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RESEARCH ARTICLE

Polish spaces of Banach spaces

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Abstract
We present and thoroughly study natural Polish spaces of separable Banach spaces. These spaces are defined as spaces of norms, respectively pseudonorms, on the countable infinite-dimensional rational vector space. We provide an exhaustive comparison of these spaces with admissible topologies recently introduced by Godefroy and Saint-Raymond and show that Borel complexities differ little with respect to these two topological approaches.
We investigate generic properties in these spaces and compare them with those in admissible topologies, confirming the suspicion of Godefroy and Saint-Raymond that they depend on the choice of the admissible topology.

Contents

1. Introduction

Banach spaces and descriptive set theory have a long history of mutual interactions. Explicit use of descriptive set theory to Banach space theory can be traced back at least to the seminal papers of Bourgain ([9, 8]), where it has become apparent that descriptive set theory is an indispensable tool for universality problems. That is a theme that has been investigated by researchers working with Banach spaces ever since (see, e.g., [2, 14] and references therein).

As it eventually turned out, 'Descriptive set theory of Banach spaces' is an interesting and rich subject and has received considerable attention in recent years. One of the starting points was the idea of Bossard of coding separable Banach spaces in [6, 7]. His approach, which can be considered standard, was to choose some universal separable Banach space X, such as $C(2^{\mathbb{N}})$, and consider the Effros-Borel space $F(X)$. Recall that this is the set of closed subsets of X equipped with a certain σ-algebra that makes $F(X)$ a standard Borel space: that is, a measurable space that is isomorphic, as a measurable space, to a Polish space equipped with the σ-algebra of Borel sets. It is then not too difficult to show that the subset $SB(X) \subseteq F(X)$ consisting of all closed linear subspaces is a Borel subset and therefore a standard Borel space itself.

Although this approach has found numerous significant applications in Banach space theory, its drawback is that there is no canonical or natural (Polish) topology on $SB(X)$. So although one can ask whether a given class of Banach spaces is Borel or not, the question about the exact complexity of that particular class is meaningless. Let us specify this. Many of the applications may be interpreted as computing co/analytic sets of Banach spaces and deriving consequences from this; this concerns especially various universality results (see, e.g., [14, Chapter 7] and [23]). Having a topology allows us to separate two classes of Banach spaces, which are both known to be Borel (we comment more on this issue in the sequel [13]).

Moreover, one of the active and ongoing research streams is to find out whether for a particular Banach space its isomorphism class is Borel or not (see, e.g., [25, 18] or the survey [17] and references therein): spaces with Borel isomorphism classes are rare and can be considered simply definable (up to isomorphism). It is then desirable, for spaces whose classes are Borel, to have a finer description of how simply definable they are (see, e.g., [18, Problem 3]).

A recent work [19] of Godefroy and Saint-Raymond addresses this general issue of associating a natural topology to the set of codes of Banach spaces. They still work with the space $SB(X)$, but among the many Polish topologies on $SB(X)$ giving the Effros-Borel structure, they select some particular subclass that is called *admissible topologies*. Although no particular admissible topology is canonical, the set of requirements put on this class guarantees that the exact Borel complexities vary little.

This paper presents an alternative approach by considering a concrete and natural Polish space (and some variants of it) of separable Banach spaces, which is convenient to work with. We have three main reasons and advantages for that in mind. The first one, which was our original motivation, is to further push the programme initiated by Godefroy and Saint-Raymond on computing precise Borel complexities of various classes of Banach spaces. It turns out that in our new space, the computations of Borel complexities are usually as straightforward as they could be, and besides many new results, we are also able to improve several estimates already obtained in [19] (see also [16] for additional results in this direction). Most of these results are contained in the sequel to this paper [13]. It should also be mentioned that computing exact Borel complexities has been a traditional research topic in analysis and topology of independent interest (see [21, Chapter 23] for a comprehensive but already outdated list of examples), and among our contributions in that regard, presented in [13], are new elegant characterizations of the Hilbert space ℓ_2. Briefly, ℓ_2 is the unique (up to isometry) infinite-dimensional separable Banach space with a closed isometry class, and it is the unique (up to isomorphism) infinite-dimensional separable Banach space with an F_σ isomorphism class. Recall that Bossard [7, Problem 2.9] originally asked whether ℓ_2 is the unique space with a Borel isomorphism class. Although this is now known to be false, we can show that no other Banach space can have such a simple isomorphism class.

The remaining two reasons have different origins; one is functional analytic, and the other comes from logic. For the former, the feature of the topology we work with is that basic open sets are essentially definitions of finite-dimensional Banach spaces up to ε-isomorphism, where $\varepsilon > 0$ is arbitrarily small. This connects this topological approach with the local theory of Banach spaces. The basic manifestation of this is that the finite representability of Banach spaces is expressed in elementary topological terms and, for example, leads to a natural reformulation of the Dvoretzky theorem that the infinite-dimensional separable Hilbert space is contained in the closure of the isometry class of every infinite-dimensional separable Banach space (see Corollary 2.11). Further applications to the local theory might be a subject of future research.

Finally, our approach brings closer the two different interactions of logic with Banach space theory that had not interacted significantly: that is, the descriptive set theory of Banach spaces and the continuous model theory of Banach spaces. Continuous model theory, or model theory of metric structures, is a generalization of classical model theory to structures that are inherently metric and has its origins, motivations as well as most of the applications in Banach space theory. We refer the reader to [4] for an introduction and more motivation. Our space of Banach spaces is closely related to how countable, respectively separable models are coded in classical, respectively continuous model theory (see, e.g., [15, Section 3.6] for the classical case and [5, Section 4] for the metric case). Moreover, the exact Borel complexities that our space allows us to compute are directly related to the López-Escobar theorem from continuous infinitary logic, which connects such complexities with the complexities of formulas that define the corresponding classes (we refer to [5, Section 6]). It may also be of interest for future research to investigate the relation, for a given Banach space, between having an isometry class of low Borel complexity and having an axiomatization in continuous first-order logic.

Having motivated our approach, let us now outline some more details and the main results contained in this paper. Informally, the space we introduce is the space of all norms, respectively pseudonorms, on the space of all finitely supported sequences of rational numbers – the unique infinite-dimensional vector space over $\mathbb{Q}$ with a countable Hamel basis. This is also, in spirit, similar to how (for instance) Vershik topologized the space of all Polish metric spaces ([26]), or how Grigorchuk topologized the space of all n-generated, respectively finitely generated, groups ([20]).

This space has already appeared in previous works of the authors in [11] and [12] as a useful coding of Banach spaces. Here we investigate it further.

Some of the main results of this paper are listed now. The first theorem below presents the main part of the comparison of the space of norms with admissible topologies, whose proof is the core of Section 3, where also all other comparison results are proved.

Theorem A. *There is a $\mathbf{\Sigma}^0_2$-measurable mapping from the space of norms to any admissible topology of Godefroy and Saint-Raymond that associates to each norm a space isometric to the space that the norm defines, and vice versa.*

Therefore, while the exact Borel complexities are more or less independent of our coding or the choice of the admissible topology, some finer topological properties, such as being meager or comeager (this is mentioned below), or the description of the topological closures, are not. We obtain a neat characterization of topological closures in the spaces of norms and pseudonorms in terms of finite representability; we refer the reader to Proposition 2.9.

Then, directly motivated by [19, Problem 5.5], we investigate the generic properties in the spaces of norms and in admissible topologies. First we show the genericity of the Gurariĭ space in the space of norms and pseudonorms.

Theorem B. *The isometry class of the Gurariĭ space is a dense G_δ-set in the space of norms and pseudonorms: that is, the Gurariĭ space is the generic separable Banach space (see Theorem 4.1).*

Then we continue with a similar investigation for admissible topologies, and among other things, we show that the Gurariĭ space is not always generic.

Theorem C. *For any isometrically universal separable Banach space X, any admissible topology τ on $SB(X)$ and any infinite-dimensional Banach spaces Y and Z such that $Y \hookrightarrow Z$ and $Z \not\hookrightarrow Y \oplus F$ for every finite-dimensional space F, there exists a finer admissible topology $\tau' \supseteq \tau$ such that the class of Banach spaces isomorphic to Z is nowhere dense in $(SB(X), \tau')$. In particular, there exists an admissible topology in which the Gurariĭ space is meager (see Theorem 4.12).*

On the other hand, the isometry class of the Gurariĭ space $\mathbb{G}$, as a subset of $SB(\mathbb{G})$, is a dense G_δ-set in the Wijsman topology (see Theorem 4.10).

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1.1. Notation

Let us conclude the introduction by setting up some notation used throughout the paper.

Throughout the paper, we usually denote the Borel classes of low complexity by the traditional notation such as F_σ and G_δ, or even $F_{\sigma\delta}$ (countable intersection of F_σ sets) and $G_{\delta\sigma}$ (countable union of G_δ-sets). However, whenever it is more convenient or necessary, we use the notation $\mathbf{\Sigma}^0_\alpha$, respectively $\mathbf{\Pi}^0_\alpha$, where $\alpha < \omega_1$ (we refer to [21, Section 11] for this notation). We emphasize that open sets, respectively closed sets, are $\mathbf{\Sigma}^0_1$, respectively $\mathbf{\Pi}^0_1$, by this notation.

Moreover, given a class $\mathbf{\Gamma}$ of sets in metrizable spaces, we say that $f : X \to Y$ is $\mathbf{\Gamma}$-measurable if $f^{-1}(U) \in \mathbf{\Gamma}$ for every open set $U \subseteq Y$.

Given Banach spaces X and Y, we denote by $X \equiv Y$ (respectively $X \simeq Y$) the fact that those two spaces are linearly isometric (respectively isomorphic). We denote by $X \hookrightarrow Y$ the fact that Y contains a subspace isomorphic to X. For $K \geq 1$, a *K-isomorphism* $T : X \to Y$ is a linear map with $K^{-1}\|x\| \leq \|Tx\| \leq K\|x\|$, $x \in X$. If $x_1, \dots, x_n$ are linearly independent elements of X and $y_1, \dots, y_n \in Y$, we write $(Y, y_1, \dots, y_n) \overset{K}{\sim} (X, x_1, \dots, x_n)$ if the linear operator $T : \operatorname{span}\{x_1, \dots, x_n\} \to \operatorname{span}\{y_1, \dots, y_n\}$ sending x_i to y_i satisfies $\max\{\|T\|, \|T^{-1}\|\} < K$. If X has a canonical basis $(x_1, \dots, x_n)$, which is clear from the context, we just write $(Y, y_1, \dots, y_n) \overset{K}{\sim} X$ instead of $(Y, y_1, \dots, y_n) \overset{K}{\sim} (X, x_1, \dots, x_n)$. Moreover, if Y is clear from the context, we write $(y_1, \dots, y_n) \overset{K}{\sim} X$ instead of $(Y, y_1, \dots, y_n) \overset{K}{\sim} X$.

Throughout the text, ℓ_p^n denotes the n-dimensional ℓ_p-space: that is, the upper index denotes dimension. Finally, in order to avoid any confusion, we emphasize that if we write that a mapping is an 'isometry' or an 'isomorphism', we do not mean it is surjective if this is not explicitly mentioned.

2. The Polish spaces $\mathcal{P}_\infty$ and $\mathcal{B}$, and their basic properties

In this section we introduce the main notions of this paper: the Polish spaces of pseudonorms $\mathcal{P}$ (and $\mathcal{P}_\infty$) representing separable (infinite-dimensional) Banach spaces, and we recall the space of norms $\mathcal{B}$ that has appeared in our previous works [11, 12]. We show some interesting features of these spaces, such as the neat relation between finite representability and topological closures in these spaces; see Proposition 2.9 and its corollaries.

Let us start with the following idea of coding the class of separable (infinite-dimensional) Banach spaces. It is based on the idea already presented in our previous papers [11, 12], where the space $\mathcal{B}$ was defined.

By V, we shall denote the vector space over $\mathbb{Q}$ of all finitely supported sequences of rational numbers: that is, the unique infinite-dimensional vector space over $\mathbb{Q}$ with a countable Hamel basis $(e_n)_{n\in\mathbb{N}}$.

Definition 2.1. Let us denote by $\mathcal{P}$ the space of all pseudonorms on the vector space V. Since $\mathcal{P}$ is a closed subset of $\mathbb{R}^V$, this gives $\mathcal{P}$ the Polish topology inherited from $\mathbb{R}^V$. The subbasis of this topology is given by sets of the form $U[v, I] := \{\mu \in \mathcal{P} : \ \mu(v) \in I\}$, where $v \in V$ and I is an open interval.

We often identify $\mu \in \mathcal{P}$ with its extension to the pseudonorm on the space c_{00}: that is, the vector space over $\mathbb{R}$ of all finitely supported sequences of real numbers.

For every $\mu \in \mathcal{P}$, we denote by X_μ the Banach space given as the completion of the quotient space X/N, where $X = (c_{00}, \mu)$ and $N = \{x \in c_{00} : \ \mu(x) = 0\}$. In what follows, we often consider V as a subspace of X_μ: that is, we identify every $v \in V$ with its equivalence class $[v]_N \in X_\mu$.

By $\mathcal{P}_\infty$, we denote the set of those $\mu \in \mathcal{P}$ for which X_μ is infinite-dimensional Banach space. As we did in [11, 12], by $\mathcal{B}$, we denote the set of those $\mu \in \mathcal{P}_\infty$ for which the extension of μ to c_{00} is an actual norm: that is, the vectors $e_1, e_2, \dots$ are linearly independent in X_μ.

We endow $\mathcal{P}_\infty$ and $\mathcal{B}$ with the topologies inherited from $\mathcal{P}$.

Our first aim is to show that the topologies on $\mathcal{P}_\infty$ and $\mathcal{B}$ are Polish (see Corollary 2.5). This can be easily verified directly: here, we obtain it as a corollary of the fact that the relation $\overset{K}{\sim}$ defined before is open in $\mathcal{P}$, a very useful fact that will prove important many times in the paper.

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We need the following background first. Given a metric space (M, d), $\varepsilon > 0$ and $N, S \subseteq M$, we say that N is *ε-dense for S* if for every $x \in S$ there is $y \in N$ with $d(x, y) < \varepsilon$ (let us emphasize that we do not assume $N \subseteq S$). For further references, we recall the following well-known approximation lemma; for a proof, see for example [1, Lemma 12.1.11].

Lemma 2.2. *There is a function $\phi_1 : [0, 1) \to [0, 1)$ continuous at zero with $\phi_1(0) = 0$ such that whenever $T : E \to X$ is a linear operator between Banach spaces, $\varepsilon \in (0, 1)$, $M \subseteq E$ is ε-dense for S_E and*

$$\forall m \in M : |\|Tm\| - 1| < \varepsilon,$$

then T is a $(1 + \phi_1(\varepsilon))$-isomorphism between E and $T(E)$.

The following definition precises the notation $\stackrel{K}{\sim}$ defined in the introduction.

Definition 2.3. If $v_1, \dots, v_n \in V$ are given, for $\mu \in \mathcal{P}$, instead of $(X_\mu, v_1, \dots, v_n) \stackrel{K}{\sim} X$, we shall write $(\mu, v_1, \dots, v_n) \stackrel{K}{\sim} X$.

Lemma 2.4. *Let X be a Banach space with $\{x_1, \dots, x_n\} \subseteq X$ linearly independent, and let $v_1, \dots, v_n \in V$. Then for any $K > 1$, the set*

$$\mathcal{N}((x_i)_i, K, (v_i)_i) = \{\mu \in \mathcal{P}\colon (\mu, v_1, \dots, v_n) \stackrel{K}{\sim} (X, x_1, \dots, x_n)\}$$

is open in $\mathcal{P}$.

In particular, the set of those $\mu \in \mathcal{P}$ for which the set $\{v_1, \dots, v_n\}$ is linearly independent in X_μ is open in $\mathcal{P}$.

Proof. Pick some $\mu \in \mathcal{N}((x_i)_i, K, (v_i)_i)$. By definition, the linear map T sending v_i to $x_i \in X$, $i \le n$, is a linear isomorphism satisfying $\max\{\|T\|, \|T^{-1}\|\} < L$ for some $L < K$. Let ϕ_1 be the function provided by Lemma 2.2, and pick $\varepsilon > 0$ such that $L(1 + \phi_1(2\varepsilon)) < K$. Let $N \subseteq V$ be a finite ε-dense set for the sphere of $\operatorname{span}\{v_1, \dots, v_n\} \subseteq X_\mu$ such that $\mu(v) \in (1 - \varepsilon, 1 + \varepsilon)$ for every $v \in N$. Then

$$U := \{\nu \in \mathcal{P}\colon \ \forall v \in N : \ |\nu(v) - \mu(v)| < \varepsilon\}$$

is an open neighborhood of μ, and $U \subseteq \mathcal{N}((x_i)_i, K, (v_i)_i)$. Indeed, for any $\nu \in U$, we have that $id : (\operatorname{span}\{v_1, \dots, v_n\}, \mu) \to (\operatorname{span}\{v_1, \dots, v_n\}, \nu)$ is a $(1 + \phi_1(2\varepsilon))$-isomorphism; hence, the linear map T considered as a map between $(\operatorname{span}\{v_1, \dots, v_n\}, \nu)$, and $\operatorname{span}\{x_1, \dots, x_n\}$ satisfies $\|T\| < L(1 + \phi_1(2\varepsilon)) < K$, and similarly $\|T^{-1}\| < K$; hence, $\nu \in \mathcal{N}((x_i)_i, K, (v_i)_i)$.

The 'In particular' part easily follows, because $v_1, \dots, v_n \in V$ are linearly independent if and only if there exists $K > 1$ with $(\mu, v_1, \dots, v_n) \stackrel{K}{\sim} \ell_1^n$. □

Corollary 2.5. *Both $\mathcal{P}_\infty$ and $\mathcal{B}$ are G_δ-sets in $\mathcal{P}$.*

Since we are interested mainly in subsets of $\mathcal{P}$ closed under isometries, we introduce the following notation.

Notation 2.6. Let Z be a separable Banach space, and let $\mathcal{I}$ be a subset of $\mathcal{P}$. We put

$$\langle Z \rangle_{\equiv}^{\mathcal{I}} := \{\mu \in \mathcal{I}\colon \ X_\mu \equiv Z\} \quad \text{and} \quad \langle Z \rangle_{\simeq}^{\mathcal{I}} := \{\mu \in \mathcal{I}\colon \ X_\mu \simeq Z\}.$$

If $\mathcal{I}$ is clear from the context, we write $\langle Z \rangle_{\equiv}$ and $\langle Z \rangle_{\simeq}$ instead of $\langle Z \rangle_{\equiv}^{\mathcal{I}}$ and $\langle Z \rangle_{\simeq}^{\mathcal{I}}$, respectively.

The important feature of the topology of the spaces $\mathcal{P}$, $\mathcal{P}_\infty$ and $\mathcal{B}$ is that basic open neighborhoods are defined using finite data: that is, finitely many vectors. That suggests that the topological properties of the aforementioned spaces should be closely related to the local theory of Banach spaces. This is certainly a point that could be investigated further in future research. Here we just observe how topological closures are related to finite representability; see Proposition 2.9. In order to formulate our results, let us consider the following generalization of the classical notion of finite representability.

Definition 2.7. Let $\mathcal{F}$ be a family of Banach spaces. We say that a Banach space X is finitely representable in $\mathcal{F}$ if, given any finite-dimensional subspace E of X and any $\varepsilon > 0$, there exists a finite-dimensional subspace F of some $Y \in \mathcal{F}$ that is $(1+\varepsilon)$-isomorphic to E.

If the family $\mathcal{F}$ consists of one Banach space Y only, we say that X is finitely representable in Y rather than in $\{Y\}$.

If $\mathcal{F} \subseteq \mathcal{P}$, by saying that X is finitely representable in $\mathcal{F}$, we mean it is finitely representable in $\{X_\mu : \mu \in \mathcal{F}\}$.

The following is an easy observation that we will use further; the proof follows, for example, immediately from [1, Lemma 12.1.7] in the case that $\mathcal{F}$ contains one Banach space only. For the more general situation, the proof is analogous.

Lemma 2.8. *Let $\mathcal{F}$ be a family of infinite-dimensional Banach spaces and $\mu \in \mathcal{P}_\infty$. Let $\{k(n)\}_{n=1}^\infty$ be a sequence such that $\{e_{k(n)} : n \in \mathbb{N}\}$ is a linearly independent set in X_μ and $\overline{\operatorname{span}}\{e_{k(n)} : n \in \mathbb{N}\} = X_\mu$. Then X_μ is finitely representable in $\mathcal{F}$ if and only if for every $n \in \mathbb{N}$ and $\varepsilon > 0$, there exists a finite-dimensional subspace F of some $Y \in \mathcal{F}$ that is $(1+\varepsilon)$-isomorphic to $(\operatorname{span}\{e_{k(1)}, \dots, e_{k(n)}\}, \mu)$.*

Proposition 2.9. *Let $\mathcal{F} \subseteq \mathcal{B}$ be such that $\langle X_\mu \rangle_\equiv^{\mathcal{B}} \subseteq \mathcal{F}$ for every $\mu \in \mathcal{F}$. Then*

$$\{\nu \in \mathcal{B} : X_\nu \text{ is finitely representable in } \mathcal{F}\} = \overline{\mathcal{F}} \cap \mathcal{B}.$$

The same holds if we replace $\mathcal{B}$ with $\mathcal{P}_\infty$ or $\mathcal{P}$.

In particular, if X is a separable infinite-dimensional Banach space, then

$$\{\nu \in \mathcal{B} : X_\nu \text{ is finitely representable in } X\} = \overline{\langle X \rangle_\equiv^{\mathcal{B}}} \cap \mathcal{B},$$

and similarly also if we replace $\mathcal{B}$ with $\mathcal{P}_\infty$ or with $\mathcal{P}$.

Proof. '$\subseteq$': Fix $\nu \in \mathcal{B}$ such that X_ν is finitely representable in $\mathcal{F}$. Pick $v_1, \dots, v_n \in V$ and $\varepsilon > 0$. We shall show there is $\mu_0 \in \mathcal{F}$ with $|\mu_0(v_i) - \nu(v_i)| < \varepsilon$, $i \le n$. Let $m \in \mathbb{N}$ be such that $\{v_1, \dots, v_n\} \subseteq \operatorname{span}_{\mathbb{Q}}\{e_j : j \le m\}$. Put $C := \max\{\nu(v_i) : i = 1, \dots, n\}$ and $Z := \operatorname{span}\{e_1, \dots, e_m\} \subseteq X_\nu$. Since X_ν is finitely representable in $\mathcal{F}$, there is $\mu \in \mathcal{F}$ and a $(1+\frac{\varepsilon}{2C})$-isomorphism $T : Z \to X_\mu$. Set $x_i := T(e_i)$, $i \le m$, and extend $x_1, \dots, x_m$ to a linearly independent sequence $(x_i)_{i=1}^\infty$ whose span is dense in X_μ. Consider $\mu_0 \in \mathcal{P}$ given by setting $\mu_0(\sum_{i \in I} \alpha_i e_i) = \mu(\sum_{i\in I} \alpha_i x_i)$, where $I \subseteq \mathbb{N}$ is finite and $(\alpha_i)_{i\in I} \subseteq \mathbb{Q}$. Clearly, $X_{\mu_0} \equiv X_\mu$ and $\mu_0 \in \mathcal{B}$, so $\mu_0 \in \mathcal{F}$. Finally, for every $i \le n$, we have $v_i = \sum_{j=1}^m \alpha_j e_j$ for some $(\alpha_j) \in \mathbb{R}^m$, and so we have

$$\mu_0(v_i) = \mu\Big(\sum_{j=1}^m \alpha_j x_j\Big) \le \big(1 + \tfrac{\varepsilon}{2C}\big)\nu\Big(\sum_{j=1}^m \alpha_j e_j\Big) = \big(1 + \tfrac{\varepsilon}{2C}\big)\nu(v_i),$$

and similarly $\mu_0(v_i) \ge (1+\frac{\varepsilon}{2C})^{-1}\nu(v_i) \ge (1-\frac{\varepsilon}{2C})\nu(v_i)$. Thus, $|\mu_0(v_i) - \nu(v_i)| \le \frac{\varepsilon}{2} < \varepsilon$ for every $i \le n$.

The case when we replace $\mathcal{B}$ with $\mathcal{P}_\infty$ or $\mathcal{P}$ is analogous; this time, we do not require $(x_i)_{i=1}^\infty$ to be linearly independent.

'$\supseteq$': Fix $\nu \in \overline{\mathcal{F}} \cap \mathcal{B}$. In order to see that X_ν is finitely representable in $\mathcal{F}$, we will use Lemma 2.8. Pick $n \in \mathbb{N}$ and $\varepsilon > 0$. Let ϕ_1 be the function from Lemma 2.2, let $\delta > 0$ be such that $\phi_1(2\delta) < \varepsilon$ and let $N \subseteq V$ be a finite set that is δ-dense for the sphere of $(\operatorname{span}\{e_1, \dots, e_n\}, \nu)$ and $\nu(v) \in (1-\delta, 1+\delta)$ for every $v \in N$. Pick $\mu \in \mathcal{F}$ such that $|\mu(v) - \nu(v)| < \delta$, $v \in N$. Then $id : (\operatorname{span}\{e_1, \dots, e_n\}, \nu) \to (\operatorname{span}\{e_1, \dots, e_n\}, \mu)$ is a $(1+\phi_1(2\delta))$-isomorphism. Thus, X_ν is finitely representable in $\mathcal{F}$. The case when we replace $\mathcal{B}$ with $\mathcal{P}_\infty$ or $\mathcal{P}$ is similar. □

This result has interesting consequences.

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Corollary 2.10. *Let X be a separable Banach space such that every Banach space is finitely representable in X. Then its isometry class is dense (in $\mathcal{P}$, $\mathcal{P}_\infty$ and also in $\mathcal{B}$).*

Proof. Follows immediately from Proposition 2.9. □

Corollary 2.11. *Let X be a separable infinite-dimensional Banach space. Then $\langle \ell_2\rangle_\equiv^{\mathcal{B}} \subseteq \overline{\langle \mathcal{B}\rangle_\equiv^X} \cap \mathcal{B}$. The same holds if we replace $\mathcal{B}$ with $\mathcal{P}_\infty$ or $\mathcal{P}$.*

Proof. By the Dvoretzky theorem, ℓ_2 is finitely representable in every separable infinite-dimensional Banach space (see, e.g., [1, Theorem 13.3.7]). So we are done by applying Proposition 2.9. □

We conclude this subsection by showing another nice feature of the above topologies on examples. We can show that the natural maps $K \mapsto C(K)$ and $\lambda \mapsto L_p(\lambda)$, where K is a compact metrizable space, and λ is a Borel probability measure on a fixed compact metric space, are continuous.

Example 2.12. (a) Let $\mathcal{K}([0,1]^{\mathbb{N}})$ denote the space of all compact subsets of the Hilbert cube $[0,1]^{\mathbb{N}}$ endowed with the Vietoris topology. Then there exists a continuous mapping $\rho\colon \mathcal{K}([0,1]^{\mathbb{N}}) \to \mathcal{P}$ such that $X_{\rho(K)} \equiv C(K)$ for every $K \in \mathcal{K}([0,1]^{\mathbb{N}})$.

(b) Let L be a compact metric space, let $p \in [1,\infty)$ be fixed, and let $\mathcal{P}rob(L)$ denote the space of all Borel probability measures on L endowed with the weak* topology (generated by elements of the Banach space $C(L)$). Then there exists a continuous mapping $\sigma\colon \mathcal{P}rob(L) \to \mathcal{P}$ such that $X_{\sigma(\lambda)} \equiv L_p(\lambda)$ for every $\lambda \in \mathcal{P}rob(L)$.

Proof. (a) Let $\{f_i\colon i \in \mathbb{N}\}$ be a linearly dense subset of $C([0,1]^{\mathbb{N}})$. For every compact subset K of $[0,1]^{\mathbb{N}}$, we define $\rho(K) \in \mathcal{P}$ by

$$\rho(K)\left(\sum_{i=1}^n r_i e_i\right) = \sup_{x\in K}\left|\sum_{i=1}^n r_i f_i(x)\right|, \qquad \sum_{i=1}^n r_i e_i \in V.$$

It is clear that $X_{\rho(K)} \equiv C(K)$, so we only need to check the continuity of ρ. It is enough to show that $\rho^{-1}(U[v,I])$ is an open subset of $\mathcal{K}([0,1]^{\mathbb{N}})$ for every $v \in V$ and every open interval I (recall that $U[v,I] = \{\mu \in \mathcal{P}\colon \mu(v) \in I\}$). So let us fix $\widetilde{K} \in \rho^{-1}(U[v,I])$, and assume that $v = \sum_{i=1}^n r_i e_i$. Fix $x_0 \in \widetilde{K}$ such that

$$\left|\sum_{i=1}^n r_i f_i(x_0)\right| = \sup_{x\in\widetilde{K}}\left|\sum_{i=1}^n r_i f_i(x)\right|.$$

Also fix $\varepsilon > 0$ such that both numbers $\left|\sum_{i=1}^n r_i f_i(x_0)\right| \pm \varepsilon$ belong to I. Now find open subsets U, V of $[0,1]^{\mathbb{N}}$ such that $x_0 \in U$ and $\widetilde{K} \subseteq V$ and such that

$$\inf_{x\in U}\left|\sum_{i=1}^n r_i f_i(x)\right| > \left|\sum_{i=1}^n r_i f_i(x_0)\right| - \varepsilon$$

and

$$\sup_{x\in V}\left|\sum_{i=1}^n r_i f_i(x)\right| < \sup_{x\in\widetilde{K}}\left|\sum_{i=1}^n r_i f_i(x)\right| + \varepsilon.$$

Then

$$\mathcal{U} := \{K \in \mathcal{K}([0,1]^{\mathbb{N}})\colon K \cap U \neq \emptyset \text{ and } K \subseteq V\}$$

is an open neighborhood of $\widetilde{K}$ such that $\rho(\mathcal{U}) \subseteq U[v,I]$.

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(b) This is similar to (a) but even easier. Let $\{g_i : i \in \mathbb{N}\}$ be a linearly dense subset of $C(L)$. For every Borel probability measure λ on L, we define $\sigma(\lambda) \in \mathcal{P}$ by

$$\sigma(\lambda)\left(\sum_{i=1}^{n} r_i e_i\right) = \left(\int_L \left|\sum_{i=1}^{n} r_i g_i\right|^p d\lambda\right)^{\frac{1}{p}}, \qquad \sum_{i=1}^{n} r_i e_i \in V.$$

It is clear that $X_{\sigma(\lambda)} \equiv L_p(\lambda)$, so we only need to check the continuity of σ. It is enough to show that $\sigma^{-1}(U[v, I])$ is an open subset of $\mathcal{P}rob(L)$ for every $v \in V$ and every open interval I. But this is clear as, for $v = \sum_{i=1}^{n} r_i e_i$, we have

$$\sigma^{-1}(U[v, I]) = \left\{\lambda \in \mathcal{P}rob(L) : \left(\int_L \left|\sum_{i=1}^{n} r_i g_i\right|^p d\lambda\right)^{\frac{1}{p}} \in I\right\}. \qquad \square$$

Remark 2.13. After the introduction of the spaces $\mathcal{P}$, $\mathcal{P}_\infty$ and $\mathcal{B}$, one faces the question of which of them is 'the right one' with which to work. For now, we leave the question undecided. Since we are mainly interested in infinite-dimensional Banach spaces, we prefer to work mainly with $\mathcal{P}_\infty$ and $\mathcal{B}$. On the other hand, it turns out that at least as far as one wants to transfer some computations performed in the space of pseudonorms directly to admissible topologies, the space $\mathcal{P}$ is useful: Theorem 3.3 below shows that whatever we compute in the space $\mathcal{P}$ also holds true in any admissible topology.

Regarding $\mathcal{P}_\infty$ and $\mathcal{B}$, in most of the arguments, it makes no difference whether we are working with the former or the latter space. However, there are a few exceptions when it seems to be convenient to work with the assumption that the sequence of vectors $\langle e_n : n \in \mathbb{N}\rangle \subseteq V$ is linearly independent, and then it might be more natural to work with $\mathcal{B}$.

3. Choice of the Polish space of separable Banach spaces

The main outcome of this section is Theorem A (denoted here as Theorem 3.11). We also prove partial converses to this result; see Theorem 3.3 and Proposition 3.6. Let us give some more details.

1. In the first subsection, we recall the coding $SB(X)$ (and $SB_\infty(X)$) of separable (infinite-dimensional) Banach spaces. We recall the notion of an admissible topology introduced in [19], which is a Polish topology corresponding to the Effros-Borel structure of $SB(X)$. We explore some basic relations between codings $\mathcal{P}$, $\mathcal{P}_\infty$, $\mathcal{B}$, $SB(X)$ and $SB_\infty(X)$. We show there is a continuous reduction from $SB(X)$ to $\mathcal{P}$, a $\boldsymbol{\Sigma}^0_2$-measurable reduction from $\mathcal{P}_\infty$ to $\mathcal{B}$, and a $\boldsymbol{\Sigma}^0_4$-measurable reduction from $\mathcal{P}$ to $SB(X)$; see Theorem 3.3, Proposition 3.6, and Theorem 3.10. Here, by a 'reduction', we mean a map Φ such that a code and its image are both codes of the same (up to isometry) Banach space.
2. The second subsection is devoted to the proof of Theorem 3.11, by which there is a $\boldsymbol{\Sigma}^0_2$-measurable reduction from $\mathcal{B}$ to $SB_\infty(X)$. Further, we note that the developed techniques also lead to a $\boldsymbol{\Sigma}^0_3$-measurable reduction from $\mathcal{P}$ to $SB(X)$, which is an improvement of the result mentioned above.

The importance of the reductions above is that there is not a big difference between Borel ranks when considered in any of the Polish spaces mentioned above.

Let us emphasize that the existence of a Borel reduction from $\mathcal{B}$ to $SB_\infty(X)$ has been essentially proved in [24, Lemma 2.4]. Going through the proof of [24, Lemma 2.4], one may obtain a reduction that is $\boldsymbol{\Sigma}^0_3$-measurable; however, the proof does not seem to give a $\boldsymbol{\Sigma}^0_2$-measurable reduction (which is the optimal result). In order to obtain this improvement (see Theorem 3.11), we have to develop a whole machinery of new ideas in combination with very technical results, and this is why we devote a whole subsection to the proof.

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Since our reductions from $SB_\infty(X)$ to $\mathcal{P}_\infty$ and from $\mathcal{B}$ to $SB_\infty(X)$ are optimal, it seems to be a very interesting open problem whether there exists a continuous reduction from $\mathcal{P}_\infty$ to $\mathcal{B}$ or at least a $\mathbf{\Sigma}^0_2$-measurable reduction from $\mathcal{P}_\infty$ to $SB_\infty(X)$; see Question 1 and Question 2.

3.1. Relations between codings $\mathcal{P}$, $\mathcal{P}_\infty$, $\mathcal{B}$, $SB(X)$ and $SB_\infty(X)$

Here we recall the approach to topologizing the class of all separable (infinite-dimensional) Banach spaces by Godefroy and Saint-Raymond from [19], which was a partial motivation for our research.

Definition 3.1. Let X be a Polish space, and let us denote by $\mathcal{F}(X)$ the set of all closed subsets of X. For an open set $U \subseteq X$, we put $E^+(U) = \{F \in \mathcal{F}(X) : U \cap F \neq \emptyset\}$. Following [19], we say that a Polish topology τ on the set $\mathcal{F}(X)$ is *admissible* if it satisfies the following two conditions:

1. For every open subset U of X, the set $E^+(U)$ is τ-open.
2. There exists a subbasis of τ such that every set from this subbasis is a countable union of sets of the form $E^+(U) \setminus E^+(V)$, where U and V are open in X.[1]

We note that Godefroy and Saint-Raymond also suggest the following optional condition that is satisfied by many natural admissible topologies.

3. The set $\{(x, F) \in X \times \mathcal{F}(X) : x \in F\}$ is closed in $X \times \mathcal{F}(X)$.

If X is a separable Banach space, we denote by $SB(X) \subseteq \mathcal{F}(X)$ the set of closed vector subspaces of X. We denote by $SB_\infty(X)$ the subset of $SB(X)$ consisting of infinite-dimensional spaces. We say that a topology on $SB(X)$ or $SB_\infty(X)$ is *admissible* if it is induced by an admissible topology on $\mathcal{F}(X)$. Both $SB(X)$ and $SB_\infty(X)$ are Polish spaces when endowed with an admissible topology; see Remark 3.2.

If Z is a separable Banach space, we put, similarly as in Notation 2.6,

$$\langle Z\rangle_\equiv := \{F \in SB(X) : F \equiv Z\} \quad \text{and} \quad \langle Z\rangle_\simeq := \{F \in SB(X) : F \simeq Z\}.$$

It will always be clear from the context whether we work with subsets of $\mathcal{P}$ or $SB(X)$.

Remark 3.2. If X is a separable Banach space and τ is an admissible topology on $\mathcal{F}(X)$, then $SB(X)$ is a G_δ-subset of $(\mathcal{F}(X), \tau)$ (see [19, Section 3]). Moreover, by [19, Corollary 4.2], $SB_\infty(X)$ is a G_δ-subset of $(SB(X), \tau)$. (In fact, [19] deals only with the case $X = C(2^\omega)$, but the generalization to any separable Banach space is easy.)

A certain connection between codings $SB(X)$ and $\mathcal{P}$ of separable Banach spaces might be deduced already from [19].

Theorem 3.3. *Let X be an isometrically universal separable Banach space, and let τ be an admissible topology on $SB(X)$. Then there is a continuous mapping $\Phi : (SB(X), \tau) \to \mathcal{P}$ such that for every $F \in SB(X)$, we have $F \equiv X_{\Phi(F)}$.*

Proof. By [19, Theorem 4.1], there are continuous functions $(f_n)_{n\in\mathbb{N}}$ on $SB(X)$ with values in X such that for each $F \in SB(X)$, we have $\overline{\{f_n(F) : n \in \mathbb{N}\}} = F$. Consider the mapping Φ given by $\Phi(F)(\sum_{n=1}^k a_n e_n) := \|\sum_{n=1}^k a_n f_n(F)\|_X$ for every $F \in SB(X)$ and $a_1, \dots, a_k \in \mathbb{Q}$. Then it is easy to see that Φ is the mapping we need. □

The following relation between various codings of Banach spaces as $SB(X)$ is easy.

Observation 3.4. *Let X, Y be isometrically universal separable Banach spaces, and let τ_1 and τ_2 be admissible topologies on $SB(X)$ and $SB(Y)$, respectively. Then there is a $\mathbf{\Sigma}^0_2$-measurable mapping $f : (SB(X), \tau_1) \to (SB(Y), \tau_2)$ such that for every $F \in SB(X)$, we have $F \equiv f(F)$. Moreover, f can be chosen such that for every open set $U \subseteq Y$ there is an open set $V \subseteq X$ such that $f^{-1}(E^+(U)) = E^+(V)$.*

[1]Note that condition (ii) is different from what is mentioned in [19]; however, as the authors have confirmed, there is a typo in the condition from [19] that makes it wrong (otherwise, no single one of the topologies mentioned in [19] would be admissible).

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Proof. Let $j : X \to Y$ be an isometry (not necessarily surjective). Then the mapping f given by $f(F) := j(F)$, $F \in SB(X)$, does the job, because $f^{-1}(E^+(U)) = E^+(j^{-1}(U))$ for every open set $U \subseteq Y$. □

Let us note the following easy fact, which we record here for a later reference. The proof is easy and so is omitted.

Lemma 3.5. *Let X be an isometrically universal separable Banach space, τ be an admissible topology on $SB(X)$, Y be a Polish space, $f : Y \to SB(X)$ be a mapping and $n \in \mathbb{N}$, $n \geq 2$, be such that $f^{-1}(E^+(U))$ is a $\mathbf{\Delta}^0_n$ set in Y for every open set $U \subseteq X$. Then f is $\mathbf{\Sigma}^0_n$-measurable.*

A straightforward idea leads to the following relation between $\mathcal{P}_\infty$ and $\mathcal{B}$.

Proposition 3.6. *There is a $\mathbf{\Sigma}^0_2$-measurable mapping $\Phi : \mathcal{P}_\infty \to \mathcal{B}$ such that for every $\mu \in \mathcal{P}_\infty$, we have $X_\mu \equiv X_{\Phi(\mu)}$.*

Moreover, Φ can be chosen such that $\Phi^{-1}(U[v, I]) \in \mathbf{\Delta}^0_2(\mathcal{P}_\infty)$ for each $v \in V$ and each open interval I.

Proof. For each $\mu \in \mathcal{P}_\infty$, let us inductively define natural numbers $(n_k(\mu))_{k\in\mathbb{N}}$ by

$$\begin{aligned} n_1(\mu) &:= \min\{n \in \mathbb{N} : \ \mu(e_n) \neq 0\}, \\ n_{k+1}(\mu) &:= \min\{n \in \mathbb{N} : \ e_{n_1(\mu)}, \dots, e_{n_k(\mu)}, e_n \text{ are linearly independent}\}. \end{aligned}$$

Consider the mapping Φ given by $\Phi(\mu)(\sum_{i=1}^k a_n e_n) := \mu(\sum_{i=1}^k a_i e_{n_i(\mu)})$ for every $\mu \in \mathcal{P}_\infty$ and $a_1, \dots, a_k \in \mathbb{Q}$. It is easy to see that $\Phi(\mu) \in \mathcal{B}$ and that X_μ is isometric to $X_{\Phi(\mu)}$ for each $\mu \in \mathcal{P}_\infty$.

For all natural numbers $N_1 < \dots < N_k$, the set $\{\mu \in \mathcal{P}_\infty : \ n_1(\mu) = N_1, \dots, n_k(\mu) = N_k\}$ is a $\mathbf{\Delta}^0_2$ set in $\mathcal{P}_\infty$. Indeed, we may prove it by induction on k because for each $k \in \mathbb{N}$ and each $\mu \in \mathcal{P}_\infty$, we have that $n_1(\mu) = N_1, \dots, n_{k+1}(\mu) = N_{k+1}$ iff

$$\begin{aligned} & n_1(\mu) = N_1, \dots, n_k(\mu) = N_k \quad \& \\ & \qquad \forall n = N_k + 1, \dots, N_{k+1} - 1 : e_{N_1}, \dots, e_{N_k}, e_n \text{ are linearly dependent} \\ & \qquad \& \quad e_{N_1}, \dots, e_{N_{k+1}} \text{ are linearly independent}, \end{aligned}$$

which is an intersection of a $\mathbf{\Delta}^0_2$-condition (by the inductive assumption) with a closed and an open condition (by Lemma 2.4).

Let us pick $v = \sum_{i=1}^k a_n e_n \in V$ and an open interval I. Then

$$\Phi^{-1}(U[v, I]) = \{\mu \in \mathcal{P}_\infty : \ \mu(\sum_{i=1}^k a_i e_{n_i(\mu)}) \in I\},$$

which is a $\mathbf{\Delta}^0_2$ set in $\mathcal{P}_\infty$. Indeed, on one hand we have $\mu \in \Phi^{-1}(U[v, I])$ iff there are natural numbers $N_1 < N_2 < \dots < N_k$ such that $n_1(\mu) = N_1, \dots, n_k(\mu) = N_k$ and $\mu(\sum_{i=1}^k a_i e_{N_i}) \in I$, which witnesses that $\Phi^{-1}(U[v, I]) \in \mathbf{\Sigma}^0_2(\mathcal{P}_\infty)$ as it is a countable union of $\mathbf{\Delta}^0_2$ sets. On the other hand, we have that $\mu \in \Phi^{-1}(U[v, I])$ iff for each $l \in \mathbb{N}$, we have that either $n_k(\mu) > l$ or there are natural numbers $N_1 < N_2 < \dots < N_k \leq l$ such that $n_1(\mu) = N_1, \dots, n_k(\mu) = N_k$ and $\mu(\sum_{i=1}^k a_i e_{N_i}) \in I$, which witnesses that $\Phi^{-1}(U[v, I]) \in \mathbf{\Pi}^0_2(\mathcal{P}_\infty)$ as it is a countable intersection of $\mathbf{\Delta}^0_2$ sets.

This proves the 'Moreover' part, from which it easily follows that Φ is $\mathbf{\Sigma}^0_2$-measurable. □

Remark 3.7. For $d \in \mathbb{N}$, let us consider the sets $\mathcal{P}_d := \{\mu \in \mathcal{P} : \ \dim X_\mu = d\}$ and

$$\mathcal{B}_d := \{\mu \in \mathcal{P} : \ e_1, \dots, e_d \text{ is a basis of } X_\mu \text{ and } \mu(e_i) = 0 \text{ for every } i > d\}.$$

A similar argument as in Proposition 3.6 shows that for every $d \in \mathbb{N}$, there is a $\mathbf{\Sigma}^0_2$-measurable mapping $\Phi : \mathcal{P}_d \to \mathcal{B}_d$ such that for every $\mu \in \mathcal{P}_d$, we have $X_\mu \equiv X_{\Phi(\mu)}$.

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Finally, let us consider the reduction from $\mathcal{P}$ to $SB(X)$. An optimal result would be to have a $\boldsymbol{\Sigma}_2^0$-reduction. This is because, as was already observed in [19], the identity map between two admissible topologies is only $\boldsymbol{\Sigma}_2^0$-measurable in general. Using the ideas of the proof of [24, Lemma 2.4], we obtain Theorem 3.10. This result is improved in the next subsection (see Theorem 3.14), but since some steps remain the same, let us give a sketch of the argument (we will be a bit sketchy at the places that will be modified later).

Lemma 3.8. *Let $n \in \mathbb{N}$, X be an isometrically universal separable Banach space and τ be an admissible topology on $SB(X)$. Let there exist $\boldsymbol{\Sigma}_n^0$-measurable mappings $\chi_k : \mathcal{B} \to X$, $k \in \mathbb{N}$, such that $X_\mu \equiv \overline{\operatorname{span}}\{\chi_k(\mu) : k \in \mathbb{N}\}$ for every $\mu \in \mathcal{B}$.*

Then there exists a $\boldsymbol{\Sigma}_{n+1}^0$-measurable mapping $\Phi : \mathcal{B} \to (SB(X), \tau)$ such that for every $\mu \in \mathcal{B}$, we have $X_\mu \equiv \Phi(\mu)$.

Proof. Consider the mapping $\Phi : \mathcal{B} \to (SB(X), \tau)$ defined as

$$\Phi(\nu) := \overline{\operatorname{span}}\{\chi_k(\nu) : k \in \mathbb{N}\}, \quad \nu \in \mathcal{B}.$$

We have $X_\nu \equiv \Phi(\nu)$. For every open set $U \subseteq X$, using the $\boldsymbol{\Sigma}_n^0$-measurability of χ_k's, it is easy to see that $\Phi^{-1}(E^+(U))$ is a $\boldsymbol{\Sigma}_n^0$-set in $\mathcal{B}$. Thus, by Lemma 3.5, the mapping Φ is $\boldsymbol{\Sigma}_{n+1}^0$-measurable. □

Remark 3.9. Similarly as in Remark 3.7, an analogous approach leads to a similar statement valid for any $\mathcal{B}_d$, $d \in \mathbb{N}$, instead of $\mathcal{B}$.

Theorem 3.10. *Let X be an isometrically universal separable Banach space, and let τ be an admissible topology on $SB(X)$. Then there is a $\boldsymbol{\Sigma}_4^0$-measurable mapping $\Phi : \mathcal{P} \to (SB(X), \tau)$ such that for every $\mu \in \mathcal{P}$, we have $X_\mu \equiv \Phi(\mu)$.*

Sketch of the proof. By Remark 3.7, it suffices to find, for every $d \in \mathbb{N} \cup \{\infty\}$, a $\boldsymbol{\Sigma}_3^0$-measurable reduction from $\mathcal{B}_d$ to $SB(X)$, where $\mathcal{B}_\infty = \mathcal{B}$. This is done for every $d \in \mathbb{N} \cup \{\infty\}$ in a similar way. Let us concentrate further only on the case of $d = \infty$; the other cases are similar. From the proof of [24, Lemma 2.4], it follows that there are Borel measurable mappings $\chi_k : \mathcal{B} \to X$, $k \in \mathbb{N}$, such that $X_\mu \equiv \overline{\operatorname{span}}\{\chi_k(\mu) : k \in \mathbb{N}\}$ for every $\mu \in \mathcal{B}$. A careful inspection of the proof actually shows that the mappings χ_k are $\boldsymbol{\Sigma}_2^0$-measurable (since this part is improved in the next subsection [see Proposition 3.12], we do not give any more details here). Thus, an application of Lemma 3.8 finishes the proof. □

3.2. *An optimal reduction from $\mathcal{B}$ to $SB(X)$*

The last subsection is devoted to the proof of the following result. The rest of the paper does not depend on it.

Theorem 3.11. *Let X be an isometrically universal separable Banach space, and let τ be an admissible topology on $SB(X)$. Then there is a $\boldsymbol{\Sigma}_2^0$-measurable mapping $\Phi : \mathcal{B} \to (SB(X), \tau)$ such that for every $\mu \in \mathcal{B}$, we have $X_\mu \equiv \Phi(\mu)$.*

The main ingredient of the proof is the following.

Proposition 3.12. *For any isometrically universal separable Banach space X, there exist continuous mappings $\chi_k : \mathcal{B} \to X$, $k \in \mathbb{N}$, such that*

$$\Big\| \sum_{k=1}^n a_k \chi_k(\nu) \Big\| = \nu\Big(\sum_{k=1}^n a_k e_k \Big)$$

for every $\sum_{k=1}^n a_k e_k \in c_{00}$ and every $\nu \in \mathcal{B}$.

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Remark 3.13. Similarly as in Remark 3.7, we may easily obtain a variant of Proposition 3.12 for $\mathcal{B}_d$, $d \in \mathbb{N}$. Indeed, let $d \in \mathbb{N}$ be given. For $\nu \in \mathcal{B}_d$, let us define $\widetilde{\nu} \in \mathcal{B}$ by

$$\widetilde{\nu}\Big(\sum_{i=1}^{\infty} a_i e_i\Big) := \nu\Big(\sum_{i=1}^{d} a_i e_i\Big) + \sum_{i=d+1}^{\infty} |a_i|, \quad \sum_{i=1}^{\infty} a_i e_i \in c_{00}.$$

If χ_k, $k \in \mathbb{N}$, are as in Proposition 3.12, then we may consider mappings $\widetilde{\chi}_k : \mathcal{B}_d \to X$, $k \leq d$, defined by $\widetilde{\chi}_k(\nu) = \chi_k(\widetilde{\nu})$, $\nu \in \mathcal{B}_d$.

We postpone the proof of Proposition 3.12 to the very end of this subsection.

Proof of Theorem 3.11. Follows immediately from Lemma 3.8 and Proposition 3.12. □

Similarly as above, we obtain also the following.

Theorem 3.14. *Let X be an isometrically universal separable Banach space, and let τ be an admissible topology on $SB(X)$. Then there is a $\mathbf{\Sigma}^0_3$-measurable mapping $\Phi : \mathcal{P} \to (SB(X), \tau)$ such that for every $\mu \in \mathcal{P}$, we have $X_\mu \equiv \Phi(\mu)$.*

Proof. This is similar to the proof of Theorem 3.10; the only modification is that we use Proposition 3.12 and Remark 3.13 instead of the reference to the proof of [24, Lemma 2.4]. □

The aim of the remainder of this subsection is now to prove Proposition 3.12. Its proof is based on two auxiliary results, namely Proposition 3.15 and Lemma 3.18; the first of them is a reformulation of our task. Although we will use only the implication (2′) ⇒ (1) for the set of norms with $\nu(e_k) = 1$, we prove a bit more and keep the formulation for a general set of (pseudo)norms. The implication (1) ⇒ (2), which is not obligatory for us, is a simple application of the Hahn-Banach theorem, and we think it is appropriate to include the proof.

In fact, we do not know if the conditions hold for the set of all pseudonorms (or, equivalently, for the set of pseudonorms with $\nu(e_k) \leq 1$). So, it is open if Proposition 3.12 holds not only for $\mathcal{B}$, but even for $\mathcal{P}$.

Proposition 3.15. *For each $\mathcal{A} \subseteq \mathcal{P}$, the following conditions are equivalent:*

(1) *There are a separable Banach space U and continuous mappings $\chi_k : \mathcal{A} \to U$, $k \in \mathbb{N}$, such that*

$$\Big\| \sum_{k=1}^{n} a_k \chi_k(\nu) \Big\| = \nu\Big(\sum_{k=1}^{n} a_k e_k\Big)$$

for every $\sum_{k=1}^{n} a_k e_k \in c_{00}$ and every $\nu \in \mathcal{A}$.

(2) *There are continuous functions $\alpha_k : \mathcal{A}^2 \to [0,\infty)$, $k \in \mathbb{N}$, such that*

$$\alpha_k(\nu,\nu) = 0$$

for every ν and k, and the following property is satisfied: If $\nu \in \mathcal{A}$ and $z^* \in (c_{00})^\#$ satisfy $|z^*(x)| \leq \nu(x)$ for every $x \in c_{00}$, then there is a mapping $\Gamma : \mathcal{A} \to (c_{00})^\#$ such that $\Gamma(\nu) = z^*$, $|\Gamma(\mu)(x)| \leq \mu(x)$ for every $\mu \in \mathcal{A}$ and $x \in c_{00}$, and

$$|\Gamma(\mu)(e_k) - \Gamma(\lambda)(e_k)| \leq \alpha_k(\mu,\lambda)$$

for every $\mu, \lambda \in \mathcal{A}$ and every k.

Moreover, if $\mathcal{A}$ consists only of pseudonorms ν with $\nu(e_k) \leq 1$ for every k, then these conditions are equivalent with:

(2') *For every $\eta \in [0,1)$, there are continuous functions $\beta_k : \mathcal{A}^2 \to [0,\infty)$, $k \in \mathbb{N}$, such that*

$$\beta_k(\nu,\nu) = 0$$

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for every ν and k, and the following property is satisfied: If $\nu \in \mathcal{A}$ and $z^* \in (c_{00})^\#$ satisfy $|z^*(x)| \leq \nu(x)$ for every $x \in c_{00}$, then there is a mapping $\Gamma : \mathcal{A} \to (c_{00})^\#$ such that $\Gamma(\nu) = \eta \cdot z^*$, $|\Gamma(\mu)(x)| \leq \mu(x)$ for every $\mu \in \mathcal{A}$ and $x \in c_{00}$, and

$$|\Gamma(\mu)(e_k) - \Gamma(\lambda)(e_k)| \leq \beta_k(\mu, \lambda)$$

for every $\mu, \lambda \in \mathcal{A}$ and every k.

Remark 3.16. The conditions (1) and (2) from Proposition 3.15 are also equivalent to the following one:
(3) *There are continuous functions $\alpha_k : \mathcal{A}^2 \to [0, \infty)$, $k \in \mathbb{N}$, such that*

$$\alpha_k(\nu, \nu) = 0$$

for every ν and k and

$$\sum_{\mu,\lambda,k} |a_{\mu,\lambda,k}| \alpha_k(\mu, \lambda) \geq \nu\Big(\sum_{\lambda,k} (a_{\nu,\lambda,k} - a_{\lambda,\nu,k}) e_k\Big) - \sum_{\mu \neq \nu} \mu\Big(\sum_{\lambda,k} (a_{\mu,\lambda,k} - a_{\lambda,\mu,k}) e_k\Big)$$

for every $\nu \in \mathcal{A}$ and every system $(a_{\mu,\lambda,k})_{\mu,\lambda \in \mathcal{A}, k \in \mathbb{N}}$ of real numbers with finite support.

The proof is similar to the proof of (1) $\Leftrightarrow$ (2) below (for (1) $\Rightarrow$ (3), the choice of α_k's is the same as in the proof of (1) $\Rightarrow$ (2); for (3) $\Rightarrow$ (1), the construction of the space U is the same as in (2) $\Rightarrow$ (1)). We omit the full proof because the details are technical, and we do not use condition (3) any further. Let us note that even though we tried to find an application of condition (3), we did not find it, and this is basically why we had to develop conditions (2) and (2′).

Proof of Proposition 3.15. (1) $\Rightarrow$ (2): Given such U and $\chi_k : \mathcal{A} \to U$, $k \in \mathbb{N}$, we put

$$\alpha_k(\nu, \mu) = \|\chi_k(\nu) - \chi_k(\mu)\|, \quad \nu, \mu \in \mathcal{A}, k \in \mathbb{N}.$$

Denote by $I_\mu : (c_{00}, \mu) \to U$ the isometry given by $e_k \mapsto \chi_k(\mu)$. Let $\nu \in \mathcal{A}$ and $z^* \in (c_{00})^\#$ satisfying $|z^*(x)| \leq \nu(x)$ be given. For $x, y \in c_{00}$ with $I_\nu x = I_\nu y$, we have $|z^*(y - x)| \leq \nu(y - x) = \|I_\nu(y - x)\| = 0$, and so $z^*(x) = z^*(y)$. Thus, the formula

$$u^*(I_\nu x) = z^*(x), \quad x \in c_{00},$$

defines a functional on $I_\nu(c_{00})$ such that $|u^*(I_\nu x)| = |z^*(x)| \leq \nu(x) = \|I_\nu x\|$. By the Hahn-Banach theorem, we can extend u^* to the whole U in the way that

$$|u^*(u)| \leq \|u\|, \quad u \in U.$$

For every $\mu \in \mathcal{A}$, let us put

$$\Gamma(\mu)(x) = u^*(I_\mu x), \quad x \in c_{00}.$$

We obtain $|\Gamma(\mu)(x)| = |u^*(I_\mu x)| \leq \|I_\mu x\| = \mu(x)$ and $|\Gamma(\mu)(e_k) - \Gamma(\lambda)(e_k)| = |u^*(I_\mu e_k) - u^*(I_\lambda e_k)| = |u^*(\chi_k(\mu) - \chi_k(\lambda))| \leq \|\chi_k(\mu) - \chi_k(\lambda)\| = \alpha_k(\mu, \lambda)$ for $\mu, \lambda \in \mathcal{A}$ and $k \in \mathbb{N}$.

(2) $\Rightarrow$ (1): Given such $\alpha_k : \mathcal{A}^2 \to [0, \infty)$, $k \in \mathbb{N}$, we define a subset of $c_{00}(\mathcal{A} \times \mathbb{N})$ by

$$\Omega = \mathrm{co}\Big(\bigcup_{\mu} \Big\{\sum_k a_k e_{\mu,k} : \mu\Big(\sum_k a_k e_k\Big) \leq 1\Big\} \cup \bigcup_{\mu,\lambda,k} \{c \cdot (e_{\mu,k} - e_{\lambda,k}) : |c| \cdot \alpha_k(\mu, \lambda) \leq 1\}\Big),$$

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and denote the corresponding Minkowski functional by ϱ. Let U be the completion of the quotient space X/N, where $X = (c_{00}(\mathcal{A}\times\mathbb{N}), \varrho)$ and $N = \{x \in c_{00}(\mathcal{A}\times\mathbb{N}) : \rho(x) = 0\}$. In what follows, we identify every $x \in c_{00}(\mathcal{A}\times\mathbb{N})$ with its equivalence class $[x]_N \in U$. Let us define

$$\chi_k : \mathcal{A} \to U, \quad \nu \mapsto e_{\nu,k}.$$

As $c \cdot (e_{\mu,k} - e_{\lambda,k}) \in \Omega$ whenever $|c| \cdot \alpha_k(\mu,\lambda) \leq 1$, we obtain $\varrho(e_{\mu,k} - e_{\lambda,k}) \leq \alpha_k(\mu,\lambda)$: that is, $\varrho(\chi_k(\mu) - \chi_k(\lambda)) \leq \alpha_k(\mu,\lambda)$. For a fixed μ, we have $\alpha_k(\mu,\lambda) \to \alpha_k(\mu,\mu) = 0$ as $\lambda \to \mu$, and consequently $\varrho(\chi_k(\mu) - \chi_k(\lambda)) \to 0$ as $\lambda \to \mu$. Therefore, χ_k is continuous on $\mathcal{A}$. It follows that the image of χ_k is separable. As these images contain all basic vectors $e_{\nu,k}$, the space U is separable.

We need to show that

$$\nu(x) = \varrho(\overline{x})$$

for fixed $\nu \in \mathcal{A}$, $x = \sum_{k\in\mathbb{N}} a_k e_k \in c_{00}$ and its image $\overline{x} = \sum_{k\in\mathbb{N}} a_k e_{\nu,k}$. The inequality $\nu(x) \geq \varrho(\overline{x})$ follows immediately from the definition of Ω (for any $c \geq \nu(x)$ with $c > 0$, we have $\nu(\frac{1}{c}x) \leq 1$, and so $\frac{1}{c}\overline{x} \in \Omega$, hence $\varrho(\frac{1}{c}\overline{x}) \leq 1$ and $\varrho(\overline{x}) \leq c$). Let us show the opposite inequality $\nu(x) \leq \varrho(\overline{x})$. Using the Hahn-Banach theorem, we can pick $z^* \in (c_{00})^\#$ satisfying $z^*(x) = \nu(x)$ and $|z^*(y)| \leq \nu(y)$ for every $y \in c_{00}$. Let $\Gamma : \mathcal{A} \to (c_{00})^\#$ be the mapping provided for ν and z^*, and let $u^* \in (c_{00}(\mathcal{A}\times\mathbb{N}))^\#$ be given by

$$u^*(e_{\mu,k}) = \Gamma(\mu)(e_k), \quad \mu \in \mathcal{A}, k \in \mathbb{N}.$$

Then $u^*(\overline{x}) = u^*(\sum_k a_k e_{\nu,k}) = \sum_k a_k u^*(e_{\nu,k}) = \sum_k a_k \Gamma(\nu)(e_k) = \Gamma(\nu)(\sum_k a_k e_k) = z^*(x) = \nu(x)$. It is sufficient to show that $u^* \leq 1$ on Ω (equivalently $|u^*(y)| \leq \varrho(y)$ for every $y \in c_{00}(\mathcal{A}\times\mathbb{N})$), since it follows that $\nu(x) = u^*(\overline{x}) \leq \varrho(\overline{x})$.

To show that $u^* \leq 1$ on Ω, we need to check that

$$\mu\Big(\sum_k b_k e_k\Big) \leq 1 \quad \Rightarrow \quad u^*\Big(\sum_k b_k e_{\mu,k}\Big) \leq 1$$

and

$$|c| \cdot \alpha_k(\mu,\lambda) \leq 1 \quad \Rightarrow \quad u^*\big(c \cdot (e_{\mu,k} - e_{\lambda,k})\big) \leq 1.$$

Concerning the first implication, we compute $u^*(\sum_k b_k e_{\mu,k}) = \sum_k b_k u^*(e_{\mu,k}) = \sum_k b_k \Gamma(\mu)(e_k) = \Gamma(\mu)(\sum_k b_k e_k) \leq \mu(\sum_k b_k e_k) \leq 1$. Concerning the second implication, we compute $u^*(c \cdot (e_{\mu,k} - e_{\lambda,k})) = cu^*(e_{\mu,k}) - cu^*(e_{\lambda,k}) = c\Gamma(\mu)(e_k) - c\Gamma(\lambda)(e_k) \leq |c|\alpha_k(\mu,\lambda) \leq 1$.

(2) $\Rightarrow$ (2'): The choice $\beta_k = \alpha_k$ works. Indeed, if Γ is provided by (2), we can take $\eta \cdot \Gamma$.

(2') $\Rightarrow$ (2): For every $n \in \mathbb{N}$, let $\beta_k^n : \mathcal{A}^2 \to [0,\infty)$, $k \in \mathbb{N}$, be provided by (2') for $\eta = (1 - 2^{-n})$. We can assume that each β_k^n is a pseudometric. Indeed, instead of β_k^n, we can take the maximal minorizing pseudometric $\widetilde{\beta}_k^n$ (in such a case, $\widetilde{\beta}_k^n$ is continuous, and since the function $(\mu,\lambda) \mapsto |\Gamma(\mu)(e_k) - \Gamma(\lambda)(e_k)|$ is a pseudometric, if it minorizes β_k^n, then it minorizes $\widetilde{\beta}_k^n$ as well). Moreover, we can assume that β_1^1 is a metric (it is possible to add a compatible metric on $\mathcal{A}$ to β_1^1).

Let us define

$$\alpha(\mu,\lambda) = \max_{n,k} \min\{\beta_k^n(\mu,\lambda), 2^{-\max\{n,k\}}\}, \quad \mu,\lambda \in \mathcal{A}.$$

It is easy to check that α is continuous. Due to our additional assumptions, α is a metric on $\mathcal{A}$. We want to show that there are some constants c_k such that the choice $\alpha_k = c_k \cdot \alpha$ works.

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Let $\nu \in \mathcal{A}$ and $z^* \in (c_{00})^\#$ satisfy $|z^*(x)| \leq \nu(x)$ for every $x \in c_{00}$. For every $n \in \mathbb{N}$, there is a mapping $\Gamma^n : \mathcal{A} \to (c_{00})^\#$ such that

$$\Gamma^n(\nu) = (1 - 2^{-n}) \cdot z^*,$$

$$|\Gamma^n(\mu)(x)| \leq \mu(x), \quad \mu \in \mathcal{A},\ x \in c_{00},$$

and, if we denote

$$\gamma_k^n(\mu) = \Gamma^n(\mu)(e_k),$$

then

$$|\gamma_k^n(\mu) - \gamma_k^n(\lambda)| \leq \beta_k^n(\mu, \lambda)$$

for every $\mu, \lambda \in \mathcal{A}$ and every k. Let us note that

$$|\gamma_k^n(\mu)| \leq 1,$$

as $|\gamma_k^n(\mu)| = |\Gamma^n(\mu)(e_k)| \leq \mu(e_k) \leq 1$ by the assumption on $\mathcal{A}$.

Now, we define the desired mapping Γ. For practical purposes, we first define

$$\gamma_k^n(\mu) = 0 \quad \text{for } n \in \mathbb{Z}, n \leq 0.$$

For every $n \in \mathbb{Z}$, let f_n denote the piecewise linear function supported by $[2^{-n-3}, 2^{-n-1}]$, which is linear on $[2^{-n-3}, 2^{-n-2}]$ and $[2^{-n-2}, 2^{-n-1}]$, and for which $f_n(2^{-n-2}) = 1$. In this way, we have $\sum_{n\in\mathbb{Z}} f_n = 1$ on $(0, \infty)$. We define

$$\gamma_k(\nu) = z^*(e_k)$$

and

$$\gamma_k(\mu) = \sum_{n\in\mathbb{Z}} f_n(\alpha(\mu, \nu))\gamma_k^n(\mu), \quad \mu \neq \nu.$$

Finally, we put $\Gamma(\mu)(e_k) = \gamma_k(\mu)$, so $\Gamma(\nu) = z^*$ and $\Gamma(\mu) = \sum_{n\in\mathbb{N}} f_n(\alpha(\mu, \nu))\Gamma^n(\mu)$ for $\mu \neq \nu$. In both cases $\mu = \nu$ and $\mu \neq \nu$, it follows that $|\Gamma(\mu)(x)| \leq \mu(x)$ for every $x \in c_{00}$. It remains to prove the inequality

$$|\gamma_k(\mu) - \gamma_k(\lambda)| \leq c_k \cdot \alpha(\mu, \lambda)$$

for some suitable constants c_k.

Let us show that the implication

$$\alpha(\mu, \lambda) < 2^{-n} \quad \Rightarrow \quad |\gamma_k^n(\mu) - \gamma_k^n(\lambda)| \leq 2^{k+1}\alpha(\mu, \lambda) \tag{3.1}$$

holds. Clearly, we can suppose that $n \geq 1$. If $\alpha(\mu, \lambda) \geq 2^{-k}$, then $2^{k+1}\alpha(\mu, \lambda) \geq 2 \geq |\gamma_k^n(\mu) - \gamma_k^n(\lambda)|$. So, let us assume that $\alpha(\mu, \lambda) < 2^{-k}$. Since $\min\{\beta_k^n(\mu, \lambda), 2^{-\max\{n,k\}}\} \leq \alpha(\mu, \lambda) < 2^{-\max\{n,k\}}$, we have $\min\{\beta_k^n(\mu, \lambda), 2^{-\max\{n,k\}}\} = \beta_k^n(\mu, \lambda)$, and so $|\gamma_k^n(\mu) - \gamma_k^n(\lambda)| \leq \beta_k^n(\mu, \lambda) = \min\{\beta_k^n(\mu, \lambda), 2^{-\max\{n,k\}}\} \leq \alpha(\mu, \lambda) \leq 2^{k+1}\alpha(\mu, \lambda)$.

Next, we show that

$$2^{-n-4} \leq \alpha(\mu, \nu) < 2^{-n} \quad \Rightarrow \quad |\gamma_k^n(\mu) - \gamma_k(\nu)| \leq (2^{k+1} + 16)\alpha(\mu, \nu). \tag{3.2}$$

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If $n \geq 1$, then $\gamma_k(\nu) - \gamma_k^n(\nu) = z^*(e_k) - (1-2^{-n})z^*(e_k) = 2^{-n}z^*(e_k)$. If $n \leq 0$, then $\gamma_k(\nu) - \gamma_k^n(\nu) = z^*(e_k)$. In both cases, $|\gamma_k(\nu) - \gamma_k^n(\nu)| \leq 2^{-n}|z^*(e_k)| \leq 2^{-n}\nu(e_k) \leq 2^{-n} \leq 2^4\alpha(\mu,\nu)$. Using formula (3.1), we can compute

$$|\gamma_k^n(\mu) - \gamma_k(\nu)| \leq |\gamma_k^n(\mu) - \gamma_k^n(\nu)| + |\gamma_k^n(\nu) - \gamma_k(\nu)| \leq (2^{k+1} + 2^4)\alpha(\mu,\nu).$$

Further, it follows from formula (3.2) that

$$|\gamma_k(\mu) - \gamma_k(\nu)| \leq (2^{k+1} + 16)\alpha(\mu,\nu). \tag{3.3}$$

Indeed, since f_n is supported by $[2^{-n-3}, 2^{-n-1}]$, we always have

$$f_n(\alpha(\mu,\nu))|\gamma_k^n(\mu) - \gamma_k(\nu)| \leq f_n(\alpha(\mu,\nu))(2^{k+1} + 16)\alpha(\mu,\nu),$$

and it is sufficient to use that $\gamma_k(\mu) - \gamma_k(\nu) = \sum_{n\in\mathbb{Z}} f_n(\alpha(\mu,\nu))(\gamma_k^n(\mu) - \gamma_k(\nu))$ for $\mu \neq \nu$.

Now, we are going to investigate the value $|\gamma_k(\mu) - \gamma_k(\lambda)|$. First, we have

$$\alpha(\lambda,\nu) \geq 2\alpha(\mu,\nu) \quad \Rightarrow \quad |\gamma_k(\mu) - \gamma_k(\lambda)| \leq 3 \cdot (2^{k+1} + 16)\alpha(\mu,\lambda). \tag{3.4}$$

Indeed, as $\alpha(\mu,\lambda) \geq \alpha(\lambda,\nu) - \alpha(\mu,\nu) \geq 2\alpha(\mu,\nu) - \alpha(\mu,\nu) = \alpha(\mu,\nu)$, we can apply inequality (3.3) and write

$$\begin{aligned}
|\gamma_k(\mu) - \gamma_k(\lambda)| &\leq |\gamma_k(\mu) - \gamma_k(\nu)| + |\gamma_k(\lambda) - \gamma_k(\nu)| \\
&\leq (2^{k+1} + 16)(\alpha(\mu,\nu) + \alpha(\lambda,\nu)) \\
&= (2^{k+1} + 16)(\alpha(\lambda,\nu) - \alpha(\mu,\nu) + 2\alpha(\mu,\nu)) \\
&\leq (2^{k+1} + 16)(1+2)\alpha(\mu,\lambda).
\end{aligned}$$

Now, we prove the last but most challenging implication:

$$\alpha(\mu,\nu) \leq \alpha(\lambda,\nu) < 2\alpha(\mu,\nu) \quad \Rightarrow \quad |\gamma_k(\mu) - \gamma_k(\lambda)| \leq [12(2^{k+1} + 16) + 2^{k+1}]\alpha(\mu,\lambda). \tag{3.5}$$

Let us compute

$$\begin{aligned}
&\gamma_k(\mu) - \gamma_k(\lambda) = \sum_{n\in\mathbb{Z}} \left[f_n(\alpha(\mu,\nu))\gamma_k^n(\mu) - f_n(\alpha(\lambda,\nu))\gamma_k^n(\lambda)\right] \\
&= \sum_{n\in\mathbb{Z}} \left[f_n(\alpha(\mu,\nu))\gamma_k^n(\mu) - f_n(\alpha(\lambda,\nu))\gamma_k^n(\mu) + f_n(\alpha(\lambda,\nu))\gamma_k^n(\mu) - f_n(\alpha(\lambda,\nu))\gamma_k^n(\lambda)\right] \\
&= \sum_{n\in\mathbb{Z}} [f_n(\alpha(\mu,\nu)) - f_n(\alpha(\lambda,\nu))]\gamma_k^n(\mu) + \sum_{n\in\mathbb{Z}} f_n(\alpha(\lambda,\nu))[\gamma_k^n(\mu) - \gamma_k^n(\lambda)] \\
&= \sum_{n\in\mathbb{Z}} [f_n(\alpha(\mu,\nu)) - f_n(\alpha(\lambda,\nu))](\gamma_k^n(\mu) - \gamma_k(\nu)) + \sum_{n\in\mathbb{Z}} f_n(\alpha(\lambda,\nu))[\gamma_k^n(\mu) - \gamma_k^n(\lambda)].
\end{aligned}$$

Hence, $|\gamma_k(\mu) - \gamma_k(\lambda)|$ is less than or equal to

$$\sum_{n\in\mathbb{Z}} |f_n(\alpha(\mu,\nu)) - f_n(\alpha(\lambda,\nu))||\gamma_k^n(\mu) - \gamma_k(\nu)| + \sum_{n\in\mathbb{Z}} f_n(\alpha(\lambda,\nu))|\gamma_k^n(\mu) - \gamma_k^n(\lambda)|.$$

Let us notice that

- $f_n(\alpha(\mu,\nu)) \neq 0$ iff $2^{-n-3} < \alpha(\mu,\nu) < 2^{-n-1}$,
- $f_n(\alpha(\lambda,\nu)) \neq 0$ iff $2^{-n-3} < \alpha(\lambda,\nu) < 2^{-n-1}$, and $2^{-n-4} < \alpha(\mu,\nu) < 2^{-n-1}$ in this case,
- the function f_n is Lipschitz with the constant 2^{n+3}.

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So, if $f_n(\alpha(\mu,\nu)) \neq 0$ or $f_n(\alpha(\lambda,\nu)) \neq 0$, then $2^{-n-4} < \alpha(\mu,\nu) < 2^{-n-1}$, and formula (3.2) can be applied. We obtain for the first sum that

$$\begin{aligned}\sum_{n\in\mathbb{Z}} |f_n(\alpha(\mu,\nu)) - f_n(\alpha(\lambda,\nu))||\gamma_k^n(\mu) - \gamma_k(\nu)| \\ &\le \sum_{2^{-n-4}<\alpha(\mu,\nu)<2^{-n-1}} 2^{n+3}|\alpha(\mu,\nu) - \alpha(\lambda,\nu)|(2^{k+1}+16)\alpha(\mu,\nu) \\ &\le \sum_{2^{-n-4}<\alpha(\mu,\nu)<2^{-n-1}} 2^{n+3}\alpha(\mu,\lambda)(2^{k+1}+16)2^{-n-1} \\ &\le 3\cdot 2^2(2^{k+1}+16)\alpha(\mu,\lambda).\end{aligned}$$

Concerning the second sum, we notice that if $f_n(\alpha(\lambda,\nu)) \neq 0$, then $\alpha(\lambda,\nu) < 2^{-n-1}$, and so $\alpha(\mu,\lambda) \le \alpha(\mu,\nu) + \alpha(\lambda,\nu) \le 2\alpha(\lambda,\nu) < 2^{-n}$. Applying formula (3.1), we obtain

$$\sum_{n\in\mathbb{Z}} f_n(\alpha(\lambda,\nu))|\gamma_k^n(\mu) - \gamma_k^n(\lambda)| \le \sum_{n\in\mathbb{Z}} f_n(\alpha(\lambda,\nu))2^{k+1}\alpha(\mu,\lambda) = 2^{k+1}\alpha(\mu,\lambda),$$

and formula (3.5) follows.

Finally, we finish the proof with the observation that formulas (3.4) and (3.5) provide

$$|\gamma_k(\mu) - \gamma_k(\lambda)| \le [12(2^{k+1}+16) + 2^{k+1}]\alpha(\mu,\lambda).$$

We can suppose that $\alpha(\mu,\nu) \le \alpha(\lambda,\nu)$. If $\alpha(\lambda,\nu) \ge 2\alpha(\mu,\nu)$, we use formula (3.4), and if $\alpha(\lambda,\nu) < 2\alpha(\mu,\nu)$, we use formula (3.5). □

Definition 3.17. Let $\mathcal{B}_{(1)}$ denote the set of all norms $\nu \in \mathcal{B}$ such that $\nu(e_k) = 1$ for each $k \in \mathbb{N}$.

Now we introduce the second auxiliary result for proving Proposition 3.12.

Lemma 3.18. *Condition (2') from Proposition 3.15 is valid for* $\mathcal{A} = \mathcal{B}_{(1)}$.

The proof of the lemma is organized according to the order of quantifiers in condition (2'). This is perhaps not the most natural order for reading, and the reader may skip the first paragraph of the proof, in which the functions β_k are introduced in a somewhat incomprehensible way, and come back later. The next part, in which functions γ_k (the coordinates of Γ) are constructed, is natural in some sense. In a recursive procedure, functionals are extended to a space with dimension 1 bigger, so there are similarities with the proof of the Hanh-Banach theorem. In our situation, the upper bound by the corresponding norm is a bit relaxed in each step (see formula (3.7)), which is allowed since we prove the relaxed condition (2'). This is needed later for having a control over the difference $|\gamma_k(\mu) - \gamma_k(\lambda)|$.

Proof. Let $\eta \in [0,1)$ be given. We fix numbers $\kappa_k < 1$ such that $\eta \le \kappa_1 < \kappa_2 < \kappa_3 < \dots$. For every $\mu,\lambda \in \mathcal{B}_{(1)}$, we define recursively

$$\beta_1(\mu,\lambda) = 0$$

and

$$\begin{aligned}\beta_{k+1}(\mu,\lambda) = \sup\Bigg\{&\left|\mu\Big(e_{k+1} + \sum_{i=1}^k a_ie_i\Big) - \lambda\Big(e_{k+1} + \sum_{i=1}^k a_ie_i\Big)\right| + \sum_{i=1}^k |a_i|\beta_i(\mu,\lambda) : \\ &a_1,\dots,a_k \in \mathbb{R},\ \min\Big\{\mu\Big(\sum_{i=1}^k a_ie_i\Big), \lambda\Big(\sum_{i=1}^k a_ie_i\Big)\Big\} < \frac{2\kappa_{k+1}}{\kappa_{k+1}-\kappa_k}\Bigg\}.\end{aligned}$$

Clearly, $\beta_k(\nu,\nu)=0$ for every $\nu\in\mathcal{B}_{(1)}$. Let us sketch a proof of continuity of the functions β_k. The function $\beta_1=0$ is obviously continuous. Assuming that β_i is continuous for every $i\le k$, we consider for $\delta>0$ the set

$$\mathcal{U}_\delta^{k+1}(\mu)=\left\{\mu'\in\mathcal{B}_{(1)}:\left(\forall x\in\operatorname{span}\{e_1,\dots,e_{k+1}\}\setminus\{0\}:(1+\delta)^{-1}<\frac{\mu'(x)}{\mu(x)}<1+\delta\right)\right\}.$$

Using Lemma 2.2, it is easy to see that $\mathcal{U}_\delta^{k+1}(\mu)$ is an open neighborhood of μ in $\mathcal{B}_{(1)}$. Given $\varepsilon>0$, we can find $\delta>0$ such that $|\beta_{k+1}(\mu',\lambda')-\beta_{k+1}(\mu,\lambda)|<\varepsilon$ for every $(\mu',\lambda')\in\mathcal{U}_\delta^{k+1}(\mu)\times\mathcal{U}_\delta^{k+1}(\lambda)$. The details are left to the reader.

Let us prove that the functions β_k work. Given $\nu\in\mathcal{B}_{(1)}$ and $z^*\in(c_{00})^\#$ satisfying $|z^*(x)|\le\nu(x)$ for every $x\in c_{00}$, we define first

$$\gamma_1(\mu)=\eta\cdot z^*(e_1),\qquad \mu\in\mathcal{B}_{(1)}.$$

Recursively, we define for every $k\in\mathbb{N}$ functions

$$u_{k+1}(\mu)=\sup_{a_1,\dots,a_k}\Big[-\kappa_{k+1}\mu\Big(-e_{k+1}+\sum_{i=1}^k a_ie_i\Big)+\sum_{i=1}^k a_i\gamma_i(\mu)\Big],$$
$$v_{k+1}(\mu)=\inf_{a_1,\dots,a_k}\Big[\kappa_{k+1}\mu\Big(e_{k+1}+\sum_{i=1}^k a_ie_i\Big)-\sum_{i=1}^k a_i\gamma_i(\mu)\Big]$$

and

$$\gamma_{k+1}(\mu)=p_{k+1}u_{k+1}(\mu)+q_{k+1}v_{k+1}(\mu),$$

where numbers $p_{k+1}\ge0,q_{k+1}\ge0$ with $p_{k+1}+q_{k+1}=1$ are chosen in the way that

$$\gamma_{k+1}(\nu)=\eta\cdot z^*(e_{k+1}).\tag{3.6}$$

Let us check that it is possible to choose such numbers. Note first that $\gamma_1(\nu)=\eta\cdot z^*(e_1)$. Assuming that the functions γ_i are already defined and satisfy $\gamma_i(\nu)=\eta\cdot z^*(e_i)$ for $i\le k$, we notice that, for every $a_1,\dots,a_k\in\mathbb{R}$,

$$\pm\eta\cdot z^*(e_{k+1})+\sum_{i=1}^k a_i\gamma_i(\nu)=\eta\cdot z^*\Big(\pm e_{k+1}+\sum_{i=1}^k a_ie_i\Big)\le\kappa_{k+1}\nu\Big(\pm e_{k+1}+\sum_{i=1}^k a_ie_i\Big),$$

and consequently

$$\eta\cdot z^*(e_{k+1})\ge-\kappa_{k+1}\nu\Big(-e_{k+1}+\sum_{i=1}^k a_ie_i\Big)+\sum_{i=1}^k a_i\gamma_i(\nu),$$
$$\eta\cdot z^*(e_{k+1})\le\kappa_{k+1}\nu\Big(e_{k+1}+\sum_{i=1}^k a_ie_i\Big)-\sum_{i=1}^k a_i\gamma_i(\nu).$$

This gives

$$u_{k+1}(\nu)\le\eta\cdot z^*(e_{k+1})\le v_{k+1}(\nu),$$

and it follows that suitable p_{k+1} and q_{k+1} do exist.

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Let us prove that

$$\sum_{i=1}^{k} a_i \gamma_i(\mu) \le \kappa_k \mu\Big(\sum_{i=1}^{k} a_i e_i\Big) \tag{3.7}$$

for every $\mu \in \mathcal{B}_{(1)}$, $k \in \mathbb{N}$ and $a_1, \dots, a_k \in \mathbb{R}$. For $k = 1$, we just write $a_1\gamma_1(\mu) = a_1\eta \cdot z^*(e_1) \le |a_1|\kappa_1\nu(e_1) = |a_1|\kappa_1 = |a_1|\kappa_1\mu(e_1) = \kappa_1\mu(a_1e_1)$. Assume that inequality (3.7) is valid for k. We show first that

$$u_{k+1}(\mu) \le \gamma_{k+1}(\mu) \le v_{k+1}(\mu).$$

Clearly, it is sufficient to show just that $u_{k+1}(\mu) \le v_{k+1}(\mu)$. Given $b_1, \dots, b_k$ and $c_1, \dots, c_k$, we need to check that

$$-\kappa_{k+1}\mu\Big(-e_{k+1} + \sum_{i=1}^{k} b_i e_i\Big) + \sum_{i=1}^{k} b_i\gamma_i(\mu) \le \kappa_{k+1}\mu\Big(e_{k+1} + \sum_{i=1}^{k} c_i e_i\Big) - \sum_{i=1}^{k} c_i\gamma_i(\mu).$$

But this is easy, as

$$\begin{aligned}\sum_{i=1}^{k} b_i\gamma_i(\mu) + \sum_{i=1}^{k} c_i\gamma_i(\mu) &= \sum_{i=1}^{k}(b_i + c_i)\gamma_i(\mu) \le \kappa_k\mu\Big(\sum_{i=1}^{k}(b_i + c_i)e_i\Big)\\ &\le \kappa_{k+1}\Big[\mu\Big(-e_{k+1} + \sum_{i=1}^{k} b_i e_i\Big) + \mu\Big(e_{k+1} + \sum_{i=1}^{k} c_i e_i\Big)\Big].\end{aligned}$$

Now, let us verify inequality (3.7) for $k + 1$. We can suppose that $a_{k+1} = \pm 1$. For $a_{k+1} = 1$, it is enough to use

$$\gamma_{k+1}(\mu) \le v_{k+1}(\mu) \le \kappa_{k+1}\mu\Big(e_{k+1} + \sum_{i=1}^{k} a_i e_i\Big) - \sum_{i=1}^{k} a_i\gamma_i(\mu),$$

and for $a_{k+1} = -1$, it is enough to use

$$\gamma_{k+1}(\mu) \ge u_{k+1}(\mu) \ge -\kappa_{k+1}\mu\Big(-e_{k+1} + \sum_{i=1}^{k} a_i e_i\Big) + \sum_{i=1}^{k} a_i\gamma_i(\mu).$$

Next, let us prove that

$$|\gamma_k(\mu) - \gamma_k(\lambda)| \le \beta_k(\mu, \lambda) \tag{3.8}$$

for every $\mu, \lambda \in \mathcal{B}_{(1)}$ and $k \in \mathbb{N}$. This is clear for $k = 1$, as γ_1 is constant. Assume that inequality (3.8) is valid for $i \le k$. To prove it for $k + 1$, it is sufficient to show the inequalities

$$|u_{k+1}(\mu) - u_{k+1}(\lambda)| \le \beta_{k+1}(\mu, \lambda) \quad \text{and} \quad |v_{k+1}(\mu) - v_{k+1}(\lambda)| \le \beta_{k+1}(\mu, \lambda).$$

We consider only the function v_{k+1}, since the inequality for u_{k+1} can be shown in the same way. Let us note first that, in the definition of $v_{k+1}(\mu)$, it is possible to take the infimum only over k-tuples with

$$\mu\Big(\sum_{i=1}^{k} a_i e_i\Big) < \frac{2\kappa_{k+1}}{\kappa_{k+1} - \kappa_k}.$$

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Indeed, for $a_1, \dots, a_k$ that do not satisfy this condition, using inequality(3.7), we obtain

$$\begin{aligned}
&\kappa_{k+1}\mu\Big(e_{k+1}+\sum_{i=1}^k a_i e_i\Big) - \sum_{i=1}^k a_i\gamma_i(\mu)\\
&\qquad \geq \kappa_{k+1}\mu\Big(\sum_{i=1}^k a_i e_i\Big) - \kappa_{k+1}\mu(e_{k+1}) - \kappa_k\mu\Big(\sum_{i=1}^k a_i e_i\Big)\\
&\qquad = (\kappa_{k+1}-\kappa_k)\mu\Big(\sum_{i=1}^k a_i e_i\Big) - \kappa_{k+1}\\
&\qquad \geq 2\kappa_{k+1} - \kappa_{k+1} = \kappa_{k+1} = \kappa_{k+1}\mu\Big(e_{k+1} + \sum_{i=1}^k 0\cdot e_i\Big) - \sum_{i=1}^k 0\cdot\gamma_i(\mu).
\end{aligned}$$

Now, inequality (3.8) is provided by the following computation, in which every sup/inf is meant over k-tuples with $\mu\Big(\sum_{i=1}^k a_i e_i\Big) < \frac{2\kappa_{k+1}}{\kappa_{k+1}-\kappa_k}$ or $\lambda\Big(\sum_{i=1}^k a_i e_i\Big) < \frac{2\kappa_{k+1}}{\kappa_{k+1}-\kappa_k}$:

$$\begin{aligned}
&|\nu_{k+1}(\mu) - \nu_{k+1}(\lambda)|\\
&= \Bigg|\inf\Big[\kappa_{k+1}\mu\Big(e_{k+1}+\sum_{i=1}^k a_i e_i\Big) - \sum_{i=1}^k a_i\gamma_i(\mu)\Big]\\
&\qquad\qquad - \inf\Big[\kappa_{k+1}\lambda\Big(e_{k+1}+\sum_{i=1}^k a_i e_i\Big) - \sum_{i=1}^k a_i\gamma_i(\lambda)\Big]\Bigg|\\
&\leq \sup\Bigg|\Big[\kappa_{k+1}\mu\Big(e_{k+1}+\sum_{i=1}^k a_i e_i\Big) - \sum_{i=1}^k a_i\gamma_i(\mu)\Big]\\
&\qquad\qquad - \Big[\kappa_{k+1}\lambda\Big(e_{k+1}+\sum_{i=1}^k a_i e_i\Big) - \sum_{i=1}^k a_i\gamma_i(\lambda)\Big]\Bigg|\\
&= \sup\Bigg|\kappa_{k+1}\Big[\mu\Big(e_{k+1}+\sum_{i=1}^k a_i e_i\Big) - \lambda\Big(e_{k+1}+\sum_{i=1}^k a_i e_i\Big)\Big] - \sum_{i=1}^k a_i\big(\gamma_i(\mu)-\gamma_i(\lambda)\big)\Bigg|\\
&\leq \sup\Bigg[\Big|\mu\Big(e_{k+1}+\sum_{i=1}^k a_i e_i\Big) - \lambda\Big(e_{k+1}+\sum_{i=1}^k a_i e_i\Big)\Big| + \sum_{i=1}^k |a_i|\beta_i(\mu,\lambda)\Bigg] = \beta_{k+1}(\mu,\lambda).
\end{aligned}$$

Finally, as usual, we put $\Gamma(\mu)(e_k) = \gamma_k(\mu)$. The required properties of Γ follow now from inequalities (3.6), (3.7) and (3.8). Thus, the functions β_k work, and the proof of the lemma is completed. □

Proof of Proposition 3.12. Let X be an isometrically universal separable Banach space. By Lemma 3.18, the condition (2') from Proposition 3.15 is valid for $\mathcal{A} = \mathcal{B}_{(1)}$. Hence, the condition (1) from this proposition is valid for $\mathcal{A} = \mathcal{B}_{(1)}$ as well. There are a separable Banach space U and continuous mappings $\chi_k : \mathcal{B}_{(1)} \to U$, $k \in \mathbb{N}$, such that

$$\Big\|\sum_{k=1}^n a_k\chi_k(\nu)\Big\| = \nu\Big(\sum_{k=1}^n a_k e_k\Big)$$

for every $\sum_{k=1}^n a_k e_k \in c_{00}$ and every $\nu \in \mathcal{B}_{(1)}$. Since X contains an isometric copy of U, we can suppose that $U \subseteq X$.

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Let us consider the continuous mapping $\Psi : \mathcal{B} \to \mathcal{B}_{(1)}$ given by

$$\Psi(\mu)\Big(\sum_{k=1}^{n} a_k e_k\Big) = \mu\Big(\sum_{k=1}^{n} \frac{a_k}{\mu(e_k)} e_k\Big).$$

If we define

$$\widetilde{\chi}_k(\mu) = \mu(e_k) \cdot \chi_k(\Psi(\mu)), \quad \mu \in \mathcal{B},$$

for each $k \in \mathbb{N}$, then we get

$$\Big\| \sum_{k=1}^{n} b_k \widetilde{\chi}_k(\mu) \Big\| = \Big\| \sum_{k=1}^{n} b_k \mu(e_k) \chi_k(\Psi(\mu)) \Big\| = \Psi(\mu)\Big(\sum_{k=1}^{n} b_k \mu(e_k) e_k\Big) = \mu\Big(\sum_{k=1}^{n} b_k e_k\Big)$$

for every $\sum_{k=1}^{n} b_k e_k \in c_{00}$ and every $\mu \in \mathcal{B}$. □

To summarize, we obtained an optimal reduction from $\mathcal{B}$ to $SB_\infty(X)$ and from $SB_\infty(X)$ to $\mathcal{P}_\infty$. However, our reduction from $\mathcal{P}_\infty$ to $\mathcal{B}$ seems not to be optimal, so one is tempted to ask the following.

Question 1. Does there exist a continuous mapping $\Phi : \mathcal{P}_\infty \to \mathcal{B}$ such that for every $\mu \in \mathcal{P}_\infty$, we have $X_\mu \equiv X_{\Phi(\mu)}$?

Note that a positive answer to Question 1 would imply a positive answer to Question 2 and that a sufficient condition for a positive solution of Question 2 is provided by Proposition 3.15.

Question 2. Let X be an isometrically universal separable Banach space, and let τ be an admissible topology on $SB(X)$. Does there exist a $\mathbf{\Sigma}^0_2$-measurable mapping $\Phi : \mathcal{P}_\infty \to (SB(X), \tau)$ such that for every $\mu \in \mathcal{P}_\infty$, we have $X_\mu \equiv \Phi(\mu)$?

4. Generic properties

As soon as one has a Polish space, or more generally a Baire space, of some objects, it is natural and often useful to find properties (of these objects) that are generic; that is, the corresponding subset of the space is comeager. In the case of the spaces $\mathcal{P}$, $\mathcal{P}_\infty$ and $\mathcal{B}$, we resolve this problem completely; see Theorem 4.1, which is a more precise description of the content of Theorem B.

In the case of the spaces $SB(X)$ with an admissible topology, this is the content of Problem 5.5 from [19]. We show that in that case, the situation is more complicated. In particular, we confirm the suspicion of Godefroy and Saint-Raymond that being meager in $SB(X)$ is not independent of the chosen admissible topology; see Theorem 4.10 and Theorem 4.12, which together imply Theorem C.

4.1. Generic objects in $\mathcal{P}$

The main result of this subsection is Theorem 4.1 below. Although it may not appear surprising to a specialist in Fraïssé theory, as the Gurariĭ space is the Fraïssé limit of finite-dimensional Banach spaces (see, e.g., [3, Theorem 4.3]), its proof is far more involved than analogous results for countable Fraïssé limits. Also, the result has several applications in estimating the complexities of isometry and isomorphism classes of other Banach spaces; see [13], where this issue is addressed.

Theorem 4.1. *Let $\mathbb{G}$ be the Gurariĭ space. Then the isometry class $\langle \mathbb{G} \rangle^{\mathcal{I}}_{\equiv}$ is a dense G_δ-set in $\mathcal{I}$ for any $\mathcal{I} \in \{\mathcal{P}, \mathcal{P}_\infty, \mathcal{B}\}$.*

Let us recall what the Gurariĭ space is. One of the characterizations of the Gurariĭ space is the following; for more details, we refer the interested reader to, for example, [10] (the characterization below is provided by [10, Lemma 2.2]).

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Definition 4.2. The Gurariĭ space is the unique (up to isometry) separable Banach space such that for every $\varepsilon > 0$ and every isometric embedding $g : A \to B$, where B is a finite-dimensional Banach space and A is a subspace of $\mathbb{G}$, there is a $(1+\varepsilon)$-isomorphism $f : B \to \mathbb{G}$ such that $\|f \circ g - id_A\| \le \varepsilon$.

In the remainder of this subsection, we prove Theorem 4.1. Let us start with the most technical part, namely that $\langle \mathbb{G} \rangle_{\equiv}^{\mathcal{P}_\infty}$ is a G_δ-set in $\mathcal{P}_\infty$.

We need two technical lemmas first.

Lemma 4.3.

1. *Given a basis $\mathfrak{b}_E = \{e_1, \dots, e_n\}$ of a finite-dimensional Banach space E, there are $C > 0$ and a function $\phi_2^{\mathfrak{b}_E} : [0, C) \to [0, \infty)$ continuous at zero with $\phi_2^{\mathfrak{b}_E}(0) = 0$ such that whenever X is a Banach space with $E \subseteq X$ and $\{x_i : i \le n\} \subseteq X$ are such that $\|x_i - e_i\| < \varepsilon$, $i \le n$, for some $\varepsilon < C$, then the linear operator $T : E \to X$ given by $T(e_i) := x_i$ is a $(1 + \phi_2^{\mathfrak{b}_E}(\varepsilon))$-isomorphism and $\|T - Id_E\| \le \phi_2^{\mathfrak{b}_E}(\varepsilon)$.*
2. *Let $\varepsilon \in (0, 1)$, $T : X \to Y$ be a surjective $(1 + \varepsilon)$-isomorphism between Banach spaces X and Y, and N be ε-dense for S_X. Then $T(N)$ is 3ε-dense for S_Y.*

Proof. 1: Pick $C > 0$ such that $C \sum_{i=1}^n |\lambda_i| \le \|\sum_{i=1}^n \lambda_i e_i\|$ for every $(\lambda_i)_{i=1}^n \in \mathbb{R}^n$. Then for any $x = \sum_{i=1}^n \lambda_i e_i$, we have

$$\|Tx - x\| \le \sum_{i=1}^n |\lambda_i| \|x_i - e_i\| < \frac{\varepsilon}{C} \|x\|.$$

Thus, $\|T - Id_E\| < \frac{\varepsilon}{C}$, $\|T\| \le 1 + \frac{\varepsilon}{C}$ and $\|Tx\| \ge (1 - \frac{\varepsilon}{C})\|x\| = (1 + \frac{\varepsilon}{C-\varepsilon})^{-1}\|x\|$. Thus, we may put $\phi_2^{\mathfrak{b}_E}(\varepsilon) := \frac{\varepsilon}{C-\varepsilon}$ for $\varepsilon \in [0, C)$.

2: Let $\varepsilon > 0$, $T : X \to Y$ and N be as in the assumptions. Then for every $y \in S_Y$, there is $x \in N$ with $\|x - \frac{T^{-1}(y)}{\|T^{-1}(y)\|}\| < \varepsilon$. Thus, we have

$$\begin{aligned} \|y - Tx\| &\le \left\| y - \frac{y}{\|T^{-1}(y)\|} \right\| + \|T\| \cdot \left\| x - \frac{T^{-1}(y)}{\|T^{-1}(y)\|} \right\| \\ &< \left| 1 - \frac{1}{\|T^{-1}y\|} \right| + (1 + \varepsilon)\varepsilon \le \varepsilon + 2\varepsilon = 3\varepsilon. \end{aligned}$$

□

Lemma 4.4. *For every $\mu \in \mathcal{P}$, finite set $A \subseteq V$ and $\varepsilon > 0$, there exists $\nu \in \mathcal{B}$ with $|\mu(x) - \nu(x)| < \varepsilon$ and $\nu(x) \in \mathbb{Q}$ for every $x \in A$.*

Proof. It suffices to define such a norm ν on $\operatorname{span} A$ since then we can easily find some extension to the whole V. We assume that $0 \notin A$, and moreover, we can assume that no two elements of A lie in the same one-dimensional subspace: that is, are scalar multiples of each other. Indeed, otherwise we would find a subset $A' \subseteq A$ where no elements are scalar multiples of each other and every element of A is a scalar multiple, necessarily rational scalar multiple, of some element from A'. Then proving the fact for A' for sufficiently small δ automatically proves it for A and ε.

We enumerate A as $\{a_1, \dots, a_n\}$ and so that the first k elements $a_1, \dots, a_k$, for some $k \le n$, are linearly independent and form a basis of $\operatorname{span} A$.

Claim. *By perturbing μ on A by an arbitrarily small $\delta > 0$, we can without loss of generality assume that for every $i \le n$, $\mu(a_i) < K_i := \inf\{\sum_{j \in J} \mu(\alpha_j a_j) : i \notin J \subseteq \{1, \dots, n\}, a_i = \sum_{j \in J} \alpha_j a_j\}$.*

Suppose the claim is proved. Then for every $i \le n$, we set $\nu'(a_i)$ to be an arbitrary positive rational number in the interval $[\mu(a_i), \min\{K_i, \mu(a_i) + \varepsilon\})$. From the assumption it is now clear that for all $i \le n$, we have

$$\nu'(a_i) \le \inf\{\sum_{j=1}^n |\alpha_j| \nu'(a_j) : a_i = \sum_{j=1}^n \alpha_j a_j\}.$$

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We extend ν' to a norm ν on span A by the formula

$$\nu(v) := \inf\{\sum_{i=1}^{n} |\alpha_i|\nu'(a_i) \colon v = \sum_{i=1}^{n} \alpha_i a_i\},$$

for $v \in \operatorname{span} A$. From the previous assumption, it follows that $\nu(a_i) = \nu'(a_i)$ for all $i \leq n$. Moreover, ν is indeed a norm since $\nu(a_i) > 0$ for all $i \leq n$, and the infimum in the definition of ν is, by compactness, always attained.

It remains to prove the claim. Let $\|\cdot\|_2$ be the ℓ_2 norm on span A with $a_1, \ldots, a_k$ the orthonormal basis. For each $m \in \mathbb{N}$, set $\mu_m := \mu + \frac{\|\cdot\|_2}{m}$. Clearly $\mu_m \to \mu$, so it suffices to show that each μ_m satisfies the condition from the claim. Suppose that for some $m \in \mathbb{N}$ and $i \leq n$, we have

$$\mu_m(a_i) = \inf\{\sum_{j \in J} \mu_m(\alpha_j a_j) \colon i \notin J \subseteq \{1, \ldots, n\}, a_i = \sum_{j \in J} \alpha_j a_j\}.$$

By compactness, the infimum is attained: that is, there exists $(\alpha_j)_{j \leq n}$, with $\alpha_i = 0$, $a_i = \sum_{j \leq n} \alpha_j a_j$ and $\mu_m(a_i) = \sum_{j \leq n} \mu_m(\alpha_j a_j)$. Indeed, if the infimum is approximated by a sequence $(\alpha_1^l, \ldots, \alpha_n^l)_{l \in \mathbb{N}} \subseteq \mathbb{R}^n$, then since each coordinate is bounded (because up to finitely many ls, we have $\sum_{j=1}^{n} \mu_m(\alpha_j^l a_j) \leq 2\mu_m(a_i)$), we may pass to a convergent subsequence and attain the infimum at the limit. The ℓ_2 norm $\|\cdot\|_2$ is strictly convex, so $\|a_i\|_2 < \sum_{j \leq n} \|\alpha_j a_j\|_2$, while μ by triangle inequality satisfies $\mu(a_i) \leq \sum_{j \leq n} \mu(\alpha_j a_j)$. Since μ_m is the sum of μ and a positive multiple of the ℓ_2 norm, we must have $\mu_m(a_i) < \sum_{j \leq n} \mu_m(\alpha_j a_j)$, a contradiction. □

Notation 4.5. For a finite set $A \subseteq V$ and P, P' partial functions on V (i.e., functions whose domains are subsets of V) with $A \subseteq \operatorname{dom}(P), \operatorname{dom}(P')$, we put $d_A(P, P') := \max_{a \in A} |P(a) - P'(a)|$.

Let T be the countable set of tuples (n, n', P, P', g) such that:

1. $n, n' \in \mathbb{N}$;
2. $P \in \mathbb{Q}^{\operatorname{dom}(P)}$, $P' \in \mathbb{Q}^{\operatorname{dom}(P')}$, where $\operatorname{dom}(P)$ and $\operatorname{dom}(P')$ are finite subsets of V;
3. there exists $\nu \in \mathcal{B}$ such that $P' = \nu|_{\operatorname{dom}(P')}$;
4. $g : \operatorname{dom}(P) \to \operatorname{dom}(P')$ is a one-to-one mapping;
5. $P = P' \circ g$;
6. whenever $\mu' \in \mathcal{P}$, $\nu' \in \mathcal{B}$ are such that $P' \subseteq \nu'$, $d_{\operatorname{dom}(P)}(P, \mu') < \frac{1}{n}$ and μ' restricted to $\operatorname{span}(\operatorname{dom}(P)) \subseteq c_{00}$ is a norm, then there exists $T_g : (\operatorname{span}(\operatorname{dom}(P)), \mu') \to (\operatorname{span}(\operatorname{dom}(P')), \nu')$ that is a $(1 + \frac{1}{n'})$-isomorphism and $T_g \supseteq g$.

For $(n, n', P, P', g) \in T$, we let $G(n, n', P, P', g)$ be the set of $\mu \in \mathcal{P}_\infty$ such that

- whenever $d_{\operatorname{dom}(P)}(P, \mu) < \frac{1}{n}$ and μ restricted to $\operatorname{span}(\operatorname{dom}(P)) \subseteq c_{00}$ is a norm, there is a $\mathbb{Q}$-linear mapping $\Phi : V \cap \operatorname{span}(\operatorname{dom}(P')) \to V$ such that $\mu(\Phi(gx) - x) < \frac{2}{n'}\mu(x)$ for every $x \in \operatorname{dom}(P)$ and $|P'(x) - \mu(\Phi(x))| < \frac{1}{n'}P'(x)$ for every $x \in \operatorname{dom}(P')$.

Proposition 4.6. *Let $\mu \in \mathcal{P}_\infty$. Then X_μ is isometric to the Gurariĭ space if and only if $\mu \in G(n, n', P, P', g)$ for every $(n, n', P, P', g) \in T$.*

Proof. In order to prove the first implication, let $\mu \in \mathcal{P}_\infty$ be such that X_μ is isometric to the Gurariĭ space, and let $(n, n', P, P', g) \in T$ be such that $d_{\operatorname{dom}(P)}(P, \mu) < \frac{1}{n}$ and μ restricted to $\operatorname{span}(\operatorname{dom}(P)) \subseteq c_{00}$ is a norm. Consider the finite-dimensional space $A := (\operatorname{span}(\operatorname{dom}(P)), \mu)$. Let $\nu \in \mathcal{B}$ be as in 3. Put $B = (\operatorname{span}(\operatorname{dom} P'), \nu)$, and pick a basis $\mathfrak{b} \subseteq V$ of B. By 6, there exists $T_g : A \to B$, which is a $(1 + \frac{1}{n'})$-isomorphism and $T_g \supseteq g$. By [22, Lemma 2.2], there is a $(1 + \frac{1}{3n'})$-isomorphism $S : B \to X_\mu$ such that $\|ST_g - Id_A\| < \frac{1}{n'}$. By Lemma 4.31, we may for every $b \in \mathfrak{b}$ find $x_b \in V$ such that the linear mapping $Q : S(B) \to X_\mu$ given by $Q(S(b)) = x_b$, $b \in \mathfrak{b}$, is a $(1 + \frac{1}{3n'})$-isomorphism with $\|Q - Id\| < \frac{1}{3n'}$. Consider $\Phi = QS|_{V \cap \operatorname{span}(\operatorname{dom} P')}$. This is indeed a $\mathbb{Q}$-linear map and since QS is a $(1 + \frac{1}{3n'})^2$-isomorphism

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and $(1+\frac{1}{3n'})^2 < 1+\frac{1}{n'}$, we have $|\mu(\Phi(x)) - \nu(x)| < \frac{1}{n'}\nu(x)$ for $x \in \operatorname{dom}(P')$. Moreover, for every $x \in \operatorname{dom}(P)$, we have

$$\begin{aligned}\mu(\Phi(gx)-x) = \mu(QST_gx - x) &\leq \mu(QST_gx - ST_gx) + \mu(ST_gx - x)\\ &< \tfrac{1}{3n'}\|ST_g\|\mu(x) + \tfrac{1}{n'}\mu(x) \leq \tfrac{2}{n'}\mu(x).\end{aligned}$$

This shows that $\mu \in G(n,n',P,P',g)$.

In order to prove the second implication, let $\mu \in \mathcal{P}_\infty$ be such that $\mu \in G(n,n',P,P',g)$ whenever $(n,n',P,P',g) \in T$. In what follows for $x \in c_{00}$, we denote by $[x]\in X_\mu$ the equivalence class corresponding to x. Pick a finite-dimensional space $A \subseteq X_\mu$ and an isometry $G : A \to B$, where B is a finite-dimensional Banach space; we may without loss of generality assume $B \subseteq X_{\mu_B}$ for some $\mu_B \in \mathcal{B}$. Let $\mathfrak{b}_A := \{a_1,\dots,a_j\}$ be a normalized basis of A, and extend $G(\mathfrak{b}_A) = \{G(a_1),\dots,G(a_j)\}$ to a normalized basis $\mathfrak{b}_B = \{b_1,\dots,b_k\}$ of B. Fix $\eta > 0$. It suffices to find a $(1+\eta)$-isomorphism $\Psi : B \to X_\mu$ with $\|\Psi G - I_A\| \leq \eta$. Consider the functions ϕ_1 and $\phi_2^{\mathfrak{b}_A}$ from Lemma 2.2 and Lemma 4.31. Pick $\delta \in (0,1)$ such that $\max\{\phi_1(t), \phi_2^{\mathfrak{b}_A}(t)\} < \eta$ whenever $t < \delta$ and $\varepsilon \in (0, \frac{1}{20})$ such that $\phi_1(5\varepsilon) < \frac{1}{20}$ and $\varepsilon + 72\max\{\varepsilon, \phi_1(5\varepsilon)\} < \delta$.

Claim 1. *There are finite sets $M, N \subseteq V$ such that μ restricted to* $\operatorname{span} N \subseteq c_{00}$ *is a norm and surjective $(1+\varepsilon)$-isomorphisms $T_A : A \to (\operatorname{span} N, \mu)$, $T_B : B \to (\operatorname{span} M, \mu_B)$ such that:*

- *N and M are ε-dense sets for $S_{T_A(A)}$ and $S_{T_B(B)}$, respectively;*
- *we have $\|[T_A a_i] - a_i\|_{X_\mu} < \varepsilon$ for every $a_i \in \mathfrak{b}_A$ and $\|(T_A)^{-1}x - [x]\|_{X_\mu} < \varepsilon$, $|\mu(x) - 1| < \varepsilon$ for every $x \in N$;*
- *$(T_B)^{-1}(M)$ is $\frac{\varepsilon}{3}$-dense for S_B and* $\max\{|\mu_B((T_B)^{-1}x) - 1|, |\mu_B(x) - 1|\} < \frac{\varepsilon}{2}$ *for every $x \in M$;*
- *$\big(T_B G(T_A)^{-1}\big)(N) \subseteq M$.*

Proof of Claim 1. By Lemma 4.31, we may pick $\{f_1,\dots,f_n\} \subseteq V$ such that the linear operator $T_A : A \to X_\mu$ given by $T_A(a_i) = [f_i]$, $i \leq j$, is a $(1+\frac{\varepsilon}{6})$-isomorphism and $\|[T_Ax] - x\|_{X_\mu} < \frac{\varepsilon}{6}\|x\|_{X_\mu}$, $x \in A$. This implies that μ restricted to $\operatorname{span}\{f_1,\dots,f_j\}$ is a norm, and since $T_A(A)$ is isometric to $(\operatorname{span}\{f_1,\dots,f_n\},\mu)$, we consider T_A a $(1+\frac{\varepsilon}{6})$-isomorphism between A and $(\operatorname{span}\{f_1,\dots,f_n\},\mu)$. Now, pick $N' \subseteq A$ a finite $\frac{\varepsilon}{6}$-dense set for S_A consisting of rational linear combinations of points from $\mathfrak{b}_A$ with $\mathfrak{b}_A \subseteq N'$ such that $|\|x\|_{X_\mu} - 1| < \frac{\varepsilon}{6}$ for every $x \in N'$. Then $\|[T_Ax] - x\|_{X_\mu} < \frac{\varepsilon}{6}\|x\|_{X_\mu} < \frac{\varepsilon}{6}(1+\frac{\varepsilon}{6}) < \frac{\varepsilon}{5}$ for every $x \in N'$. Put $N := T_A(N') \subseteq V$. Then we easily obtain $|\mu(x) - 1| < \frac{\varepsilon}{2}$ for every $x \in N$ and, by Lemma 4.32, N is $\frac{\varepsilon}{2}$-dense in $S_{T_A(A)}$. Similarly as above, we may pick $\{g_1,\dots,g_k\} \subseteq V$ such that the linear operator $T_B : B \to X_{\mu_B}$ given by $T_B(b_i) = g_i$, $i \leq k$, is a $(1+\frac{\varepsilon}{6})$-isomorphism, and we find $M' \subseteq B$ a finite $\frac{\varepsilon}{6}$-dense set for S_B consisting of rational linear combinations of points from $\mathfrak{b}_B$ with $M' \supseteq \{G(x) :\ x \in N'\} \cup \mathfrak{b}_B$ and $|\mu_B(x) - 1| < \frac{\varepsilon}{6}$ for $x \in M'$. Put $M := T_B(M')$; then similarly as above, $|\mu_B(x) - 1| < \frac{\varepsilon}{2}$ for every $x \in M$ and M is $\frac{\varepsilon}{2}$-dense in $S_{T_B(B)}$. Finally, we obviously have $\big(T_BG(T_A)^{-1}\big)(N) = T_B(G(N')) \subseteq T_B(M') = M$. □

By Lemma 4.4, there is $\nu \in \mathcal{B}$ having rational values on M with $d_M(\nu, \mu_B \circ (T_B)^{-1}) < \frac{\varepsilon}{2}$. Put $P' = \nu|_M$, consider the one-to-one map $g : N \to M$ given by $g := T_BG(T_A)^{-1}|_N$ and put $P = P' \circ g$. Let $n \in \mathbb{N}$ be the integer part of $\frac{2}{3\varepsilon}$ and $n' \in \mathbb{N}$ be the integer part of $\frac{1}{9\max\{\varepsilon,\phi_1(5\varepsilon)\}}$. Easy computations show that $\frac{3}{2}\varepsilon \leq \frac{1}{n} < 2\varepsilon$ and $9\max\{\varepsilon,\phi_1(5\varepsilon)\} \leq \frac{1}{n'} < 18\max\{\varepsilon,\phi_1(5\varepsilon)\}$ (in the last inequality, we are using that $\max\{\varepsilon,\phi_1(5\varepsilon)\} < \frac{1}{20}$).

Note that for every $x \in M$, we have

$$\max\{|\nu(T_Bx) - 1|, |\nu(x) - 1|\} \leq \frac{\varepsilon}{2} + \max\{|\mu_B(x) - 1|, |\mu_B((T_B)^{-1}x) - 1|\} < \varepsilon. \tag{4.1}$$

Claim 2. *We have $(n,n',P,P',g) \in T$ and $d_N(P,\mu) < \frac{1}{n}$.*

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Proof of Claim 2. In order to see that $d_N(P,\mu) < \frac{1}{n}$, pick $x \in N$. Then

$$|P(x) - \mu(x)| \le \frac{\varepsilon}{2} + |\mu_B(G(T_A)^{-1}(x)) - \mu(x)| = \frac{\varepsilon}{2} + |\|(T_A)^{-1}(x)\|_{X_\mu} - \|[x]\|_{X_\mu}| < \frac{3}{2}\varepsilon.$$

In order to see that $(n,n',P,P',g) \in T$, let us verify condition 6. Let $\mu' \in \mathcal{P}$, $\nu' \in \mathcal{B}$ be such that $P' \subseteq \nu'$, $d_N(P,\mu') < \frac{1}{n} < 2\varepsilon$ and μ' restricted to $\operatorname{span} N \subseteq c_{00}$ is a norm. Note that $|\mu'(x) - 1| < 5\varepsilon$ for every $x \in N$, and so, since N is ε-dense for the sphere of $T_A(A) = (\operatorname{span} N, \mu)$, the mapping $id : (\operatorname{span} N, \mu) \to (\operatorname{span} N, \mu')$ is a $(1+\phi_1(5\varepsilon))$-isomorphism. Further, $|\nu'(x)-1| = |\nu(x)-1| < \varepsilon$ for every $x \in M$, and so the mapping $id : (\operatorname{span} M, \mu_B) \to (\operatorname{span} M, \nu')$ is a $(1+\phi_1(5\varepsilon))$-isomorphism as well. Finally, since $T_B G(T_A)^{-1}$ is a $(1+\varepsilon)^2$-isomorphism between $(\operatorname{span} N, \mu)$ and $(\operatorname{span} g(N), \mu_B)$ and

$$(1+\phi_1(5\varepsilon))^2(1+\varepsilon)^2 \le (1+3\phi_1(5\varepsilon))(1+3\varepsilon) \le 1 + 9\max\{\varepsilon, \phi_1(5\varepsilon)\} \le 1 + \frac{1}{n'},$$

we have that $T_g := id \circ T_B \circ G \circ (T_A)^{-1} \circ id : (\operatorname{span} N, \mu') \to (\operatorname{span} M, \nu')$ is a $(1+\frac{1}{n'})$-isomorphism. □

Since $\mu \in G(n,n',P,P',g)$, there is a $\mathbb{Q}$-linear mapping $\Phi : V \cap (\operatorname{span} M, \nu) \to V$ such that $\mu(\Phi(gx) - x) < \frac{2}{n'}\mu(x)$ for every $x \in N$ and $|\nu(x) - \mu(\Phi(x))| < \frac{1}{n'}\nu(x)$ for every $x \in M$. It is easy to see that Φ extends to a bounded linear operator $\Phi' : (\operatorname{span} M, \nu) \to X_\mu$. Finally, consider $\Psi := \Phi' \circ T_B : B \to X_\mu$.

For every $x \in M$, we have

$$|\mu(\Phi(x)) - 1| \le |\mu(\Phi(x)) - \nu(x)| + |\nu(x) - 1| \overset{(4.1)}{\le} \frac{1}{n'}\nu(x) + \varepsilon \overset{(4.1)}{\le} \frac{1}{n'}(1+\varepsilon) + \varepsilon < \delta;$$

thus, $|\|\Psi(x)\|_{X_\mu} - 1| < \delta$ for every $x \in (T_B)^{-1}(M)$ and so Ψ is a $(1+\eta)$-isomorphism.

Further, we have

$$\begin{aligned}\|\Psi G(a_i) - a_i\|_{X_\mu} &\le \|\Phi(g(T_A a_i)) - [T_A a_i]\|_{X_\mu} + \|[T_A a_i] - a_i\|_{X_\mu}\\ &< \frac{2}{n'}(1+\varepsilon) + \varepsilon \le \frac{4}{n'} + \varepsilon < \delta;\end{aligned}$$

hence, by Lemma 4.31, we have $\|\Phi' T_B G - I_A\| \le \phi_2^{\mathfrak{b}_A}(\frac{2}{n'}(1+\varepsilon)+\varepsilon) < \eta$. □

Theorem 4.7. *Let $\mathbb{G}$ be the Gurariĭ space. Then the isometry class $\langle \mathbb{G} \rangle_{\equiv}^{\mathcal{P}_\infty}$ is a G_δ-set in $\mathcal{P}_\infty$.*

Proof. By Proposition 4.6, we have for the countable set T defined before Proposition 4.6 that

$$\langle \mathbb{G} \rangle_{\equiv}^{\mathcal{P}_\infty} = \bigcap_{(n,n',P,P',g)\in T} G(n,n',P,P',g),$$

where $G(n,n',P,P',g)$ is the union of a closed and an open set in $\mathcal{P}_\infty$ (here we use the observation that the set $\{\mu \in \mathcal{P}_\infty : \mu \text{ restricted to } \operatorname{span}(\operatorname{dom} P) \subseteq c_{00} \text{ is a norm}\}$ is open due to Lemma 2.4); thus it is the countable intersection of G_δ-sets. □

Proof of Theorem 4.1. Let us recall that $\mathcal{P}_\infty$ and $\mathcal{B}$ are G_δ-sets in $\mathcal{P}$; see Corollary 2.5. Thus, since we have $\langle \mathbb{G} \rangle_{\equiv}^{\mathcal{B}} = \langle \mathbb{G} \rangle_{\equiv}^{\mathcal{P}_\infty} \cap \mathcal{B}$, it follows from Proposition 4.6 that $\langle \mathbb{G} \rangle_{\equiv}^{\mathcal{I}}$ is a G_δ-set in any $\mathcal{I} \in \{\mathcal{P}, \mathcal{P}_\infty, \mathcal{B}\}$.

By Corollary 2.10, we also have that $\langle \mathbb{G} \rangle_{\equiv}^{\mathcal{I}}$ is dense in $\mathcal{I}$ for every $\mathcal{I} \in \{\mathcal{P}, \mathcal{P}_\infty, \mathcal{B}\}$. □

4.2. *Generic objects in* $SB(X)$

In this subsection, we address Problem 5.5 from [19], which suggests investigating generic properties of admissible topologies. We have both positive and negative results. The positive result is Theorem 4.10, which shows that the isometry class of the Gurariĭ space, as a subset of $SB(\mathbb{G})$, is a dense G_δ-set in the Wijsman topology. The negative results are Propositions 4.11 and 4.13 and Theorem 4.12.

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Definition 4.8. Given a closed set H in X, we denote by $E^-(H)$ the set $SB(X) \setminus E^+(X \setminus H)$: that is, $E^-(H) = \{F \in SB(X) : F \subseteq H\}$. Obviously, this is a closed set in any admissible topology on $SB(X)$.

Definition 4.9. Let X be an isometrically universal separable Banach space. By τ_W, we denote the restriction of *the Wijsman topology* from $\mathcal{F}(X)$ to $SB(X)$: that is, the minimal topology on $SB(X)$ such that the mappings $SB(X) \ni F \mapsto \mathrm{dist}_X(x, F)$ are continuous for every $x \in X$. Note that τ_W is admissible; see [19, Section 2].

Theorem 4.10. *The isometry class $\langle \mathbb{G} \rangle_\equiv$ is a dense G_δ-set in $(SB(\mathbb{G}), \tau_W)$.*

Proof. The isometry class $\langle \mathbb{G} \rangle_\equiv$ is a G_δ-set in $(SB(\mathbb{G}), \tau_W)$ since it is a G_δ-set in $\mathcal{P}$ (by Theorem 4.1) and there is a continuous reduction from $(SB(\mathbb{G}), \tau_W)$ to $\mathcal{P}$ by Theorem 3.3. So we must show that it is dense.

Choose a basic open set N in τ_W that is given by some closed subspace $X \subseteq \mathbb{G}$, finitely many points $x_1, \dots, x_n \in \mathbb{G}$ and $\varepsilon > 0$ so that

$$N = \{Z \in SB(\mathbb{G}) \colon \forall i \leq n \ (|\,\mathrm{dist}_{\mathbb{G}}(x_i, X) - \mathrm{dist}_{\mathbb{G}}(x_i, Z)| < \varepsilon)\}.$$

Let us find a space G isometric to $\mathbb{G}$ such that $G \in N$. Let Y be $\mathrm{span}\{X \cup \{x_i \colon i \leq n\}\}$. Since X embeds into both Y and $\mathbb{G}$, we can consider the push-out of that diagram: that is, the amalgamated sum of Y and $\mathbb{G}$ along the common subspace X. Recall this is nothing but the quotient $(\mathbb{G} \oplus_1 Y)/Z$, where $Z = \{(z, -z) \colon z \in X\}$. Denote this space by G', and notice that $\mathbb{G}$ is naturally embedded into G'. It is straightforward to verify that for each $i \leq n$, $\mathrm{dist}_{G'}(x_i, \mathbb{G}) = \mathrm{dist}_{\mathbb{G}}(x_i, X)$. Since $\mathbb{G}$ is universal, there is a linear isometric embedding $\iota : G' \hookrightarrow \mathbb{G}$. As there is a linear isometry $\phi : \iota[\mathrm{span}\{x_i \colon i \leq n\}] \to \mathrm{span}\{x_i \colon i \leq n\}$, by [22, Theorem 1.1], there is a bijective linear isometry $\Phi : \mathbb{G} \to \mathbb{G}$ such that $\|\Phi \circ \iota(x_i) - x_i\| < \varepsilon$ for each $i \leq n$. By the triangle inequality, it follows that $G := \Phi \circ \iota[\mathbb{G} \subseteq G']$ satisfies for each $i \leq n$, $|\,\mathrm{dist}_{\mathbb{G}}(x_i, G) - \mathrm{dist}_{\mathbb{G}}(x_i, X)| < \varepsilon$, so it is the desired space isometric to $\mathbb{G}$ lying in the open set N. □

The rest of the section is devoted to negative results. They show that the definition of an admissible topology allows a lot of flexibility by which one can alter which properties should be meager or not.

Proposition 4.11. *Let X be an isometrically universal separable Banach space, and let τ be an admissible topology on $SB(X)$. Then there exists an admissible topology $\tau' \supseteq \tau$ on $SB(X)$ such that the set $\langle \mathbb{G} \rangle_\simeq$ is nowhere dense in $(SB_\infty(X), \tau')$.*

Proof. By the definition of an admissible topology, we may pick $(U_n)_{n \in \mathbb{N}}$, a basis of the topology τ, such that for every $n \in \mathbb{N}$, there are nonempty open sets V^n_k, $k = 1, \dots, N_n$, and W_n in X such that the set U'_n defined by

$$U'_n = \bigcap_{k=1}^{N_n} E^+(V^n_k) \setminus E^+(W_n)$$

is a nonempty subset of U_n.

We *claim* that for every $n \in \mathbb{N}$, there is $F_n \in U'_n$ such that $\mathbb{G} \not\hookrightarrow F_n$. Indeed, pick an arbitrary $Z \in U'_n$. We may without loss of generality assume there is $H_0 \subseteq Z$ with $H_0 \simeq \mathbb{G}$, and since $\mathbb{G}$ is isometrically universal, there is $H_1 \subseteq H_0$ with $H_1 \simeq \ell_2$. Now, pick points $v_k \in Z \cap V^n_k$, $k = 1, \dots, N_n$. Then we put $F_n := \overline{\mathrm{span}}\{v_1, \dots, v_{N_n}, u : u \in H_1\}$. Since F_n is a subset of Z, we have $F_n \notin E^+(W_n)$, and since it contains the points $v_1, \dots, v_{N_n}$, we have $F_n \in U'_n$. Moreover, it is a space isomorphic to ℓ_2, and so $\mathbb{G} \not\hookrightarrow F_n$.

Thus, for every $n \in \mathbb{N}$, there is a closed subspace F_n of X such that $U_n \cap E^-(F_n)$ is a nonempty set disjoint from $\langle \mathbb{G} \rangle_\simeq$.

It is a classical fact (see, e.g., [21, Lemma 13.2 and Lemma 13.3]) that the topology τ' generated by $\tau \cup \{E^-(F_n) : n \in \mathbb{N}\}$ is Polish. It is easy to check it is admissible. Moreover, for every $n \in \mathbb{N}$, we have

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that $U_n \cap E^-(F_n)$ is a nonempty τ'-open set in U_n disjoint from $\langle \mathbb{G} \rangle_{\simeq}$. It follows that nonempty sets of the form $U_n \cap \bigcap_{m\in I} E^-(F_m)$, for finite $I \subseteq \mathbb{N}$, give us a π-basis of τ'. Since obviously each element of the form $U_n \cap \bigcap_{m\in I} E^-(F_m)$ is disjoint from $\langle \mathbb{G} \rangle_{\simeq}$, the set $\langle \mathbb{G} \rangle_{\simeq}$ is τ'-nowhere dense. □

Actually, one may observe that the same proof gives the following more general result, where the pair $(\mathbb{G}, \ell_2)$ is replaced by a more general pair of Banach spaces.

Theorem 4.12. *Let X be an isometrically universal separable Banach space, and let τ be an admissible topology on $SB(X)$. Let Y and Z be infinite-dimensional Banach spaces such that $Y \hookrightarrow Z$ and $Z \not\hookrightarrow Y \oplus F$ for every finite-dimensional space F.*

Then there exists an admissible topology $\tau' \supseteq \tau$ on $SB(X)$ such that the set $\langle Z \rangle_{\simeq}$ is nowhere dense in $(SB_\infty(X), \tau')$.

It is even possible to find an admissible topology τ such that $\langle \ell_2 \rangle_{\equiv}$ is not a meager set in $(SB_\infty(X), \tau)$, which is an immediate consequence of the following more general observation (property (P) below would be 'X is isometric to ℓ_2').

Proposition 4.13. *Let X be an isometrically universal separable Banach space and τ be an admissible topology on $SB_\infty(X)$. Let (P) be a non-void property (i.e., there are spaces with such a property) of infinite-dimensional Banach spaces closed under taking subspaces. Then there is an admissible topology $\tau' \supseteq \tau$ such that the set $\{Y \in SB_\infty(X) : Y \text{ has } (P)\}$ has non-empty interior in $(SB_\infty(X), \tau')$.*

Proof. Pick $F \in SB_\infty(X)$ with (P). Using again the classical fact (see, e.g., [21, Lemma 13.2]) that the topology τ' generated by $\tau \cup \{E^-(F)\}$ is Polish, it is easy to check it is admissible. Then the τ'-open set $E^-(F)$ is a subset of $\{Y \in SB_\infty(X) : Y \text{ has } (P)\}$. □

Conflicts of Interest. None

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