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EXTENDED SEMI-HEREDITARY RINGS

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Abstract

A ring R for which every finitely generated right submodule of S_R, the left flat epimorphic hull of R, is projective is termed an extended semi-hereditary ring. It is shown that several of the characterizing properties of Prufer domains may be generalized to give characterizations of extended semi-hereditary rings. A suitable class of PP rings is introduced which in this case serves as a generalization of commutative integral domains.

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A ring will be said to be a *right perfect PP ring* if S, the left flat epimorphic hull of R is a regular ring and every cyclic R-submodule of S_R is projective. A ring R will be said to be a *right extended semi-hereditary* ring if S, the left flat epimorphic hull of R, is regular and every finitely generated right R-submodule of S_R is projective.

Clearly every right extended semi-hereditary ring is a right perfect PP ring. *A semi-hereditary right perfect PP ring need not be a right extended semi-hereditary ring*. Commutative PP rings are right perfect PP rings as are right Ore domains. Commutative semi-hereditary rings are right extended semi-hereditary rings and also the $n \times n$ matrix ring over a commutative semi-hereditary ring is such a ring. Further examples are presented in Sections 3 and 4.

In this paper characterizations of both of these classes of rings are given and the ideal structure is discussed. In Section 3 characterizations are given for right extended semi-hereditary rings which are analogous to the "classical" characterizations of Prüfer domains.

In Hattori (1960) a right R-module A_R was defined to be torsion-free if for all $a \in A$ and $x \in R$, $ax = 0$ implies $a \in Al_R(x)$. Throughout this paper torsion-free will be meant in this sense. Let $\mathscr{S}_R$ denote the class of torsion-free R-modules. If $\varphi: R \to T$ is a left flat epimorphism there is a corresponding perfect topology F for $\mathscr{M}_R$, the category of right R-modules, such that T is the quotient ring of R with respect to this topology. Let $(\mathscr{T}_T, \mathscr{F}_T)$ be the corresponding torsion theory and $C_T(R)$ the set of right ideals I of R such that $R/I \in \mathscr{F}_T$. For all details of torsion theories the reader is referred to Stenström (1975).

Throughout this paper R is an associative ring with identity. If $A \in \mathscr{M}_R$, $E_R(A)$ is the right injective hull of A. Q will always denote the maximal right quotient ring of R. In general, the notation will be that of Evans (1977).

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2. Perfect PP rings

In this section several characterizations of right perfect PP rings are given and the closed ideal structure of such rings is described. The following theorem establishes the equivalence of right perfect PP rings with the class of PP rings previously discussed in Evans (1977).

THEOREM 2.1. *For a ring R, the following are equivalent:*

(i) *R is a right perfect PP ring.*

(ii) *R is a right PP ring, S is regular and $S_R \in \mathscr{S}_R$.*

(iii) *The right torsion-free modules of Hattori are the torsion-free class of a perfect torsion theory for $\mathscr{M}_R$.*

PROOF. (i) $\Rightarrow$ (ii). Every cyclic R-submodule of S_R is projective and thus torsion-free. Hence S_R is torsion-free.

(ii) $\Rightarrow$ (i). Since $S_R \in \mathscr{S}_R$ and R is a right PP ring every cyclic submodule of S_R is torsion-free and hence flat. Furthermore if $x \in S$, $xR \otimes S \cong xS$ and xS is projective since S is regular. Theorem 3.1 of Jøndrup (1970) gives the result.

(ii) $\Leftrightarrow$ (iii). Theorem 3.2 of Evans (1977).

It was shown in Evans (1977) that the corresponding topology G, of the perfect torsion theory mentioned in condition (iii) of the above theorem, is the class of right ideals of R which contain a finitely generated dense right ideal. S_R, the left flat epimorphic hull of a right extended PP ring R, is the ring of quotients of R with respect to this topology. The torsion radical of this torsion theory will be denoted by τ.

A right perfect PP ring is not necessarily a left perfect PP ring. For an example, let R be a right Ore domain which is not a left Ore domain. However, a right perfect PP ring is a left PP ring.

THEOREM 2.2. *For a ring R the following are equivalent:*

(i) *R is a perfect PP ring.*

(ii) *Every principal right ideal of S is generated by an idempotent of R.*

PROOF. (i) $\Rightarrow$ (ii). S is a regular ring and thus every principal right ideal of S is generated by an idempotent. Let eS be such an ideal. Then S/eS is flat as a right S-module and as S is torsion free as an R-module, S/eS is torsion free as an R-module. Now since S is the left flat epimorphic hull of R, $(eS \cap R)\,S = eS$ (Stenström (1975), p. 32) and thus $R/(eS\cap R)$ may be considered as an R-submodule of S/eS. Since $\mathscr{S}_R$ is closed under taking submodules $R/(eS\cap R)\in\mathscr{S}_R$ and hence is flat. Furthermore $R/(eS\cap R)\otimes S \simeq S/eS \simeq (1-e)\,S$ which is a projective S-module. Hence $R/(eS\cap R)$ is a projective R-module (Jøndrup (1970), Theorem 3.1). This implies $eS\cap R$ is a direct summand of R and thus there exists $f^2 = f\in R$ such that $eS = (eS\cap R) = fRS = fS$.

(ii) $\Rightarrow$ (i). Clearly S is a regular ring. Let $x\in R$. Then $r_R(x) = r_S(x)\cap R$ and since S is regular there exists $e^2 = e\in S$ such that $r_S(x) = eS$. By the assumption there is an $f^2 = f\in R$ such that $eS = fS$. Hence $r_R(x) = fS\cap R = fR$. Thus R is a right PP ring. Finally, we show $S\in\mathscr{S}_R$. Let $e\in S$. Then there exists $f^2 = f\in R$ such that $xS = fS$ and $l_S(x) = l_S(f) = S(1-f) = S(R(1-f)) = S(l_R(f)) = S(l_R(x))$.

REMARK. (i) Not every idempotent of S need be an idempotent of R as seen by Example 2 of Evans (1977). However, from the result, it is easily seen that every central idempotent of S must be an idempotent of R. This result therefore generalizes the result of Bergman (1971) and Evans (1972).

(ii) If T is a ring with $R\subseteq T\subseteq S$ and R a right perfect PP ring, then T is also a right perfect PP ring.

LEMMA 2.3. *If T is a left flat epimorphic extension of R, a right ideal $J\in C_T(R)$ if and only if $J = JT\cap R$.*

A right ideal J of a right perfect PP ring R is closed if $R/J\in\mathscr{S}_R$, that is $J\in C_S(R)$.

PROPOSITION 2.4. *For a right perfect PP ring R, the following are equivalent:*

(i) *J is a right closed ideal.*

(ii) *For any finite subset $\{x_i\}_{i=1}^{n}\subseteq J$ there exists $g^2 = g\in J$ such that $x_i = gx_i$ for all $1\leqslant i\leqslant n$.*

(iii) *For any $x\in J$ there exists $e^2 = e\in J$ such that $x = ex$.*

(iv) *R/J is flat.*

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Proof. (i) $\Rightarrow$ (ii). Let $\{x_i\}_{i=1}^n \subseteq J \subseteq JS$. Then there exists $f^2 = f \in S$ such that $\sum_{i=1}^n x_i S = fS \subseteq JS$ and hence by Theorem 2.2 there exists $g^2 = g \in R$ such that $\sum_{i=1}^n x_i S = gS$. Clearly $x_i = gx_i$ for all $1 \leqslant i \leqslant n$.

(ii) $\Rightarrow$ (iii). This is trivial.

(iii) $\Rightarrow$ (iv). Let I be a left ideal of R. Then if $x \in J \cap I$, $x = ex \in JI$ and hence $J \cap I = JI$ which implies J is right flat (Lambek (1966), Proposition 3, p. 133).

(iv) $\Rightarrow$ (i). Since R/J is right flat we have the exact sequence $0 \to R/J \to R/J \otimes_R S$ and hence the exact sequence $0 \to R/J \to S/JS$. Thus $R/J \in \mathscr{S}_R$, as every S-module is right torsion free and $\mathscr{S}_R$ is closed under taking submodules.

When R is a commutative PP ring these closed ideals are the Baer ideals as described in Speed (1972). The closed ideals of the form $V \cap R$, where V is a maximal right ideal of S, are the minimal prime ideals of R in this case.

The following is a form of converse to the above result.

Corollary 2.5. *For a ring R with left flat epimorphic hull S the following are equivalent:*

(i) *R is a right perfect PP ring.*

(ii) *For every $J \in C_S(R)$, $x \in J$ implies that there is $e^2 = e \in J$ such that $x = ex$.*

(iii) *For every $J \in C_S(R)$, R/J is right flat.*

Proof. The proof of (iii) $\Rightarrow$ (i) is routine by using Lemma 2.3 and Theorem 2.2.

It is easily seen that every right complement ideal of R is a closed right ideal. However, in general not every closed ideal is of this form.

Proposition 2.6. *For a ring R, with $Z(R) = 0$ and left flat epimorphic hull S the following are equivalent:*

(i) *Q is semi-simple.*

(ii) *If $J \in C_S(R)$, J is a right complement ideal.*

An ideal J is said to be a *right torsion ideal* of R if there exists a torsion radical σ for $\mathscr{M}_R$ such that $\sigma(R) = J$. Several characterizations of torsion ideals are given in Lambek (1971), p. 33.

Proposition 2.7. *For a right PP ring R the following are equivalent for a two-sided ideal J:*

(i) *J is a right torsion ideal.*

(ii) *For all $x \in J$, $l_R r_R(x) \subseteq J$.*

(iii) *R/J is left flat as an R-module.*

PROOF. (i) $\Rightarrow$ (ii). Let $x \in J$ and assume $l_R r_R(x) \nsubseteq J$. Then there exists $y \in l_R r_R(x) \setminus J$ and since J is a right torsion ideal, there exists $z \in R$ such that $xz = 0$ and $yz \notin J$ (Lambek (1971), p. 33). But $y \in l_R r_R(x)$ and $yz = 0$.

(ii) $\Rightarrow$ (iii). Let $x \in J$. Then since R is a right PP ring there exists $e^2 = e \in R$ such that $l_R r_R(x) = Re$ and $x = xe$ with $e \in J$. Hence R/J is flat.

(iii) $\Rightarrow$ (i). Lambek (1971), p. 35.

REMARK. From this result it may be seen that if R is a right perfect PP ring, J a two-sided ideal, then J is a closed right ideal of R if and only if J is a left torsion ideal of R.

3. Extended semi-hereditary rings

The following theorem simply establishes the equivalence of extended semi-hereditary rings with the class of semi-hereditary ring previously discussed in Evans (1977).

THEOREM 3.1. *For a ring R, with maximal right quotient ring Q, the following are equivalent:*

(i) *R is a right extended semi-hereditary ring.*

(ii) *R is right semi-hereditary, S_R is regular and right flat as an R-module.*

(iii) *R is right semi-hereditary and Q_R is flat.*

(iv) *R is right semi-hereditary and every finitely generated non-singular right R-module is flat.*

PROOF. (i) $\Rightarrow$ (ii). Since every finitely generated right R-submodule of S_R is flat, S_R is flat.

(ii) $\Leftrightarrow$ (iii). This is Theorem 4.4 of Evans (1977).

(iii) $\Leftrightarrow$ (iv). If A_R is a non-singular R-module, $E_R(A)$ is a Q-module and since Q is regular, $E_R(A)$ is flat as a Q-module. Thus $E_R(A)$ is flat as an R-module and as R is right semi-hereditary A_R is right flat as an R-module. The converse is clear.

(iv) $\Rightarrow$ (i). Let I_R be a finitely generated R-submodule of S_R. By condition (iv), I_R is flat and $I \otimes_R S \cong IS$, which is a projective S-module. Thus I_R is projective by Theorem 3.1 of Jøndrup (1970).

From condition (ii) it can be seen that S_R is the right flat epimorphic hull of R and R is left semi-hereditary. Thus right extended semi-hereditary implies left extended semi-hereditary and henceforth these rings will be referred to as extended semi-hereditary rings. The equivalence of (iii) and (iv) is known (Stenström (1975), p. 261).

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In the following R_n denotes the $n \times n$ matrix ring over the ring R.

PROPOSITION 3.2. *For a ring R, the following are equivalent:*
(i) *R is an extended semi-hereditary ring.*
(ii) *R_n is an extended semi-hereditary ring for all $n \geqslant 1$.*
(iii) *R_n is a right perfect PP ring for all $n \geqslant 1$.*

PROOF. This follows immediately from Small (1967) and Theorem 3.8 of Stone (1970), which establishes that if Q_n is right torsion-free as an R_n-module for all $n \geqslant 1$, then Q is right flat as an R-module.

If S is the left flat epimorphic hull of R, an extended semi-hereditary ring, then S_n is the left flat epimorphic hull of R_n. If every finitely generated left ideal of R, a right perfect PP ring, is principal (that is, R is left Bezout) then R is an extended semi-hereditary ring.

A right R-module M_R is said to be essentially finitely generated (M_R is EFG) if M contains a finitely generated essential submodule. A right R-module M is said to be essentially finitely related (M_R is EFR) if there exists an exact sequence $0 \to K_R \to F_R \to M_R \to 0$ with F_R finitely generated and free and K_R is EFG (Cateforis (1969)).

THEOREM 3.3. *For a right perfect PP ring R, the following are equivalent:*
(i) *R is an extended semi-hereditary ring.*
(ii) *Every right torsion-free R-module is flat.*
(iii) *Every right torsion-free EFR module is projective.*
(iv) *For every EFR module A, $A/\tau(A)$ is projective.*

PROOF. (i) $\Rightarrow$ (ii). (The equivalent conditions of Theorem 3.1 are used.) Let $A \in \mathscr{S}_R$. Then $E_R(A)$ is an S-module and is thus flat. The transitivity of flatness gives that $E_R(A)$ is right flat as an R-module. Since $\text{w.gl.}\dim R \leqslant 1$, A_R is flat.

(ii) $\Rightarrow$ (iii). Let $A \in \mathscr{S}_R$ and be essentially finitely related. Then by the assumption, A_R is flat and by Proposition 1.7 of Cateforis (1969), $A \otimes_R S$ is projective. Thus A is projective.

(iii) $\Leftrightarrow$ (iv). Since the torsion theory is perfect it may be assumed that $A/\tau(A)$ is a submodule of $A \otimes_R S$ (Stenström (1975), Proposition 3.4). From the sequence $0 \to A/\tau(A) \to A \otimes_R S$ we have $0 \to A/\tau(A) \otimes_R S \to A \otimes_R S$, since S is left flat and $S \otimes_R S \simeq S$. Now A is EFR and thus $A \otimes_R S$ is projective and as S is regular $A/\tau(A) \otimes S$ is a direct summand of $A \otimes_R S$ and is therefore projective. Proposition 1.7 of Cateforis (1969) gives that $A/\tau(A)$ is EFR and hence by the assumption projective. The converse is clear.

(iv) $\Rightarrow$ (i). Let I_R be a finitely generated R-submodule of S. Then $I \otimes_R S \simeq IS$ which is a projective S-module. Hence I_R is EFR and clearly torsion-free. Thus I_R is projective.

REMARK. If R is a right perfect PP ring, every finitely generated torsion-free right R-module is projective if and only if R is an extended semi-hereditary ring and S_R is a semi-simple ring.

Stenström (1975) gives several characterizations of right semi-hereditary rings when the ring is a right order in a semi-simple ring.

Let $(\mathscr{T}, \mathscr{F})$ be a hereditary torsion theory, F the corresponding topology and $R_R \in F$. Then a right submodule I of R_F is said to be invertible if there exists $q_1, q_2, \ldots, q_n \in R_F$ and $x_1, x_2, \ldots, x_n \in I$ such that $q_i I \subseteq R$ for all $1 \leqslant i \leqslant n$ and $i = \sum_{i=1}^n x_i q_i$. This definition appears on p. 209 of Stenström (1975).

THEOREM 3.4. *For a right perfect PP ring R, the following are equivalent:*

(i) *Every finitely generated dense R-submodule of S is invertible in S.*

(ii) *Every finitely generated dense R-submodule of S is projective.*

(iii) *R is an extended semi-hereditary ring.*

PROOF. (i) $\Leftrightarrow$ (ii) is established in Proposition 5.2 of Stenström (1975) and (iii) $\rightarrow$ (ii) is immediate. Thus it only remains to show (ii) $\Rightarrow$ (iii). Let $\{y_i\}_{i=1}^n$ be a generating set for a finitely generated R-submodule I of S. Then since S is regular $IS = fS$ for some idempotent f. By Theorem 2.2 it may be assumed $f \in R$. Now $IS \cap (1-f)S = 0$. Consider the R-submodule J of S generated by $\{y_i\}_{i=1}^n$ and $1-f$. Then if $q \in Q$ and $qJ = 0$, $q \in \bigcap_{i=1}^n l_Q(y_i) \cap l_Q(1-f) = Q(1-f) \cap Qf = 0$. Hence J is dense and by the assumption projective. Also $J = I \oplus (1-f)R$ which implies I is a direct summand of a projective R-submodule of S. Hence I_R is projective and R is an extended semi-hereditary ring.

In general a reduced right PP ring does not have a regular classical quotient ring (for example a non-Ore domain). In the following it is shown that being perfect is a necessary and sufficient condition for this to be true. A non-reduced right perfect PP ring need not have a regular classical quotient ring (Evans (1977), Example 2).

LEMMA 3.5. *If R is a reduced PP ring every finitely generated dense right ideal of R contains a non-zero divisor.*

PROOF. Suppose I is a finitely generated right ideal with generating set $\{x_i\}_{i=1}^n$. Then $\bigcap_{i=1}^n l_R(x_i) = 0$. For each x_i there exists $e_i = e_i^2$ such that $l_R(x_i) = Re_i$ and it is easily seen that x_i+e_i is a non-zero divisor for each i. Hence $\prod_{i=1}^n(x_i+e_i)$ is a non-zero divisor and as $\prod_{i=1}^n e_i\in\bigcap_{i=1}^n l_R(x_i)=0$, $\prod_{i=1}^n(x_i+e_i)\in I$.

THEOREM 3.6. *If R is a reduced PP ring then the following are equivalent:*
(i) *R is a right perfect PP ring.*
(ii) *S is the right classical quotient ring of R.*
(iii) *There exists a right classical quotient ring K of R which is regular.*

PROOF. (i) $\Rightarrow$ (ii). Let $x\in S$. Then there exists a finitely generated dense right ideal I of R such that $xI\subseteq R$ and by the lemma I contains a non-zero divisor. Clearly every non-zero divisor of R is a unit in S. If $x\in S$ there exists a non-zero divisor d such that $xd=y\in R$ and thus $x=yd^{-1}$.

(ii) $\Rightarrow$ (iii). This is immediate.

(iii) $\Rightarrow$ (i). It is easily shown that K is reduced and every idempotent is central and a member of R. If $a\in K$ and $ax=0$ where $x\in R$ then $a\in l_K(x)=Ke$ for some idempotent $e\in R$. Thus $l_R(x)=Re$ and $a\in Kl_R(x)$. Thus K is right torsion free and is clearly the left flat epimorphic hull of R.

COROLLARY 3.7. *If R is a reduced semi-hereditary ring, then the following are equivalent:*
(i) *R is an extended semi-hereditary ring.*
(ii) *S is the left and right classical quotient ring of R.*

EXAMPLE. *A right and left semi-hereditary right perfect PP ring which is not an extended semi-hereditary ring.* Let R be a right and left semi-hereditary ring which is a right Ore domain but which does not satisfy the left Ore condition. Then by the above corollary R is not an extended semi-hereditary ring. For a "concrete" example of such a ring see Bergman (1972), p. 49.

4. Remarks and examples

If R is an extended semi-hereditary ring and $\varphi: R\to T$ a ring epimorphism such that ${}_RT_R$ is left and right flat, then T is also an extended semi-hereditary ring. For an example of such a ring, let J be an ideal of S. Then $R/J\cap R$ is left and right flat as an R-module (Proposition 2.4).

If $R\subseteq T\subseteq S$, then the inclusion map $\varphi: R\to T$ is a ring epimorphism and ${}_RT_R$ is both left and right flat and hence T is also an extended semi-hereditary ring.

EXAMPLES. 1. If R is a right and left semi-hereditary ring which has a two-sided maximal ring of quotients then R is an extended semi-hereditary ring (Sandomierski (1968), Theorem 2.10). A semi-prime semi-hereditary PI ring satisfies this property (Martindale (1973)).

2. If T is a regular ring $R = \begin{pmatrix} TT \\ OT \end{pmatrix}$ is an extended semi-hereditary ring. It is not regular as $\begin{pmatrix} 00 \\ 01 \end{pmatrix}\begin{pmatrix} ab \\ oc \end{pmatrix}\begin{pmatrix} 01 \\ 00 \end{pmatrix} = \begin{pmatrix} 00 \\ 00 \end{pmatrix}$. The left flat epimorphic hull of R is $\begin{pmatrix} TT \\ TT \end{pmatrix}$ and $Q' = \begin{pmatrix} QQ \\ QQ \end{pmatrix}$ is the maximal right quotient ring of R, where Q is the maximal right quotient ring of R. If Q is not a left ring of quotients of T then Q' is not a left ring of quotients of R.

3. Let T be a right self-injective simple regular ring which is not Artinian (Goodearl (1974)). Then there exists a maximal right essential ideal V of R which is not two-sided. Consider the idealizer ring $R = \{t \in T\colon tV \subseteq V\}$. Then (i) $T \otimes_R T \simeq T$; (ii) ${}_R T$ is flat; (iii) T_R is finitely generated and projective (for example Robson (1972), Goodearl (1973)). By Theorem 2.3 of Goodearl (1975) $\text{w.gl.dim}\, R \leqslant 1 + \text{w.gl.dim}\, T$. Hence $\text{w.gl.dim}\, R = 1$. Also T is the maximal right quotient ring of R and thus R is right semi-hereditary. (Sandomierski (1968), Theorem 2.10). Thus R is an extended semi-hereditary ring with S_R finitely generated and projective and R is semi-prime.

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