MATHEMATICAL MODEL OF A THREE-DIMENSIONAL NON-ISOTHERMAL GLACIER*

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Abstract. A mathematical model is constructed for land glaciers with the thickness much less than the horizontal dimensions and radii of curvature of large bottom irregularities by means of the method of a thin boundary layer in dimensionless orthogonal coordinates. The dynamics are described by a statically determinate system of equations, so the solution for stresses is found. For the general non-isothermal case the interrelated velocity and temperature distributions are calculated by means of the iteration of solutions for velocity and for temperature. Temperature distribution is determined by a parabolic equation with a small parameter at the senior derivative. Its solution is reduced to the solution of a system of recurrent non-uniform differential equations of the first order by means of a series expansion of the small parameter. A relatively thin conducting boundary layer adjoins the upper and lower surfaces of a glacier, playing the role of a temperature damper in the ablation area. For ice divides, the statically indeterminate problem is solved, so the result for stresses depends on the temperature distribution.

Résumé. *Modèle mathematique d'un glacier à trois dimensions non isotherme.* Le modèle mathématique est construit pour des glaciers terrestres d'épaisseur très inférieure aux dimensions horizontales et que les rayons de courbures des grandes irrégularités du fond, par la méthode de la fine couche limite en coordonnées orthogonales sans dimensions. La dynamique est décrite par un système d'équations déterminé statique-ment, ainsi est résolu le problème des contraintes. Pour le cas général non isotherme, le calcul des distribu-tions interdépendantes de la vitesse et de la température est réalisé par itération des solutions pour la vitesse et la température. La distribution de la température est déterminée par une équation parabolique avec un petit paramètre pour la dérivée seconde. La solution se ramène à la résolution d'un système d'équations différentielles non uniforme reccurentes du premier ordre par le biais d'un développement en série du petit paramètre. Une couche limite conductrice relativement mince s'ajoute aux surfaces supérieures et inférieures du glacier, jouant le rôle d'un égalisateur de température dans le zone d'ablation. Pour les crêtes de glaces, un problème statiquement indéterminé est résolu, si bien que les résultats sur les contraintes dépendent de la distribution de la température.

Zusammenfassung. *Mathematisches Modell eines dreidimensionalen nicht-isothermen Gletschers.* Für Land-Gletscher, deren Dicke bedeutend geringer ist als ihre horizontalen Abmasse und die Krümmungsradien grosser Unregelmässigkeiten am Untergrund, wird mit Hilfe der Methode der dünnen Grenzschicht ein mathematisches Modell in dimensionslosen orthogonalen Koordinaten entworfen. Die Bewegungsvorgänge werden durch das statisch bestimmte Gleichungssystem erfasst, wobei sich die Lösung für den Spannungs-zustand ergibt. Für den allgemeinen nicht-isothermen Fall werden die gegenseitig abhängige Geschwindig-keit und Temperaturverteilung mit Hilfe von Iterationslösungen für Geschwindigkeit und Temperatur berechnet. Die Temperaturverteilung ergibt sich aus der parabolischen Gleichung mit einem kleinen Parameter für die bestimmende Ableitung. Ihre Lösung lässt sich durch Reihenentwicklung nach dem kleinen Parameter auf die Lösung des Systems rekurrenter nicht-uniformer Differentialgleichungen erster Ordnung reduzieren. Die Anpassung an die Oberfläche und den Untergrund eines Gletschers geschieht durch eine relativ dünne, leitende Grenzschicht, der die Rolle eines Temperaturdämpfers im Ablationsgebiet zukommt. Für Eisscheiden ist ein statisch unbestimmtes Problem zu lösen, wobei der Spannungszustand von der Temperaturverteilung abhängt.

Introduction

In Grigoryan and Shumskiy (1975) the simplest model of a three-dimensional non-stationary glacier was suggested. It is based on the method of a thin boundary layer neglecting the curvatures of the bed and top surfaces. In such an approximation the glacier stress field is described by a statically defined system of equations for which a closed solution is given. In the isothermal case a solution was also derived for the velocity field which is uniquely determined by the stress field, the rheological equations, and the boundary conditions. Thus, the validity of this simplest model is limited by the negligibly small curvature of the glacier bed and top surfaces and by isothermality.

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* This paper was accepted for the International Symposium on the Thermal Regime of Glaciers and Ice Sheets, Burnaby, Canada, April 1975, but was not presented because of the absence of the authors.

In the general case of a non-isothermal glacier, the equations of dynamics should be integrated together with the thermal-conduction–heat-transfer–heat-generation equation because of the dependence (1) of strain-rate and motion of ice on its temperature and (2) the dependence of ice temperature on the rate of advective heat transfer by the moving ice and on the heat generation intensity during deformation. In the case of a sub-isothermal glacier, whose temperature is near the melting point and which exhibits small temperature gradients, the mathematical model falls into two independent dynamic and thermal parts.

The parameters of the rheological law should be refined by using field data taking due account of the effect of anisotropic structures formed during the flow and the role of rupture movements. The boundary conditions are known (can be found experimentally) only on the free glacier surface. So far methods are not available for determining the conditions at the bed, therefore the lower boundary conditions (at the glacier bed) have to be calculated. Refinement of the flow-law parameters, and the determination of lower boundary conditions from field observations (mostly on free surfaces) with the help of mathematical models, are necessary before models can be used to calculate the regimes, shapes, and sizes of glaciers, based on given conditions without the need for field observations, and thus to reconstruct and forecast the changes of glaciers.

It is not desirable to construct a universal mathematical model of a glacier since the system of equations for the glacier thermodynamics is complicated. Models should be simplified, as far as is allowed by peculiar features of various glaciers. Therefore, let us first consider the principal differences in the glacier dynamics arising from the specific properties of the underlying substratum, morphology, and thermal conditions.

1. DYNAMIC CLASSIFICATION OF GLACIERS

Distinctions in the properties of the underlying substratum lead to the radical differences between *land* and *floating* glaciers. In land glaciers static or dynamic friction exists between the bottom surface and the rock bed. Such a friction is not observed with floating glaciers if we discard the negligibly small turbulent friction of moving water. Therefore, the free lower surface of floating glaciers and the upper surface of all glaciers coincide with one of the principal planes of the stress surface, whereas the bottom of land glaciers coincides with the plane of maximum shear stress or the envelope of such planes. The absence of horizontal shear stresses in floating glaciers is connected with the transfer of tensile stress from the steep edge to whole body, their flat shape, and their high velocity.

From the dynamical point of view land glaciers can be morphologically differentiated into *narrow mountain glaciers* and *ice sheets*. In the former case the width is comparable to the thickness, in the latter case the horizontal dimensions far exceed the thickness. Wide mountain glaciers with a width considerably larger than their thickness are intermediate between these limiting cases.

The bed of land glaciers always has an irregular topography, but the size of most irregularities is very small compared with the glacier thickness. A rock bed with irregularities comparable with the glacier thickness creates particular dynamic conditions: the shear and the normal stresses are of the same order of magnitude. The dynamics of such sections of a glacier can only be described by a very complicated complete model. However, for the greater part of land glaciers the model can be considerably simplified. In the particular case of ice sheets and wide mountain glaciers, the two shear stresses parallel to the bed far exceed the normal stress and the third shear stress. As a result, their dynamics can be described by a statically determinate system of equations. In this case we can approximate their dynamics using the method of a thin boundary layer, which we shall improve in the present paper as compared with the simplest model, with due regard for the curvatures of the bed and the free surface. The model for ice sheets should include, besides the general dynamic solution, a

particular solution for ice divides which represent singular points or lines of the stress and velocity fields. The thin-boundary-layer method cannot be applied to narrow mountain glaciers. Their model has to include the solution of a statically indeterminate system of equations though it remains much simpler than that for the portions with large bed irregularities.

Floating glaciers can be dynamically differentiated into *external* and *internal ice shelves*. The former adjoin the land along one edge and spread freely over the water surface, whereas the latter are surrounded on three sides by land with which they interact dynamically.

Because of the strong temperature dependence of the strain-rate of ice, the dynamics of *isothermal*, or "temperate" glaciers are essentially distinct from those of *non-isothermal*, "cold" glaciers. In the former case, heat of melting plays the role of an unlimited potential heat sink which absorbs the heat generated by the sources and prevents any heat exchange (the only exception is heat run off with liquid phase). The volume variations of the ice due to internal melting and liquid run-off are negligible. Hence, the dynamics are appreciably simplified due to the uniform properties of the ice. On the other hand, they are complicated by the possible sliding of glaciers along the rock bed due to liquid lubricant and aqueous cavitation. For non-isothermal glaciers, the mathematical model should include, as has been already pointed out above, a joint integration of the dynamics equations and non-linear equations of heat transfer involving second-order partial derivatives for calculating the interrelated velocity and temperature fields. The dynamics of a non-isothermal glacier with a bottom temperature at the melting point are very complicated. Intermediate sub-isothermal conditions develop in glaciers whose top is at a temperature slightly lower than the melting point, as well as in small "cold" glaciers where the temperature gradient is generally small due to their small size and low heat generation because of their weak activity.

The natural conditions for the existence of glaciers are such that ice sheets, as a rule, are non-isothermal while the floating ice shelves might exclusively be non-isothermal as they exist only in cold regions. On the contrary, narrow mountain glaciers belong, as a rule, either to isothermal or sub-isothermal types. Wide mountain glaciers may have any temperature regime.

Based on this classification we have developed mathematical models for land ice sheets and wide mountain glaciers with a smooth bed and different temperature regimes as well as for ice shelves. For narrow mountain glaciers a statically indeterminate system of equations is considered and differential equations have been derived for describing the velocity field. They can be solved by numerical methods; the solution for the temperature field in the sub-isothermal case is not directly related to the dynamic solution. Thus, the models suggested cover all the types of glacier, with the exception of some special sections having big irregularities at the bottom.

2. SYMBOLS

a Dimensionless specific rate of accumulation or ablation on the surface (negative in the case of ablation) related to the dimensional value $\tilde{a}$ by $\tilde{a} = v_0 a$

a_0 Specific rate of accumulation or ablation at the bed

$A_\xi = 1 + k_\xi{}^\eta(\xi, \eta)\eta/\delta_\eta$

$A_\eta = 1 + k_\eta{}^\xi(\xi, \eta)\xi/\delta_\xi$

$A, B, C, D, E, a, b, c, d, e$ Parameters of the rheological equation of ice in the hyperbolic approximation

B Also characteristic width of the glacier (constant)

c Specific heat capacity of ice

d Boundary layer thickness

g_i ith component of gravitational acceleration

g_{ij} Fundamental metric tensor component

h x_3 coordinate of free glacier surface

h_i ith Lamé coefficient

H Characteristic thickness of the glacier

$\Delta\mathcal{H}$ Activation heat of strain and mechanical relaxation

$\Delta\mathcal{H}_m$ Latent heat of melting of ice

$\mathcal{J}$ Mechanical equivalent of heat

$k_i{}^j$ Curvature of coordinate line Ox_i in a plane normal to the coordinate surface Ox_ix_j, dimensional $(k_i{}^j = 1/R_i{}^j)$ if $i,j = 1, 2, 3$ and dimensionless $(k_i{}^j = H/R_i{}^j)$ if $i,j = \xi, \eta, \zeta$

K Parameter of the rheological equation of ice in the power approximation

k Thermal diffusivity of ice

$K_i{}^\zeta$ Dimensionless curvature of the upper surface of the glacier in the direction of coordinate line i

l_a, l_c Characteristic lengths for advective and conductive heat transport respectively

L Characteristic length of the glacier (constant)

n Parameter of the rheological equation of ice in the power approximation

$\mathcal{N} = \mathcal{J}H\sigma_0{}^2/\delta_\xi \rho c v_0 T_0$

p Dimensionless pressure, related to dimensional pressure $\check{p}$ by $\check{p} = \rho g H p$

p_a Atmospheric pressure (under water, hydrostatic pressure)

q Density of geothermal heat flow

r Cylindrical polar coordinate

R Gas constant

R_{τ_i} Radius of torsion

$R_i{}^j$ Radius of curvature of coordinate line Ox_i in a plane normal to the coordinate surface Ox_ix_j

$s = -p -$ mean normal stress

$s_{ik} = s\delta_{ik} + \sigma_{ik}$ Components of the stress tensor

t Time

T Dimensionless absolute temperature related to the dimensional temperature $\check{T}$ by $T = \check{T}/T_0$

T_0 Absolute melting point of ice

T_i Dimensionless torsion of coordinate line i projected on the upper surface of the glacier

v_i ith component of the dimensionless velocity vector related to the dimensional $\check{v}$ by $\check{v} = v_0 v$

v_0 Characteristic velocity (constant)

x, y, z Rectangular Cartesian coordinates

x_i ith orthogonal curvilinear coordinate $(i = 1, 2, 3)$

$\mathcal{Z} = \mathcal{Z}(\xi, \eta, \tau)$ Dimensionless coordinate of free surface

$\alpha = \alpha(\xi, \eta)$ Angle of bottom slope

$\beta = \beta(\xi, \eta, \tau)$ Angle of upper surface slope

$\gamma = \beta - \alpha$ Relative angle between upper surface and bed

$\delta_c = k/\delta_\xi v_0 H$

$\delta_\xi = H/L$, $\delta_\eta = H/B$, $\delta_0 = B/L = \delta_\xi/\delta_\eta$

δ_{ik} Kronecker delta $(\delta_{ik} = 0$ if $i \neq k$, $\delta_{ik} = 1$ if $i = k)$

$\check{\varepsilon}$ Dimensional shear strain-rate intensity $\check{\varepsilon} = [-(\check{\varepsilon}_1\check{\varepsilon}_2 + \check{\varepsilon}_2\check{\varepsilon}_3 + \check{\varepsilon}_3\check{\varepsilon}_1)]^{\frac{1}{2}}$ where $\check{\varepsilon}_1, \check{\varepsilon}_2$ and $\check{\varepsilon}_3$ are the principal strain-rates.

$\check{\varepsilon}_{ik}$ Component of strain-rate tensor

ζ see ξ

η see ξ

$\kappa = \Delta\mathcal{H}/RT_0$

λ Thermal conductivity of ice

ν Specific power of mass sources (water freezing)

ξ, η, ζ Dimensionless affine orthogonal curvilinear coordinates

ρ Density of ice

ρ_w Density of water

σ Dimensionless shear stress intensity $\sigma = [-(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)]^{\frac{1}{2}}$ where $\sigma_1, \sigma_2,$ and σ_3 are the principal stresses related to the dimensional $\bar{\sigma}$ by $\bar{\sigma} = \sigma_0\sigma$

σ_{ik} Component of dimensionless stress deviator

σ_0 Characteristic shear stress (constant)

$\tau = v_0 t/L$ Dimensionless time

$\tau_i = H/R_{\tau_i}$ Dimensionless torsion of coordinate line

ϕ Cylindrical polar coordinate

$\phi_\xi = \sin\alpha_\xi + \cos\alpha_\xi \cos\alpha_\eta \tan\gamma_\xi/A_\xi$

$\phi_\eta = \cos\alpha_\xi (\sin\alpha_\eta + \cos\alpha_\eta \tan\gamma_\eta/A_\eta)$

$\Phi = (\phi_\xi{}^2 + \phi_\eta{}^2)^{\frac{1}{2}}$

$\psi_\xi = (k_\xi{}^\zeta \sin\alpha_\xi \cos\alpha_\eta + \tau_\xi \cos\alpha_\xi \sin\alpha_\eta)/2A_\xi$

$\psi_\eta = (\tau_\eta \sin\alpha_\xi \cos\alpha_\eta + k_\eta{}^\zeta \cos\alpha_\xi \sin\alpha_\eta)/2A_\eta$

3. Equations of thermodynamics of glaciers in orthogonal coordinates

In the simplest model (Grigoryan and Shumskiy, 1975) the coordinates x, y, z with the origin at the glacier bed, and the Ox-axis parallel to the smoothed bed and to the longitudinal axis of the glacier and the Oz-axis directed upward normal to the bed are virtually local rectangular coordinates because the space metrics remains Euclidian (the metric tensor components $g_{ii} = 1$).

In order to account for the curvature of the bed, we shall substitute the coordinates x, y, z by the orthogonal curvilinear coordinates x_1, x_2, x_3 having the same orientation and Lamé coefficients

$$h_i = \sqrt{g_{ii}} = \left[\left(\frac{\partial x}{\partial x_i}\right)^2 + \left(\frac{\partial y}{\partial x_i}\right)^2 + \left(\frac{\partial z}{\partial x_i}\right)^2\right]^{\frac{1}{2}},$$

which in the most complicated case of a thin boundary layer approximation are equal to

$$\left.\begin{aligned} h_1 &= [1 + k_1{}^2(x_1, x_2) x_2][1 + k_1{}^3(x_1, x_2) x_3], \\ h_2 &= [1 + k_2{}^1(x_1, x_2) x_1][1 + k_2{}^3(x_1, x_2) x_3], \\ h_3 &= 1, \end{aligned}\right\} \tag{1}$$
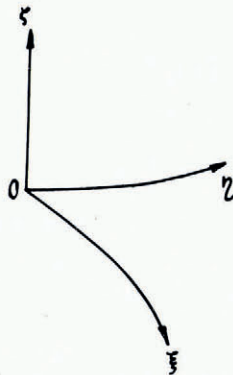

Fig. 1. *Orthogonal curvilinear coordinates. Dimensionless coordinates ξ, η, ζ as shown are related to dimensional coordinates x_1, x_2, x_3.*

where $k_i{}^j = 1/R_i{}^j$ is the curvature of the coordinate axis Ox_i in the plane tangential to the coordinate surface Ox_ix_j at the point (x_i, x_j), $R_i{}^j$ is the corresponding radius of curvature and the coordinate axis Ox_3 is a straight line normal to the smoothed bed (Fig. 1). For example, in the particular case of a circular cylindrical bed with generating line Ox_1 and radius R one has

$$x_1 = x, \qquad x_2 = R\phi, \qquad x_3 = r, \qquad k_1{}^2 = k_1{}^3 = k_2{}^1 = 0, \qquad k_2{}^3 = 1/R,$$
$$h_1 = h_3 = 1, \qquad h_2 = 1 + r/R,$$

and the square of the length of an element of the radius vector is

$$ds^2 = h_1{}^2\, dx_1{}^2 + h_2{}^2\, dx_2{}^2 + h_3{}^2\, dx_3{}^2 = dx^2 + \left(1 + \frac{r}{R}\right)^2 (R\, d\phi)^2 + dr^2$$
$$= dx^2 + (R + r)^2\, d\phi^2 + dr^2.$$

We shall write a closed system of equations describing the stress, velocity, and temperature (or melting rate) fields for a body of macroscopically isotropic (polycrystalline) incompressible ice in the field of gravity, using orthogonal coordinates and symbols of Grigoryan and Shumskiy (1975) listed in Section 2.

The equations of quasi-static equilibrium are

$$-\frac{1}{h_i}\frac{\partial \tilde{p}}{\partial x_i} + \frac{1}{h_i}\frac{\partial \tilde{\sigma}_{ii}}{\partial x_i} + \frac{1}{h_j}\frac{\partial \tilde{\sigma}_{ij}}{\partial x_j} + \frac{1}{h_k}\frac{\partial \tilde{\sigma}_{ik}}{\partial x_k} + \left(\frac{2}{h_ih_j}\frac{\partial h_i}{\partial x_j} + \frac{1}{h_jh_k}\frac{\partial h_k}{\partial x_j}\right)\tilde{\sigma}_{ij} +$$
$$+ \left(\frac{2}{h_ih_k}\frac{\partial h_i}{\partial x_k} + \frac{1}{h_jh_k}\frac{\partial h_j}{\partial x_k}\right)\tilde{\sigma}_{ik} + \frac{1}{h_ih_jh_k}\frac{\partial}{\partial x_i}(h_jh_k)\,\tilde{\sigma}_{ii} -$$
$$- \frac{1}{h_ih_j}\frac{\partial h_j}{\partial x_i}\tilde{\sigma}_{jj} - \frac{1}{h_ih_k}\frac{\partial h_k}{\partial x_i}\tilde{\sigma}_{kk} + \rho g_i = 0, \qquad (i \neq j \neq k), \qquad (2)$$

in which the convention of summing repeated suffices is *not* observed.

In what follows, the rheological equations

$$\tilde{\epsilon}_{ik} = f(\tilde{\sigma})\,\tilde{\sigma}_{ik} f_1(T), \qquad f_1(T) = \exp\left[-\kappa(T_0/\tilde{T} - 1)\right], \qquad (3)$$

will be used in two variants:

(a) in a power approximation, where

$$f(\tilde{\sigma}) = K\tilde{\sigma}^{n-1}, \qquad (3a)$$

(b) in hyperbolic approximation, where

$$f(\tilde{\sigma}) = A + B/\tilde{\sigma} + (C + D/\tilde{\sigma} + E/\tilde{\sigma}^2)^{\frac{1}{2}}. \qquad (3b)$$

For ice the equation of continuity

$$\frac{\partial \rho}{\partial t} + \frac{1}{h_1h_2h_3}\frac{\partial}{\partial x_i}(\rho h_jh_k\tilde{v}_i) = \nu, \qquad (4)$$

reduces to the equation of incompressibility

$$\frac{\partial}{\partial x_i}(h_jh_k v_i) = 0. \qquad (4a)$$

The equations of components of the strain-rate tensor are

$$\left.\begin{array}{l}
\tilde{\epsilon}_{ii} = \dfrac{1}{h_i}\dfrac{\partial \tilde{v}_i}{\partial x_i} + \dfrac{\tilde{v}_j}{h_ih_j}\dfrac{\partial h_i}{\partial x_j} + \dfrac{\tilde{v}_k}{h_ih_k}\dfrac{\partial h_i}{\partial x_k}, \\[3mm]
\tilde{\epsilon}_{ij} = \dfrac{1}{2}\left[\dfrac{h_j}{h_i}\dfrac{\partial}{\partial x_i}\left(\dfrac{\tilde{v}_j}{h_j}\right) + \dfrac{h_i}{h_j}\dfrac{\partial}{\partial x_j}\left(\dfrac{\tilde{v}_i}{h_i}\right)\right], \qquad i \neq j \neq k.
\end{array}\right\} \qquad (5)$$

where again the summation convention is not used.

The equation of thermal conduction–heat transport–heat generation (Fourier–Kirchhof–Poisson) is

$$\frac{J}{\rho c}\tilde\sigma_{ij}\tilde\epsilon_{ij} = \begin{cases} \dfrac{\partial \tilde T}{\partial t} + \dfrac{v_i}{h_i}\dfrac{\partial \tilde T}{\partial x_i} - \dfrac{1}{h_1 h_2 h_3}\dfrac{\partial}{\partial x_i}\left(\dfrac{h_j h_k}{h_i}k\dfrac{\partial \tilde T}{\partial x_i}\right), & \tilde T < T_0, \quad (6)\\[2ex] -\Delta \mathcal{H}_m v, & \tilde T = T_0. \quad (7) \end{cases}$$

As was pointed out by Grigoryan and Shumskiy (1975), the solution of this system of equations should satisfy the initial conditions

$$t = 0, \qquad h = h(x_1, x_2, 0), \qquad \tilde T = \tilde T(x_1, x_2, x_3, 0); \tag{8}$$

and the boundary conditions on free surface $x_3 = h(x_1, x_2)$

$$s_{33} = -p_a, \qquad \tilde\sigma_{i3} = 0, \qquad \tilde T = T(x_1, x_2, t), \tag{9}$$

$$\frac{\partial h}{\partial t} + \frac{\tilde v_i(h)}{h_i}\frac{\partial h}{\partial x_i} - \tilde v_3(h) = \tilde a_3(x_1, x_2, t), \qquad i = 1, 2; \tag{10}$$

and at the bed $x_3 = 0$

$$\frac{\partial T}{\partial x_3} = -\frac{1}{\lambda}(q_3 + J\tilde\sigma_{i3}\tilde v_i + \Delta\mathcal{H}_m\rho\tilde v_3), \tag{11}$$

$$\mathbf{v} = \begin{cases} 0, & \tilde T < T_0, \quad (12)\\[2ex] \mathcal{F}(\sigma_{i3}, s_{33}), & \tilde T = T_0. \quad (13) \end{cases}$$

The normal component of velocity at the bottom $\tilde v_3$ must be equal to ice freezing rate $\tilde a_0$ (a negative value indicating a melting rate). The form and the parameters of function $\mathcal{F}$ in Equation (13) are to be determined.

4. Thermodynamic equations of a glacier in dimensionless orthogonal coordinates

To estimate the magnitude of the terms in the equations of the previous section, we shall rewrite them in dimensionless coordinates

$$\xi = x_1/L, \qquad \eta = x_2/B, \qquad \zeta = x_3/H, \tag{14}$$

which are the dimensionless affine analogues of the orthogonal coordinates x_1, x_2, x_3. The characteristic size of a glacier (L, B, H) will be measured along the coordinate axes x_1, x_2, x_3 (see Section 2 for notation).

Let us reduce the dimensional values (with a tilde above the same symbols) to a dimensionless form as follows:

$$\left.\begin{array}{ccc} \tilde p = \rho g H p, & \tilde\sigma = \sigma_0\sigma, & t = \dfrac{L}{v_0}\tau, \\[2ex] \tilde v = v_0 v, & \tilde T = T_0 T, & \tilde a = v_0 a. \end{array}\right\} \tag{15}$$

Lamé coefficients for the dimensionless coordinates will be

$$h_\xi = A_\xi(\xi, \eta)[1 + k_\xi{}^\zeta(\xi, \eta)\zeta], \qquad h_\eta = A_\eta(\xi, \eta)[1 + k_\eta{}^\zeta(\xi, \eta)\zeta], \tag{16}$$

where

$$A_\xi = 1 + \frac{k_\xi{}^\eta(\xi, \eta)}{\delta_\eta}\eta, \qquad A_\eta = 1 + \frac{k_\eta{}^\xi(\xi, \eta)}{\delta_\xi}\xi, \tag{17}$$

and the dimensionless curvature is

$$k_i{}^j = H/R_i{}^j, \qquad i = \xi, \eta; \quad j = \xi, \eta, \zeta; \quad i \neq j. \tag{18}$$

The equations expressing the thermodynamics of glaciers can be written as follows. The equations of quasi-static equilibrium

$$
\begin{aligned}
-\frac{\delta_\xi}{h_\xi}\frac{\partial p}{\partial \xi}&+\frac{\sigma_0}{\rho g H}\left\{\frac{\delta_\xi}{h_\xi}\frac{\partial \sigma_{\xi\xi}}{\partial \xi}+\frac{\delta_\eta}{h_\eta}\frac{\partial \sigma_{\xi\eta}}{\partial \eta}+\frac{\partial \sigma_{\xi\zeta}}{\partial \zeta}+\frac{2}{h_\xi h_\eta}\left[\frac{\partial}{\partial \eta}(k_\xi{}^\eta)(1+k_\xi{}^\zeta\zeta)+\right.\right.\\
&\left.+A_\xi\delta_\eta\frac{\partial k_\xi{}^\zeta}{\partial \eta}\zeta\right]\sigma_{\xi\eta}+\left(2\frac{A_\xi}{h_\xi}k_\xi{}^\zeta+\frac{A_\eta}{h_\eta}k_\eta{}^\zeta\right)\sigma_{\xi\zeta}+\\
&\left.+\frac{1}{h_\xi h_\eta}\left[\frac{\partial}{\partial \xi}(k_\eta{}^\xi)(1+k_\eta{}^\zeta\zeta)+A_\eta\delta_\xi\frac{\partial k_\eta{}^\zeta}{\partial \xi}\zeta\right](\sigma_{\xi\xi}-\sigma_{\eta\eta})\right\}-\sin\alpha_\xi=0,\\[2ex]
-\frac{\delta_\eta}{h_\eta}\frac{\partial p}{\partial \eta}&+\frac{\sigma_0}{\rho g H}\left\{\frac{\delta_\xi}{h_\xi}\frac{\partial \sigma_{\eta\xi}}{\partial \xi}+\frac{\delta_\eta}{h_\eta}\frac{\partial \sigma_{\eta\eta}}{\partial \eta}+\frac{\partial \sigma_{\eta\zeta}}{\partial \zeta}+\frac{2}{h_\xi h_\eta}\left[\frac{\partial}{\partial \xi}(k_\eta{}^\xi)(1+k_\eta{}^\zeta\zeta)+\right.\right.\\
&+A_\eta\delta_\xi\frac{\partial k_\eta{}^\zeta}{\partial \xi}\zeta\Big]\sigma_{\eta\xi}+\left(\frac{A_\xi}{h_\xi}k_\xi{}^\zeta+2\frac{A_\eta}{h_\eta}k_\eta{}^\zeta\right)\sigma_{\eta\zeta}-\frac{1}{h_\xi h_\eta}\left[\frac{\partial}{\partial \eta}(k_\xi{}^\eta)(1+k_\xi{}^\zeta\zeta)+\right.\\
&\left.\left.+A_\xi\delta_\eta\frac{\partial k_\xi{}^\zeta}{\partial \eta}\zeta\right](\sigma_{\eta\eta}-\sigma_{\xi\xi})\right\}-\cos\alpha_\xi\sin\alpha_\eta=0,\\[2ex]
-\frac{\partial p}{\partial \zeta}&+\frac{\sigma_0}{\rho g H}\left\{\frac{\delta_\xi}{h_\xi}\frac{\partial \sigma_{\zeta\xi}}{\partial \xi}+\frac{\delta_\eta}{h_\eta}\frac{\partial \sigma_{\zeta\eta}}{\partial \eta}+\frac{\partial \sigma_{\zeta\zeta}}{\partial \zeta}+\frac{1}{h_\xi h_\eta}\left[\frac{\partial}{\partial \xi}(k_\eta{}^\xi)(1+k_\eta{}^\zeta\zeta)+\right.\right.\\
&+A_\eta\delta_\xi\frac{\partial k_\eta{}^\zeta}{\partial \xi}\zeta\Big]\sigma_{\zeta\xi}+\frac{1}{h_\xi h_\eta}\left[\frac{\partial}{\partial \eta}(k_\xi{}^\eta)(1+k_\xi{}^\zeta\zeta)+A_\xi\delta_\eta\frac{\partial k_\xi{}^\zeta}{\partial \eta}\zeta\right]\sigma_{\zeta\eta}+\\
&\left.+\frac{A_\xi}{h_\xi}k_\xi{}^\zeta(\sigma_{\zeta\zeta}-\sigma_{\xi\xi})+\frac{A_\eta}{h_\eta}k_\eta{}^\zeta(\sigma_{\zeta\zeta}-\sigma_{\eta\eta})\right\}-\cos\alpha_\xi\cos\alpha_\eta=0.
\end{aligned}\right\}\ (19)
$$

The rheological equations

$$
\bar{\varepsilon}_{ik}=f(\sigma)\,\sigma_0\sigma_{ik}f_1(T),\qquad f_1(T)=\exp\left[-\kappa\left(\frac{1}{T}-1\right)\right],\tag{20}
$$

where in the power approximation

$$
f(\sigma)=K\sigma_0{}^{n-1}\sigma^{n-1},\tag{20a}
$$

and in the hyperbolic approximation

$$
f(\sigma)=A+B/\sigma_0\sigma+[C+D/\sigma_0\sigma+E/\sigma_0{}^2\sigma^2]^{\frac{1}{2}}.\tag{20b}
$$

The equation of incompressibility

$$
\begin{aligned}
\frac{\delta_\xi}{h_\xi}\frac{\partial v_\xi}{\partial \xi}&+\frac{\delta_\eta}{h_\eta}\frac{\partial v_\eta}{\partial \eta}+\frac{\partial v_\zeta}{\partial \zeta}+\frac{1}{h_\xi h_\eta}\left\{\left[\frac{\partial}{\partial \xi}(k_\eta{}^\xi)(1+k_\eta{}^\zeta\zeta)+\right.\right.\\
&\left.+A_\eta\delta_\xi\frac{\partial k_\eta{}^\zeta}{\partial \eta}\zeta\right]v_\xi+\left[\frac{\partial}{\partial \eta}(k_\xi{}^\eta)(1+k_\xi{}^\zeta\zeta)+A_\xi\delta_\eta\frac{\partial k_\xi{}^\zeta}{\partial \eta}\zeta\right]v_\eta\Bigg\}+\\
&+\left(\frac{A_\xi}{h_\xi}k_\xi{}^\zeta+\frac{A_\eta}{h_\eta}k_\eta{}^\zeta\right)v_\zeta=0.
\end{aligned}\right\}\ (21)
$$

The equations of the strain-rate components

$$\tilde{\epsilon}_{\xi\xi} = \frac{v_0}{H}\left\{\frac{\delta_\xi}{h_\xi}\frac{\partial v_\xi}{\partial \xi} + \frac{1}{h_\xi h_\eta}\left[\frac{\partial}{\partial \eta}(k_\xi\eta)(1+k_\xi\zeta\zeta) + A_\xi\delta_\eta\frac{\partial k_\xi^\zeta}{\partial \eta}\zeta\right]v_\eta + \frac{A_\xi}{h_\xi}k_\xi\zeta v_\zeta\right\},$$

$$\tilde{\epsilon}_{\eta\eta} = \frac{v_0}{H}\left\{\frac{\delta_\eta}{h_\eta}\frac{\partial v_\eta}{\partial \eta} + \frac{1}{h_\xi h_\eta}\left[\frac{\partial}{\partial \xi}(k_\eta\xi)(1+k_\eta\zeta\zeta) + A_\eta\delta_\xi\frac{\partial k_\eta^\zeta}{\partial \eta}\zeta\right]v_\xi + \frac{A_\eta}{h_\eta}k_\eta\zeta v_\zeta\right\},$$

$$\tilde{\epsilon}_{\zeta\zeta} = \frac{v_0}{H}\frac{\partial v_\zeta}{\partial \zeta},$$

$$\tilde{\epsilon}_{\xi\eta} = \frac{v_0}{2H}\left\{\frac{\delta_\xi}{h_\xi}\frac{\partial v_\eta}{\partial \xi} + \frac{\delta_\eta}{h_\eta}\frac{\partial v_\xi}{\partial \eta} - \frac{1}{h_\xi h_\eta}\left[\frac{\partial}{\partial \xi}(k_\eta\xi)(1+k_\xi\zeta\zeta) + A_\xi\delta_\eta\frac{\partial k_\xi^\zeta}{\partial \eta}\zeta\right]v_\eta - \right.$$
$$\left. - \frac{1}{h_\xi h_\eta}\left[\frac{\partial}{\partial \eta}(k_\xi\eta)(1+k_\eta\zeta\zeta) + A_\eta\delta_\xi\frac{\partial k_\eta^\zeta}{\partial \xi}\zeta\right]v_\xi\right\},$$

$$\tilde{\epsilon}_{\xi\zeta} = \frac{v_0}{2H}\left(\frac{\delta_\xi}{h_\xi}\frac{\partial v_\zeta}{\partial \xi} + \frac{\partial v_\xi}{\partial \zeta} - \frac{A_\xi}{h_\xi}k_\xi\zeta v_\xi\right),$$

$$\tilde{\epsilon}_{\eta\zeta} = \frac{v_0}{2H}\left(\frac{\delta_\eta}{h_\eta}\frac{\partial v_\zeta}{\partial \eta} + \frac{\partial v_\eta}{\partial \zeta} - \frac{A_\eta}{h_\eta}k_\eta\zeta v_\eta\right).$$

$$(22)$$

The equation of thermal conduction-heat transport-heat generation

$$\frac{J}{\rho c}\bar{\sigma}_{ij}\tilde{\epsilon}_{ij} = \begin{cases} -\Delta\mathscr{H}_m\nu, \qquad T=1, & (23)\end{cases}$$

$$\frac{v_0 T_0}{H}\left(\delta_\xi\frac{\partial T}{\partial \tau} + \frac{\delta_\xi}{h_\xi}v_\xi\frac{\partial T}{\partial \xi} + \frac{\delta_\eta}{h_\eta}v_\eta\frac{\partial T}{\partial \eta} + v_\zeta\frac{\partial T}{\partial \zeta}\right) -$$
$$- \frac{kT_0}{H^2}\left\{\frac{\delta_\xi^2}{h_\xi^2}\frac{\partial^2 T}{\partial \xi^2} + \frac{\delta_\eta^2}{h_\eta^2}\frac{\partial^2 T}{\partial \eta^2} + \frac{\partial^2 T}{\partial \zeta^2} + \frac{\delta_\xi}{h_\xi^2}\left[\frac{1}{h_\eta}\frac{\partial}{\partial \xi}(k_\eta\xi)(1+k_\eta\zeta\zeta) + \right.\right.$$
$$+ \frac{A_\eta}{h_\eta}\delta_\xi\frac{\partial k_\eta^\zeta}{\partial \xi}\zeta - \frac{\delta_0}{h_\xi}\frac{\partial k_\xi^\eta}{\partial \xi}\eta(1+k_\xi\zeta\zeta) - \frac{A_\xi}{h_\xi}\delta_\xi\frac{\partial k_\xi^\zeta}{\partial \xi}\zeta\right]\frac{\partial T}{\partial \xi} +$$
$$+ \frac{\delta_\eta}{h_\eta^2}\left[\frac{1}{h_\xi}\frac{\partial}{\partial \eta}(k_\xi\eta)(1+k_\xi\zeta\zeta) + \frac{A_\xi}{h_\xi}\delta_\eta\frac{\partial k_\xi^\zeta}{\partial \eta}\zeta - \frac{\xi}{\delta_0 h_\eta}\frac{\partial k_\eta^\xi}{\partial \eta}(1+k_\eta\zeta\zeta) - \right.$$
$$\left.\left. - \frac{A_\eta}{h_\eta}\delta_\eta\frac{\partial k_\eta^\zeta}{\partial \eta}\zeta\right]\frac{\partial T}{\partial \eta} + \left(\frac{A_\xi}{h_\xi}k_\xi\zeta + \frac{A_\eta}{h_\eta}k_\eta\zeta\right)\frac{\partial T}{\partial \zeta}\right\}, \qquad T<1.$$

$$(24)$$

The initial conditions

$$\tau = 0, \qquad \mathscr{Z} = f(\xi, \eta), \qquad T = \Psi(\xi, \eta, \zeta). \qquad (25)$$

The boundary conditions on free surface $\zeta = \mathscr{Z}(\xi, \eta, \tau)$

$$p = p_a, \qquad \sigma_{i\zeta} = 0, \qquad T = T(\xi, \eta, \tau), \qquad (i = \xi, \eta), \qquad (26)$$

$$\frac{\partial \mathscr{Z}}{\partial \tau} + \frac{v_\xi(\mathscr{Z})}{h_\xi}\frac{\partial \mathscr{Z}}{\partial \xi} + \frac{v_\eta(\mathscr{Z})}{\delta_0 h_\eta}\frac{\partial \mathscr{Z}}{\partial \eta} = \frac{v_\zeta(\mathscr{Z}) + a_\zeta(\mathscr{Z})}{\delta_\xi}. \qquad (27)$$

The boundary conditions at the bottom

$$\frac{\partial T}{\partial \zeta} = -\frac{H}{\lambda T_0}(q\cos\alpha_\xi\cos\alpha_\eta + J\sigma_0 v_0\sigma_{i\zeta}v_i + \Delta\mathscr{H}_m\rho v_0 v_\zeta), \qquad (28)$$

$$\mathbf{v} = \begin{cases} 0, & T < 1, \\ \mathscr{F}(\sigma_{i\zeta}, s_{\zeta\zeta}), & T = 1. \end{cases} \tag{29}$$

5. THE CONDITIONS OF THE VALIDITY OF APPROXIMATE SOLUTION BY THE THIN BOUNDARY LAYER METHOD AND SIMPLIFIED SYSTEM OF EQUATIONS

The simple mathematical model suggested by Grigoryan and Shumskiy (1975) is applicable to glaciers in which (1) the characteristic length L and the characteristic width B greatly exceed the characteristic thickness H:

$$H \ll L, \qquad H \ll B, \tag{30}$$

and (2) the radius of curvature R of large irregularities of the bed (except roughnesses) is so great that the curvature of the bed is negligible:

$$k_i{}^3 = 1/R_i{}^3 \to 0, \qquad R_i{}^3 \to \infty, \qquad i = 1, 2. \tag{31}$$

The first condition is satisfied by most glaciers, except narrow mountain glaciers. But the second condition is usually violated in real glaciers. A consideration of the bed curvature gives an opportunity to extend the applicability of the thin boundary-layer method to the limits of validity of the condition

$$H \ll R_i{}^3, \qquad k_i{}^3 = 1/R_i{}^3 \ll 1/H, \tag{32}$$

which is necessary and sufficient for converting the equation of quasi-static equilibrium, in a first approximation, to a statically defined system due to the fact that the normal stress deviator components are the values of a much higher order of smallness compared to the shear stresses. On the whole, the conditions of applicability of the model built up by the thin boundary-layer method can be written as

$$\delta_i \ll 1, \qquad k_i{}^j \ll 1, \qquad i = \xi, \eta, \tag{33}$$

where $k_i{}^j$ is the dimensionless curvature as in Equation (18). These conditions are satisfied by ice sheets and wide mountain glaciers with a smooth bed. Here,

$$A_i = O(1), \qquad h_i \approx A_i. \tag{34}$$

The derivative of the function with respect to dimensionless coordinates is of the same order of magnitude as the function itself.

In the equations of Section 4, if the terms of the higher-order of small quantities are discarded, we obtain the following simplified system of equations:

The equations of quasi-static equilibrium

$$\left. \begin{aligned} -\frac{\delta_\xi}{A_\xi} \frac{\partial p}{\partial \xi} + \frac{\sigma_0}{\rho g H} \frac{\partial \sigma_{\xi\xi}}{\partial \zeta} - \sin \alpha_\xi &= 0, \\[2mm] -\frac{\delta_\eta}{A_\eta} \frac{\partial p}{\partial \eta} + \frac{\sigma_0}{\rho g H} \frac{\partial \sigma_{\eta\zeta}}{\partial \zeta} + \cos \alpha_\xi \sin \alpha_\eta &= 0, \\[2mm] -\frac{\partial p}{\partial \zeta} + \frac{\sigma_0}{\rho g H} \frac{\partial \sigma_{\zeta\zeta}}{\partial \zeta} - \cos \alpha_\xi \cos \alpha_\eta &= 0. \end{aligned} \right\} \tag{35}$$

The equation of incompressibility

$$\frac{\delta_\xi}{A_\xi} \frac{\partial v_\xi}{\partial \xi} + \frac{\delta_\eta}{A_\eta} \frac{\partial v_\eta}{\partial \eta} + \frac{\partial v_\zeta}{\partial \zeta} + \frac{1}{A_\xi A_\eta} \left[\frac{\partial}{\partial \xi} (k_\eta{}^\xi \xi) v_\xi + \frac{\partial}{\partial \eta} (k_\xi{}^\eta \eta) v_\eta \right] = 0. \tag{36}$$

The equations of components of strain-rate tensor

$$\tilde{\epsilon}_{\xi\xi} = \frac{v_0}{H}\left[\frac{\delta_\xi}{A_\xi}\frac{\partial v_\xi}{\partial \xi} + \frac{1}{A_\xi A_\eta}\frac{\partial}{\partial \eta}\,(k_\xi{}^\eta\eta)\,v_\eta\right],$$

$$\tilde{\epsilon}_{\eta\eta} = \frac{v_0}{H}\left[\frac{\delta_\eta}{A_\eta}\frac{\partial v_\eta}{\partial \eta} + \frac{1}{A_\xi A_\eta}\frac{\partial}{\partial \xi}\,(k_\eta{}^\xi\xi)\,v_\xi\right],$$

$$\tilde{\epsilon}_{\zeta\zeta} = \frac{v_0}{H}\frac{\partial v_\xi}{\partial \zeta},$$

$$\tilde{\epsilon}_{\xi\eta} = \frac{v_0}{2H}\left\{\frac{\delta_\xi}{A_\xi}\frac{\partial v_\eta}{\partial \xi} + \frac{\delta_\eta}{A_\eta}\frac{\partial v_\xi}{\partial \eta} - \frac{1}{A_\xi A_\eta}\left[\frac{\partial}{\partial \eta}\,(k_\xi{}^\eta\eta)\,v_\xi + \frac{\partial}{\partial \xi}\,(k_\eta{}^\xi\xi)\,v_\eta\right]\right\},$$

$$\tilde{\epsilon}_{\xi\zeta} = \frac{v_0}{2H}\frac{\partial v_\xi}{\partial \zeta},$$

$$\tilde{\epsilon}_{\eta\zeta} = \frac{v_0}{2H}\frac{\partial v_\eta}{\partial \zeta}.$$

(37)

The equation of thermal conduction–heat transport–heat generation

$$\frac{\mathscr{J}}{\rho c}\tilde{\sigma}_{ij}\tilde{\epsilon}_{ij} = \begin{cases} -\Delta\mathscr{H}_\mathrm{m}v, & T = 1, \hspace{2cm} (38) \\[2ex] \dfrac{v_0 T_0}{H}\left(\delta_\xi\dfrac{\partial T}{\partial \tau} + \dfrac{\delta_\xi}{A_\xi}v_\xi\dfrac{\partial T}{\partial \xi} + \dfrac{\delta_\eta}{A_\eta}v_\eta\dfrac{\partial T}{\partial \eta} + v_\zeta\dfrac{\partial T}{\partial \zeta}\right) - \dfrac{kT_0}{H^2}\dfrac{\partial^2 T}{\partial \zeta^2}, & T < 1. \hspace{0.5cm} (39) \end{cases}$$

The other equations of Section 4 remain unchanged.

In Equations (37) and (39) it is taken into account that, as follows from Equation (21), the products of δ_i and the dimensionless normal velocity derivatives with respect to coordinates, and hence, the products of δ_i and the dimensionless velocities themselves are of the same order of magnitude:

$$\delta_\xi\frac{\partial v_\xi}{\partial \xi} \approx \delta_\eta\frac{\partial v_\eta}{\partial \eta} \approx \frac{\partial v_\zeta}{\partial \zeta}, \qquad \delta_\xi v_\xi \approx \delta_\eta v_\eta \approx v_\zeta, \qquad v_\zeta \ll v_\xi, \qquad v_\zeta \ll v_\eta. \qquad (40)$$

For longitudinal axes of ice sheets, or coordinate axes $O\xi$, we shall take the lines of maximum slope of the upper surface diverging from the ice divide to the periphery. Ice divides are singular points or lines of the stress and velocity fields to which a general solution is not applicable. In the case of ice sheets $\delta_0 = O(1)$ and in the case of mountain glaciers $\delta_0 \ll 1$.

6. General solution in stresses

A relationship similar to Equation (40) exists between the stress deviator components and the strain-rate tensor components, i.e. $\sigma_{\zeta\zeta}$ is a value of a higher order of small quantities than $\sigma_{\xi\zeta}$ and $\sigma_{\eta\zeta}$ and the term with its derivative can be neglected not only as compared to other terms of the last Equation (35), but also to the terms of the other two Equations (35). By integrating the last Equation (35) without this term under boundary conditions (26) we obtain

$$p = \cos\alpha\,(\mathcal{Z} - \zeta), \qquad \cos\alpha = \cos\alpha_\xi\cdot\cos\alpha_\eta. \qquad (41)$$

By differentiating this value of p with respect to ξ and η, by substituting the derivative into the first two Equations (35), and by integrating under the boundary conditions (26) we obtain an expression for the bed-parallel shear-stress components

$$\sigma_{i\zeta} = \frac{\rho g H}{\sigma_0}\left[\phi_i(\mathcal{Z}-\zeta)-\psi_i(\mathcal{Z}-\zeta)^2\right], \qquad i = \xi, \eta, \tag{42}$$

where ϕ_i, ψ_i are defined as in Section 2. The condition $k_i\zeta \ll 1$ is equivalent to condition $\tau_j \ll 1$ $(i, j = \xi, \eta)$. Thus, if $\sin \alpha_i \ll 1$ and $\tan \gamma_i \ll 1$ are of the same order of magnitude then

$$\psi_i \ll \phi_i.$$

Therefore, in the general solution instead of Equation (42) we can use the following equation

$$\sigma_{i\zeta} = \frac{\rho g H}{\sigma_0}\phi_i(\mathcal{Z}-\zeta). \tag{43}$$

On neglecting the stress deviator components $\sigma_{\xi\xi}$, $\sigma_{\eta\eta}$, $\sigma_{\zeta\zeta}$, $\sigma_{\xi\eta}$, which are values of higher-order of smallness as compared to $\sigma_{\xi\zeta}$ and $\sigma_{\eta\zeta}$, the shear stress intensity is

$$\sigma = (\sigma_{\xi\zeta}{}^2 + \sigma_{\eta\zeta}{}^2)^{\frac{1}{2}} = \frac{\rho g H}{\sigma_0}\,\Phi(\mathcal{Z}-\zeta). \tag{44}$$

7. General solution in velocities in non-isothermal case

In order to calculate the velocity field for $T < 1$, we have to integrate a system of equations which, on the basis of the above results, can be reduced to

$$v_i = \frac{2H}{v_0}\int f(\sigma)\,\sigma_{i\zeta}\exp\left[-\kappa\left(\frac{1}{T}-1\right)\right]\mathrm{d}\zeta, \qquad . \tag{45}$$

$$v_\zeta = -\frac{2H}{v_0}\int\left\{\frac{\delta_\xi}{A_\xi}\frac{\partial}{\partial\xi}\int f(\sigma)\,\sigma_{\xi\zeta}\exp\left[-\kappa\left(\frac{1}{T}-1\right)\right]\mathrm{d}\zeta+\right.$$

$$+\frac{\delta_\eta}{A_\eta}\frac{\partial}{\partial\eta}\int f(\sigma)\,\sigma_{\eta\zeta}\exp\left[-\kappa\left(\frac{1}{T}-1\right)\right]\mathrm{d}\zeta+$$

$$+\frac{1}{A_\xi A_\eta}\left[\frac{\partial}{\partial\xi}(k_\eta{}^\xi\xi)\int f(\sigma)\,\sigma_{\xi\zeta}\exp\left[-\kappa\left(\frac{1}{T}-1\right)\right]\mathrm{d}\zeta+\right.$$

$$\left.\left.+\frac{\partial}{\partial\eta}(k_\xi{}^\eta\eta)\int f(\sigma)\,\sigma_{\eta\zeta}\exp\left[-\kappa\left(\frac{1}{T}-1\right)\right]\mathrm{d}\zeta\right]\right\}\mathrm{d}\zeta, \tag{46}$$

$$\frac{\partial T}{\partial\tau}+\frac{v_\xi}{A_\xi}\frac{\partial T}{\partial\xi}+\frac{v_\eta}{\delta_0 A_\eta}\frac{\partial T}{\partial\eta}+\frac{v_\zeta}{\delta_\xi}\frac{\partial T}{\partial\zeta} = \delta_c\frac{\partial^2 T}{\partial\zeta^2}+\mathcal{N}f(\sigma)\,\sigma^2\exp\left[-\kappa\left(\frac{1}{T}-1\right)\right], \tag{47}$$

where $\sigma_{i\zeta}$ is defined by Equation (43), σ by Equation (44), and $f(\sigma)$ is defined by Equation (20a), or more exactly (20b). The system should be integrated under the initial and boundary conditions (Equations (25)–(29)).

A joint solution can be obtained by iterating the solutions of Equations (45) and (46), under a given temperature field and solving Equation (47) for a given velocity field.

To integrate Equation (47) for a given velocity, one should bear in mind that for most glaciers $k = 36$ m²/year, $\delta_\xi = 0.01$ to 0.1, $v_0 = 10$ to 500 m/year and $H = 100$ to $1\,000$ m, hence the dimensionless parameter δ_c is of order 10^{-2} to 10^{-3} and only in the case of small sub-isothermal glaciers it amounts to 10^{-1} but always remains less than 1. Consequently, for the non-isothermal glaciers concerned, the left-hand terms in Equation (47) characterizing the advective heat transfer are considerably larger than the first item on the right which describes the conductive heat transfer. Equation (47) is a parabolic one with a small parameter for the highest derivative. It degenerates into the first-order equation as $\delta_c \to 0$.

The general solution of Equation (47) can be represented as a power series in the small parameter δ_c (Cole, 1968):

$$T = T^{(0)}+\delta_c T^{(1)}+ \ldots +\delta_c{}^i T^{(i)}+ \ldots . \tag{48}$$

By substituting the expansion (48) into Equation (47) and by equating the coefficients of the same powers of δ_c, we obtain a system of recurrent differential equations:

$$\frac{\partial T^{(0)}}{\partial \tau}+\frac{v_\xi}{A_\xi}\frac{\partial T^{(0)}}{\partial \xi}+\frac{v_\eta}{\delta_0 A_\eta}\frac{\partial T^{(0)}}{\partial \eta}+\frac{v_\zeta}{\delta_\xi}\frac{\partial T^{(0)}}{\partial \zeta} = Nf(\sigma)\,\sigma^2 \exp\left[-\kappa\left(\frac{1}{T^{(0)}}-1\right)\right], \quad (49)$$

$$\frac{\partial T^{(i)}}{\partial \tau}+\frac{v_\xi}{A_\xi}\frac{\partial T^{(i)}}{\partial \xi}+\frac{v_\eta}{\delta_0 A_\eta}\frac{\partial T^{(i)}}{\partial \eta}+\frac{v_\zeta}{\delta_\xi}\frac{\partial T^{(i)}}{\partial \zeta} = \frac{\partial^2 T^{(i-1)}}{\partial \zeta^2}, \quad i = 1, 2, \dots. \quad (50)$$

The highest-order term in the expansion (48) satisfies the non-uniform differential first-order equation with the heat-source function in the right-hand side. The other expansion terms satisfy similar equations with the second derivative with respect to ζ from the previous expansion term on the right.

Thus, the heat transfer in a glacier is mainly due to the ice motion and is described by Equation (49). Its solution satisfies the following relations

$$\int d\tau = \int \frac{A_\xi\,d\xi}{v_\xi(\xi,\eta,\zeta,\tau)} = \delta_0 \int \frac{A_\eta\,d\eta}{v_\eta(\xi,\eta,\zeta,\tau)} = \delta_\xi \int \frac{d\zeta}{v_\zeta(\xi,\eta,\zeta,\tau)} \quad (51)$$

and, by integration along the characteristic,

$$= \frac{1}{N}\int \frac{dT}{f(\sigma)\,\sigma^2 \exp\left[-\kappa(1/T-1)\right]}, \quad (52)$$

i.e. in a glacier the temperature propagates along the particle trajectories (51) according to law (52). Conductive heat transfer causes small variations in the temperature field, which can be calculated to any degree of accuracy with the help of Equations (50) having a similar solution.

But Equations (49), (50), initial conditions (25), and boundary conditions (26) and (28) present a mathematically incompatible problem: one of the boundary conditions proceeds from the accumulation region along trajectory (51) according to law (52) slightly changed by thermal conductivity, and at the exit, where the line comes out on the outer surface in the region of ablation, the temperature value will be incompatible with that determined by the surface conditions. This incompatibility is due to different mechanisms of heat transfer inside the glacier and in the thin layer adjacent to the outer surface (upper or lower).

Temperature propagates from the outer surface into the interior in time t by advective transport at a depth of order

$$l_a \approx v_0 t, \quad (53)$$

and by thermal conduction at a depth of order

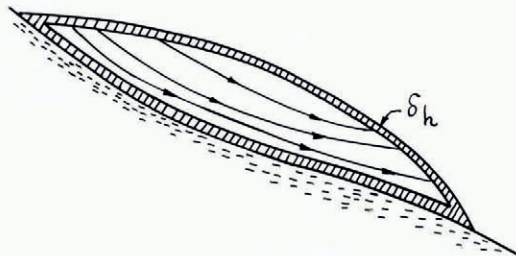$$l_c \approx (kt)^{\frac{1}{2}}. \qu(54)$$

Fig. 2. Diagram to indicate how the convective and conductive terms vary with depth.

These dependences are plotted in Fig. 2. It is evident that in a short time $t < t^\star$ the boundary condition is transferred to the thin near-surface layer mainly due to conductive heat transfer. Thus, a thin boundary layer lies adjacent to the outer surface of a glacier and this layer acquires the boundary conditions (26) and (28) by conduction. Beyond it, inside the glacier, heat propagates along the particle trajectories (Fig. 3). The limiting thickness d of the conductive boundary layer is determined by the condition of equality of the advective and the conductive terms of Equation (47)

$$d \lesssim k/v_0, \qquad k = 36 \text{ m}^2/\text{year}. \tag{55}$$

The thickness of the conductive boundary layer is not constant and depends on the form of the boundary conditions (26) and (28) and on the magnitude of the velocity component normal to the surface.
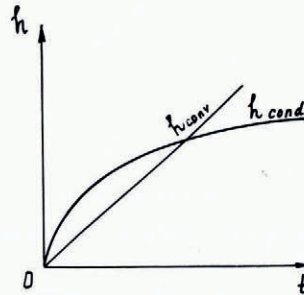

Fig. 3. *Heat transfer within the glacier. The thin boundary layer shown shaded acquires the boundary conditions essentially by conduction, while in the rest of the glacier heat primarily moves by mass transfer (advection).*

Because of the conductive boundary layer, the mathematical problem of temperature-field calculations becomes quite definite. The glacier outer surface (including the lower one) is divided into the accumulation region, where particle trajectories enter into the glacier, and the ablation region, where they leave the glacier. In the accumulation region, conductive heat transfer and water infiltration directly transmit the boundary condition to advective heat transfer, i.e. heat propagation is described by the boundary problem for the first-order hyperbolic equations (25), (26), (49), (50). In the ablation region covering also the bed at the melting point, the conductive boundary layer serves as a temperature damper. As this layer is not thick we have to solve the equation of thermal conductivity (near free surface) or thermal conduction-heat generation (near the bottom):

$$\frac{\partial T}{\partial \tau} = \delta_c \frac{\partial^2 T}{\partial \zeta^2} + \mathcal{N} f(\sigma) \ \sigma^2 \exp\left[-\kappa\left(\frac{1}{T}-1\right)\right], \tag{56}$$

under condition (25), one of conditions (26) (for the free surface) or (28) (for the bed) and under the condition of joining the solution with that of the problem for hyperbolic equations (25), (26), (49), (50) on the inner boundary of the conductive boundary layer:

$$[T]_{\zeta-d, \ d} = 0, \tag{57}$$

where the brackets represent a jump in the function.

If the temperature field varies slowly, when $\partial T/\partial t \to 0$, the model becomes quasi-stationary, i.e. time is contained in the equation of motion and heat transfer in a parametric form, but it is contained explicitly in the equation of motion of the upper glacier surface (27). The solution given holds good for the quasi-stationary conditions as well.

8. SOLUTION FOR ICE DIVIDES

The points on the rock under the summits of ice domes and saddles of diffluent glaciers and lines under ice divides are singular points and lines of the stress and velocity fields characterized by zero stress deviator $\sigma_{ik} = 0$ and velocity vector $\mathbf{v} = 0$ (if we discard the rate of subsidence due to bottom melting $v = v_\zeta = a_0$). On the normal to the bottom ice divide lines and planes connecting the singular points and lines with the upper surface, the tangential components of the stress deviator and the horizontal components of the velocity vector retain zero value

$$\sigma_{i\zeta} = 0, \qquad v_i = 0, \qquad i = \xi, \eta. \tag{58}$$

Hence, by virtue of Equation (42), we obtain an equation for the ice divide line

$$\zeta = Z - \frac{\phi_i}{\psi_i} \qquad (i = \xi, \eta).$$

If the bed is not horizontal and not flat ($\alpha_i \neq 0$, $k_i{}^\zeta \neq 0$) the ice divide can slightly deviate from the highest point (line) of the ice sheet or from the saddle of the diffluent glacier ($\beta_i = 0$).

The coordinate axis $O\zeta$ at the ice divides coincides with the third main axis of the stress surface (the only compression axis) and the shear stress intensity is determined, instead of by Equation (44), by the formula

$$\sigma = \{\tfrac{1}{6}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]\}^{\frac{1}{2}} = (\sigma_1{}^2 + \sigma_1\sigma_2 + \sigma_2{}^2)^{\frac{1}{2}}. \tag{59}$$

If the $O\xi$-axis subtends an angle ω with the horizontal direction of greatest curvature of the surface Ox_1, then

$$\left.\begin{aligned}
\sigma_1 &= \tfrac{1}{2}[\sigma_{\xi\xi}(1 + \cos 2\omega) + \sigma_{\eta\eta}(1 - \cos 2\omega)], \\
\sigma_2 &= \tfrac{1}{2}[\sigma_{\xi\xi}(1 - \cos 2\omega) + \sigma_{\eta\eta}(1 + \cos 2\omega)], \\
\sigma &= \tfrac{1}{2}[3(\sigma_{\xi\xi} + \sigma_{\eta\eta})^2 + (\sigma_{\xi\xi} - \sigma_{\eta\eta})^2 \cos^2 2\omega]^{\frac{1}{2}}.
\end{aligned}\right\} \tag{60}$$

But the stresses on the ice divides are much smaller than the usual values, therefore, the flow law of ice is almost linear (Shumskiy, 1975).

By integrating the equations of quasi-static equilibrium (19) at $\sigma_{i\zeta} = 0$ we obtain only the trivial result that the vertical normal component of the stress deviator is equal to the difference between the ice pressure and weight

$$\sigma_{\zeta\zeta} = \frac{\rho g H}{\sigma_0}[p - \cos \alpha (Z - \zeta)], \tag{61}$$

and cannot distinguish p from $\sigma_{\zeta\zeta}$. The problem is solved by calculating the derivative of the bed-parallel shear stresses with respect to ξ and η.

On differentiating the equation of incompressibility (36) with respect to ζ we get

$$\frac{\partial^2 v_\zeta}{\partial \zeta^2} = -\left(\frac{\delta_\xi}{A_\xi}\frac{\partial^2 v_\xi}{\partial \xi \partial \zeta} + \frac{\delta_\eta}{A_\eta}\frac{\partial^2 v_\eta}{\partial \eta \partial \zeta}\right). \tag{62}$$

On differentiating Equations (37) in the same way we see that the right-hand terms in Equation (62) can be expressed through the derivatives of the strain-rate tensor components with respect to ζ in two ways:

$$\left.\begin{aligned}
\frac{\delta_\xi}{A_\xi}\frac{\partial^2 v_\xi}{\partial \xi \partial \zeta} &= \frac{H}{v_0}\frac{\partial \tilde{\epsilon}_{\xi\xi}}{\partial \zeta} = 2\frac{H}{v_0}\frac{\delta_\xi}{A_\xi}\frac{\partial \tilde{\epsilon}_{\xi\xi}}{\partial \xi}, \\
\frac{\delta_\eta}{A_\eta}\frac{\partial^2 v_\eta}{\partial \eta \partial \zeta} &= \frac{H}{v_0}\frac{\partial \tilde{\epsilon}_{\eta\eta}}{\partial \zeta} = 2\frac{H}{v_0}\frac{\delta_\eta}{A_\eta}\frac{\partial \tilde{\epsilon}_{\eta\zeta}}{\partial \eta}.
\end{aligned}\right\} \tag{63}$$

Hence,

$$\frac{\partial^2 v_\zeta}{\partial \zeta^2} = -\frac{H}{v_0}\left(\frac{\partial \tilde{\epsilon}_{\xi\xi}}{\partial \zeta} + \frac{\partial \tilde{\epsilon}_{\eta\eta}}{\partial \zeta}\right) = -2\frac{H}{v_0}\left(\frac{\delta_\xi}{A_\xi}\frac{\partial \tilde{\epsilon}_{\xi\xi}}{\partial \xi} + \frac{\delta_\eta}{A_\eta}\frac{\partial \tilde{\epsilon}_{\eta\zeta}}{\partial \eta}\right). \tag{64}$$

From Equations (20), (42), (59) and (64) we obtain

$$\frac{\partial^2 v_\zeta}{\partial \zeta^2} = \frac{H}{v_0}\,\sigma_0\,\frac{\partial}{\partial \zeta}\left[f(\sigma)\,\sigma_{\zeta\zeta}f_1(T)\right] = -\frac{H}{v_0}\,\sigma_0\,\frac{\partial}{\partial \zeta}\left[f(\sigma)(\sigma_{\xi\xi}+\sigma_{\eta\eta})f_1(T)\right]$$

$$= 2\,\frac{H}{v_0}\,\rho g H f(\sigma)f_1(T)\left[\frac{\delta_\xi}{A_\xi}\frac{\partial\phi_\xi}{\partial\xi}+\frac{\delta_\eta}{A_\eta}\frac{\partial\phi_\eta}{\partial\eta}-\left(\frac{\psi_\xi}{A_\xi}\tan\gamma_\xi+\frac{\psi_\eta}{A_\eta}\tan\gamma_\eta\right)\right](\mathcal{Z}-\zeta), \qquad (65)$$

$$\sigma_{\zeta\zeta} = -(\sigma_{\xi\xi}+\sigma_{\eta\eta}) = \frac{\rho g H}{\sigma_0}\frac{2}{f(\sigma)f_1(T)}\int_{\mathcal{Z}}^{z-\zeta} f(\sigma)f_1(T)\left[\frac{\delta_\xi}{A_\xi}\frac{\partial\phi_\xi}{\partial\xi}+\frac{\delta_\eta}{A_\eta}\frac{\partial\phi_\eta}{\partial\eta}\right]-$$

$$-\left(\frac{\psi_\xi}{A_\xi}\tan\gamma_\xi+\frac{\psi_\eta}{A_\eta}\tan\gamma_\eta\right)\right](\mathcal{Z}-\zeta)\,\mathrm{d}(\mathcal{Z}-\zeta), \qquad (66)$$

$$v_\zeta(\zeta) = v_\zeta(0)+\frac{H}{v_0}\,\sigma_0\int_0^\zeta \sigma_{\zeta\zeta}(\zeta)f(\sigma)f_1(T)\,\mathrm{d}\zeta = v_\zeta(\mathcal{Z})-\frac{H}{v_0}\,\sigma_0\int_\zeta^z \sigma_{\zeta\zeta}(\zeta)f(\sigma)f_1(T)\,\mathrm{d}\zeta. \quad (67)$$

As the stress is small, in a first-order approximation

$$n = 1, \qquad f(\sigma) = K. \qquad (68)$$

In the non-isothermal case the solutions of Equations (46) and (67) are integrated, and they take the form

$$\frac{\partial T}{\partial\tau}+\frac{v_\zeta}{\delta_\xi}\frac{\partial T}{\partial\zeta} = \delta_c\frac{\partial^2 T}{\partial\zeta^2}+\mathcal{N}f(\sigma)(\sigma_1{}^2+\sigma_2{}^2+\sigma_3{}^2)\exp\left[-\kappa\left(\frac{1}{T}-1\right)\right], \qquad (69)$$

and generate a system of recurrent equations

$$\frac{\partial T^{(0)}}{\partial\tau}+\frac{v_\zeta}{\delta_\xi}\frac{\partial T^{(0)}}{\partial\zeta} = \mathcal{N}f(\sigma)(\sigma_1{}^2+\sigma_2{}^2+\sigma_3{}^2)\exp\left[-\kappa\left(\frac{1}{T^{(0)}}-1\right)\right], \qquad (70)$$

$$\frac{\partial T^{(i)}}{\partial\tau}+\frac{v_\zeta}{\delta_\xi}\frac{\partial T^{(i)}}{\partial\zeta} = \frac{\partial^2 T^{(i-1)}}{\partial\zeta^2}, \qquad i = 1, 2, 3, \dots . \qquad (71)$$

This system is solved similarly to Equations (49) and (50) under initial and boundary conditions (25) to (29).

Ice divides are surrounded by regions with a low-slope convex surface. The greater the horizontal size of the ice sheet, the larger these spaces. Transient conditions hold in this central region

$$\phi_\xi+\phi_\eta \approx \frac{\delta_\xi}{A_\xi}\frac{\partial\phi_\xi}{\partial\xi}+\frac{\delta_\eta}{A_\eta}\frac{\partial\phi_\eta}{\partial\eta},$$

or over the horizontal bed

$$\phi_\xi+\phi_\eta \approx k_\xi\zeta+k_\eta\zeta,$$

and ice subsidence under horizontal stress $\sigma_{\xi\xi}$, $\sigma_{\eta\eta}$ is gradually replaced by its centrifugal diffluence under shear stress $\sigma_{\zeta\zeta}$ parallel to the bed. In this region all stresses play more or less the same role, the system of equations of quasi-static equilibrium and incompressibility is not closed; each of these solutions taken separately is not acceptable and they have to be combined together.

MS. received 26 March 1975

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