s \equiv 0 (\mathrm{mod}\ 4)$	$(A2,1); (A3,1)$	$1;1$	$\frac{1}{2}\phi(s); \frac{1}{2}\phi(s)$	$\phi(s); \phi(s)$
$\mathbf{Z_s}$, $s \equiv 2 (\mathrm{mod}\ 4)$	$(A1,2); (A2,1); (A3,1)$	$1;1;1$	$\frac{1}{2}\phi(s); \frac{1}{2}\phi(s); \frac{1}{2}\phi(s)$	$\phi(s); \phi(s); \phi(s)$
$\mathbf{Z_2} \times \mathbf{Z_s}$, $s \equiv 2 (\mathrm{mod}\ 4)$	$(A2,2)$	1	$3\phi(s)$	$6\phi(s)$
$\mathbf{Z_2} \times \mathbf{Z_s}$, $s \equiv 0 (\mathrm{mod}\ 4)$	$(A2,2)$	1	$2\phi(s)$	$4\phi(s)$
$\mathbf{D_s}$, s odd	$(B1,1); (B2,1)$	$1;1$	$\frac{1}{2}s\phi(s); \frac{1}{2}\phi(s)$	$s\phi(s); \phi(s)$
$\mathbf{D_s}$, $s \equiv 0 (\mathrm{mod}\ 4)$	$(B6,1); (B7,1); (B8,1)$	$1;1;1$	$\frac{1}{2}s\phi(s); \frac{1}{2}s\phi(s); \frac{1}{2}\phi(s)$	$s\phi(s); s\phi(s); \phi(s)$
$\mathbf{D_s}$, $s \equiv 2 (\mathrm{mod}\ 4)$	$(B1,2); (B6,1);$ $(B7,1); (B8,1)$	$1;1;$ $1;1$	$\frac{1}{2}s\phi(s); \frac{1}{2}s\phi(s);$ $\frac{1}{2}s\phi(s); \frac{1}{2}\phi(s)$	$s\phi(s); s\phi(s);$ $s\phi(s); \phi(s)$
$\mathbf{Z_2} \times \mathbf{D_s}$, $s \equiv 2 (\mathrm{mod}\ 4)$	$(B6,2)$	1	$6s\phi(s)$	$12s\phi(s)$
$\mathbf{Z_2} \times \mathbf{D_s}$, $s \equiv 0 (\mathrm{mod}\ 4)$	$(B6,2)$	1	$4s\phi(s)$	$8s\phi(s)$

TABLE 3. The G-actions on the solid Klein bottle $\tilde{V}_1$

PROOF. If G acts on $\tilde{V}_1$ then the quotient type is one of the nonorientable types $\mathcal{Q}$ given by Theorem 3.3. Furthermore, $\widetilde{\mathcal{K}}(\mathcal{Q})$ is nonempty so there is a group extension of the form

$$1 \longrightarrow N \longrightarrow \Pi(\mathcal{Q}) \longrightarrow G \longrightarrow 1$$

where $N \cong \mathbf{Z}$ is not contained in $\Pi^+(\mathcal{Q})$. By considering the groups $\Pi^+(\mathcal{Q}) \subset \Pi(\mathcal{Q})$ as described in Table 2, it is straightforward to determine all such extensions which can satisfy these conditions. The possibilities are listed in Table 4.

To illustrate the derivation of this table consider the type $(A1,k)$. Let $\rho\colon \Pi(A1,k) \longrightarrow \mathbf{D_k}$ be the epimorphism whose kernel is $\langle t^2 \rangle$. Since both t^2 and a are in $\Pi^+(A1,k)$, N must project to a nontrivial cyclic normal subgroup of $\mathbf{D_k}$ generated by an element of $\mathbf{D_k} - \rho(\langle a \rangle)$. However no such element generates a normal subgroup of $\mathbf{D_k}$ unless $k = 1$ or 2. In the latter cases we have $(\Pi(A1,1), \Pi^+(A1,1)) \cong (\mathbf{Z_k} \times \mathbf{Z}, \mathbf{Z_k} \times 2\mathbf{Z})$. This describes the only possible $(A1,k)$ quotient types which may arise and leads to the first 3 rows of Table 4. The remainder of the table is determined by straightforward arguments of a similar nature.

Q	N	$G = \Pi(Q)/N$
$(A1, 1)$	$\langle t^s \rangle, s$ odd	$\mathbf{Z_s}$, s odd
$(A1, 2)$	$\langle t^s \rangle, s$ odd $\langle at^s \rangle, s$ odd	$\mathbf{Z_{2s}}$, s odd $\mathbf{Z_{2s}}$, s odd
$(A2, 1)$	$\langle bt^s \rangle$	$\mathbf{Z_{2s}}$
$(A2, 2)$	$\langle bt^s \rangle$ $\langle abt^s \rangle$	$\mathbf{Z_2} \times \mathbf{Z_{2s}}$ $\mathbf{Z_2} \times \mathbf{Z_{2s}}$
$(A3, 1)$	$\langle b(bt)^s \rangle$	$\mathbf{Z_{2s}}$
$(B1, 1)$	$\langle (bc)^s \rangle, s$ odd	$\mathbf{D_s}$, s odd
$(B1, 2)$	$\langle (bc)^s \rangle, s$ odd $\langle a(bc)^s \rangle, s$ odd	$\mathbf{D_{2s}}$, s odd $\mathbf{D_{2s}}$, s odd
$(B2, 1)$	$\langle (ab)^s \rangle, s$ odd	$\mathbf{D_s}$, s odd
$(B6, 1)$	$\langle b(ad)^s \rangle$	$\mathbf{D_{2s}}$
$(B6, 2)$	$\langle b(ad)^s \rangle$ $\langle c(ad)^s \rangle$	$\mathbf{Z_2} \times \mathbf{D_{2s}}$ $\mathbf{Z_2} \times \mathbf{D_{2s}}$
$(B7, 1)$	$\langle c(ba)^s \rangle$	$\mathbf{D_{2s}}$
$(B8, 1)$	$\langle b(ac^{-1})^s \rangle$	$\mathbf{D_{2s}}$

TABLE 4. The infinite cyclic normal subgroups $N \not\subset \Pi^+(Q)$

It follows from Table 4 that any group G acting on $\tilde{V}_1$ is a subgroup of $\mathbf{Z_2} \times \mathbf{D_s}$ for some s, and a rearrangement of this information yields the first 2 columns of Table 3. From Table 4 we also see that for each G the order of $\mathcal{K}^Q(G)$ is at most 2 and only equals 2 when Q is either $(A1, 2), (A2, 2), (B1, 2)$, or$(B6, 2)$. In these four cases there is a realizable automorphism carrying one N to the other. In fact using the notation of Lemma 4.1 we have: $\phi_2(t^s) = at^s$ for $(A1, 2)$; $\phi_2(bt^s) = abt^s$ for $(A2, 2)$; $\phi_2((bc)^s) = a(bc)^s$ for $(B1, 2)$; and $\phi_2(b(ad)^s) = c(ad)^s$ for $(B6, 2)$. Thus $\mathrm{Aut}_R(\Pi(Q))$ acts trivially on $\mathcal{K}^Q(G)$ and this verifies the third column of Table 3 by Theorem 1.3(a).

To determine the number of equivalence classes for the fourth column of Table 3 we use Corollary 1.2 (or rather its analogue for actions on the solid Klein bottle). For each group G and each G-admissible quotient type Q we need to compute the index $[\mathrm{Aut}(G) : \mathrm{Aut}_R^N(G)]$ where N is given in Table 4. It remains to determine $|\,\mathrm{Aut}(G)|$ and $|\,\mathrm{Aut}_R^N(G)|$. For the former we have the following isomorphisms:

(1) $\mathrm{Aut}(\mathbf{Z_s}) \cong \mathbf{Z_s}^\star$

(2) $\mathrm{Aut}(\mathbf{Z_2} \times \mathbf{Z_{2s}}) \cong \begin{cases} \mathbf{Z_s}^\star \times \mathbf{D_3} & \text{if } s \text{ is odd} \\ \mathbf{Z_{2s}}^\star \times \mathbf{D_2} & \text{if } s \text{ is even} \end{cases}$

(3) $\mathrm{Aut}(\mathbf{D_s}) \cong \begin{cases} \mathbf{D_3} & \text{if } s = 2 \\ \mathbf{Z_s} \circ \mathbf{Z_s}^\star & \text{if } s > 2 \end{cases}$

(4) $\mathrm{Aut}(\mathbf{Z_2} \times \mathbf{D_{2s}}) \cong \begin{cases} SL(2,3) & \text{if } s = 1 \\ (\mathrm{Aut}(\mathbf{D_{2s}}) \circ \mathbf{D_2}) \circ \mathbf{Z_2} & \text{if } s \text{ is even} \\ (\mathrm{Aut}(\mathbf{D_{2s}}) \circ \mathbf{Z_2}) \circ \mathbf{D_3} & \text{if } s > 1 \text{ is odd.} \end{cases}$

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These descriptions are straightforward to derive. Case (1) is immediate and (3) is well-known. For (2), when s is odd $\mathbf{Z_2} \times \mathbf{Z_{2s}} \cong \mathbf{D_2} \times \mathbf{Z_s}$ and $\mathrm{Aut}(\mathbf{Z_2} \times \mathbf{Z_{2s}}) \cong \mathrm{Aut}(\mathbf{Z_s}) \times \mathrm{Aut}(\mathbf{D_2})$; when s is even it is readily seen that the subgroup $\mathrm{Aut}(\mathbf{Z_2}) \times \mathrm{Aut}(\mathbf{Z_{2s}}) \cong \mathbf{Z_{2s}}^\star$ of automorphisms leaving both $\mathbf{Z_2} \times 1$ and $1 \times \mathbf{Z_{2s}}$ invariant has index 4 in $\mathrm{Aut}(\mathbf{Z_2} \times \mathbf{Z_{2s}})$. Moreover the automorphisms $(1,0) \longmapsto (1,s), (0,1) \longmapsto (0,1)$ and $(1,0) \longmapsto (1,0), (0,1) \longmapsto (1,1)$ generate a $\mathbf{D_2}$ complement for this subgroup. Case (4) is proved in similar fashion by considering the subgroup of automorphisms which leave both $\mathbf{Z_2} \times 1$ and $1 \times \mathbf{D_{2s}}$ invariant (this subgroup is isomorphic to $\mathrm{Aut}(\mathbf{D_{2s}})$).

The determination of $|\mathrm{Aut}_R^N(G)|$ requires a case-by-case analysis. For each case it should also be checked whether any element of $\ker(\hat{\lambda}_N)$ inverts N as this gives the number of strong equivalence classes which is needed to completely fill out the table.

We illustrate the arguments by working out the case $\mathcal{Q} = (A1,k)$ in detail. Since $k = 1$ or 2, we have $\Pi(A1,k) = \langle a,t \mid a^k = 1, a \leftrightarrow t \rangle \cong \mathbf{Z_k} \times \mathbf{Z}$. From Table 4 we see that $G \cong \mathbf{Z_{ks}}$ where s is an odd integer. Choose $N = \langle t^s \rangle$ as the representative for $\widetilde{\mathcal{K}}^{(A1,k)}(G)/\mathrm{Aut}_R(\Pi(A1,k))$, which is a one element set by the above. Referring to Lemma 4.1(1) we find that $\mathrm{Aut}_R^N(\Pi(A1,k))$ is generated by ϕ_1 (if $k = 1$ then ϕ_2 is the identity, if $k = 2$ then ϕ_2 does not stabilize N). Observe that ϕ_1 induces inversion on $G = \Pi(A1,k)/N \cong \mathbf{Z_{ks}}$ and this is a nontrivial automorphism if and only if $s \neq 1$. We now apply the analogue of Corollary 1.2 for actions on $\tilde{V}_1$. If $s > 1$ then $\ker(\hat{\lambda}_N)$ is trivial (so "j_i" is 2). It follows that

$$|\widetilde{\mathcal{E}}^{(A1,k)}(\mathbf{Z_{ks}})| = \frac{|\mathrm{Aut}(\mathbf{Z_{ks}})|}{2} = \frac{\phi(s)}{2} \quad \text{and} \quad |\widetilde{\mathcal{SE}}^{(A1,k)}(\mathbf{Z_{ks}})| = 2|\widetilde{\mathcal{E}}^{(A1,k)}(\mathbf{Z_{ks}})|.$$

On the other hand if $s = 1$ then $k = 2$ (since $G \cong \mathbf{Z_k}$). In this case $\mathrm{Aut}_R^N(G) = 1$ and $\phi_1 \in \ker(\hat{\lambda}_N)$ inverts N. Therefore we have

$$|\widetilde{\mathcal{E}}^{(A1,2)}(\mathbf{Z_2})| = |\mathrm{Aut}(\mathbf{Z_2})| = 1 = |\widetilde{\mathcal{SE}}^{(A1,2)}(\mathbf{Z_2})|. \qquad \blacksquare$$

5. **Orientation reversing actions on the solid torus.** In this section we classify the orientation-reversing G-actions on V_1 using an approach similar to that of Section 4.

THEOREM 5.1. *The finite groups G which act on the solid torus V_1 reversing orientation are those listed in Table 5. For each nonorientable quotient type $\mathcal{Q}$ the number $|\mathcal{WE}^{\mathcal{Q}}(G)|$ of weak equivalence classes is given and for each $N \in \mathcal{K}^{\mathcal{Q}}(G)$ the order of $\mathrm{Aut}_R^N(G)$ is also given. If k is a positive integer then $\delta(k)$ equals 1 if $k \leq 2$ and it equals 2 if $k > 2$. (Equivalently, $\delta(k) = (2, \phi(k))$.)*

Q	N	Conditions	$G = \Pi(Q)/N$	$\lvert G\rvert$	$\lvert\mathcal{WE}^{Q}(G)\rvert$	$\lvert \mathrm{Aut}_R^N(G)\rvert$
(A1,k)	$\langle t^{2s}\rangle$ $\langle a^{\frac{k}{2}} t^{2s}\rangle$	– k=0(2)	$\mathbf{Z_k \circ Z_{2s}}$ $\mathbf{Z_k \circ Z_{4s}}/\sim$	$2ks$ $2ks$	1	$k\delta(k)\delta(2s)$ $k\delta(k)\delta(2s)$
(A2,k)	$\langle t^{s}\rangle$ $\langle a^{\frac{k}{2}} t^{s}\rangle$	– k=0(2)	$\mathbf{D_k \times Z_s}$ $\mathbf{D_k \times Z_{2s}}/\sim$	$2ks$ $2ks$	2 if $k = 0$ (2) & $s = 1$ (2) 1 otherwise	$\delta(2k)\delta(s)$ $2\delta(2s)$
$(A3,k)$	$\langle t^{2s}\rangle$ $\langle (bt)^k t^{s-k}\rangle$	– s=k (2)	$\mathbf{D_k \circ Z_{2s}}$ $\mathbf{D_k \circ Z_{2s}}/\sim$	$4ks$ $2ks$	1	$\delta(2k)\delta(2s)$ $\delta(2k)\delta(2s)$
$(B1,k)$	$\langle (bc)^{2s}\rangle$ $\langle a^{\frac{k}{2}} (bc)^{2s}\rangle$	– k=0 (2)	$\mathbf{Z_k \circ D_{2s}}$ $\mathbf{Z_k \circ D_{4s}}/\sim$	$4ks$ $4ks$	1	$k\delta(2s)\delta(k)$ $2k\delta(k)$
$(B2,k)$	$\langle (ab)^{2s}\rangle$ $\langle a^{k} (ab)^{2s}\rangle$	– k=0 (2)	$\mathbf{(Z_k \circ Z_{2s})\circ Z_2}$ $\mathbf{(Z_k \circ Z_{4s}}/\sim\mathbf{)\circ Z_2}$	$4ks$ $4ks$	1	$ks\delta(2k)\delta(2s)$ $2ks\delta(k)\delta(2s)$
$(B3,k)$	$\langle (ac)^{s}\rangle$ $\langle b^{\frac{k}{2}} (ac)^{s}\rangle$	– k=0 (2)	$\mathbf{Z_k \times D_s}$ $\mathbf{Z_k \times D_{2s}}/\sim$	$2ks$ $2ks$	2 if $k = 0$ (2) & $s = 1$ (2) 1 otherwise	$\delta(k)\delta(2s)$ $2\delta(k)$
$(B4,k)$	$\langle (abab^{-1})^{s}\rangle$ $\langle b^{k-s} (ab)^{s}\rangle$	– s=k (2)	$\mathbf{D_s \circ Z_{2k}}$ $\mathbf{D_s \circ Z_{2k}}/\sim$	$4ks$ $2ks$	1	$\delta(2k)\delta(2s)$ $\delta(2k)\delta(2s)$
$(B5,k)$	$\langle (ab^{-1})^{s}\rangle$ $\langle b^{k} (ab^{-1})^{s}\rangle$	– k=0 (2)	$\mathbf{Z_s \circ Z_{2k}}$ $\mathbf{Z_{2s} \circ Z_{2k}}/\sim$	$2ks$ $2ks$	2 if $k = 0$ (2) & $s = 1$ (2) 1 otherwise	$s\delta(2k)\delta(s)$ $2s\delta(k)\delta(2s)$
$(B6,k)$	$\langle (ad)^{s}\rangle$ $\langle (bc)^{\frac{k}{2}} (ad)^{s}\rangle$	– k=0 (2)	$\mathbf{D_k \times D_s}$ $\mathbf{D_k \times D_{2s}}/\sim$	$4ks$ $4ks$	2 if $k = 0$ (2) & $s = 1$ (2) 1 otherwise	$\delta(2k)\delta(2s)$ $2\delta(2k)$
$(B7,k)$	$\langle (abab^{-1})^{s}\rangle$ $\langle b^{k-s} (ab)^{s}\rangle$	– s=k (2)	$\mathbf{D_s \circ D_{2k}}$ $\mathbf{D_s \circ D_{2k}}/\sim$	$8ks$ $4ks$	1	$\delta(2k)\delta(2s)$ $\delta(2k)\delta(2s)$
$(B8,k)$	$\langle (ac^{-1})^{s}\rangle$ $\langle a^{k} (ac^{-1})^{s}\rangle$	– k=0 (2)	$\mathbf{Z_s \circ D_{2k}}$ $\mathbf{Z_{2s} \circ D_{2k}}/\sim$	$4ks$ $4ks$	1	$s\delta(2k)\delta(s)$ $2s\delta(2k)\delta(s)$

TABLE 5. The orientation reversing G-actions on V_1.

In the table there are two infinite families of finite groups G for each Q and they will be denoted by $\{G^{Q}(s)\}$ and $\{\tilde{G}^{Q}(s)\}$. Each group $G^{Q}(s)$ in the first family has a simple description as a semidirect product of cyclic and/or dihedral groups while each group $\tilde{G}^{Q}(s)$ in the second family is a quotient of a group from the first family obtained by identifying a pair of involutions from the two semidirect product factors—this construction is indicated by "$/\sim$". In most cases the semidirect product action is given by "inversion" on the first factor; if the first factor is cyclic this is the involution which inverts each element, and if it is dihedral this is the involution which interchanges a pair of order two generators. In each case precise presentations for G can be obtained by referring to the corresponding presentation in Table 2.

PROOF. An orientation reversing G-action on V_1 with nonorientable quotient type Q corresponds to an infinite cyclic normal subgroup N in $\Pi(Q)$ which is contained in $\Pi^+(Q)$. Using the descriptions of $\Pi(Q)$ and $\Pi^+(Q)$ given in Table 2, it is straightforward to find all of the possible subgroups N for each nonorientable quotient type Q. In this way the information in all but the last two columns of the table can be determined.

To compute $|\mathcal{WE}^{Q}(G)|$ we consider $\mathcal{K}^{Q}(G)$. If $G^{Q}(s) \cong G^{Q}(s')$ or $\tilde{G}^{Q}(s) \cong \tilde{G}^{Q}(s')$ then by comparing orders of the groups (and since k is fixed) it follows that $s = s'$. Therefore $\mathcal{K}^{Q}(G)$ has cardinality at most two, and the cardinality equals two if and only if $G^{Q}(s) \cong \tilde{G}^{Q}(s')$. When k is even and s is odd such isomorphisms are easily found if Q is one of $(A2,k)$, $(B3,k)$, $(B5,k)$ or $(B6,k)$:

$$\tilde{G}^{(A2,k)}(s) \cong \mathbf{D_k} \times \mathbf{Z_{2s}}/\sim \;\cong \mathbf{D_k} \times (\mathbf{Z_2} \times \mathbf{Z_s})/\sim \;\cong \mathbf{D_k} \times \mathbf{Z_s} \cong G^{(A2,k)}(s)$$
$$\tilde{G}^{(B3,k)}(s) \cong \mathbf{Z_k} \times \mathbf{D_{2s}}/\sim \;\cong \mathbf{Z_k} \times (\mathbf{Z_2} \times \mathbf{D_s})/\sim \;\cong \mathbf{Z_k} \times \mathbf{D_s} \cong G^{(B3,k)}(s)$$
$$\tilde{G}^{(B5,k)}(s) \cong \mathbf{Z_{2s}} \circ \mathbf{Z_{2k}}/\sim \;\cong (\mathbf{Z_s} \times \mathbf{Z_2}) \circ \mathbf{Z_{2k}}/\sim \;\cong \mathbf{Z_s} \circ \mathbf{Z_{2k}} \cong G^{(B5,k)}(s)$$
$$\tilde{G}^{(B6,k)}(s) \cong \mathbf{D_k} \times \mathbf{D_{2s}}/\sim \;\cong \mathbf{D_k} \times (\mathbf{Z_2} \times \mathbf{D_s})/\sim \;\cong \mathbf{D_k} \times \mathbf{D_s} \cong G^{(B6,k)}(s).$$

In these four special cases it is readily seen (using Lemma 4.1) that $\mathrm{Aut}_R(\Pi(\mathcal{Q}))$ acts trivially on the two element set $\mathcal{K}^{\mathcal{Q}}(G)$, and therefore $|\mathcal{WE}^{\mathcal{Q}}(G)| = 2$ by Theorem 1.1. In each of the remaining cases it can be shown that $G^{\mathcal{Q}}(s) \ncong \tilde{G}^{\mathcal{Q}}(s')$ so that both $\mathcal{K}^{\mathcal{Q}}(G)$ and $\mathcal{WE}^{\mathcal{Q}}(G)$ consist of a single element. We illustrate this by considering case $(B6,k)$: If k and s are even then the center of $G^{(B6,k)}(s) \cong \mathbf{D_k} \times \mathbf{D_s}$ has double the order of the center of $\tilde{G}^{(B6,k)}(s) \cong \mathbf{D_k} \times \mathbf{D_{2s}}/\sim$. It follows that $G^{(B6,k)}(s) \cong \tilde{G}^{(B6,k)}(s)$ only if s is odd (and k is even), and this is one of the four special cases discussed above. The other quotient types are dealt with by ad hoc arguments generally somewhat more complicated than this.

The information in the last column of the table is obtained by projecting the generators for $\mathrm{Aut}_R(\Pi(\mathcal{Q}))$ given in Lemma 4.1 into $\mathrm{Aut}(G)$ and using them to determine the order of the image. For example, $\mathrm{Aut}_R(\Pi(A2,k)) \cong \mathbf{Z_2} \times \mathbf{Z}_{\delta(\mathbf{2k})}$ and its image in $\mathrm{Aut}(G^{(A2,k)}(s))$ is readily seen to be $\mathbf{Z}_{\delta(\mathbf{s})} \times \mathbf{Z}_{\delta(\mathbf{2k})}$ (ϕ_1 projects to the trivial automorphism if and only if $s \leq 2$). Thus, for $G = G^{(A2,k)}(s)$ it follows that $|\mathrm{Aut}_N^R(G)| = \delta(s)\delta(2k)$. Also, for $G = \tilde{G}^{(A2,k)}(s)$ we find that $\mathrm{Aut}_R^N(G) \cong \mathbf{Z}_{\delta(\mathbf{2s})} \times \mathbf{Z_2}$. For the other cases $\mathrm{Aut}_R(\Pi(\mathcal{Q}))$ is somewhat more complicated to describe as is evident from the table. ∎

The description of the orientation-reversing actions on V_1 given in Table 5 is less explicit than the description of actions on $\tilde{V}_1$ given in Table 3 in two ways. First we have not tabulated the occurences of the isomorphisms classes of the groups G. The reason for this is that there are now many more of these isomorphisms classes and the various isomorphisms between groups with different quotient types are too numerous to categorize. Here are a few examples:

(i) $G^{(A1,k)}(s) \cong G^{(B5,s)}(k)$
(ii) $G^{(A3,k)}(s) \cong \tilde{G}^{(A3,2k)}(s)$ if k is even and s is odd
(iii) $G^{(B7,k)}(s) \cong \tilde{G}^{(B6,k)}(2s)$ if k is odd
(iv) $G^{(A3,k)}(s) \cong \tilde{G}^{(A2,2k)}(s)$ if s is odd

In general, all of the possible isomorphisms are easy to find: some, such as (i), are obtained by interchanging k and s; some, such as (ii), are based on the isomorphism $\mathbf{Z_{2k}} \cong \mathbf{Z_2} \times \mathbf{Z_k}$ for k odd; some are based on the isomorphism $\mathbf{D_{2k}} \cong \mathbf{Z_2} \times \mathbf{D_k}$ for k odd, such as (iii); and some, such as (iv), are based on the isomorphism $\mathbf{D_k} \circ \mathbf{Z_2} \cong \mathbf{D_{2k}}$ where the semidirect product action is given by inversion. Also, in Table 5 we have not enumerated the equivalence and strong equivalence classes of actions. The reason for this is that the automorphism groups of the groups G are more complicated and their orders are not suitable for listing. However, for any of the groups G which arise, if the order of Aut(G) is known then the number of equivalence classes are easily determined using $|\operatorname{Aut}_R^N(G)|$ as given in Table 5. The strong equivalence classes of actions can be determined using the results of Lemma 4.1. We now give corollaries carrying out this procedure for two special classes of groups.

COROLLARY 5.2. *If V_1 admits an orientation reversing G-action where $|G| \equiv 2 (\mathrm{mod}\ 4)$ then $G \cong \mathbf{D_k} \times \mathbf{Z_s}$ for some odd integers k and s. The quotient types and equivalence classes of such actions are enumerated in Table 6.*

G	Q	$\lvert \mathcal{WE}^Q(G)\rvert$	$\lvert \mathcal{E}^Q(G)\rvert$	$\lvert \mathcal{SE}^Q(G)\rvert$
$\mathbf{D_k} \times \mathbf{Z_s}$	$(A1,k)$; $(A2,k)$;	1; 1;	$\frac{\phi(k)\phi(s)}{\delta(k)\delta(s)}$; $\frac{k\phi(k)\phi(s)}{\delta(k)\delta(s)}$;	$\frac{\phi(k)\phi(s)}{\delta(k)}$; $\frac{k\phi(k)\phi(s)}{\delta(k)}$;
	$(A3,k)$; $(B3,s)$;	1; 1;	$\frac{k\phi(k)\phi(s)}{\delta(k)\delta(s)}$; $\frac{k\phi(k)\phi(s)}{\delta(k)\delta(s)}$;	$\frac{k\phi(k)\phi(s)}{\delta(k)}$; $\frac{k\phi(k)\phi(s)}{\delta(s)}$;
	$(B4,s)$; $(B5,s)$	1; 1	$\frac{k\phi(k)\phi(s)}{\delta(k)\delta(s)}$; $\frac{\phi(k)\phi(s)}{\delta(k)\delta(s)}$	$\frac{k\phi(k)\phi(s)}{\delta(s)}$; $\frac{\phi(k)\phi(s)}{\delta(s)}$

TABLE 6. Orientation reversing G-actions on V_1 where $|G| = 2ks$, k and s odd.

PROOF. Let the order of G be $2ks$ where k and s are odd. If G acts reversing orientation on V_1 then a quick study of Table 5 shows that the only possible quotient types are those listed in Table 6 and that G is isomorphic to $\mathbf{D_k} \times \mathbf{Z_s}$. Moreover $|\mathcal{WE}^Q(G)| = 1$, so the number of equivalence classes in $\mathcal{E}^Q(G)$ is $[\operatorname{Aut}(G) : \operatorname{Aut}_R^N(G)]$ by Corollary 1.2. In this case it is easy to see that $\mathbf{D_k} \times \{1\}$ and $\{1\} \times \mathbf{Z_s}$ are characteristic subgroups of $\mathbf{D_k} \times \mathbf{Z_s}$ so that $\operatorname{Aut}(G) \cong \operatorname{Aut}(\mathbf{D_k}) \times \operatorname{Aut}(\mathbf{Z_s})$. Thus $|\operatorname{Aut}(G)| = k\phi(k)\phi(s)$ and using the values of $|\operatorname{Aut}_R^N(G)|$ from Table 5 we obtain $|\mathcal{E}^Q(G)|$. To compute $|\mathcal{SE}^Q(G)|$ we again refer to Corollary 1.2 and to Lemma 4.1. For each of the six quotient types arising here, the automorphism ϕ_1 of $\operatorname{Aut}_R(\Pi(Q))$ (as described in Lemma 4.1) can be seen to invert the infinite cyclic subgroup N. If $k \leq 2$ this automorphism is in the kernel of $\hat{\lambda}_N$, otherwise it can be seen that no element of $\ker(\hat{\lambda}_N)$ can invert N. Therefore, for the quotient types $(A1,k)$, $(A2,k)$ and $(A3,k)$ the number of strong equivalence classes is obtained by multiplying the number of equivalence classes by $\delta(s)$, and for the quotient types $(B3,k)$,$(B4,k)$ and $(B5,k)$ one multiplies by $\delta(k)$. ∎

COROLLARY 5.3. *The quotient types and equivalence classes of all orientation reversing actions of finite abelian groups on V_1 are enumerated in Table 7.*

PROOF. A perusal of Table 5 readily leads to a description of the abelian groups G which can act on V_1 and the corresponding quotient types and numbers of weak equivalence classes. The computations for the equivalence and strong equivalence classes are carried out exactly as in the previous corollary. We only remark that $\operatorname{Aut}(\mathbf{Z_2} \times \mathbf{Z_2} \times \mathbf{Z_{2s}})$

G	Q	$\lvert \mathcal{WE}^{Q}(G)\rvert$	$\lvert \mathcal{E}^{Q}(G)\rvert$	$\lvert \mathcal{SE}^{Q}(G)\rvert$
$\mathbf{Z}_2$	$(A1,1);(A2,1);(A3,1);$ $(B3,1);(B4,1);(B5,1)$	$1;1;1;$ $1;1;1$	$1;1;1;$ $1;1;1$	$1;1;1;$ $1;1;1$
$\mathbf{Z}_2\times\mathbf{Z}_2$	$(A1,2);(A2,1);(A2,2);(A3,1);$ $(B1,1);(B2,1);(B3,1);(B3,2);$ $(B4,1);(B5,1);$ $(B6,1);(B7,1);(B8,1)$	$1;1;2;1;$ $1;1;1;2;$ $1;1;$ $1;1;1$	$3;6;6;6;$ $6;6;3;9;$ $6;3;$ $6;6;6$	$3;6;6;6;$ $6;6;3;9;$ $6;3;$ $6;6;6$
$\mathbf{Z}_2\times\mathbf{Z}_2\times\mathbf{Z}_2$	$(A2,2);(B1,2);(B3,2);$ $(B6,1);(B6,2);(B7,1);(B8,2)$	$1;1;1$ $1;2;1;1;$	$84;84;84;$ $84;126;168;84$	$84;84;84;$ $84;126;168;84$
$\mathbf{Z}_2\times\mathbf{Z}_2\times\mathbf{Z}_2\times\mathbf{Z}_2$	$(B6,2)$	1	$5,040$	$5,040$
$\mathbf{Z}_{2s}$, s even	$(A1,1);(A1,2);$ $(B5,s)$	$1;1;$ 2	$\phi(s);\frac{\phi(s)}{\delta(s)};$ $\phi(s)+\frac{\phi(s)}{\delta(s)}$	$2\phi(s);\phi(s);$ $3\phi(s)$
$\mathbf{Z}_{2s}$, s odd	$(A1,1);(A2,1);(A3,1);$ $(B3,s);(B4,s);(B5,s)$	$1;1;1;$ $1;1;1$	$\frac{1}{2}\phi(s);\frac{1}{2}\phi(s);\frac{1}{2}\phi(s);$ $\frac{1}{2}\phi(s);\frac{1}{2}\phi(s);\frac{1}{2}\phi(s)$	$\phi(s);\phi(s);\phi(s);$ $\frac{1}{2}\phi(s);\frac{1}{2}\phi(s);\frac{1}{2}\phi(s)$
$\mathbf{Z}_2\times\mathbf{Z}_{2s}$, s even	$(A1,2);(A2,1);$ $(A2,2);(A3,1);$ $(B3,2s);(B4,s);(B5,s)$	$1;1;$ $1;1;$ $2;1;1$	$2\phi(s);4\phi(s);$ $2\phi(s);4\phi(s);$ $6\phi(s);4\phi(s);2\phi(s)$	$4\phi(s);8\phi(s);$ $4\phi(s);8\phi(s);$ $8\phi(s);4\phi(s);2\phi(s)$
$\mathbf{Z}_2\times\mathbf{Z}_{2s}$, s odd	$(A1,2);(A2,1);(A2,2);$ $(A3,1);(B3,2s);$ $(B3,s);(B4,s);(B5,s)$	$1;1;2;$ $1;2;$ $1;1;1$	$\frac{3}{2}\phi(s);3\phi(s);3\phi(s);$ $3\phi(s);\frac{9}{2}\phi(s);$ $\frac{3}{2}\phi(s);3\phi(s);\frac{3}{2}\phi(s)$	$3\phi(s);6\phi(s);6\phi(s);$ $6\phi(s);6\phi(s);$ $3\phi(s);3\phi(s);\frac{3}{2}\phi(s)$
$\mathbf{Z}_2\times\mathbf{Z}_2\times\mathbf{Z}_{2s}$, s even	$(A2,2);(B3,2s)$	$1;1$	$48\phi(s);48\phi(s)$	$96\phi(s);96\phi(s)$
$\mathbf{Z}_2\times\mathbf{Z}_2\times\mathbf{Z}_{2s}$, s odd	$(A2,2);(B3,2s)$	$1;1$	$42\phi(s);42\phi(s)$	$84\phi(s);84\phi(s)$

TABLE 7. Orientation reversing actions of abelian groups on V_1. ($s \geq 2$.)

has order $168\phi(s)$ if s is odd, and order $192\phi(s)$ if s is even. (This is relevant for the bottom two rows of the table.) ■

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