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NOTE ON THE CONVOLUTION OF HARMONIC MAPPINGS

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Abstract

Dorff *et al.* ['Convolutions of harmonic convex mappings', *Complex Var. Elliptic Equ.* **57**(5) (2012), 489–503] formulated a question concerning the convolution of two right half-plane mappings, where the normalisation of the functions was considered incorrectly. In this paper, we reformulate the problem correctly and provide a solution to it in a more general form. We also obtain two new theorems which correct and improve related results.

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1. Introduction

Harmonic mappings are amongst the most studied areas of geometric function theory of one and several complex variables (see the monographs [2, 9] and the problems in [4, 16]). In [1], Aleman and Constantin used harmonic mappings to provide a new approach towards obtaining explicit solutions to the incompressible two-dimensional Euler equations. While the general solution is not available in explicit form, they used structural properties of the system to identify several families of explicit solutions. More recently, Constantin and Martin [6] used Lagrangian coordinates to investigate these solutions and improved the work of [1].

In this article, we consider complex-valued harmonic mappings f defined on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, which have a canonical representation of the form $f = h + \overline{g}$, where h and g are analytic in $\mathbb{D}$. This representation is unique with the condition $g(0) = 0$. The Jacobian J_f of $f = h + \overline{g}$ is given by $J_f(z) = |h'(z)|^2 - |g'(z)|^2$. According to the inverse mapping theorem, if the Jacobian of a C^1 mapping from $\mathbb{D}$

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to $\mathbb{C}$ is different from zero, then the function is locally univalent. The classical result of Lewy implies that the converse of this statement also holds for harmonic mappings. Thus, every harmonic function f on $\mathbb{D}$ is locally one-to-one and sense preserving on $\mathbb{D}$ if and only if $J_f(z) > 0$ in $\mathbb{D}$, that is, $|h'(z)| > |g'(z)|$ in $\mathbb{D}$. The condition $J_f(z) > 0$ is equivalent to the existence of an analytic function ω_f in $\mathbb{D}$ such that

$$|\omega_f(z)| < 1 \quad \text{for } z \in \mathbb{D},$$

where $\omega_f(z) = g'(z)/h'(z)$ is called the dilatation of f. When there is no risk of confusion, we use ω instead ω_f. Let $\mathcal{H} = \{f = h + \overline{g} : h(0) = g(0) = 0 \text{ and } h'(0) = 1\}$. The class $\mathcal{H}_0$ consists of those functions $f \in \mathcal{H}$ with $g'(0) = 0$.

The family of all sense-preserving univalent harmonic mappings in $\mathcal{H}$ will be denoted by $\mathcal{S}_H$ and $\mathcal{S}_H^0 = \mathcal{S}_H \cap \mathcal{H}_0$. Clearly, the familiar class $\mathcal{S}$ of normalised analytic univalent functions in $\mathbb{D}$ is contained in $\mathcal{S}_H^0$. The class $\mathcal{S}_H$ together with its geometric subclasses have been studied extensively by Clunie and Sheil-Small [5] and investigated subsequently by several others (see [3, 9] and the survey article [17]). In particular, we consider the convolution properties of the class $\mathcal{K}_H$ (respectively, $\mathcal{K}_H^0$) of functions $\mathcal{S}_H$ (respectively, $\mathcal{S}_H^0$) that map the unit disk $\mathbb{D}$ onto a convex domain.

1.1. Preliminaries and convex harmonic mappings. One of the important and interesting geometric subclasses of $\mathcal{S}_H$ is the class of univalent harmonic functions f for which the range $D = f(\mathbb{D})$ is convex in the direction α $(0 \le \alpha < \pi)$, meaning that the intersection of D with each line parallel to the line through 0 and $e^{i\alpha}$ is an interval or the empty set. Convex in the direction $\alpha = 0$ (respectively, $\alpha = \pi/2$) is referred to as convex in the horizontal (respectively, vertical) direction.

It is known [5] that a harmonic mapping $f = h + \overline{g}$ belongs to $\mathcal{K}_H^0 := \mathcal{K}_H \cap \mathcal{H}_0$ if and only if, for each $\alpha \in [0, \pi)$, the function $F = h - e^{2i\alpha} g$ belongs to $\mathcal{S}$ and is convex in the direction α.

Definition 1.1. A function $f = h + \overline{g} \in \mathcal{S}_H$ is said to be a *slanted half-plane mapping with* γ $(0 \le \gamma < 2\pi)$ if f maps $\mathbb{D}$ onto $H_\gamma := \{w : \mathrm{Re}(e^{i\gamma} w) > -(1 + a)/2\}$, where $-1 < a < 1$.

Using the shearing method due to Clunie and Sheil-Small [5] and the Riemann mapping theorem, it is easy to see that such a mapping has the form

$$h(z) + e^{-2i\gamma} g(z) = \frac{(1 + a)z}{1 - e^{i\gamma} z}.$$

Note that $h(0) = g(0) = h'(0) - 1 = 0$ and $g'(0) = e^{2i\gamma} a$. The class of all slanted half-plane mappings with γ is denoted by $\mathcal{S}(H_\gamma)$, and we denote by $\mathcal{S}^0(H_\gamma)$ the subclass of $\mathcal{S}(H_\gamma)$ with $a = 0$. Obviously, each $f \in \mathcal{S}(H_\gamma)$ (respectively, $\mathcal{S}^0(H_\gamma)$) belongs to the convex family $\mathcal{K}_H$ (respectively, $\mathcal{K}_H^0$). Evidently, there are infinitely many slanted half-plane mappings with a fixed γ. Functions $f \in \mathcal{S}(H_\gamma)$ with $\gamma = 0$ are usually referred to as *right half-plane mappings*, especially when $a = 0$. For example, if $f_0 = h_0 + \overline{g_0}$, where

$$h_0(z) = \frac{z - \frac{1}{2}z^2}{(1 - z)^2} = \frac{1}{2}\left(\frac{z}{1 - z} + \frac{z}{(1 - z)^2}\right) \tag{1.1}$$

and

$$g_0(z) = \frac{-\frac{1}{2}z^2}{(1-z)^2} = \frac{1}{2}\Big(\frac{z}{1-z} - \frac{z}{(1-z)^2}\Big), \tag{1.2}$$

then

$$h_0(z) + g_0(z) = \frac{z}{1-z},$$

showing that $f_0 = h_0 + \overline{g_0} \in \mathcal{S}^0(H_0)$ with the dilatation $\omega_0(z) = -z$. The function f_0 plays the role of extremal in many extremal problems for the convex family $\mathcal{K}_H^0$.

1.2. Convolution of harmonic mappings. For two harmonic mappings $f = h + \overline{g}$ and $F = H + \overline{G}$ in $\mathcal{H}$ with power series of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n} \overline{z}^n \quad \text{and} \quad F(z) = z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} \overline{B_n} \overline{z}^n,$$

we define the harmonic convolution (or Hadamard product) by

$$(f * F)(z) = (h * H)(z) + \overline{(g * G)(z)} = z + \sum_{n=2}^{\infty} a_n A_n z^n + \sum_{n=1}^{\infty} \overline{b_n B_n} \overline{z}^n.$$

Clearly, the space $\mathcal{H}$ is closed under the operation $*$, that is, $\mathcal{H} * \mathcal{H} \subset \mathcal{H}$. In the case of conformal mappings, the literature about convolution theory is exhaustive (see, for example, [19]). Unfortunately, most of these results do not necessarily carry over to the class of univalent harmonic mappings in $\mathbb{D}$. It is surprising that even if $f, F \in \mathcal{K}_H$, the convolution $f * F$ is not necessarily locally univalent in $\mathbb{D}$. Little was achieved on the convolution of harmonic univalent mappings until the recent progress initiated by Dorff [7].

1.3. Reformulation of the problem. In 2012, Dorff *et al.* [8] proved the following result.

THEOREM A [8, Theorem 2]. *If $f_k \in \mathcal{S}^0(H_{\gamma_k})$ for $k = 1, 2$, and $f_1 * f_2$ is locally univalent in $\mathbb{D}$, then $f_1 * f_2$ is convex in the direction $-(\gamma_1 + \gamma_2)$.*

By similar reasoning, we can generalise the result to the setting $\mathcal{S}(H_\gamma)$.

LEMMA 1.2. *If $f_k \in \mathcal{S}(H_{\gamma_k})$ for $k = 1, 2$ and $f_1 * f_2$ is locally univalent in $\mathbb{D}$, then $f_1 * f_2$ is convex in the direction $-(\gamma_1 + \gamma_2)$.*

Dorff *et al.* [8] also considered the situation where $f_1 * f_2$ is locally univalent and sense preserving.

THEOREM B [8, Theorem 4]. *Suppose that $f = h + \overline{g} \in \mathcal{S}^0(H_0)$ with dilatation $\omega(z) = (z + a)/(1 + az)$, where $a \in (-1, 1)$. Then $f_0 * f \in \mathcal{S}_H^0$ and is convex in the horizontal direction.*

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In 2010, Bshouty and Lyzzaik [4] published a collection of open problems and conjectures on planar harmonic mappings, proposed by many colleagues throughout the past quarter of a century. In particular, Dorff *et al.* [4, Problem 3.26(a)] posed the following open question.

PROBLEM C. Let $f = h + \overline{g} \in \mathcal{S}^0(H_0)$ with dilatation $\omega(z) = (z + a)/(1 + \overline{a}z)$, $|a| < 1$. Determine other values of $a \in \mathbb{D}$ for which the result of Theorem B holds.

Observe that for functions $f \in \mathcal{S}^0(H_0)$, the corresponding dilatation ω must satisfy the condition $\omega(0) = 0$ which forces $a = 0$ in Theorem B. Thus, Theorem B and Problem C are meaningful only when $\omega(z) = z$. In other words, the normalisation was not taken care of properly. It is necessary to reconsider the above problem in the setting $\mathcal{S}(H_\gamma)$, taking into account the correct normalisation condition.

PROBLEM 1.3. Let $f = h + \overline{g} \in \mathcal{S}(H_\gamma)$ such that

$$h(z) + e^{-2i\gamma} g(z) = \frac{(1 + a)z}{1 - e^{i\gamma} z} \quad \text{and} \quad \omega(z) = e^{2i\gamma} \frac{ze^{i\theta} + a}{1 + aze^{i\theta}}.$$

Determine the values of a and θ such that $f_0 * f \in \mathcal{S}_H^0$ is univalent in $\mathbb{D}$.

Without realising the error in Theorem B and Problem C, the present authors [13, 14] investigated the convolution properties of $f_0 = h_0 + \overline{g_0}$ with slanted half-plane mappings $f \in \mathcal{S}(H_\gamma)$ and obtained [13, Theorem 2.2] and [14, Theorem 1.3]. In a recent article, Liu and Ponnusamy [15] obtained the following corrected version of Theorem B.

THEOREM D [15, Theorem 1]. *Let $f = h + \overline{g} \in \mathcal{S}(H_0)$ with*

$$h + g = \frac{(1 + a)z}{1 - z} \quad \textit{and} \quad \omega(z) = \frac{z + a}{1 + az},$$

where $-1 < a < 1$, and $f_1 = h_1 + \overline{g_1} \in \mathcal{S}^0(H_0)$ with dilatation $\omega_1(z) = e^{i\theta} z$ ($\theta \in \mathbb{R}$). Then $f_1 * f$ is locally univalent and convex in the horizontal direction.

In this paper, we determine a family of values a and θ such that the condition in Problem 1.3 is satisfied (see Theorem 2.1 which corrects [13, Theorem 2.2] and [14, Theorem 1.3]). We also state and prove two new results, Theorems 2.2 and 2.3, which correct and improve some related results. Motivation and statements of these results are discussed in Section 2 and their proofs will be given in Section 3.

2. Main results

Our first result is the following theorem.

THEOREM 2.1. *Let $f = h + \overline{g} \in \mathcal{S}(H_\gamma)$ with*

$$h(z) + e^{-2i\gamma} g(z) = \frac{(1 + a)z}{1 - e^{i\gamma} z} \quad \textit{and} \quad \omega(z) = e^{2i\gamma} \frac{ze^{i\theta} + a}{1 + aze^{i\theta}},$$

where $\theta \in \mathbb{R}$ and $a \in (-1, 1)$. If one of the following conditions holds, then $f_0 * f \in \mathcal{S}_H^0$ and is convex in the direction $-\gamma$:

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(1) $\cos(\theta-\gamma) = -1$ *and* $-1/3 \le a < 1$*;*
(2) $-1 < \cos(\theta-\gamma) \le 1$ *and* $a^2 < 1/(5-4\cos(\theta-\gamma))$.

In order to state and prove our next two results, we consider the class $\mathcal{S}^0(\Omega_\beta)$ of functions $f \in \mathcal{S}_H^0$ such that f maps $\mathbb{D}$ onto the asymmetric vertical strip domains

$$\Omega_\beta = \left\{w : \frac{\beta-\pi}{2\sin\beta} < \operatorname{Re} w < \frac{\beta}{2\sin\beta}\right\},$$

where $0 < \beta < \pi$. Each $f = h + \overline{g} \in \mathcal{S}^0(\Omega_\beta)$ has the form

$$h(z) + g(z) = \psi(z), \quad \psi(z) = \frac{1}{2i\sin\beta}\log\left(\frac{1+ze^{i\beta}}{1+ze^{-i\beta}}\right).$$

Kumar *et al.* [10–12] considered mappings $F_a = H_a + \overline{G_a}$ with

$$H_a(z) + G_a(z) = \frac{z}{1-z} \quad \text{and} \quad \frac{G_a'(z)}{H_a'(z)} = \frac{a-z}{1-az} \tag{2.1}$$

and obtained convolution results of such mappings with mappings in $\mathcal{S}^0(H_0) \cup \mathcal{S}^0(\Omega_\beta)$. We recall one of their results.

Theorem E [10, Theorem 2.2]. *Let $f = h + \overline{g} \in \mathcal{S}^0(H_0)$ with $h + g = z/(1-z)$ and dilatation $\omega(z) = e^{i\theta}z^n$, where $\theta \in \mathbb{R}$ and n is a positive integer. If a satisfies $(n-2)/(n+2) \le a < 1$, then $f * F_a$ is convex in the horizontal direction.*

They gave similar results in [11, Theorem 2.4] and [12, Theorems 2.3, 2.5 and 2.6] for $f = h + \overline{g} \in \mathcal{S}^0(\Omega_\beta)$ with dilatation $\omega(z) = e^{i\theta}z^n$.

Again, from (2.1), we see that $H_a'(0) + G_a'(0) = 1$, which leads to $G_a'(0) = 0$ and contradicts the second condition in (2.1) unless $a = 0$.

In order to reformulate these results in the correct form, we may consider harmonic mappings $f_0^a = h_0^a + \overline{g_0^a}$ such that

$$h_0^a(z) + g_0^a(z) = \frac{(1+a)z}{1-z} \quad \text{and} \quad \frac{(g_0^a)'(z)}{(h_0^a)'(z)} = \frac{a-z}{1-az}.$$

In fact, we consider a more general form of f_0^a. Let $f_\gamma^a = h_\gamma^a + \overline{g_\gamma^a} \in \mathcal{S}(H_\gamma)$ with the dilatation

$$\omega(z) = -e^{2i\gamma}\frac{e^{i\gamma}z-a}{1-ae^{i\gamma}z} \quad \text{and} \quad h_\gamma^a(z) + e^{-2i\gamma}g_\gamma^a(z) = \frac{(1+a)z}{1-e^{i\gamma}z}.$$

Then a computation gives slanted half-plane mappings with γ as $f_\gamma^a = h_\gamma^a + \overline{g_\gamma^a}$, where

$$h_\gamma^a(z) = \frac{(1+a)I_\gamma(z) + (1-a)zI_\gamma'(z)}{2},$$
$$g_\gamma^a(z) = e^{2i\gamma}\frac{(1+a)I_\gamma(z) - (1-a)zI_\gamma'(z)}{2}$$

and

$$I_\gamma(z) = \frac{z}{1 - e^{i\gamma} z}.$$

Obviously, when $\gamma = 0$, f_γ^a coincides with the f_0^a that was defined above. For any $f = h + \overline{g} \in \mathcal{H}$, the above representation for f_γ^a quickly gives

$$(h_\gamma^a * h)(z) = \frac{(1+a)e^{-i\gamma} h(e^{i\gamma} z) + (1-a) z h'(e^{i\gamma} z)}{2}$$

and

$$(g_\gamma^a * g)(z) = e^{2i\gamma} \frac{(1+a)e^{-i\gamma} g(e^{i\gamma} z) - (1-a) z g'(e^{i\gamma} z)}{2}.$$

Then, by a computation, we see that the dilatation $\widetilde{\omega}$ of $f_\gamma^a * f$ is given by

$$\widetilde{\omega(z)} = e^{2i\gamma} \frac{2a g'(ze^{i\gamma}) - (1-a) e^{i\gamma} z g''(ze^{i\gamma})}{2h'(ze^{i\gamma}) + (1-a) z e^{i\gamma} h''(ze^{i\gamma})}. \tag{2.2}$$

For such slanted half-plane mappings f_γ^a, we obtain the following convolution theorems.

Theorem 2.2. *Let $f = h + \overline{g} \in \mathcal{S}^0(H_{\gamma_1})$ with*

$$h(z) + e^{-2i\gamma_1} g(z) = \frac{z}{1 - e^{i\gamma_1} z}$$

and dilatation $\omega(z) = e^{i\theta} z^n$, where n is a positive integer and $\theta \in \mathbb{R}$. If a satisfies $(n-2)/(n+2) \le a < 1$, then $f * f_\gamma^a$ is convex in the direction $-(\gamma_1 + \gamma)$.

Theorem 2.3. *Let $f = h + \overline{g} \in \mathcal{S}^0(\Omega_\beta)$ with dilatation $\omega(z) = e^{i\theta} z^n$, where $0 < \beta < \pi$, $\theta \in \mathbb{R}$ and n is a positive integer. If $a \in [(n-2)/(n+2), 1)$, then $f * f_\gamma^a$ is convex in the direction $-\gamma$.*

Remark 2.4. Theorem 2.2 is the corrected version of Theorem E. Theorem 2.3 is not only the corrected version of [11, Theorem 2.4] and [12, Theorems 2.3, 2.5 and 2.6] but also a generalisation of these theorems.

3. The proofs of Theorems 2.1, 2.2 and 2.3

3.1. Two lemmas. For the proof of Theorem 2.1, we need two lemmas.

Lemma 3.1. *Let $f = h + \overline{g} \in \mathcal{S}(H_\gamma)$ with dilatation $\omega(z) = g'(z)/h'(z)$ and*

$$h(z) + e^{-2i\gamma} g(z) = \frac{(1+a)z}{1 - e^{i\gamma} z}. \tag{3.1}$$

Then the dilatation $\widetilde{\omega}$ of $f_0 * f$ is

$$\widetilde{\omega}(z) = -z e^{-i\gamma} \left(\frac{\omega^2(z) + e^{2i\gamma}[\omega(z) - \frac{1}{2} z\omega'(z)] + \frac{1}{2} e^{i\gamma} \omega'(z)}{1 + e^{-2i\gamma}[\omega(z) - \frac{1}{2} z\omega'(z)] + \frac{1}{2} e^{-i\gamma} z^2 \omega'(z)} \right). \tag{3.2}$$

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PROOF. Assume that $f = h + \overline{g} \in \mathcal{S}(H_\gamma)$ with $\omega(z) = g'(z)/h'(z)$. Then

$$g'(z) = \omega(z)h'(z) \quad \text{and} \quad g''(z) = \omega'(z)h'(z) + \omega(z)h''(z).$$

Moreover, as h and g are related by the condition (3.1), the first equality gives

$$h'(z) = \frac{1+a}{(1+e^{-2i\gamma}\omega(z))(1-e^{i\gamma}z)^2} \tag{3.3}$$

and, therefore,

$$h''(z) = (1+a)\Big[\frac{-(1-e^{i\gamma}z)e^{-2i\gamma}\omega'(z) + 2(1+e^{-2i\gamma}\omega(z))e^{i\gamma}}{(1+e^{-2i\gamma}\omega(z))^2(1-e^{i\gamma}z)^3}\Big]. \tag{3.4}$$

From the representation of h_0 and g_0 given by (1.1) and (1.2),

$$(h_0 * h)(z) = \frac{h(z) + zh'(z)}{2} \quad \text{and} \quad (g_0 * g)(z) = \frac{g(z) - zg'(z)}{2}.$$

Therefore, as $f_0 * f = h_0 * h + \overline{g_0 * g}$, the dilatation $\widetilde{\omega}$ of $f_0 * f$ is given by

$$\widetilde{\omega}(z) = \frac{(g_0 * g)'(z)}{(h_0 * h)'(z)} = -\frac{zg''(z)}{2h'(z) + zh''(z)} = -\frac{z\omega'(z)h'(z) + \omega(z)zh''(z)}{2h'(z) + zh''(z)}. \tag{3.5}$$

In view of (3.3) and (3.4), after some computation, (3.5) takes the desired form. □

LEMMA 3.2. *Let $f = h + \overline{g} \in \mathcal{S}(H_\gamma)$ with*

$$h(z) + e^{-2i\gamma}g(z) = \frac{(1+a)z}{1-e^{i\gamma}z} \quad \textit{and} \quad \omega(z) = e^{2i\gamma}\frac{ze^{i\theta} + a}{1 + aze^{i\theta}},$$

where $\theta \in \mathbb{R}$. Then the dilatation $\widetilde{\omega}$ of $f_0 * f$ is given by

$$\widetilde{\omega}(z) = -ze^{3i\gamma}e^{2i\theta} \cdot \frac{(z+A)(z+B)}{(1+\overline{A}z)(1+\overline{B}z)},$$

where

$$t(z) = z^2 + \frac{3a+1}{2}e^{-i\theta}z + ae^{-2i\theta} + \frac{1-a}{2}e^{-i\gamma}e^{-i\theta}, \tag{3.6}$$

and $-A, -B$ are the two roots of $t(z) = 0$. (Here, A and B may be equal.)

PROOF. From the definition of ω, it follows that

$$\omega'(z) = e^{2i\gamma}e^{i\theta}\frac{1-a^2}{(1+aze^{i\theta})^2}$$

and thus, by a computation, the expression for $\widetilde{\omega}(z)$ in (3.2) takes the form

$$\widetilde{\omega}(z) = -ze^{3i\gamma}e^{2i\theta}\left(\frac{z^2 + \frac{3a+1}{2}e^{-i\theta}z + ae^{-2i\theta} + \frac{1-a}{2}e^{-i\gamma}e^{-i\theta}}{1 + \frac{3a+1}{2}e^{i\theta}z + ae^{2i\theta}z^2 + \frac{1-a}{2}e^{i\gamma}e^{i\theta}z^2}\right) = -ze^{3i\gamma}e^{2i\theta}\frac{t(z)}{t^*(z)},$$

where $t(z)$ is given by (3.6) and

$$t^*(z) = 1 + \frac{3a+1}{2}e^{i\theta}z + ae^{2i\theta}z^2 + \frac{1-a}{2}e^{i\gamma}e^{i\theta}z^2.$$

Suppose that $-A$ and $-B$ are the two roots of $t(z) = 0$. Again, by a simple calculation, it can be easily seen that

$$t^*(z) = z^2 \cdot \overline{t(1/\overline{z})} = z^2 \cdot \overline{(1/\overline{z} + A)(1/\overline{z} + B)} = (1 + \overline{A}z)(1 + \overline{B}z).$$

Therefore, the dilatation $\widetilde{\omega}$ of $f_0 * f$ has the desired form. □

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3.2. The proof of Theorem 2.1. By Lemma 1.2, it suffices to prove that $f_1 * f_2$ is locally univalent and sense preserving. So we only need to show that the dilatation $\widetilde{\omega}$ of $f_0 * f$ satisfies that

$$|\widetilde{\omega}(z)| < 1 \quad \text{for all } z \in \mathbb{D}.$$

Using the notation of Lemma 3.2, we write

$$t(z) = z^2 + \frac{3a+1}{2}e^{-i\theta}z + ae^{-2i\theta} + \frac{1-a}{2}e^{-i\gamma}e^{-i\theta} = (z+A)(z+B).$$

Case 1. $\cos(\theta-\gamma) = -1$ and $-1/3 \le a < 1$.

For this case,

$$t(z) = z^2 + \frac{3a+1}{2}e^{-i\theta}z + \frac{3a-1}{2}e^{-2i\theta} = (z+e^{-i\theta})\Big(z + \frac{3a-1}{2}e^{-i\theta}\Big),$$

so that the roots of $t(z)$ are $-A$ and $-B$, where $A = e^{-i\theta}$ and $B = ((3a-1)/2)e^{-i\theta}$. Moreover,

$$t^*(z) = (1+e^{i\theta}z)\Big(1 + \frac{3a-1}{2}e^{i\theta}z\Big)$$

and, from Lemma 3.2,

$$\widetilde{\omega}(z) = -ze^{3i\gamma}e^{2i\theta}\Big(\frac{z+e^{-i\theta}}{1+e^{i\theta}z}\Big)\Big(\frac{z+\frac{3a-1}{2}e^{-i\theta}}{1+\frac{3a-1}{2}e^{i\theta}z}\Big),$$

which clearly implies that $|\widetilde{\omega}(z)| < 1$ for $z \in \mathbb{D}$, since $-1/3 \le a < 1$.

Case 2. $-1 < \cos(\theta-\gamma) \le 1$ and $a^2 < 1/(5-4\cos(\theta-\gamma))$.

Let $a_0 = ae^{-2i\theta} + ((1-a)/2)e^{-i\gamma}e^{-i\theta}$, $a_1 = ((3a+1)/2)e^{-i\theta}$ and $a_2 = 1$. Then $t(z) = a_0 + a_1z + a_2z^2$. By a calculation,

$$|a_2|^2 - |a_0|^2 = \frac{1-a}{4}\cdot[a(5-4\cos(\theta-\gamma))+3] > 0$$

provided that $-1 < \cos(\theta-\gamma) \le 1$ and $a^2 < 1/(5-4\cos(\theta-\gamma))$. Now consider

$$t_1(z) =: \frac{\overline{a_2}t(z) - a_0t^*(z)}{z} = \frac{1-a}{4}\cdot[a(5-4\cos(\theta-\gamma))+3]\cdot(z-z_0),$$

where

$$z_0 = \frac{(3a+1)(e^{-i\gamma}-2e^{-i\theta})}{a(5-4\cos(\theta-\gamma))+3} =: \frac{u(a)}{v(a)},$$

with

$$u(a) = (3a+1)(e^{-i\gamma}-2e^{-i\theta}) \quad \text{and} \quad v(a) = a(5-4\cos(\theta-\gamma))+3.$$

A tedious calculation and the assumption yield

$$|v(a)|^2 - |u(a)|^2 = 4(1+\cos(\theta-\gamma))[1-a^2(5-4\cos(\theta-\gamma))],$$

which is positive by the assumption of Case 2. By using Cohn's rule (see, for instance, [18]), the conclusion follows in this case. This completes the proof.

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3.3. The proof of Theorem 2.2. As in the proof of the previous theorem, by Lemma 1.2, it suffices to show that the dilatation $\widetilde{\omega}$ of $f * f_\gamma^a$ given by (2.2) satisfies $|\widetilde{\omega}(z)| < 1$ for $z \in \mathbb{D}$. By (2.2), we see that $\widetilde{\omega(z)} = S(e^{i\gamma}z)$, where

$$S(z) = e^{2i\gamma}\Big(\frac{2ag'(z) - (1-a)zg''(z)}{2h'(z) + (1-a)zh''(z)}\Big).$$

Consequently, to complete the proof, it is enough to prove that $|S(z)| < 1$ for $z \in \mathbb{D}$. The assumption that

$$h(z) + e^{-2i\gamma_1}g(z) = \frac{z}{1 - e^{i\gamma_1}z}$$

and the dilatation $\omega(z) = e^{i\theta}z^n$ yield

$$g'(z) = e^{i\theta}z^n h'(z) \quad \text{and} \quad g''(z) = ne^{i\theta}z^{n-1}h'(z) + e^{i\theta}z^n h''(z)$$

so that $S(z)$, defined above, takes the form

$$S(z) = e^{(2\gamma+\theta)i}z^n\Big(\frac{2a - (1-a)n - (1-a)u[h(z)]}{2 + (1-a)u[h(z)]}\Big), \tag{3.7}$$

where

$$u[h(z)] =: z\frac{h''(z)}{h'(z)} = 2\frac{ze^{i\gamma_1}}{1 - ze^{i\gamma_1}} - n\frac{e^{(\theta-2\gamma_1)i}z^n}{1 + e^{(\theta-2\gamma_1)i}z^n}.$$

Let $X = \operatorname{Re} u[h(z)]$. Then $X > -1 - n/2$ and thus $2 + n + 2X > 0$ for all $z \in \mathbb{D}$.

By (3.7), it suffices to prove that, for all $z \in \mathbb{D}$,

$$T(z) = [2 + (1-a)X]^2 - [2a - (1-a)n - (1-a)X]^2 \geq 0.$$

By simplification, $T(z)$ reduces to

$$T(z) = (1-a)[2(1+a) - (1-a)n][2 + n + 2X].$$

Finally, because $1 - a > 0$ and $2 + n + 2X > 0$, we conclude that $T(z) \geq 0$ if and only if $a \in [(n-2)/(n+2), 1)$. The desired conclusion follows.

3.4. The proof of Theorem 2.3. Recall that $f * f_\gamma^a$ is convex in the direction $-\gamma$ provided $f * f_\gamma^a$ is locally univalent and sense preserving. By similar reasoning to that in the proof of Theorem 2.2, we only need to prove that, for all $z \in \mathbb{D}$,

$$T(z) = (1-a)[2(1+a) - (1-a)n][2 + n + 2X] \geq 0,$$

where $X = \operatorname{Re} u[h(z)]$ and $u[h(z)]$ is defined again by

$$u[h(z)] = z\frac{h''(z)}{h'(z)}.$$

On the other hand, the assumption

$$h(z) + g(z) = \frac{1}{2i\sin\beta}\log\Big(\frac{1 + ze^{i\beta}}{1 + ze^{-i\beta}}\Big)$$

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and the dilatation $\omega(z) = e^{i\theta}z^n$ yield

$$h'(z) = \frac{1}{(1+\omega(z))(1+ze^{i\beta})(1+ze^{-i\beta})},$$

so that

$$\begin{aligned} u[h(z)] &= -\frac{2(z+\cos\beta)z}{(1+ze^{i\beta})(1+ze^{-i\beta})} - \frac{z\omega'(z)}{1+\omega(z)} \\ &= -\frac{2(z+\cos\beta)z}{(1+ze^{i\beta})(1+ze^{-i\beta})} - n\frac{e^{i\theta}z^n}{1+e^{i\theta}z^n} \\ &= -\frac{e^{-i\beta}z}{1+e^{-i\beta}z} - \frac{e^{i\beta}z}{1+e^{i\beta}z} - n\frac{e^{i\theta}z^n}{1+e^{i\theta}z^n}. \end{aligned}$$

Clearly, the last equation implies that $2X > -2-n$ for $z \in \mathbb{D}$. Finally, because $1-a > 0$ and $(2+n)+2X > 0$, it follows that $T(z) \geq 0$ if and only if $a \in [(n-2)/(n+2), 1)$. The desired conclusion follows.

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