THE FREQUENCIES OF ALMOST PERIODIC SOLUTIONS OF ALMOST PERIODIC DIFFERENTIAL EQUATIONS

Dedicated to the memory of Hanna Neumann

G. C. O'BRIEN

(Received 30 June 1972)

Communicated by M. F. Newman

Abstract: Almost periodic solutions of a first order almost periodic differential equation in R^p are shown to have less than p basic frequencies additional to the basic frequencies of the almost periodic right hand of the equation.

1. Introduction

This paper is concerned with the frequency basis of an almost periodic solution of the differential equation

$$x' = \psi(x, t) \tag{1}$$

where $x' = (dx/dt)$, $x \in R^p$ and $\psi(x, t)$ is almost periodic in t, uniformly for x in any bounded subset of R^p. When $\psi(x, t)$ satisfies sufficient conditions for the solutions of (1) to be uniquely determined by their initial conditions we show that the codimension of the frequency basis of $\psi(x, t)$ with respect to the frequency basis of any almost periodic solution of (1) is less than p. This includes two special cases of particular interest. When equation (1) is autonomous the frequency basis of an almost periodic solution contains no more than $(p - 1)$ elements, and when $\psi(x, t)$ is periodic, and hence has only one element in its frequency basis, the frequency basis of any almost periodic solution contains no more than p elements.

This problem, with equation (1) Lipschitzian, has been considered recently by Cartwright, [2] and [3]. The autonomous and periodic cases are dealt with in the former paper while the latter is devoted to the general almost periodic case. Cartwright uses the techniques of topological dynamics (see [6]) and examines

The author thanks W. A. Coppel for his generous help in the preparation of this paper.

the properties of certain almost periodic flows. In [3], the technical difficulties encountered in using flows for non-autonomous equations as developed by Sell, [10] and [11], and others, are overcome by finding a particular flow associated with an almost periodic solution of (1) which allows the general case to be reduced to an application of the results of [2]. Further, Lerman and Shilnikov have recently announced some related results in [9]. They consider quasi-periodic solutions of (1) and construct an r-parameter family of solutions which form an integral manifold homeomorphic to the tube $T^r \times R$, where T^r is the r-dimensional torus.

Here, by extending a technique due to W. A. Coppel we are able to give a short, direct proof of Cartwright's results. Our method incidentally generalises the construction of Lerman and Shilnikov to the almost periodic case. It has the additional advantages of not using topological dynamics or the concept of translation numbers of an almost periodic function. Furthermore the means by which we avoid considering all equations in the closed hull of (1) can be applied to other problems associated with almost periodic solutions of equation (1).

In Section 2 we give precise definitions and a statement of the general theorem. Section 3 is devoted to the statement and proof of the theorem for the autonomous case. While the general result is not proved by reducing it to the autonomous case, as in [3], the simpler autonomous case provides a concise introduction to our method and shortens the proof of the general theorem in Section 4. We make some concluding remarks in Section 5.

2. Definitions and Theorem 1

Since translation numbers play no part in this paper we adopt Bochner's criterion as our definition of an almost periodic function. A continuous (vector valued) function $\phi(t)$ is said to be *almost periodic* if every sequence $\{h_v\}$ of real numbers contains a subsequence $\{k_v\}$ such that $\phi(t + k_v)$ converges uniformly on the whole real axis R.

The set of all almost periodic functions is a Banach space with respect to the uniform norm

$$\|\phi\| = \sup_{-\infty<t<\infty} |\phi(t)|.$$

The *closed hull* of an almost periodic function $\phi(t)$ is the set of all almost periodic functions $\gamma(t)$ such that $\phi(t + k_v) \to \gamma(t)$, with respect to the above norm, for some real sequence $\{k_v\}$, that is, uniformly on R.

To an almost periodic function $\phi(t)$ in R^p there corresponds a unique Fourier expansion

$$\phi(t) \sim \sum_{v=1} c_v e^{i\lambda_v t} \tag{2}$$

where the Fourier coefficients c_ν are non-zero vectors in R^p, the numbers λ_ν are real and distinct, and the number of terms is finite or countably infinite.

The numbers λ_ν are called the *frequencies*. A set of real numbers $\beta_1, \beta_2, \cdots$, is called a *basis* for the set of frequencies $\{\lambda_\nu\}$ or a *frequency basis* for the almost periodic function $\phi(t)$ if they form a basis for the vector space generated by the frequencies λ_ν over the field of rational numbers. Thus each λ_ν can be uniquely expressed in the form

$$\lambda_\nu = \sum_{\mu=1}^{p_\nu} r_{\nu\mu}\beta_\mu, \; r_{\nu\mu} \text{ rational, } r_{\nu p_\nu} \neq 0. \tag{3}$$

Each member of the closed hull of an almost periodic function $\phi(t)$ has the same frequencies as $\phi(t)$. An almost periodic function does not have a unique frequency basis, but for definiteness we select a *standard basis* as follows.

Let $\psi(t)$ be an almost periodic function with Fourier expansion

$$\psi(t) \sim \sum_{\nu=1} c_\nu^* e^{i\lambda_\nu^* t} \tag{4}$$

where the frequencies λ_ν^* are real and distinct and the coefficients c_ν^* are non-zero vectors in R^p. Put $\nu_1 = 1$ and $\beta_1^* = \lambda_{\nu_1}^*$. Let ν_2 be the least integer $\nu > 1$ such that λ_ν^* is not a rational multiple of β_1^* and put $\beta_2^* = \lambda_{\nu_2}^*$. In general, having defined $\beta_1^* = \lambda_{\nu_1}^*, \cdots, \beta_n^* = \lambda_{\nu_n}^*$, let ν_{n+1} be the least integer $\nu > \nu_n$ such that λ_ν^* is rationally independent of $\beta_1^*, \cdots, \beta_n^*$ and put $\beta_{n+1}^* = \lambda_{\nu_{n+1}}^*$. In this way we define a finite or infinite sequence $\{\beta_k^*\}$ of rationally independent frequencies such that every frequency λ_ν^* can be uniquely expressed in the form

$$\lambda_\nu^* = s_{\nu_1}^*\beta_1^* + s_{\nu_2}^*\beta_2^* + \cdots + s_{\nu q_\nu}^*\beta_{q_\nu}^* \tag{5}$$

with rational coefficients $s_{\nu\mu}^*$ and $s_{\nu q_\nu}^* \neq 0$. That is, we obtain a basis for the set of frequencies, each member of the basis being itself a frequency.

We extend the standard frequency basis for $\psi(t)$ to a frequency basis for $\phi(t)$ and define the *standard additional basis* of $\{\lambda_\nu\}$ with respect to $\{\lambda_\nu^*\}$ as follows. Let ν_1 be the least integer ν such that λ_ν is rationally independent of $\beta_1^*, \beta_2^*, \cdots$, and put $\beta_1 = \lambda_{\nu_1}$. Let ν_2 be the least integer $\nu > \nu_1$ such that λ_ν is rationally independent of $\beta_1, \beta_1^*, \beta_2^*, \cdots$ and put $\beta_2 = \lambda_{\nu_2}$. In general, having defined $\beta_1 = \lambda_{\nu_1}, \cdots, \beta_n = \lambda_{\nu_n}$ let ν_{n+1} be the least integer $\nu > \nu_n$ such that λ_ν is rationally independent of $\beta_1, \beta_2, \cdots, \beta_n, \beta_1^*, \beta_2^*, \cdots$ and put $\beta_{n+1} = \lambda_{\nu_{n+1}}$. In this way we define a finite or infinite sequence $\{\beta_k\}$ of rationally independent frequencies, which, together with the sequence $\{\beta_k^*\}$, form a basis for $\{\lambda_\nu\}$, $\{\lambda_\nu^*\}$. Moreover every frequency $\{\lambda_\nu\}$ can be uniquely expressed in the form

$$\lambda_\nu = \sum_{\mu=1}^{p_\nu} r_{\nu\mu}\beta_\mu + \sum_{\mu=1}^{p_\nu^*} r_{\nu\mu}^*\beta_\mu^* \tag{6}$$

https://doi.org/10.1017/S1446788700017092 Published online by Cambridge University Press

with rational coefficients $r_{\nu\mu}$, $r^*_{\nu\mu}$ and $r_{\nu p_\nu} \neq 0$ for each λ_ν which is not rationally dependent on the elements of the set $\{\beta^*_k\}$. It should be observed that the standard additional basis $\{\beta_k\}$ alone is not, in general, a basis for $\{\lambda_\nu\}$ without the elements of the basis $\{\beta^*_k\}$ for $\{\lambda^*_\nu\}$.

All of the above applies without change to almost periodic functions depending uniformly on parameters (see, for example, Corduneanu [5], Chapter II). We consider the differential equation (1) and suppose that $\psi(x,t)$ has the Fourier series

$$\psi(x,t) \sim \sum_{\nu=1} c^*_\nu(x) e^{i\lambda^*_\nu t} \tag{7}$$

where the coefficients $c^*_\nu(x)$ are not identically zero. We now state the main theorem.

THEOREM 1. *Let the differential equation*

$$x' = \psi(x,t),$$

where $x' = dx/dt$, $x \in R^p$ and $\psi(x,t)$ is almost periodic in t, uniformly for x in any bounded subset of R^p, satisfy conditions sufficient for its solutions to be uniquely determined by their initial values for all $(x,t) \in R^p \times R$. Let $\phi(t)$ be an almost periodic solution of this differential equation. Then the codimension of the frequency basis of $\psi(x,t)$ with respect to the frequency basis of $\phi(t)$ is at most $p-1$.

We prove this result in Section 4 by showing that the standard additional basis has less than p elements. Thus, without loss of generality we assume that $r_{\nu p_\nu} \neq 0$ in (6) for at least one value of ν.

3. The autonomous case

For the autonomous equation

$$x' = \psi(x)$$

where $x' = dx/dt$, $x \in R^p$, let $\psi(x)$ satisfy sufficient conditions for the solutions of this equation to be uniquely determined by their initial conditions. Let $\phi(t)$ be an almost periodic solution with Fourier series

$$\phi(t) \sim \sum_{\nu=1} c_\nu e^{i\lambda_\nu t}.$$

Then the set $\{\beta^*_k\}$ is empty and the standard additional basis is just the standard basis. Thus we need only show that this basis contains less than p elements.

Since the equation is autonomous translates of $\phi(t)$ are also solutions of the equation. We use this property to show that all of the functions $\phi(t, \tau_1, \tau_2, \cdots)$ with Fourier series

$$\phi(t, \tau_1, \tau_2, \cdots) \sim \sum_{\nu=1} c_\nu e^{i(r_{\nu 1}\beta_1\tau_1 + \dots + r_{\nu p}\,\beta_{p_\nu}\tau_{p_\nu})} e^{i\lambda_\nu t}$$

which are in the closed hull of $\phi(t)$, are also solutions. In particular, by considering the initial values of these solutions we then show that R^p contains a homeomorphic image of the "cube"

$$\mathscr{C}: 0 \leqq \tau_k \leqq \frac{\pi}{\beta_k}, \qquad k = 1, 2, \cdots, m,$$

for any $m \leqq \sup_\nu p_\nu$. Hence $\sup_\nu p_\nu \leqq p$. A further argument shows that equality cannot hold.

THEOREM 2. *Let $\phi(t)$ be an almost periodic solution of the autonomous differential equation*

$$x' = \psi(x), \tag{8}$$

where $x' = dx/dt$, $x \in R^p$, with Fourier expansion

$$\phi(t) \sim \sum_{\nu=1} c_\nu e^{i\lambda_\nu t}.$$

Let $\psi(x)$ satisfy sufficient conditions for the solutions of this equation to be uniquely determined by their initial values. Then for any sequence of real numbers $\{\tau_n\}$ there exists an almost periodic function $\phi(t, \tau_1, \tau_2, \cdots)$ in the closed hull of $\phi(t)$ with Fourier expansion

$$\phi(t, \tau_1, \tau_2, \cdots) \sim \sum_{\nu=1} c_\nu e^{i(r_{\nu 1}\beta_1\tau_1 + \dots + r_{\nu p_\nu}\beta_{p_\nu})} e^{i\lambda_\nu t},$$

where

$$\lambda_\nu = r_{\nu 1}\beta_1 + r_{\nu 2}\beta_2 + \cdots + r_{\nu p_\nu}\beta_{p_\nu}, \qquad \nu = 1, 2, \cdots.$$

Moreover, $\phi(t, \tau_1, \tau_2, \cdots)$ is also a solution of equation (8).

PROOF. For each positive integer N, we write $P_N = \max_{\nu=1,2,\dots,N} p_\nu$ and f_j for the lowest common multiple of the denominators of the rational numbers $r_{\nu j}$, $\nu = 1, 2, \cdots, N$. The numbers $[(2\pi/\beta_j)f_j]^{-1}$, $j = 1, 2, \cdots, P_N$ are linearly independent over the integers. Thus according to Kronecker's Theorem ([7], Theorem 444, p. 370) to each integer N and any $\delta > 0$ there corresponds a real number t_N and a set of integers k_j such that

$$\left| t_N - \tau_j - \frac{2\pi}{\beta_j} f_j k_j \right| < \delta, \qquad j = 1, 2, \cdots, P_N.$$

That is

$$|t_N - \tau_j| < \delta \left(\operatorname{mod} \frac{2\pi f_j}{\beta_j}\right), \quad j = 1, 2, \cdots, P_N.$$

https://doi.org/10.1017/S1446788700017092 Published online by Cambridge University Press

By selecting δ small enough we obtain

$$\left| e^{i(r_{v1}\beta_1\tau_1+\dots+r_{vp_v}\beta_{p_v}\tau_{p_v})} - e^{i\lambda_v t_N} \right| \leqq \frac{1}{N}, \quad v = 1, 2, \cdots, N.$$

Since these inequalities continue to hold when the sequence $\{t_N\}$ is replaced by any subsequence we can suppose that $\phi(t + t_N)$ converges uniformly for all real t as $N \to \infty$. The limit $\chi(t)$, say, is in the closed hull of $\phi(t)$ and has the Fourier series

$$\chi(t) \sim \sum_{v=1} a_v e^{i\lambda_v t}$$

where

$$a_v = \lim_{N\to\infty} c_v e^{i\lambda_v t_N} = c_v e^{i(r_{v1}\beta_1\tau_1+\dots+r_{vp_v}\beta_{p_v}\tau_{p_v})}.$$

It remains now to prove that

$$\chi(t) = \phi(t, \tau_1, \tau_2, \cdots)$$

is also a solution of (8). It is sufficient to show, that if the real sequence $\{t_N\}$ is such that $\xi_N = \phi(t_N) \to \xi$ as $N \to \infty$ then $\phi(t + t_N)$ converges uniformly to the solution $\omega(t)$ of (8) with initial value ξ at $t = 0$.

In fact $\phi(t + t_N)$ is the almost periodic solution of (8) which takes the value ξ_N for $t = 0$. Any subsequence of $\{t_N\}$ contains a further subsequence $\{t'_N\}$ such that $\phi(t + t'_N)$ converges uniformly for all real t. Because of the continuous dependence of solutions on initial values (Coppel [4], Theorem 3) the limit function must be $\omega(t)$. Since the limit is independent of the choice of subsequence the whole sequence $\phi(t + t_N)$ must converge uniformly to $\omega(t)$.

Thus if we denote by $\mathscr{M}$ the closure of the range of $\phi(t)$ there is a function in the closed hull of $\phi(t)$ which is a solution of (8) and has as initial value any point of $\mathscr{M}$. The next result shows that for integers $m \leqq \sup_v p_v$ there is a local homeomorphism between the initial values of the solutions $\phi(t, \tau_1, \tau_2, \cdots, \tau_m, 0, 0, \cdots)$ and the points $(\tau_1, \tau_2, \cdots, \tau_m)$.

LEMMA 1. *For any $m \leqq \sup_v p_v$ the set $\mathscr{M}$ contains a homeomorphic image of the cube*

$$\mathscr{C}: 0 \leqq \tau_k \leqq \pi/\beta_k, \qquad (k = 1, 2, \cdots, m).$$

PROOF. If

$$\phi(0, \tau'_1, \tau'_2, \cdots) = \phi(0, \tau''_1, \tau''_2, \cdots)$$

then the solutions $\phi(t, \tau'_1, \tau'_2, \cdots)$ and $\phi(t, \tau''_1, \tau''_2, \cdots)$ of (8) coincide because they have the same initial value. Therefore they have the same Fourier coefficients. That is,

$$(9) \qquad r_{\nu 1}\beta_1(\tau_1' - \tau_1'') + \cdots + r_{\nu p_\nu}\beta_{p_\nu}(\tau_{p_\nu}' - \tau_{p_\nu}'') \equiv 0 \pmod{2\pi}$$

for all ν. Since $\beta_n = \lambda_{\nu_n}$ is rationally independent of the preceding β's we have

$$r_{\nu_n n} = 1,\ r_{\nu_n k} = 0 \text{ for } k \neq n.$$

Therefore taking $\nu = \nu_n$ in the preceding congruence we obtain

$$\tau_n' - \tau_n'' \equiv 0 \pmod{2\pi/\beta_n}.$$

Let m be a fixed positive integer not exceeding the number of elements in the basis β_n, that is $m \leqq \sup_\nu p_\nu$. Suppose $0 \leqq \tau_k', \tau_k'' \leqq \pi/\beta_k$ for $k = 1, \cdots, m$ and $\tau_k' = \tau_k'' = 0$ for $k > m$. Then if (9) holds we must have

$$\tau_k' = \tau_k'' \text{ for } k = 1, \cdots, m.$$

Thus the map

$$T : (\tau_1, \cdots, \tau_m) \mapsto \phi(0, \tau_1, \tau_2, \cdots, \tau_m, 0, 0, \cdots)$$

of the "cube"

$$\mathscr{C} : 0 \leqq \tau_k \leqq \pi/\beta_k \qquad (k = 1, \cdots, m)$$

into R^p is one-to-one. That this map is also continuous may be seen in the following way.

For any $\varepsilon > 0$, there exists a trigonometric polynomial (e.g. the Bochner-Fejér polynomial, see Besicovitch [1], pp. 46–51)

$$p(t) = \sum_{\nu=1}^{M} \rho_\nu c_\nu e^{i\lambda_\nu t}$$

with rational coefficients ρ_ν depending only on ε and the λ's and satisfying $0 \leqq \rho_\nu \leqq 1$, such that

$$|\phi(t) - p(t)| \leqq \varepsilon \text{ for } -\infty < t < \infty.$$

Moreover if we replace $\phi(t)$ by any function $\chi(t)$ in its closed hull and the Fourier coefficients c_ν of $\phi(t)$ by the corresponding Fourier coefficients $\chi(t)$ then this inequality continues to hold. We can now choose $\delta = \delta(\varepsilon) > 0$, so that if

$$|\tau_k' - \tau_k''| \leqq \delta, \qquad (k = 1, \cdots, m)$$

the first M Fourier coefficients of $\phi(t, \tau_1', \tau_2', \cdots, \tau_m', 0, 0, \cdots)$ and $\phi(t, \tau_1'', \tau_2'', \cdots, \tau_m'', 0, 0, \cdots)$ satisfy

$$|c_\nu' - c_\nu''| \leqq \frac{\varepsilon}{M} \qquad (\nu = 1, 2, \cdots, M).$$

Then the corresponding trigonometric polynomials satisfy

$$|p'(t) - p''(t)| \leqq \varepsilon, \text{ for } -\infty < t < \infty,$$

and hence

$$|\phi(t, \tau_1', \tau_2', \cdots, \tau_m', 0, 0, \cdots) - \phi(t, \tau_1'', \tau_2'', \cdots, \tau_m'', 0, 0, \cdots)| \leqq 3\varepsilon \text{ for } -\infty < t < \infty.$$

Since the cube $\mathscr{C}$ is compact, the inverse map T^{-1} is also continuous. Therefore the set $\mathscr{M}$ contains a homeomorphic image of $\mathscr{C}$.

We can now proceed to the statement and the proof of Theorem 1 for the autonomous case.

THEOREM 3. *Let $\phi(t)$ be an almost periodic solution of the autonomous differential equation*

$$x' = \psi(x)$$

where $x' = dx/dt$, $x \in R^p$. Let $\psi(x)$ satisfy sufficient conditions for all of the solutions of this equatiion to be uniquely determined by their initial conditions. Then the frequency basis of $\phi(t)$ contains less than p elements.

PROOF. It follows from Lemma 1 that the dimensions† of the sets $\mathscr{C}$ and $\mathscr{M}$ satisfy the inequality

$$m = \dim \mathscr{C} \leqq \dim \mathscr{M}.$$

On the other hand $\dim \mathscr{M} \leqq p$ since $\mathscr{M} \subseteq R^p$. Therefore $m \leqq p$. Thus $\{\beta_k\}$ the frequency basis of $\phi(t)$ contains at most p elements.

It remains to show that this basis contains strictly less than p elements. Otherwise, if $\dim \mathscr{C} = p$, $\mathscr{C}$ contains a non empty open subset of R^p. Therefore $\mathscr{M}$ contains a non empty open subset G of R^p. Let η be a point on the boundary of the compact set $\mathscr{M}$ and let $\chi(t)$ be the solution of (8) which takes the value η for $t = 0$. Since the range of $\chi(t)$ is dense in $\mathscr{M}$ there is an $\xi_0 \in G$ and some $t_0 \in R$ such that $\chi(t_0) = \xi_0$. For any $\xi \in G$ let $\phi(t; \xi)$ denote the solution of (8) which takes the value ξ for $t = 0$. In particular $\phi(t; \xi_0) = \chi(t + t_0)$.The map $\xi \mapsto \phi(-t_0; \xi)$ is a homeomorphism of a neighbourhood of ξ_0 onto a neighbourhood of $\phi(-t_0, \xi_0) = \eta$. But this contradicts our choice of η as a boundary point of $\mathscr{M}$.

This completes the proof of Theorem 3.

4. The proof of Theorem 1

As in the preceding section we consider a whole family of solutions which belong to the closed hull of the given almost periodic solution. However, in this case these solutions are generated from the additional frequency basis of the solution. The following analogue of Theorem 2, while in principle unchanged,

† For the properties of dimension which we require see Hurewicz and Wallman [8], in particular Theorems 3.1, 4.1 and 4.3.

becomes more complicated to prove, since only particular sequences of translates of the given solution converge to solutions of the equation.

THEOREM 4. *Let $\phi(t)$ be an almost periodic solution of the almost periodic differential equation*

$$x' = \psi(x,t) \tag{10}$$

where $x' = dx/dt$, $x \in R^p$, $\psi(x,t)$ is almost periodic in t, uniformly with respect to x for x in any bounded subset of R^p and $\phi(t)$ and $\psi(x,t)$ have Fourier series

$$\phi(t) \sim \sum_{\nu=1} c_\nu e^{i\lambda_\nu t}$$

and

$$\psi(x,t) \sim \sum_{\nu=1} c^*_\nu(x) e^{i\lambda^*_\nu t},$$

where

$$\lambda_\nu = \sum_{\mu=1}^{p_\nu} r_{\nu\mu}\beta_\mu + \sum_{\mu=1}^{p^*_\nu} r^*_{\nu\mu}\beta^*_\mu ,$$

$$\lambda^*_\nu = \sum_{\mu=1}^{q_\nu} s^*_{\nu\mu}\beta^*_\mu ,$$

$\{\beta^*_k\}$ is the standard frequency basis of $\psi(x,t)$, and $\{\beta_k\}$ is the standard additional basis of $\phi(t)$. Let $\psi(x,t)$ satisfy sufficient conditions for the solutions of (10) *to be uniquely determined by their initial values. Then, for any real numbers $\{\tau_n\}$ there exists an almost periodic function $\phi(t,\tau_1,\tau_2,\cdots)$ in the closed hull of $\phi(t)$ with Fourlier expansion*

$$\phi(t,\tau_1,\tau_2,\cdots) \sim \sum_{\nu=1} c_\nu e^{i(r_{\nu 1}\beta_1\tau_1 + \dots + r_{\nu p_\nu}\beta_{p_\nu}\tau_{p_\nu})} e^{i\lambda_\nu t}.$$

Moreover $\phi(t,\tau_1,\tau_2,\cdots)$ is also a solution of (10).

PROOF. Write $P_N = \max_{\nu=1,2,\dots,N} p_\nu$, $P^*_N = \max_{\nu=1,2,\dots,N}\{p^*_\nu, q_\nu\}$, f_j for the lowest common multiple of the quotients of the rational numbers $r_{\nu j}$, $\nu = 1,2,\cdots,N$ and g_j for the lowest common multiple of the denominators of the rational numbers $r^*_{\nu j}$, $s^*_{\nu j}$, $\nu = 1,2,\cdots,N$. Set

$$n = P_N + P^*_N$$

$$\alpha_i = \begin{cases} \tau_i & i = 1,2,\cdots,P_N \\ 0 & i = P_N+1,\cdots,P_N+P^*_N \end{cases}$$

https://doi.org/10.1017/S1446788700017092 Published online by Cambridge University Press

$$q_i = \begin{cases} \dfrac{2\pi}{\beta_i} f_i & i = 1, 2, \cdots, P_N \\ \dfrac{2\pi}{\beta^*_{i-PN}} g_{i-P_N} & i = P_N + 1, \cdots, P_N + P^*_N. \end{cases}$$

The numbers $q_1^{-1}, q_2^{-1}, \cdots, q_n^{-1}$ are linearly independent over the integers. From Kronecker's Theorem ([7], Theorem 444, p. 370) corresponding to any $\delta > 0$ there exist integers $k_1, \cdots, k_n$ and a real number ξ such that

$$|\xi - \alpha_i - k_i q_i| < \delta, \qquad i = 1, 2, \cdots, n.$$

Writing $\xi = t_N$ and $k_i' = k_{i+P_N}$ we have

$$\left| t_N - \tau_i - \frac{2\pi}{\beta_i} f_i k_i \right| < \delta, \text{ for } i = 1, 2, \cdots, P_N,$$

and

$$\left| t_N - \frac{2\pi}{\beta_i^*} g_i k_i' \right| < \delta, \text{ for } i = 1, 2, \cdots, P_N^*.$$

That is

$$|t_N - \tau_i| < \delta \left(\operatorname{mod} \frac{2\pi}{\beta_i} f_i \right), \text{ for } i = 1, 2, \cdots, P_N,$$

and

$$|t_N| < \delta \left(\operatorname{mod} \frac{2\pi}{\beta^*} g_i \right), \text{ for } i = 1, 2, \cdots, P_N^*.$$

By combining these inequalities, and by selecting δ small enough

(11) $$\left| e^{i(r_{\nu 1}\beta_1 \tau_1 + \cdots + r_{\nu p_\nu}\beta_{p_\nu}\tau_{p_\nu}) - i\lambda_\nu t_N} - 1 \right| < \frac{1}{N}$$

for $\nu = 1, 2, \cdots, N$

and

(12) $$\left| e^{i\lambda_\nu^* t_N} - 1 \right| < \frac{1}{N} \text{ for } \nu = 1, 2, \cdots, N.$$

Since these inequalities continue to hold when the sequence $\{t_N\}$ is replaced by any subsequence we can suppose that both $\psi(x, t + t_N)$ and $\phi(t + t_N)$ converge uniformly for all real t as $N \to \infty$. The function $\psi(x, t + t_N)$ converges to $\psi(x, t)$. For, the Fourier series of $\psi(x, t + t_N)$ is

$$\sum_{\nu=1} c_\nu^*(x) e^{i\lambda_\nu^* t_N} e^{i\lambda_\nu^* t}$$

and the Fourier series of $\lim_{N \to \infty} \psi(x, t + t_N)$ is

$$\sum_{\nu=1} c_\nu^*(x) e^{i\lambda_\nu^* t}$$

therefore, since the Fourier series of an almost periodic function uniquely determines the function

$$\psi(x, t+t_N) \to \psi(x, t), \text{ uniformly.}$$

Similarly

$$\phi(t+t_N) \to \chi(t)$$

uniformly, where $\chi(t)$ has the Fourier series

$$\chi(t) \sim \sum_{\nu=1} a_\nu e^{i\lambda_\nu t},$$

$$a_\nu = \lim_{N\to\infty} c_\nu e^{i\lambda_\nu t_N} = c_\nu e^{i(r_{\nu 1}\beta_1\tau_1 + \dots + r_{\nu p_\nu}\beta_{p_\nu}\tau_{p_\nu})}.$$

Evidently we have

$$\phi(t, 0, 0, \cdots) = \phi(t).$$

Let $\{t_N\}$ be any real sequence such that $\psi(x, t+t_N) \to \psi(x, t)$ uniformly as $N \to \infty$. We will show that if $\xi_N = \phi(t_N) \to \xi$ as $N \to \infty$ then $\phi(t+t_N)$ converges uniformly on R to $\omega(t)$ where $\omega(t)$ is the solution of (10) such that $\omega(0) = \xi$.

In fact $\phi(t+t_N)$ is the almost periodic solution of

$$x' = \psi(x, t+t_N)$$

which takes the value ξ_N for $t = 0$. It follows from a standard theorem (Coppel [4], Theorem 3) that $\phi(t+t_N)$ converges uniformly on every compact interval of R. But any subsequence $\{t'_N\}$ of $\{t_N\}$ contains a further subsequence $\{t''_N\}$ such that $\phi(t+t_N)$ converges uniformly for all real t and since the limit $\omega(t)$ is independent of the choice of subsequences the whole sequence $\phi(t+t_N)$ must converge to $\omega(t)$ uniformly on R.

Hence the functions $\phi(t, \tau_1, \tau_2, \cdots)$ are all almost periodic solutions of (10). Observe that these functions in general form a smaller set than in the autonomous case, since we only admit functions obtained from $\phi(t)$ by sequences $\{t_N\}$ for which $\psi(x, t+t_N) \to \psi(x, t)$ uniformly with respect to x, for x in any bounded subset of R^p.

As in Section 3 we denote by $\mathscr{M}$ the closure of the range of $\phi(t)$. Then the initial values of the solutions $\phi(t, \tau_1, \tau_2, \cdots)$ for all sequences $(\tau_1, \tau_2, \cdots)$ form a subset of $\mathscr{M}$. The statement and proof of Lemma 1 now hold for this situation.

We now proceed to the proof of Theorem 1. We obtain from Lemma 1 that

$$m = \dim \mathscr{C} \leqq \dim \mathscr{M}.$$

On the other hand $\dim \mathscr{M} \leqq p$, since $\mathscr{M} \subseteq R^p$. Therefore

$$m \leqq p.$$

It remains to show that $m \neq p$. The argument of Theorem 3 applies unchanged. This completes the proof of Theorem 1.

5. Conclusion

In the article [2] Cartwright shows that in the autonomous case when $p = n - 1$ the frequency basis is an integral basis. Thus

$$\lambda_\nu = n_{\nu 1}\beta_1 + n_{\nu 2}\beta_2 + \cdots + n_{\nu, p-1}\beta_{p-1},$$
$$\nu = 1, 2, \cdots, \text{ with } n_{\nu\mu} \text{ integers,}$$

and the almost periodic solution $\phi(t)$ is quasiperiodic. In other words,

$$\phi(t) = \Phi(t, t, \cdots, t)$$

where $\Phi(t_1, t_2, \cdots, t_{p-1})$ is periodic in t_i with period $2\pi/\beta_i$, $i = 1, 2, \cdots, p-1$. Since the proof of the result for the almost periodic equation (8) is reduced to the autonomous case this implies that in [3] when $p = n - 1$ the additional frequency basis is integral and we have for the frequencies of the almost periodic solution,

$$\lambda_\nu = \sum_{\mu=1}^{p-1} n_{\nu\mu}\beta_\mu + \sum_{\mu=1}^{p_\nu^*} r_{\nu\mu}^*\beta_\mu^*.$$

This result has an independent proof in [2] which can be applied directly here.

If $\psi(x, t)$ satisfies a Lipschitz condition with Lipschitz constant independent of t then every $\Psi(x, t)$ in the closed hull of $\psi(x, t)$ satisfies a Lipschitz condition and the solutions of each differential equation

$$x' = \Psi(x, t)$$

are uniquely determined by their initial values. Cartwright's version of Theorem 1 assumes that $\psi(x, t)$ is Lipschitzian, but not in an essential way.

References

[1] A. S. Besicovitch, *Almost Periodic Functions* (Dover Publ. Inc. 1954).

[2] M. L. Cartwright, 'Almost periodic flows and the solutions of differential equations', *Proc. London Math. Soc.* (3) 17 (1967), 355–380. Corrigenda: *Proc. London Math. Soc.* (3) 17 (1967), 768.

[3] M. L. Cartwright, 'Almost periodic differential equations and almost periodic flows', *J. Differential Equations* 5 (1969), 167–181.

[4] W. A. Coppel, *Stability and Asymptotic Behaviour of Ordinary Differential Equations* (D. C. Heath, Boston, 1965).
[5] C. Corduneanu, *Almost Periodic Functions* (Translated by Gitta Berstein and Eugene Tomer. Interscience [John Wiley and Sons], New York, London, Sydney, Toronto, 1968).
[6] Walter Helbig Gottschalk and Gustav Arnold Hedlund, *Topological Dynamics* (Colloquium Publ. 36. Amer. Math. Soc., Providence, Rhode Island, 1955).
[7] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 2nd ed. (Clarendon Press, Oxford, 1945).
[8] Witold Hurewicz and Henry Wallman, *Dimension Theory* (Princeton Univ. Press, Princeton, 1948).
[9] L. M. Lerman and L. P. Shilnikov, 'On the existence and stability of almost periodic tubes (Russian)', *Proc. Fifth International Conf. on Nonlinear Oscillations* (Kiev 1969) 2, 292–297, Izd. Inst. Mat. USSR, Kiev, 1970.
[10] George R. Sell, 'Non autonomous differential equations and topological dynamics. I. The basic theory', *Trans. Amer. Math. Soc.* 127 (1967), 241–262.
[11] George R. Sell, 'Non autonomous differential equations and topological dynamics. II. Limiting Equations', *Trans. Amer. Math. Soc.* 127 (1967), 263–283.

Department of Economics
La Trobe University
Bundoora, 3083
Australia

https://doi.org/10.1017/S1446788700017092 Published online by Cambridge University Press