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A JORDAN-HÖLDER THEOREM FOR FINITARY GROUPS

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ABSTRACT. Let V be any left vector space over any division ring D and let G be any group of finitary linear maps of V. Then the $D-G$ bimodule V satisfies a Jordan-Hölder theorem. Specifically, there is a bijection between the G-nontrivial factors in two composition series of V such that corresponding factors are isomorphic as $D-G$ bimodules. This cannot be extended to cover the G-trivial factors.

Below D denotes a division ring and V a left vector space over D. The finitary general linear group $\mathrm{FGL}(V)$ or $\mathrm{FAut}_D\, V$ over V is the subgroup of $\mathrm{Aut}_D\, V$ of D-automorphisms g of V such that $[V,g]=V(g-1)$ has finite (left) dimension over D. By a finitary skew linear group we mean any subgroup G of $\mathrm{FGL}(V)$ for any D and V. Simultaneously in North America, Western Europe and Russia, work on finitary (linear and skew-linear) groups has mushroomed during the last couple of years or so. Here we prove a very general but simple result that so far seems to have been overlooked.

With the notation above let G be a subgroup of $\mathrm{FGL}(V)$. Then V is a $D-G$ (bi)module in the obvious way. This $D-G$ module V need not satisfy the Jordan-Hölder Theorem; for let D be any division ring (including a field) and let $V=\prod_{1\leq i\leq\infty} Dv_i$ be the cartesian product of the $\aleph_0$ one-dimensional left D-spaces and set $G=\langle 1\rangle$. A composition series of the $D-G$ module V is simply a series of subspaces of V with 1-dimensional factors. (We use the word 'series' in the very general sense of P. Hall, see §1.2 of [2].) Let B be any basis of V. Then $|B|=2^{\aleph_0}$. Well order B and set $V_\sigma=\sum_{\alpha<\sigma} Db_\alpha$, where $B=\{b_\alpha:\alpha<\rho\}$ and α, σ and ρ are ordinals. Then $\{V_\sigma : 0\leq\sigma\leq\rho\}$ is an ascending composition series of V with $2^{\aleph_0}$ factors. Similarly $\{\sum_{\alpha\geq\sigma} Db_\alpha : 0\leq\sigma\leq\rho\}$ is a descending composition series of V with $2^{\aleph_0}$ factors. Now set $W_i=\prod_{j>i} Dv_j$. Then the intersection of the W_i is $\{0\}$ and $\{W_i : 0\leq i\leq\omega\}$ is a descending composition series of V with only $\aleph_0$ factors. (Clearly there is no ascending composition series of V with $\aleph_0$ factors.) This example exhibits essentially the only reason that the Jordan-Hölder 'Theorem' here breaks down.

THEOREM. *Let G be any subgroup of* $\mathrm{FGL}(V)$, *the notation being as above. Then for any two composition series of the $D-G$ module V there is a one-to-one correspondence between the sets of non-trivial factors of the two series such that corresponding factors are isomorphic as $D-G$ modules.*

For G a subgroup of $\mathrm{FGL}(V)$, by a trivial factor (or section) of V we mean one upon which G acts as the trivial group, not necessarily a zero factor. The example above shows that the word 'non-trivial' in the theorem cannot be deleted.

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PROOF. Let X be a finitely generated subgroup of G. Then $d = \dim_D[V, X]$ is finite, see [3] §1, so X can act non-trivially on at most d non-zero factors of any $D - G$ series of V. Hence let

$$(1) \qquad \{0\} = N_0 \leq M_1 < N_1 \leq M_2 < N_2 \leq \cdots \leq M_r < N_r \leq M_{r+1} = V \quad \text{and}$$

$$(2) \qquad \{0\} = Q_0 \leq P_1 < Q_1 \leq P_2 < Q_2 \leq \cdots \leq P_s < Q_s \leq P_{s+1} = V$$

by parts of the two given composition series of V, where the N_i/M_i and the Q_j/P_j are exactly the factors of the given series upon which X does not act as a stability group (see [3] §2 for definition). Necessarily they are $D - G$ irreducible. Further X stabilizes $D - X$ series in all the factors M_i/N_{i-1} and P_j/Q_{j-1} and $r, s \leq d$ are finite. The $D - G$ series (1) and (2) have isomorphic refinements. Thus $r = s$ and there is an element σ of $\mathrm{Sym}(r)$ such that N_i/M_i and $Q_{i\sigma}/P_{i\sigma}$ are $D - G$ isomorphic for each i.

Let Θ and Φ index the non-trivial factors in the two given composition series, and let Θ_X and Φ_X be the subsets of Θ and Φ, respectively, indexing the factors upon which X does not act as a stability group. We have constructed above a bijection $\psi_X\colon \Theta_X \longrightarrow \Phi_X$ such that corresponding factors are $D - G$ isomorphic. Thus the set Ψ_X of all such bijections is finite (trivially) and non-empty. If $Y \supseteq X$ is also a finitely generated subgroup of G, then clearly $\Theta_Y \supseteq \Theta_X$ and $\Phi_Y \supseteq \Phi_X$, and restriction $\mathrm{res}\,^Y{\downarrow}_X$ maps Ψ_Y to Ψ_X. Apply [3] 2.2a) to the composition factors of V. It follows that $\Theta = \bigcup_X \Theta_X$ and $\Phi = \bigcup_X \Phi_X$. The Ψ_X and the maps $\mathrm{res}\,^Y{\downarrow}_X$ form an inverse system of non-empty finite sets over a directed set, and hence the corresponding inverse limit is not empty (*e.g.* [1] §1.K). Let (ψ_X) lie in this inverse limit, where $\psi_X \in \Psi_X$ for each X as above. Define a map $\psi\colon \Theta \longrightarrow \Phi$ by setting $\theta\psi = \theta\psi_X$ for any θ in Φ_X. By the above ψ is a well-defined bijection of the required type.

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