References

1. H. Dorrie, Fuss' problem of the chord-tangent quadrilateral, §39 in 100 *Great Problems of Elementary Mathematics*, Dover (1965) pp. 188-193.
2. F. G-M., *Exercises de geometrie* (6th edn, 1920), J. Gabay reprint, Paris (1991) pp. 837-839.
3. J. C. Salazar, Fuss' theorem, *Math. Gaz.*, **90** (July 2006), pp. 306-307.

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108.47 Thoughts on the Fermat point of a triangle

Introduction

Much has been written about the Fermat point of a triangle, and here we provide an alternative arrangement of the existing material which, we suggest, has certain advantages over the usual developments. First, a little of the history. According to [1], in 1638 Descartes invited Fermat to investigate the locus of a point X such that, for a given set $\{A, B, C, D\}$ of distinct points, the sum $XA + XB + XC + XD$ of the four distances is constant. Later, in 1643, Fermat asked Torricelli for the point X which minimises the sum of the distances $XA + XB + XC$ to three given points A, B and C. Subsequently, Torricelli found several solutions to the problem, and then, in 1659, his pupil Viviani published a solution. Briefly, there is a unique point P (now called the *Fermat*, or *Fermat-Torricelli*, point of the triangle $\triangle ABC$) which minimizes $XA + XB + XC$ over all points X in the plane. In fact, P must lie inside, or on the boundary of, $\triangle ABC$ for otherwise (by relabelling the triangle if necessary) it would lie on the opposite side of the line ℓ through A and B to the vertex C. Now let Q be the reflection of P in the line ℓ. Then ℓ is given by $\{X : XP = XQ\}$, and C lies on the same side of ℓ as Q does, namely in $\{X : XQ < XP\}$; thus $QC < PC$. Since A and B lie on ℓ, we have $QA = PA$, $QB = PB$, so that

$$QA + QB + QC < PA + PB + PC$$

which is a contradiction. Thus, as illustrated in Figure 1, P must lie in the closed triangle $\triangle ABC$. Further, a search through the literature shows that not only does the Fermat point P exist within the closed triangle $\triangle ABC$, it lies strictly inside this triangle if each angle of the triangle is less than 120°; otherwise, it lies at the vertex with the largest angle.

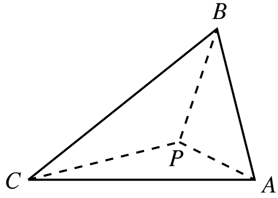

FIGURE 1: Fermat's problem

Next, the vast majority of the many solutions of Fermat's problem in the literature show that, in the cases where each angle of ΔABC is less than 120°, the Fermat point P is a point such that each two of the three rays PA, PB and PC make an angle of 120° with each other at P. If such a point exists, *irrespective of any knowledge of, or concern about, the Fermat point*, we shall call it an *equi–angular point* of the triangle. This raises the question, without any reference to the Fermat point, of *whether or not, given any triangle* ΔABC *there is a point* X, *inside* ΔABC, *such that each of the three rays* XA, XB *and* XC *make an angle of* 120° *with each other*. In effect, this independent question is equivalent to Fermat's question, but with an emphasis on angles rather than lengths and, we suggest, there are good reasons to consider this question seriously. Mathematics thrives on seeing ideas from different perspectives and certainly, at an elementary level, questions about angles are more likely to be easier to handle than a question about the sum of three square roots. Moreover, it is much easier to see that there is no equi-angular point inside a triangle that has an angle greater than 120° than it is to see that there is no Fermat point inside such a triangle, so the dichotomy (in the solution) between having, or not having, an angle of at least 120° is immediately more engaging in the amended problem than it is in Fermat's original problem. Finally, as we show in the next paragraph, our alternative problem arises in an (idealised) physical situation.

Suppose that we have a horizontal, triangular, table top with small holes at each of the vertices A, B and C of the table. Each of three equal masses is suspended on a piece of string (one for each mass), and these hang freely with one string passing through the hole at A, another at B, and the third at C. The strings are tied together (above the table) at P and the system allowed to find its equilibrium (if, indeed, there is one): see Figure 2. Physical considerations suggest that, at least for an acute–angled triangle, there will be a unique equilibrium position of P inside the triangle, and that at this position, each string will meet the other two strings at an angle of 120° (since otherwise the equal tensions in the three strings will exert a non-zero force on P). Thus this physical situation provides a strong, intuitive, expectation of the existence of an equi-angular point of an acute-angled triangle.

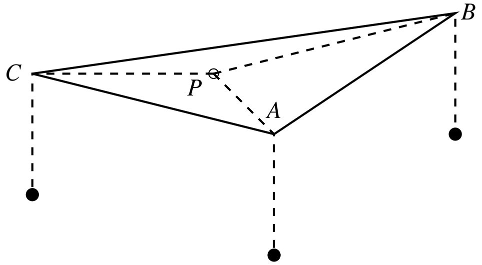

FIGURE 2: A triangular table with equal weights hanging from A, B and C

Now it is known that if there is an equi-angular point of a triangle, then it is necessarily a Fermat point of the triangle, and here we cannot do better than reproduce the elegant argument due to T. Andreescu and O. Mushkarov (see [2]). First, place the triangle ΔABC so that its equi-angular point P is at origin O. Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be the vectors from O to A, B and C, respectively, and let $\mathbf{n}_A$, $\mathbf{n}_B$ and $\mathbf{n}_C$ be the unit vectors in these same directions. Since P is the equi-angular point, $\mathbf{n}_A + \mathbf{n}_B + \mathbf{n}_C = \mathbf{0}$. Now consider any point X, and let $\mathbf{x}$ be the vector from O to X. Since

$$|\mathbf{a}| = \mathbf{a}\cdot\mathbf{n}_A = (|\mathbf{a} - \mathbf{x}| + \mathbf{x})\cdot\mathbf{n}_A \leqslant |\mathbf{a} - \mathbf{x}| + \mathbf{x}\cdot\mathbf{n}_A,$$

and similarly for $\mathbf{b}$ and $\mathbf{c}$, we find that

$$\begin{aligned} OA + OB + OC &= |\mathbf{a}| + |\mathbf{b}| + |\mathbf{c}| \\ &\leqslant |\mathbf{a} - \mathbf{x}| + |\mathbf{b} - \mathbf{x}| + |\mathbf{c} - \mathbf{x}| + \mathbf{x}\cdot(\mathbf{n}_A + \mathbf{n}_B + \mathbf{n}_C) \\ &= |\mathbf{a} - \mathbf{x}| + |\mathbf{b} - \mathbf{x}| + |\mathbf{c} - \mathbf{x}| \\ &= XA + XB + XC, \end{aligned}$$

so that O is the Fermat point of the triangle.

The equi-angular point of a triangle

We shall now give a direct proof (without any reference to the Fermat point) that *if each angle of* ΔABC *is less than* $120°$, *then there is an equi-angular point strictly inside the triangle*, and, as we shall see, this is easier than proving the existence of a Fermat point directly. We may suppose that the largest angle, say θ, of ΔABC is at A, so (after identifying θ with its measure in degrees, and ignoring the equilateral triangle) we have $60 < \theta < 120$.

Now consider Figure 3, where the line through T and V is the tangent to the circle at V, and note that $\angle TVU = 60°$. We now construct the two arcs that are equivalent to the circular arc VXU in the two cases when VU is each of the sides AB and AC of the triangle. Since $60 < \theta < 120$, these two circular arcs (based on the segments AB and AC) must intersect at a point P (other than A) inside the triangle and, by construction, $\angle APB = 120° = \angle APC$. It follows that $\angle BPC = 120°$ so P is an equi-angular point of the triangle.

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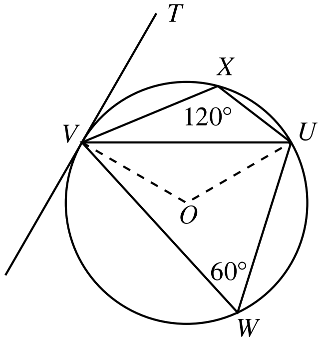

FIGURE 3: $\angle TVU = 60°$

It is worth noting that, in this proof, there is no need to consider a third arc, nor any of the more elaborate constructions that occur in proofs of the existence of a Fermat point, nor proofs that these constructions are valid.

Uniqueness

Our proof of the existence of an equi-angular point of a triangle makes it clear that such a point is unique. The Fermat point is also unique and here we prefer a proof based on an important mathematical idea (the convexity of the Euclidean distance function) rather than on a specific construction peculiar to this problem. Let $A = (0, 1)$ and $X(t) = (t, 0)$, where t is real. Then the distance between A and $X(t)$ is $\sqrt{1 + t^2}$, which we denote by $g(t)$. As $g''(t) > 0$, we see that g is a *convex* function, with $g'(t)$ an increasing function, so that, for any r and any positive s, we have

$$g(s) - g(r - s) = \int_{r-s}^{r} g'(t)\,dt < \int_{r}^{r+s} g'(t)\,dt = g(r + s) - g(r),$$

so $g(r - s) + g(r + s) > 2g(r)$. Now this argument does not depend on the particular choices of A and the real axis, and it shows that if A is a point not on a line ℓ, and if X, Y and Z are on ℓ, with Y the midpoint of the segment XZ, then $AX + AZ > 2AY$. It follows that there cannot be two Fermat points, say X and Z, for then

$$2(AX + BX + CX) = (AX + BX + CX) + (AZ + BZ + CZ) > 2(AY + BY + CY)$$

which contradicts the claim that X is a Fermat point.

Concluding remarks

We have shown that, by re-arranging the material in the usual approach, there is a simpler route to the Fermat point: first we define, and establish the existence of, an equi-angular point of the triangle; then we show that this is the unique Fermat point of the triangle. The literature on the Fermat point is extensive, so here we select a few items which may interest the reader. For more general information on the Fermat point see, for example, [1, 2, 3]. We

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also mention [4, pp. 354-358] (on Steiner's work which is particularly relevant to this article), [5] (for weighted distances), [6] (for the problem in higher dimensions), and [7] (for the problem in spherical geometry).

References

1. Fermat-Torricelli problem, Encyclopedia of Mathematics, http://encyclopediaofmath.org/index.php?title=Fermat-Torricelli_problem&oldid=49960
2. https://artofproblemsolving.com/wiki/index.php/Fermat_point
3. https://www.cut-the-knot.org/Generalization/fermat_point.shtml
4. R. Courant and H. Robbins, *What is Mathematics*?, OUP, London, Eleventh Edn. (1961).
5 Nguyen Minh Ha and N. Dergiades, An elementary proof of the generalised Fermat problem, *Math. Gaz.* **92** (March 2008) pp. 141-147.
6. L. Dalla, A note on the Fermat-Torricelli point of a d-simplex, *J. Geom.* **70** (2001) pp. 38-43.
7. Khuloud Ghalieh and Mowaffaq Hajja, The Fermat point of a spherical triangle, *Math. Gaz.* **80** (November 1996) pp. 561-564.

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