Can. Math Bull. Vol. **39** (3), 1996 pp. 330–345

PSEUDO-HOMOGENEOUS COORDINATES FOR HUGHES PLANES

PETER MAIER AND MARKUS STROPPEL

ABSTRACT. Among the projective planes, the class of Hughes planes has received much interest, for several good reasons. However, the existing descriptions of these planes are somewhat unsatisfactory. We introduce pseudo-homogeneous coordinates which at the same time are easy to handle and give insight into the action of the group that is generated by all elations of the desarguesian Baer subplane of a Hughes plane. The information about the orbit decomposition is then used to give a description in terms of coset spaces of this group. Finally, we exhibit a non-closing Desargues configuration in terms of coordinates.

1. **Introduction.** In [13], D. R. Hughes constructed a class of finite projective planes $\mathcal{P}$ with the following property:

(H) There exists a desarguesian Baer subplane $\mathcal{B}$ of $\mathcal{P}$ such that every axial collineation of $\mathcal{B}$ is induced by an axial collineation of $\mathcal{P}$.

In particular, these planes are semi-translation planes in the sense of T. G. Ostrom [18], compare [6, p. 136]. Among the finite projective planes, there are two classes of *proper projective planes* (in the sense of C. Hering [12]: the group of collineations fixes neither a point nor a line); namely, the Figueroa planes (see [10] and the references given there), and the Hughes planes (including the desarguesian planes). Moreover, the group that is generated by the set of all axial collineations of $\mathcal{B}$ is quite large, compared to the order of $\mathcal{P}$, and it contains a large simple subgroup (generated by the set of all elations). This group may be used for characterizations of Hughes planes, see [8], [15], [23], [28]. The finite Hughes planes are self-dual, see [21], [19], [20]; therefore, they contain interesting ovals and unitals, *cf.* [22]. Thus there are several reasons for the interest in the class of Hughes planes.

For his construction, D. R. Hughes used a finite nearfield N with the property that the kernel K of N coincides with the center of N, and that $\dim_K N = 2$; recall that the distributive law $a(x+y) = ax+ay$ implies that N is a right vector space over the skewfield K. The latter condition is equivalent to the existence of some $a \in N \setminus K$ such that $N = K + aK$. Subsequently, the construction was generalized until P. Dembowski claimed in [7] that one obtains a projective plane whenever starting with a nearfield N of dimension 2 over its kernel K. However, M. Biliotti [3] observed that, if K is not contained in the center

Received by the editors October 14, 1994.

AMS subject classification: Primary 51A35, secondary 51A25, 51A10.

Key words and phrases: Hughes planes, projective planes, special linear groups, nearfileds, orbit decompositions.

of N, an additional condition is necessary (and sufficient), namely, that there exists some $t \in N$ such that $N = K + Kt$. This condition follows immediately from the assumption that $\dim_K N = 2$ if K is contained in the center of N, or if N is finite. In the general case, however, the dimension assumption alone is not sufficient: There exist skewfield extensions (N, K) where the dimension of N is 2, if N is considered as a right vector space over K, but the dimension is different if N is considered as a left vector space [4], see [5, 11.5] for an overview and [25] for exhaustive results.

In order to give a relatively easy (and elegant) description of the Hughes planes, and to give a lucid proof of the fact that the Hughes planes are projective planes, we introduce "pseudo-homogeneous" coordinates over N, and we study the action of the groups GL_3K and SL_3K, where K is a suitable sub-skewfield of the kernel of N. Recall that the special linear group SL_nK is the (normal) subgroup of GL_nK that is generated by the set of all transvections. The special linear group may also be described by means of J. Dieudonné's determinant function, which takes its values in the commutator factor group of the multiplicative group of K, see [2, Ch. IV, §1], [9, Ch. II, §1], [5, 11.2] or [16, Kap. 12].

Pseudo-homogeneous coordinates provide a convenient general construction which exhibits many common features for the class of planes that comprises the planes originally constructed by D. R. Hughes as well as the generalizations to the infinite case and to the case where K is not contained in the center of N. We believe that there is no more reason to distinguish between Hughes planes and "generalized Hughes planes", as it was done by P. Dembowski [7]. Note that there is still a more general notion of "Hughes plane", which just requires condition **(H)**. For special situations (*e.g.*, compact connected planes), this point of view has proved to be fruitful, compare [24, Section 86]. However, this leads to planes which can no longer be described by coordinates over a nearfield; one has to employ rather involved group theoretical arguments in order to show that these planes are projective planes.

The present article developed from a treatment of the Hughes planes in a series of lectures by the second author. Among other sources, these lectures drew some inspiration from [1], although this happened in a rather sub-conscious manner. Improvements of both the results and the exposition were achieved in the first author's thesis [17].

2. **Pseudo-homogeneous Coordinates.** Recall that a *nearfield* $N = (N, +, 0, \cdot, 1)$ satisfies all axioms for a skewfield except that we stipulate only one of the distributive laws, namely

(D) $$a(x+y) = ax + ay \text{ for all } a, x, y \in N.$$

The *kernel* $K(N)$ of N is formed by all elements $k \in N$ that satisfy

($\mathbf{D_k}$) $$(x+y)k = xk + yk \text{ for all } x, y \in N.$$

We remark that the element -1 is always contained in $K(N)$, since it is even central in N, *cf.* [14]. With the induced operations, the subset $K(N)$ forms a skewfield, and N is a right vector space over $K(N)$. See [14] for an account of the theory of nearfields.

DEFINITION 2.1. Assume that N is a nearfield, and that K is a sub-skewfield of $K(N)$. The pair (N, K) is called *H-suited*, if the following hold.

(**A**) There exists some $a \in N \setminus K$ such that $N = K + aK$. This means that N, considered as a right vector space over K, has dimension 2.

(**B**) There exists some $t \in N \setminus K$ such that $N = K + Kt$.

REMARK 2.2. (1) If (**A**) and (**B**) are satisfied, then the representation $x = \xi + \xi' t$, where $\xi, \xi' \in K$, is of course unique for every $x \in N$.

(2) Using (**D**), one easily infers that if condition (**B**) is valid for some $t \in N \setminus K$, then it is valid for every $s \in N \setminus K$.

LEMMA 2.3. *Condition* (**B**) *of 2.1 is equivalent to the following.*

(**B**$'$) *For every pair* $(\alpha', \beta') \in K^2$ *there exists a pair* $(\alpha, \beta) \in K^2$ *such that* $(\alpha + t\alpha')t = \beta + t\beta'$.

PROOF. Assume first that (**B**) holds. Then we find $\alpha, \beta \in K$ such that $t\beta' = -\beta + \alpha t$, and assertion (**B**$'$) holds for the case where $\alpha' = 0$. If $\alpha' \neq 0$, put $s := (t - \alpha'^{-1}\beta')^{-1}$. Since $s \in N \setminus K$, there are $\gamma, \gamma' \in K$ such that $t\alpha' = \gamma + \gamma' s$, *cf.* 2.2. For $\alpha := -\gamma$ and $\beta := \gamma' - \gamma\alpha'^{-1}\beta'$ we obtain that $\alpha + t\alpha' = \alpha + \gamma + \gamma' s = (\beta - \alpha\alpha'^{-1}\beta')s$, and therefore $(\alpha + t\alpha')(t - \alpha'^{-1}\beta') = \beta - \alpha\alpha'^{-1}\beta'$. Using (**D**) and ($\mathbf{D}_{\alpha'^{-1}\beta'}$), we obtain (**B**$'$).

Now assume that (**B**$'$) holds. Considering pairs of the form $(0, \beta')$, we infer that $tK \subseteq K + Kt$. Since $K + Kt$ is closed with respect to addition, we infer that $N = K + tK \subseteq K + Kt \subseteq N$, whence $N = K + Kt$. ∎

DEFINITION 2.4. Let N be a nearfield, let K be a sub-skewfield of $K(N)$, and fix an element $t \in N \setminus K$. We consider the action ω of GL_3K on $N^3 \setminus \{\mathbf{0}\}$ that is obtained from the restriction of the usual linear action of GL_3K on N^3 (from the right), and the action of the multiplicative group $N^\times$ on $N^3 \setminus \{\mathbf{0}\}$ via $\big((x, y, z), f\big) \mapsto (f^{-1}x, f^{-1}y, f^{-1}z)$. The orbit of (x, y, z) under the latter action will be denoted by $[x, y, z]$. For every matrix

$$A = \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{pmatrix} \in K^{3\times 2} \setminus \{\mathbf{0}\},$$

we define

$$[\![A]\!] := \left[\!\!\left[\begin{matrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{matrix}\right]\!\!\right] := \Big\{[x_1, x_2, x_3] \,\Big|\, \textstyle\sum_\mu x_\mu\alpha_\mu + \big(\sum_\mu x_\mu\beta_\mu\big)\, t = 0\Big\}.$$

We write

$$P := \Big\{[x, y, z] \,\Big|\, (x, y, z) \in N^3 \setminus \{\mathbf{0}\}\Big\} \text{ and } \mathcal{L} := \Big\{[\![A]\!] \,\Big|\, A \in K^{3\times 2} \setminus \{\mathbf{0}\}\Big\},$$

and set $H(N, K) := (P, \mathcal{L})$.

Our aim is to show that for every H-suited pair (N, K) the incidence structure $H(N, K)$ is a projective plane with the property (**H**).

As an immediate consequence of the definitions, we observe the following.

REMARK 2.5. The action ω of $\Gamma := \mathrm{GL}_3K$ is in fact an action by collineations of $H(N, K)$. The application of $M \in \mathrm{GL}_3K$ to a line is given by $[\![A]\!]^M := [\![M^{-1}A]\!]$.

We shall mainly be interested in the restriction of ω to the subgroup $\Delta := \mathrm{SL}_3K$.

NOTATION 2.6. For $\boldsymbol{x} = (x_1, x_2, x_3) \in N^3$ and $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3) \in K^3$, we use the simplified notation

$$\boldsymbol{x\alpha} = \sum_{\mu=1}^{3} x_\mu \alpha_\mu \text{ and } [\![\boldsymbol{\alpha}, \boldsymbol{\beta}]\!] = \left[\!\!\left[\begin{matrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{matrix}\right]\!\!\right].$$

In particular, we have $[\![\boldsymbol{\alpha}, \boldsymbol{\beta}]\!] = \{[\boldsymbol{x}] \mid \boldsymbol{x\alpha} + (\boldsymbol{x\beta})t = 0\}$. By $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ we denote the standard basis $\big((1, 0, 0), (0, 1, 0), (0, 0, 1)\big)$ of K^3.

Note that, for every $f \in N$, we have that $(f\boldsymbol{x})\boldsymbol{\alpha} = f(\boldsymbol{x\alpha})$ by **(D)**. Therefore, the incidence relation $[\boldsymbol{x}] \in [\![\boldsymbol{\alpha}, \boldsymbol{\beta}]\!]$ does not depend on the representative $\boldsymbol{x}$. Note also that, in general, the equation $(\boldsymbol{x\alpha})\kappa = \boldsymbol{x}(\boldsymbol{\alpha}\kappa)$ is valid only for $\kappa \in K(N)$.

The following observation is due to M. Biliotti [3].

PROPOSITION 2.7. *Assume that N is a nearfield, and that K is a sub-skewfield of K(N) such that assertion* **(A)** *of 2.1 is valid. If H(N, K) is a projective plane, then condition* **(B)** *of 2.1 is also satisfied; i.e., (N, K) is an H-suited pair.*

PROOF. Assume that $H(N, K)$ is a projective plane. As in the Proof of 2.3, it suffices to show that $t\gamma \in K + Kt$ for arbitrary $\gamma \in K^\times$. We consider the points $[1, t, 0]$ and $[0, t\gamma, 1]$ together with their joining line $[\![\boldsymbol{\alpha}, \boldsymbol{\beta}]\!]$.

(i) If $\beta_2 = 0$, we obtain the equations

$$\alpha_1 + t\alpha_2 + \beta_1 t = 0 \text{ and } t\gamma\alpha_2 + \alpha_3 + \beta_3 t = 0.$$

From $\beta_1 = 0$ we could deduce that $\alpha_1 = 0 = \alpha_2$, and that $\alpha_3 = \beta_3 = 0$, contradicting the fact that $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \neq (\boldsymbol{0}, \boldsymbol{0})$. Similarly, we infer that $\alpha_2 \neq 0$. We multiply the first of the equations by $\beta_3\beta_1^{-1}$ from the left and subtract this from the second equation. This leads to

$$t\gamma = (\beta_3\beta_1^{-1}\alpha_1 - \alpha_3)\alpha_2^{-1} + \beta_3\beta_1^{-1}t \in K + Kt.$$

(ii) If $\beta_2 \neq 0$, we have

$$\alpha_1 + t\alpha_2 + (\beta_1 + t\beta_2)t = 0 \text{ and } t\gamma\alpha_2 + \alpha_3 + (t\gamma\beta_2 + \beta_3)t = 0. \tag{1}$$

Putting $\lambda := \beta_2^{-1}\alpha_2$, $\mu := \alpha_1 - \beta_1\beta_2^{-1}\alpha_2$ and $\nu := \alpha_3 - \beta_3\beta_2^{-1}\alpha_2$, we obtain

$$(\beta_1 + t\beta_2)\lambda + \mu = (\beta_1 + t\beta_2)\beta_2^{-1}\alpha_2 + \alpha_1 - \beta_1\beta_2^{-1}\alpha_2 = t\alpha_2 + \alpha_1 \tag{2}$$

and

$$(t\gamma\beta_2 + \beta_3)\lambda + \nu = (t\gamma\beta_2 + \beta_3)\beta_2^{-1}\alpha_2 + \alpha_3 - \beta_3\beta_2^{-1}\alpha_2 = t\gamma\alpha_2 + \alpha_3. \tag{3}$$

Substituting (2) and (3) in (1) and using **(D)**, we infer

$$0 = (\beta_1 + t\beta_2)(\lambda + t) + \mu \text{ and } 0 = (t\gamma\beta_2 + \beta_3)(\lambda + t) + \nu.$$

This gives two expressions for $\lambda + t$, and we obtain

$$(\beta_1 + t\beta_2)^{-1}\mu = (t\gamma\beta_2 + \beta_3)^{-1}\nu,$$

which leads to

$$\mu^{-1}(\beta_1 + t\beta_2) = \nu^{-1}(t\gamma\beta_2 + \beta_3);$$

recall that $\mu \neq 0$ since $\beta_2 \neq 0$. Now $t\gamma = (\nu\mu^{-1}\beta_1 - \beta_3)\beta_2^{-1} + \nu\mu^{-1}t \in K + Kt$. ∎

The next Lemma justifies our seemingly arbitrary choice of t, and shows that our line set contains the line set of the Hughes planes, as defined in [7]. Of course, this means that the definitions coincide, if $H(N, K)$ is a projective plane.

LEMMA 2.8. *If (N, K) is an H-suited pair, then the set $\mathcal{L}$ of lines is independent of the choice of $t \in N \setminus K$.*

PROOF. Let s be an element of $N \setminus K$. According to **(B)**, there are $\sigma, \sigma' \in K$ such that $s = \sigma + \sigma' t$. Now one computes easily that $\{[\boldsymbol{x}] \in P \mid \boldsymbol{x\alpha} + (\boldsymbol{x\beta})s = 0\} = \left[\!\left[\boldsymbol{\alpha} + \boldsymbol{\beta}\sigma, \boldsymbol{\beta}\sigma'\right]\!\right]$. ∎

Another easy computation yields the following.

LEMMA 2.9. *If $(\beta, \gamma) \in K^2 \setminus \{\mathbf{0}\}$ and $\boldsymbol{\alpha} \in K^3 \setminus \{\mathbf{0}\}$, then $\left[\!\left[\boldsymbol{\alpha}\beta, \boldsymbol{\alpha}\gamma\right]\!\right] = \left[\!\left[\boldsymbol{\alpha}, \mathbf{0}\right]\!\right]$.*

The sets $P^\circ := \left\{[\boldsymbol{\xi}] \,\middle|\, \boldsymbol{\xi} \in K^3 \setminus \{\mathbf{0}\}\right\}$ and $\mathcal{L}^\circ := \left\{\left[\!\left[\boldsymbol{\alpha}, \mathbf{0}\right]\!\right] \,\middle|\, \boldsymbol{\alpha} \in K^3 \setminus \{\mathbf{0}\}\right\}$ of *inner points* and *inner lines* form an ω-invariant substructure $H^\circ(N, K)$ of $H(N, K)$. For every line $l \in \mathcal{L}$, we shall write $l^\circ := l \cap P^\circ$. Obviously, we have:

THEOREM 2.10. *The incidence structure $H^\circ(N, K)$ is a projective plane; it is isomorphic to the desarguesian plane over K, and the action of Δ is equivalent to the usual action. In particular, this action is twofold transitive both on P° and on $\mathcal{L}^\circ$.*

LEMMA 2.11. *The sets $P \setminus P^\circ$ and $\mathcal{L} \setminus \mathcal{L}^\circ$ of outer points and lines are orbits under Δ.*

PROOF. If $[\boldsymbol{x}]$ is an outer point, we may assume that $x_1 = 1$, and that $x_2 \in N \setminus K$. We write $x_\mu = \xi_\mu + t\xi'_\mu$, where $\xi_\mu, \xi'_\mu \in K$. Then the matrix

$$M := \begin{pmatrix} 1 & \xi_2 & \xi_3 \\ 0 & \xi'_2 & \xi'_3 \\ 0 & 0 & \xi_2'^{-1} \end{pmatrix}$$

belongs to Δ, and $[\boldsymbol{x}] = [(1, t, 0)M]$ belongs to the Δ-orbit of the point $[1, t, 0]$.

If $\left[\!\left[A\right]\!\right]$ is an outer line, then the columns of A are linearly independent by 2.9. Therefore, we find a matrix $M \in \Delta$ such that $\left[\!\left[M^{-1}A\right]\!\right] = \left[\!\left[\boldsymbol{e}_2, -\boldsymbol{e}_1\right]\!\right]$, and find that $\left[\!\left[A\right]\!\right]$ belongs to the Δ-orbit of $\left[\!\left[\boldsymbol{e}_2, -\boldsymbol{e}_1\right]\!\right]$. ∎

NOTATION 2.12. In the sequel, we shall use the following points and lines:

$$p := [\boldsymbol{e}_1], \quad p' := [\boldsymbol{e}_3], \quad q := [1, t, 0]$$
$$g := \left[\!\left[\boldsymbol{e}_3, \mathbf{0}\right]\!\right], \quad g' := \left[\!\left[\boldsymbol{e}_2, \mathbf{0}\right]\!\right], \quad h := \left[\!\left[\boldsymbol{e}_2, -\boldsymbol{e}_1\right]\!\right].$$

Note that p, p', g, g' are inner elements, while q and h are outer. We remark that $g = \{[\boldsymbol{x}] \in P \mid x_3 = 0\}$, $g' = \{[\boldsymbol{x}] \in P \mid x_2 = 0\}$, and $h = \{[1, t, f] \mid f \in N\} \cup \{p'\}$. In particular, we have that $p, q \in g$, $p, p' \in g'$, and $p', q \in h$.

https://doi.org/10.4153/CMB-1996-040-9 Published online by Cambridge University Press

LEMMA 2.13. *Every outer point is incident with exactly one of the inner lines. Dually, every outer line l is incident with exactly one of the inner points.*

PROOF. By 2.11, it suffices to consider the point q and the line h. Assume that $m = [\![\boldsymbol{\alpha}, \mathbf{0}]\!]$ is an inner line through q. From condition (**A**) we infer that $\alpha_1 = \alpha_2 = 0$, and $m = g$. If $\boldsymbol{\xi} \in K^3$ describes a point $[\boldsymbol{\xi}] \in h$, then $\xi_2 - \xi_1 t = 0$, whence $\xi_1 = \xi_2 = 0$ and $[\boldsymbol{\xi}] = p'$. ∎

LEMMA 2.14. *If l and m are outer lines such that $l \subseteq m$, then $l = m$.*

PROOF. We may assume that $m = h$. By 2.13, we have that $p' \in l$. If $l = [\![\boldsymbol{\alpha}, \boldsymbol{\beta}]\!]$, this implies that $\alpha_3 + \beta_3 t = 0$, whence $\alpha_3 = \beta_3 = 0$. Since $l \subseteq h$, every outer point of l is of the form $[1, t, f]$. This yields that $\alpha_1 + t\alpha_2 + (\beta_1 + t\beta_2)t = 0$, and that $l \supseteq h$. ∎

In order to show that joining lines and intersection points exist and are unique, we shall use the following Lemma, which is of course of its own interest as well.

LEMMA 2.15. *The stabilizer Δ_q acts transitively on $P^\circ \setminus g$. Dually, the stabilizer Δ_h acts transitively on $\mathcal{L}^\circ \setminus \mathcal{L}_{p'}$.*

PROOF. Every point in $P^\circ \setminus g$ is of the form $[\boldsymbol{\xi}]$, and every line $l \in \mathcal{L}^\circ \setminus \mathcal{L}_{p'}$ has the form $[\![\boldsymbol{\xi}, \mathbf{0}]\!]$, where $\boldsymbol{\xi} = (\xi_1, \xi_2, 1) \in K^3$. The group

$$\Phi := \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \xi_1 & \xi_2 & 1 \end{pmatrix} \middle| \, \xi_1, \xi_2 \in K \right\}$$

acts transitively on $P^\circ \setminus g$, and the transposed group $\Phi^T \leq \Delta_h$ acts transitively on $\mathcal{L}^\circ \setminus \mathcal{L}_{p'}$. ∎

LEMMA 2.16. *The stabilizer Δ_p acts transitively both on $\mathcal{L}_p^\circ$ and on $\mathcal{L}_p \setminus \mathcal{L}^\circ$. Dually, the stabilizer Δ_g acts transitively both on g° and on $g \setminus g^\circ$.*

PROOF. Transitivity of Δ_p on $\mathcal{L}_p^\circ$ and of Δ_g on g° is obvious from 2.10.

For $[\![\boldsymbol{\alpha}, \boldsymbol{\beta}]\!] \in \mathcal{L}_p \setminus \mathcal{L}^\circ$, we have that $\alpha_1 = \beta_1 = 0$, and that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are linearly independent, see 2.9. Thus there exists a matrix $A' \in \mathrm{GL}_2 K$ such that the matrix $A := \begin{pmatrix} \gamma & 0 & 0 \\ 0 & & \\ 0 & \multicolumn{2}{c}{A'} \end{pmatrix}$ satisfies $[\![\boldsymbol{\alpha}, \boldsymbol{\beta}]\!]^A = [\![\boldsymbol{e}_2, \boldsymbol{e}_3]\!]$, and we may choose $\gamma \in K$ such that $A \in \mathrm{SL}_3 K = \Delta$.

If $[\boldsymbol{x}]$ is an outer point of g, then we find a representation $[\boldsymbol{x}] = [1, \xi + t\xi', 0]$, where $\xi, \xi' \in K$ and $\xi' \neq 0$. Then the matrix $\begin{pmatrix} 1 & \xi & 0 \\ 0 & \xi' & 0 \\ 0 & 0 & \xi'^{-1} \end{pmatrix}$ belongs to Δ_g, and maps q to $[\boldsymbol{x}]$. ∎

LEMMA 2.17. *The set $h \setminus \{p'\}$ contains a set of representatives for the orbits under Δ_q in $P \setminus (P^\circ \cup g)$. Dually, the set $\mathcal{L}_q \setminus \{g\}$ contains a set of representatives for the orbits under Δ_h in $\mathcal{L} \setminus (\mathcal{L}^\circ \cup \mathcal{L}_{p'})$.*

PROOF. Every point $[\boldsymbol{x}] \in P \setminus (P^\circ \cup g)$ has a representation $[\boldsymbol{x}] = [\xi_1 + t\xi_1', \xi_2 + t\xi_2', 1]$, where $\xi_i, \xi_i' \in K$ and $(\xi_1', \xi_2') \neq (0, 0)$. According to (**B**), there exist elements $\eta_1, \eta_2 \in K$ such that $(\eta_1 + t\xi_1')t = \eta_2 + t\xi_2'$. This implies that $\eta_1 + t\xi_1' \neq 0$. Now $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \eta_1 - \xi_1 & \eta_2 - \xi_2 & 1 \end{pmatrix} \in \Delta_q$ gives $[x]^A = [\eta_1 + t\xi_1', \eta_2 + t\xi_2', 1] = [1, t, (\eta_1 + t\xi_1')^{-1}] \in h$.

For every line $[\![\boldsymbol{\alpha}, \boldsymbol{\beta}]\!] \in \mathcal{L} \setminus (\mathcal{L}^\circ \cup \mathcal{L}_{p'})$ we have that $(\alpha_3, \beta_3) \neq (0, 0)$. We discuss the two cases:

(i) If $\beta_3 = 0$, then $\alpha_3 \neq 0$, and we may achieve by matrices of the form $\begin{pmatrix} 1 & 0 & \xi \\ 0 & 1 & \eta \\ 0 & 0 & 1 \end{pmatrix}$ in Δ_h that $\alpha_1 = \alpha_2 = 0$. We find $\gamma, \gamma' \in K$ such that $\gamma + t\gamma' = (\beta_1 + t\beta_2)t\alpha_3^{-1}$, and put $A := \begin{pmatrix} 1 & 0 & \gamma \\ 0 & 1 & \gamma' \\ 0 & 0 & 1 \end{pmatrix} \in \Delta_h$. Then $[\![\boldsymbol{\alpha}, \boldsymbol{\beta}]\!]^A = \left[\!\!\left[\begin{matrix} -\gamma\alpha_3 & \beta_1 \\ -\gamma'\alpha_3 & \beta_2 \\ \alpha_3 & 0 \end{matrix}\right]\!\!\right] \in \mathcal{L}_q$.

(ii) If $\beta_3 \neq 0$, we may assume that $\beta_1 = \beta_2 = 0$. We find $\gamma, \gamma' \in K$ such that $\gamma + t\gamma' = (\alpha_1 + t\alpha_2)(\alpha_3 + \beta_3 t)^{-1}$. Again, $A = \begin{pmatrix} 1 & 0 & \gamma \\ 0 & 1 & \gamma' \\ 0 & 0 & 1 \end{pmatrix} \in \Delta_h$, and $[\![\boldsymbol{\alpha}, \boldsymbol{\beta}]\!]^A = \left[\!\!\left[\begin{matrix} \alpha_1 - \gamma\alpha_3 & -\gamma\beta_3 \\ \alpha_2 - \gamma'\alpha_3 & -\gamma'\beta_3 \\ \alpha_3 & \beta_3 \end{matrix}\right]\!\!\right] \in \mathcal{L}_q$. ∎

THEOREM 2.18. *Assume that the pair (N, K) is H-suited. Then $H(N, K)$ is a projective plane.*

PROOF. The existence of joining lines and intersection points for pairs of outer points resp. outer lines is a consequence of 2.11 and 2.17. If one of the elements in question is inner, we obtain the existence from 2.10 or 2.15. Since every partial linear space is also a dual partial linear space, it suffices to show uniqueness of joining lines.

Let a and x be two points. If both a and x are inner, then every joining line is inner by 2.13, and unique by 2.10. So assume that x is an outer point.

If a is inner, then either there is an outer joining line, and we may assume by 2.11 and 2.15 that $(a, x) = (p', q)$, or there is an inner joining line, and we may assume by 2.10 that $a = p$, and then by 2.16 that $x \in g$. If a is an outer point, we infer from 2.13 that for the problem of uniqueness we need only consider the case where there exists an outer joining line. In this case, we may assume that $a = q$ and $x \in h \setminus \{p', q\}$. Thus it remains to consider the following three cases.

(i) Let $l = [\![\boldsymbol{\alpha}, \boldsymbol{\beta}]\!]$ be a line through p' and q. Then l is an outer line in view of 2.13. Since $p' \in l$, we have that $\alpha_3 = \beta_3 = 0$. Now $q \in l$ yields that $h \subseteq l$, and $h = l$ by 2.14.

(ii) Let $l = [\![\boldsymbol{\alpha}, \boldsymbol{\beta}]\!]$ be a line joining p and $x \in g \setminus P^\circ$. Then $\alpha_1 = \beta_1 = 0$. Since x is an outer point, we have that $x = [f, 1, 0]$, where $f \in N \setminus K$. This leads to $\alpha_2 = \beta_2 = 0$, and $l = g$ by 2.9 and 2.13.

(iii) Finally, let $l = [\![\boldsymbol{\alpha}, \boldsymbol{\beta}]\!]$ be a line through q and $x \in h \setminus \{p', q\}$. Then l is an outer line. If $l \neq h$, then the unique inner point r of l is different from p' by step (i). From (ii) we know that r is not contained in g. Therefore, $r = [\rho_1, \rho_2, 1]$ for suitable $\rho_1, \rho_2 \in K$.

https://doi.org/10.4153/CMB-1996-040-9 Published online by Cambridge University Press

Now the line $m = \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ -\rho_2 & \rho_1 \end{bmatrix}$ is a line through r and q, and $m = l$ by step (ii). Writing $x = [1, t, f]$, where $f \in N \setminus \{0\}$, we obtain the equation $f\rho_2 = t + (f\rho_1 - 1)t$. If $\rho_1 \neq 0$, we set $s := f\rho_1 - 1$ and obtain $s(\rho_1^{-1}\rho_2 - t) = t - \rho_1^{-1}\rho_2$, which yields $s = -1$ and $\rho_1 = 0$, a contradiction. If $\rho_1 = 0$, it follows immediately that $\rho_2 = 0$. But this means that $r = p'$, again a contradiction. ∎

3. **Orbit Decompositions.** Throughout this section, let (N, K) be an H-suited pair, and let t be an element of $N \setminus K$. We also retain the names p, p', q, g, h for special points and lines in the projective plane $H(N, K)$ as in 2.12.

NOTATION 3.1. For every $a \in N$, the mapping $\lambda_a : x \mapsto ax$ is an endomorphism of N, considered as a right vector space over K. With respect to the basis $(1, t)$, this endomorphism is described by the matrix

$$L_a = \begin{pmatrix} \alpha & \beta \\ \alpha' & \beta' \end{pmatrix},$$

where $\alpha, \alpha', \beta, \beta' \in K$ are defined by $a = \alpha + t\alpha'$ and $at = \beta + t\beta'$. Consequently, $L_a \in \mathrm{GL}_2 K$ if, and only if, $a \neq 0$.

LEMMA 3.2. *The stabilizer of q in Γ is*

$$\Gamma_q = \left\{ \begin{pmatrix} L_a & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} \xi & \xi' \end{matrix} & \gamma \end{pmatrix} \middle| a \in N \setminus \{0\}, \xi, \xi' \in K, \gamma \in K \setminus \{0\} \right\}.$$

Dually, we have

$$\Gamma_h = \left\{ \begin{pmatrix} L_a & \begin{matrix} \xi \\ \xi' \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \gamma \end{pmatrix} \middle| a \in N \setminus \{0\}, \xi, \xi' \in K, \gamma \in K \setminus \{0\} \right\}.$$

PROOF. Quite obviously, the given sets are contained in Γ_q resp. Γ_h; recall that $h = \{[1, t, f] \mid f \in N\} \cup \{p'\}$. Conversely, assume that q is fixed by a matrix $M \in \Gamma$. Then M fixes the unique inner line g through q, and we obtain that M is a block matrix $\begin{pmatrix} A & \begin{matrix} 0 \\ 0 \end{matrix} \\ C & \delta \end{pmatrix}$, where $\delta \in K \setminus \{0\}$, $C \in K^2$, and $A \in \mathrm{GL}_2 K$ such that $(1, t)A = a(1, t)$ for some $a \in N \setminus \{0\}$. This implies that $A = L_a$.

Since any matrix in Γ_h fixes the unique inner point p' of h, we obtain that such a matrix is of the form $\begin{pmatrix} A & B \\ \begin{matrix} 0 & 0 \end{matrix} & \delta \end{pmatrix}$, and that $A = L_a$ for some $a \in N \setminus \{0\}$, as before. ∎

LEMMA 3.3. *The stabilizer Δ_q acts transitively both on $\mathcal{L}_q \setminus \{g\}$ and on g°. Dually, the stabilizer Δ_h acts transitively both on $h \setminus \{p'\}$ and on $\mathcal{L}_{p'}^\circ$.*

PROOF. (i) Let l be a line in $\mathcal{L}_q \setminus \{g\}$, and let x be the unique inner point on l. Since x does not belong to g, we may write $x = [\xi_1, \xi_2, 1]$ for suitable $\xi_1, \xi_2 \in K$. Thus $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \xi_1 & \xi_2 & 1 \end{pmatrix} \in \Delta_q$ maps p' to x and, therefore, h to l.

If x is a point of g°, then $x = [\xi_1, \xi_2, 0]$ for some $(\xi_1, \xi_2) \in K^2 \setminus \{\mathbf{0}\}$. According to 2.3, there exist $\eta_1, \eta_2 \in K$ such that $(\eta_1 + t\xi_1)t = \eta_2 + t\xi_2$. Putting $z = \eta_1 + t\xi_1$, we find some $\zeta \in K$ such that $\begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{L_z}} & 0 \\ & & 0 \\ 0 & 0 & \zeta \end{pmatrix} \in \Delta$. Obviously, this matrix fixes q and maps $[e_2]$ to x.

(ii) If x is a point on $h \setminus \{p'\}$, then we find $\xi, \xi' \in K$ such that $x = [1, t, \xi + t\xi']$. The matrix $\begin{pmatrix} 1 & 0 & \xi \\ 0 & 1 & \xi' \\ 0 & 0 & 1 \end{pmatrix} \in \Delta_h$ maps q to x.

Finally, let l be a line in $\mathcal{L}_{p'}^\circ$. Then $l = [\![\boldsymbol{\alpha}, \mathbf{0}]\!]$ for $\boldsymbol{\alpha} = (\alpha, \alpha', 0)^T \in K^3 \setminus \{\mathbf{0}\}$. Putting $a = \alpha + t\alpha'$, we find $\gamma \in K$ such that $\begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{L_a}} & 0 \\ & & 0 \\ 0 & 0 & \gamma \end{pmatrix} \in \Delta$. Obviously, this matrix fixes h and maps l to $[\![e_1, \mathbf{0}]\!]$. ∎

Combining 3.3 with 2.11 and 2.13, we obtain the following.

PROPOSITION 3.4. *Each of the following sets is a Δ-orbit:*

$$\{(x, y) \mid x \in P^\circ, y \in P \setminus P^\circ, x \vee y \in \mathcal{L}^\circ\},$$
$$\{(l, m) \mid l \in \mathcal{L}^\circ, m \in \mathcal{L} \setminus \mathcal{L}^\circ, l \wedge m \in P^\circ\},$$
$$\{(x, y) \mid x \in P^\circ, y \in P \setminus P^\circ, x \vee y \in \mathcal{L} \setminus \mathcal{L}^\circ\},$$
$$\{(l, m) \mid l \in \mathcal{L}^\circ, m \in \mathcal{L} \setminus \mathcal{L}^\circ, l \wedge m \in P \setminus P^\circ\}.$$

LEMMA 3.5. *The stabilizer $\Delta_{p,g}$ acts transitively both on $\mathcal{L}_p \setminus \mathcal{L}^\circ$ and on $g \setminus P^\circ$.*

PROOF. (i) Let l be a line in $\mathcal{L}_p \setminus \mathcal{L}^\circ$. Then l meets $m = [\![e_1, \mathbf{0}]\!]$ in an outer point x, since $x \neq p$ and l is an outer line. It follows that $x = [0, \xi + t\xi', 1]$ for suitable $\xi, \xi' \in K$ such that $\xi' \neq 0$. The matrix $A = \begin{pmatrix} \xi'^{-1} & 0 & 0 \\ 0 & \xi' & 0 \\ 0 & \xi & 1 \end{pmatrix} \in \Delta_{p,g}$ maps the point $[0, t, 1]$ to x. Therefore, the line l is mapped by A to the line that joins p and $[0, t, 1]$.

(ii) Every point $x \in g \setminus P^\circ$ is of the form $x = [\xi + t\xi', 1, 0]$. The matrix $\begin{pmatrix} \xi' & 0 & 0 \\ \xi & 1 & 0 \\ 0 & 0 & \xi'^{-1} \end{pmatrix} \in \Delta_{p,g}$ maps the point $[t, 1, 0]$ to x. ∎

The following assertion is, in general, no longer true if we replace Γ by Δ; consider, *e.g.*, the pair $(N, K) = \big(\mathbb{Q}(t), \mathbb{Q}\big)$ for $t = \sqrt{2}$.

LEMMA 3.6. *The stabilizer $\Gamma_{q,h}$ acts transitively on $\mathcal{L}_q \setminus \{g, h\}$ and on $h \setminus \{p', q\}$.*

PROOF. (i) Let l be a line in $\mathcal{L}_q \setminus \{g, h\}$, and let x be the unique inner point of l. Then $x \in P^\circ \setminus (g \cup \{p'\})$, whence $x = [\xi_1, \xi_2, 1]$ for some $(\xi_1, \xi_2) \in K^2 \setminus \{\mathbf{0}\}$. According to 2.3, we find $\eta_1, \eta_2 \in K$ such that $(\eta_1 + t\xi_1)t = \eta_2 + t\xi_2$. Putting $z = \eta_1 + t\xi_1$, we find that $A = \begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{L_z}} & 0 \\ & & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \Gamma_{g,h}$ maps $[0, 1, 1]$ to x. Since A fixes q, we obtain that l is the image of the line that joins q and $[0, 1, 1]$.

https://doi.org/10.4153/CMB-1996-040-9 Published online by Cambridge University Press

(ii) Every point in $h \setminus \{p', q\}$ is of the form $[1, t, f]$ for some $f \in N \setminus \{0\}$. This point is mapped to $[1, t, 1]$ by the matrix $\begin{pmatrix} & L_f & & 0 \\ & & & 0 \\ 0 & & 0 & 1 \end{pmatrix}$. ∎

Combining 3.6 and 2.11, we obtain the following.

PROPOSITION 3.7. *The following sets are Γ-orbits:*

$$\{(x, y) \mid x, y \in P \setminus P^\circ, x \neq y, x \vee y \in \mathcal{L} \setminus \mathcal{L}^\circ\},$$
$$\{(l, m) \mid l, m \in \mathcal{L} \setminus \mathcal{L}^\circ, l \neq m, l \wedge m \in P \setminus P^\circ\}.$$

THEOREM 3.8: ORBITS OF PAIRS OF POINTS OR LINES. *Assume that the pair (N, K) is H-suited, and let $\Gamma = \mathrm{GL}_3 K$, $\Delta = \mathrm{SL}_3 K$. Then the following hold in the Hughes plane $H(N, K)$.*

(1) *If p is an inner point, then the sets*

$$\{p\}, \quad P^\circ \setminus \{p\}, \quad \bigcup\{l \setminus P^\circ \,\big|\, l \in \mathcal{L}_p^\circ\}, \quad \bigcup\{l \setminus P^\circ \mid l \in \mathcal{L}_p \setminus \mathcal{L}^\circ\}$$

are the point orbits under Δ_p.

(2) *If q is an outer point, and g is the unique inner line through q, then the sets*

$$\{q\}, \quad g \setminus (P^\circ \cup \{q\}), \quad P^\circ \setminus g, \quad P \setminus (P^\circ \cup g)$$

form a Γ_q-invariant partition of P. The set $P \setminus (P^\circ \cup g)$ is a Γ_q-orbit, the set $P^\circ \setminus g$ is even a Δ_q-orbit.

(1*) *If g is an inner line, then the sets*

$$\{g\}, \quad \mathcal{L}^\circ \setminus \{g\}, \quad \bigcup\{\mathcal{L}_s \setminus \mathcal{L}^\circ \mid s \in g^\circ\}, \quad \bigcup\{\mathcal{L}_s \setminus \mathcal{L}^\circ \mid s \in g \setminus P^\circ\}$$

are the line orbits under Δ_g.

(2*) *If h is an outer line, and p' is the unique inner point on h, then the sets*

$$\{h\}, \quad \mathcal{L}_{p'} \setminus (\mathcal{L}^\circ \cup \{h\}), \quad \mathcal{L}^\circ \setminus \mathcal{L}_{p'}, \quad \mathcal{L} \setminus (\mathcal{L}^\circ \cup \mathcal{L}_{p'})$$

form a Γ_h-invariant partition of $\mathcal{L}$. The set $\mathcal{L} \setminus (\mathcal{L}^\circ \cup \mathcal{L}_{p'})$ is a Γ_h-orbit, the set $\mathcal{L}^\circ \setminus \mathcal{L}_{p'}$ is even a Δ_h-orbit.

It is easy to find examples where Γ_q is not transitive on the set $g \setminus (P^\circ \cup \{q\})$, and Γ_h is not transitive on $\mathcal{L}_{p'} \setminus (\mathcal{L}^\circ \cup \{h\})$; *e.g.*, consider the H-suited pair $(\mathbb{C}, \mathbb{R})$, and put $t = i$.

THEOREM 3.9: ORBITS OF FLAGS AND ANTI-FLAGS. *Assume that the pair (N, K) is H-suited, and let $\Gamma = \mathrm{GL}_3 K$, $\Delta = \mathrm{SL}_3 K$. Then the following hold in the Hughes plane $H(N, K)$.*

(1) *If p is an inner point, then the sets*

$$\mathcal{L}_p^\circ, \quad \mathcal{L}_p \setminus \mathcal{L}^\circ, \quad \mathcal{L}^\circ \setminus \mathcal{L}_p, \quad \mathcal{L} \setminus (\mathcal{L}^\circ \cup \mathcal{L}_p)$$

are the line orbits under Δ_p.

(2) *If* q *is an outer point, then the sets*

$$\mathcal{L}_q^\circ, \quad \mathcal{L}_q \setminus \mathcal{L}^\circ, \quad \mathcal{L}^\circ \setminus \mathcal{L}_q, \quad \mathcal{L} \setminus (\mathcal{L}^\circ \cup \mathcal{L}_q)$$

form a Γ_q*-invariant partition of* $\mathcal{L}$. *The sets* $\mathcal{L}_q^\circ$, $\mathcal{L}_q \setminus \mathcal{L}^\circ$, *and* $\mathcal{L}^\circ \setminus \mathcal{L}_q$ *are* Δ_q*-orbits. The set* $\mathcal{L} \setminus (\mathcal{L}^\circ \cup \mathcal{L}_q)$ *is never an orbit under* Γ_q.

(1*) *If* g *is an inner line, then the sets*

$$g^\circ, \quad g \setminus P^\circ, \quad P^\circ \setminus g, \quad P \setminus (P^\circ \cup g)$$

are the point orbits under Δ_g.

(2*) *If* h *is an outer line, then the sets*

$$h^\circ, \quad h \setminus P^\circ, \quad P^\circ \setminus h, \quad P \setminus (P^\circ \cup h)$$

form a Γ_h*-invariant partition of* P. *The sets* h°, $h \setminus P^\circ$, *and* $P^\circ \setminus h$ *are* Δ_h*-orbits. The set* $P \setminus (P^\circ \cup h)$ *is never an orbit under* Γ_h.

PROOF. (i) Assume first that $(x, l) \in P \times \mathcal{L}$ is a flag, *i.e.*, $x \in l$. If x is an inner point, we may assume that $x = p$, and obtain from 3.8 (1) that Δ_p has the orbits $\mathcal{L}_p^\circ$ and $\mathcal{L}_p \setminus \mathcal{L}^\circ$ in $\mathcal{L}_p$. If x is an outer point, then we may assume that $x = q$. Since every element of $\mathcal{L}_q \setminus \{g\}$ is incident with exactly one point in $P^\circ \setminus g$, we obtain from 3.8 (2) that the Δ_q-orbits in $\mathcal{L}_q$ are $\mathcal{L}_q \setminus \{g\}$ and $\{g\}$.

(ii) Now we consider the case where (x, l) is not a flag. Assume first that x is inner. From 2.10, we infer that $\mathcal{L}^\circ \setminus \mathcal{L}_p$ is an orbit under Δ_p. If l is an outer line, we may assume that $l = h$. Then x and p' are joined by a unique inner line, and we may assume by 3.3 that this line is $[\![e_2, \mathbf{0}]\!]$. This implies that x is of the form $[1, 0, \xi]$ for some $\xi \in K$, and $x \in p^{\Delta_h}$, see 3.2.

In order to prove assertions (1) and (2), it remains to consider the case where x is an outer point and $l \in \mathcal{L}^\circ \setminus \mathcal{L}_x$. In this case, we may assume that $x = q$. According to 3.3, we may assume that l meets g in the point p. Then l is of the form $l = \left[\!\!\left[\begin{matrix} 0 & 0 \\ \lambda_2 & 0 \\ \lambda_3 & 0 \end{matrix}\right]\!\!\right]$, where λ_2, $\lambda_3 \in K$ and $\lambda_2 \neq 0$. Therefore, we find an element $\delta \in K$ such that $A = \begin{pmatrix} \multicolumn{2}{c}{L_{t\lambda_2}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \lambda_3 & 0 & \delta \end{pmatrix}$ belongs to Δ. Now $A \in \Delta_q$, and $l^A = [\![e_1, \mathbf{0}]\!]$.

(iii) The dual assertions (1*) and (2*) follow by the dual arguments, using the dual statements 3.8 (1*), (2*). ∎

Even the group Γ does not act transitively on the set

$$\{(x, l) \mid x \in P \setminus P^\circ, l \in \mathcal{L} \setminus (\mathcal{L}^\circ \cup \mathcal{L}_x)\}$$

of antiflags consisting of outer elements. In fact, transitivity on this set would imply that Γ_q acts transitively on the set X of intersection points of g with lines from $\mathcal{L} \setminus (\mathcal{L}^\circ \cup \mathcal{L}_q)$. This is impossible, since X contains both inner and outer points. Of course, the dual statement holds as well.

COROLLARY 3.10. *The group Δ acts transitively on the following sets:*
(1) *Flags consisting of inner elements.*
(2) *Flags consisting of outer elements.*
(3) *Flags consisting of one inner and one outer element.*
(4) *Anti-flags consisting of inner elements.*
(5) *Anti-flags consisting of one inner and one outer element.*

Even the group Γ is not transitive on the set of anti-flags consisting of outer elements.

4. **Group-theoretical Description.** In this section, we comment on the possibility to reconstruct a Hughes plane $H(N, K)$ from the action of the group SL_3K. Group-theoretical descriptions of the Hughes planes were used in several papers in order to characterize certain subclasses of the class of Hughes planes; *e.g.*, see [15], [8], [23], [28].

Let (A, B, I) be an incidence structure, and let $\alpha\colon G \longrightarrow \mathrm{Aut}(A,B,I)$ be an action of a group G on (A, B, I). We recall from [27] that the geometry $\big(G, (A, B, I)\big)$ is called *sketched*, if there exist sets of representatives R_A, R_B for the G-orbits in A resp. B such that $R_A \times R_B$ forms a set of representatives for the G-orbits in I. (If one of the sets R_A, R_B has exactly one element, this is equivalent to conditions (R 1) and (R 2) in [26]). It has been shown in [27] that sketched geometries can be reconstructed from the group action as unions of coset spaces, if only the residues of different elements in R_A and R_B are different. For partial linear spaces (in particular, for subgeometries of projective planes), this latter condition is always satisfied. In the reconstructed geometry (A', B'), elements $a \in A'$ and $b \in B'$ are incident if, and only if, they have non-empty intersection.

From 3.9, we infer the following.

THEOREM 4.1. *Assume that the pair (N, K) is H-suited, and let $H(N,K) = (P, \mathcal{L})$ be the corresponding Hughes plane, with the action of $\Delta = \mathrm{SL}_3K$ as in 2.5. Then the subgeometries $(P^\circ, \mathcal{L}^\circ)$, $(P \setminus P^\circ, \mathcal{L})$ and $(P, \mathcal{L} \setminus \mathcal{L}^\circ)$ are sketched; in the notation of 2.12, suitable systems of orbit representatives are $(\{p\}, \{g\})$, $(\{q\}, \{g, h\})$ and $(\{p', q\}, \{h\})$, respectively. Consequently, we have isomorphisms $(P^\circ, \mathcal{L}^\circ) \cong \big(\frac{\Delta}{\Delta_p}, \frac{\Delta}{\Delta_g}\big)$, $(P \setminus P^\circ, \mathcal{L}) \cong \big(\frac{\Delta}{\Delta_q}, \frac{\Delta}{\Delta_g} \cup \frac{\Delta}{\Delta_h}\big)$ and $(P, \mathcal{L} \setminus \mathcal{L}^\circ) \cong \big(\frac{\Delta}{\Delta_{p'}} \cup \frac{\Delta}{\Delta_q}, \frac{\Delta}{\Delta_h}\big)$.*

According to [27, 2.7], it is pointless to try to extend the description in 4.1 to the whole plane $H(N, K)$. In fact, this would imply that there exist sets of representatives R_P and $R_{\mathcal{L}}$ such that $R_P \times R_{\mathcal{L}}$ consists of flags. Since Δ is neither transitive on P nor on $\mathcal{L}$, this would yield a pair of points with more than one joining line. However, there is the following description (generalized from a method used by H. Hähl in [11, 3.15], see [24, Section 86]).

DEFINITION 4.2. Assume that the pair (N, K) is H-suited, and let p, p', q, g, h as in 2.12. We write $\iota = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in \Delta$; note that $p^\iota = p'$. Using the corresponding

https://doi.org/10.4153/CMB-1996-040-9 Published online by Cambridge University Press

stabilizers in Δ, we define an incidence structure $(\hat{P}, \hat{\mathcal{L}})$ as follows.

$$\hat{P}^\circ = \frac{\Delta}{\Delta_p}, \quad \hat{\mathcal{L}}^\circ = \frac{\Delta}{\Delta_g}$$

$$\hat{P} = \frac{\Delta}{\Delta_p} \cup \frac{\Delta}{\Delta_q}, \quad \hat{\mathcal{L}} = \frac{\Delta}{\Delta_g} \cup \frac{\Delta}{\Delta_h}.$$

"Inner" elements $\Delta_p\delta$ and $\Delta_g\gamma$ are incident if, and only if, $\Delta_p\delta \cap \Delta_g\gamma \neq \emptyset$.
"Outer" elements $\Delta_q\delta$ and $\Delta_h\gamma$ are incident if, and only if, $\Delta_q\delta \cap \Delta_h\gamma \neq \emptyset$.
Elements $\Delta_g\delta$ and $\Delta_q\gamma$ are incident if, and only if, $\Delta_q\delta \cap \Delta_g\gamma \neq \emptyset$.
Elements $\Delta_p\delta$ and $\Delta_h\gamma$ are incident if, and only if, $\Delta_p\delta = \Delta_p\iota\gamma$, (*i.e.*, $p^{\delta\gamma^{-1}} = p'$).

Note that, in order to define incidence of an inner point and an outer line, we have used two cosets of Δ_p rather than a coset of Δ_p and a coset of Δ_h. We could have defined incidence of outer points and inner lines similarly. However, we need this deviation from the method used in 4.1 only in one of the cases.

THEOREM 4.3. *Assume that the pair (N, K) is H-suited. Then the incidence structures $H(N, K)$ and $(\hat{P}, \hat{\mathcal{L}})$ are isomorphic via the mapping $\pi: P \cup \mathcal{L} \longrightarrow \hat{P} \cup \hat{\mathcal{L}}$ that is given by $x^\delta \longmapsto \Delta_x\delta$ for $x \in \{p, q, g, h\}$.*

PROOF. We note first that the mapping π is a bijection; in fact we need only to verify that the stabilizers Δ_p, Δ_g, Δ_q and Δ_h are all different. This is clear from 2.10 and 3.2. It remains to show that π preserves and reflects incidence. From 4.1, we infer that π induces isomorphisms from $(P^\circ, \mathcal{L}^\circ)$ onto $(\hat{P}^\circ, \hat{\mathcal{L}}^\circ)$ and from $(P \setminus P^\circ, \mathcal{L})$ onto $(\hat{P} \setminus \hat{P}^\circ, \hat{\mathcal{L}})$. Therefore, it remains to show that $p^\delta \in h^\gamma \iff p^{\delta\gamma^{-1}} = p'$. This equivalence is a consequence of the fact that p' is the unique inner point of h. ■

Theorem 4.3 will be used in a forthcoming paper for a study of dualities.

REMARK 4.4. In order to perform the description of a Hughes plane as in 4.2, it is not necessary to use exactly those representatives that were chosen. In fact, one only has to choose p, q, g, h and ι such that the following hold.

(1) p and g are inner elements, q and h are outer elements, and ι is an element of Δ.
(2) (p, g), (q, g), (q, h) and (p^ι, h) are incident pairs.

5. **A non-closing Desargues Configuration.** Using pseudo-homogeneous coordinates, we shall exhibit in this section a non-closing Desargues configuration for every Hughes plane $H(N, K)$ defined by a H-suited pair (N, K) where N is not a skewfield. Thus we give a proof for the fact that these Hughes planes are non-desarguesian planes; a fact which was hitherto obtained by somewhat tedious arguments with ternary fields. In fact, the ternary field for $H(N, K)$ with respect to an inner quadrangle has been completely determined in [17], the derived binary operations of addition and multiplication coincide with those of the nearfield N, but the ternary operation is not linear. However, we believe that the affine description via ternary fields does not fit well with proper projective planes (in the sense of C. Hering [12]).

We start with a lemma that says that non-distributivity is apparent in rather nice cases already.

LEMMA 5.1. *If N is a proper nearfield, then there exist elements $w, s \in N \setminus K(N)$ such that* $(1+w)s \neq s + ws$.

PROOF. Since N is a proper nearfield, there exist elements $u, v \in N \setminus \{0\}$ and $s \in N \setminus K(N)$ such that $(u+v)s \neq us + vs$, whence $(1+u^{-1}v)s \neq s + u^{-1}vs$. Assume that $(1+w)s = s + ws$ for every $w \in N \setminus K(N)$. Then $\omega := u^{-1}v \in K(N)$, and

$$(1+\omega+s)s = s + (\omega+s)s = s + (1+s\omega^{-1})\omega s = s + \omega s + s^2$$

in contradiction to

$$(1+\omega+s)s = (1+\omega)\big(1+(1+\omega)^{-1}s\big)s = (1+\omega)s + s^2.$$

Therefore, there exists $w \in N \setminus K(N)$ such that $(1+w)s \neq s + ws$. ∎

The following is verified by a straightforward computation.

LEMMA 5.2. *Let (N, K) be H-suited. For every $f \in N \setminus K$, we define $\tau_f, \tau_f' \in K$ by the equation $\tau_f + f\tau_f' = (1+t)^{-1}$. Then $f\tau_f' + \tau_f - 1 + (f\tau_f' + \tau_f)t = 0$.*

Let (N, K) be H-suited, and assume that w, s are elements of $N \setminus K$ such that $(1+w)s \neq s+ws$. Since t may be chosen arbitrarily in $N \setminus K$ for the definition of $H(N, K)$, see 2.8, we may assume that $s = (1+t)^{-1}$. We define $\alpha, \beta \in K$ by the equation $\alpha + w\beta = s + ws$, and set $a := s - \alpha$, $b := (s-\beta)^{-1}$. Then we infer that $wb^{-1} = ws - w\beta = ws - (ws+s-\alpha) = -a$, whence $ab = -w$. Therefore, we have that $(1-ab)s \neq s - abs$. Finally, we put $c := (1+w)s - \alpha - w\beta \neq 0$. We claim that the triangles (p_1, p_2, p_3) and (q_1, q_2, q_3) given by

$$\begin{aligned} p_1 &= [b,1,1], & p_2 &= [0,1,1], & p_3 &= [b,0,1]; \\ q_1 &= [ab,a,1], & q_2 &= [0,a,1], & q_3 &= [ab,c,1] \end{aligned}$$

are in perspective from the line $[\![e_3, \mathbf{0}]\!]$, but not in perspective from any point. In order to show this, we simply state descriptions for the lines $g_k := p_i \vee p_j$ and $h_k = q_i \vee q_j$, their intersection points $s_k = g_k \wedge h_k$ and the lines $l_i = p_i \wedge q_i$; here $\{i,j,k\} = \{1,2,3\}$. Although it was quite tedious to find these descriptions, the verification is very easy; one just has to check incidence (using 5.2).

$$g_1 = \left[\!\!\left[\begin{matrix} \tau_b' & \tau_b' \\ 1-\tau_b & -\tau_b \\ \tau_b - 1 & \tau_b \end{matrix}\right]\!\!\right], \quad g_2 = \left[\!\!\left[\begin{matrix} \tau_b' & \tau_b' \\ 0 & 0 \\ \tau_b - 1 & \tau_b \end{matrix}\right]\!\!\right], \quad g_3 = \left[\!\!\left[\begin{matrix} 0 & 0 \\ 1 & 0 \\ -1 & 0 \end{matrix}\right]\!\!\right];$$

$$h_1 = \left[\!\!\left[\begin{matrix} 1-\beta & -\beta \\ 1 & 1 \\ \alpha - 1 & \alpha \end{matrix}\right]\!\!\right], \quad h_2 = \left[\!\!\left[\begin{matrix} \tau_{ab}' & \tau_{ab}' \\ 0 & 0 \\ \tau_{ab} - 1 & \tau_{ab} \end{matrix}\right]\!\!\right], \quad h_3 = \left[\!\!\left[\begin{matrix} 0 & 0 \\ \tau_a' & \tau_a' \\ \tau_a - 1 & \tau_a \end{matrix}\right]\!\!\right];$$

$$s_1 = [b,-1,0], \quad s_2 = [0,1,0], \quad s_3 = [1,0,0];$$

$$l_1 = \left[\!\!\left[\begin{matrix} \tau_b' & \tau_b' \\ \tau_b - 1 & \tau_b \\ 0 & 0 \end{matrix}\right]\!\!\right], \quad l_2 = \left[\!\!\left[\begin{matrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix}\right]\!\!\right].$$

Quite obviously, the points s_k belong to the line $[\![e_3, \mathbf{0}]\!]$. The intersection point of l_1 and l_2 is $z = [0, 0, 1]$. The line $[\![e_2, \mathbf{0}]\!]$ joins z and p_3, but does not pass through q_3. Therefore, the triangles (p_1, p_2, p_3) and (q_1, q_2, q_3) are not in perspective from any point.

We have proved

THEOREM 5.3. *If (N, K) is H-suited, and if N is a proper nearfield, then $H(N, K)$ is a non-desarguesian projective plane.*

Of course, $H(N, K)$ is just a description of the projective plane over N, if (N, K) is H-suited and N is a skewfield. However, if N is a skewfield, but (N, K) is not H-suited, then $H(N, K)$ is the projective plane over N, with some lines missing, *cf.* 2.7.

REFERENCES

1. J. André, *Nichtkommutative Geometrie und verallgemeinerte Hughes-Ebenen*, Math. Z. **177**(1981), 449–462.

2. E. Artin, *Geometric Algebra*, Interscience Tracts in Pure and Applied Mathematics, **3**, Interscience Publishers, New York (1957).

3. M. Biliotti, *Su una generalizzazione di Dembowski dei piani di Hughes*, Bolletino U.M.I. (5) **16-B**(1979), 674–693.

4. P. M. Cohn, *Quadratic extensions of skewfields*, Proc. London Math. Soc. **11**(1961), 531–556.

5. ———, *Algebra.*, **3**, Wiley, Chichester, (2nd ed.) (1991).

6. P. Dembowski, *Finite Geometries*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer, Berlin **44**(1968).

7. ———, *Generalized Hughes planes*, Canad. J. Math. **23**(1971), 481–494.

8. ———, *Gruppenerhaltende quadratische Erweiterungen endlicher desarguesscher projektiver Ebenen*, Arch. Math. **22**(1971), 214–220.

9. J. Dieudonné, *La Géométrie des Groupes Classiques*, Ergebnisse der Mathematik und ihrer Grenzgebiete, **5**, Springer, Berlin, (2nd ed.) (1963).

10. T. Grundhöfer, *A synthetic construction of the Figueroa planes*, J. Geom. **26**(1986), 191–201.

11. H. Hähl, *Charakterisierung der kompakten, zusammenhängenden Moufang-Hughes-Ebenen anhand ihrer Kollineationen*, Math. Z. **191**(1986), 117–136.

12. C. Hering, H.-J. Schaeffer, *On the new projective planes of R. Figueroa*, Proceedings Rauischholzhausen, Lecture notes in Mathematics **969**(1982), 187–190.

13. D. R. Hughes, *A class of non-desarguesian projective planes*, Canad. J. Math. **9**(1957), 378–388.

14. H. Karzel, *Bericht über projektive Inzidenzgruppen*, Jahresber. Deutsche Math. **67**(1965), 58–92.

15. H. Lüneburg, *Characterizations of the generalized Hughes planes*, Canad. J. Math. **28**(1976), 376–402.

16. ———, *Vorlesungen über Lineare Algebra, versehen mit der zu ihrem Verständnis nötigen Algebra sowie einigen Bemerkungen zu ihrer Didaktik*, BI Wissenschaftsverlag, Mannheim, 1993.

17. P. Maier, *Die speziellen linearen Gruppen über Schiefkörpern und Hughes-Ebenen*, Thesis, Technische Hochschule Darmstadt, Darmstadt, 1994.

18. T. G. Ostrom, *Semi-translation planes*, Trans. Amer. Math. Soc. **111**(1964), 1–18.

19. F. Piper, *Polarities in the Hughes plane*, Bull. London Math. Soc. **2**(1970), 209–213.

20. T. G. Room, *Polarities and ovals in the Hughes plane*, J. Austral. Math. Soc. **13**(1972), 196–204.

21. L. A. Rosati, *Unicità e autodualità dei piani di Hughes*, Rendic. Sem. Mat. Padova **30**(1960), 316–327.

22. ———, *Disegni unitari nei piani di Hughes*, Geom. Dedicata **27**(1988), 295–299.

23. H. Salzmann, *Kompakte, 8-dimensionale projektive Ebenen mit großer Kollineationsgruppe*, Math. Z. **176**(1981), 345–357.

24. H. Salzmann, D. Betten, T. Grundhöfer, H. Hähl, R. Löwen, M. Stroppel, *Compact projective planes*, De Gruyter, Berlin, 1996.

25. A. H. Schofield, *Artin's problem for skew field extensions*, Math. Proc. Cambridge Phil. Soc. **97**(1985), 1–6.

26. M. Stroppel, *Reconstruction of incidence geometries from groups of automorphisms*, Arch. Math. (Basel) **58**(1992), 621–624.
27. ———, *A categorical glimpse at the reconstruction of geometries*, Geom. Dedicata **46**(1993), 47–60.
28. ———, *Locally compact Hughes planes*, Canad. Math. Bull. **37**(1994), 112–123.

Fachbereich Mathematik der TH
Schlossgartenstr. 7
D-64289 Darmstadt