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Multifractal analysis of homological growth rates for hyperbolic surfaces

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Dedicated to Professor Masato Tsujii on the occasion of his 60th birthday

Abstract. We perform a multifractal analysis of homological growth rates of oriented geodesics on hyperbolic surfaces. Our main result provides a formula for the Hausdorff dimension of level sets of prescribed growth rates in terms of a generalized Poincaré exponent of the Fuchsian group. We employ symbolic dynamics developed by Bowen and Series, ergodic theory and thermodynamic formalism to prove the analyticity of the dimension spectrum.

Key words: Fuchsian group, Bowen–Series map, thermodynamic formalism, multifractal analysis
2020 Mathematics Subject Classification: 37C45, 37D35, 37F32, 37D25 (Primary); 37E05, 37D40 (Secondary)

1. *Introduction*

A Fuchsian group is a discrete group of orientation-preserving isometries acting in the Poincaré disk model $(\mathbb{D}, d)$ of hyperbolic space. Fuchsian groups play an important role in the uniformization of hyperbolic surfaces and geometric group theory. For the background on Fuchsian groups, we refer the reader to [3].

Throughout this paper, G denotes a finitely generated non-elementary Fuchsian group. Having fixed a finite set of generators of G, for $g \in G$, we denote by $|g|$ the minimal number of generators needed to represent g, called the word length of g. It follows from the triangle inequality that there exists $\alpha_+ > 0$ such that $d(0, g0) \le \alpha_+|g|$ for all $g \in G$. If $\mathbb{D}/G$ has no cusps, the Švarc–Milnor lemma implies the existence of $\alpha_- > 0$ such that $d(0, g0) \ge \alpha_-|g|$. If $\mathbb{D}/G$ has cusps, there exists $C > 0$ such that

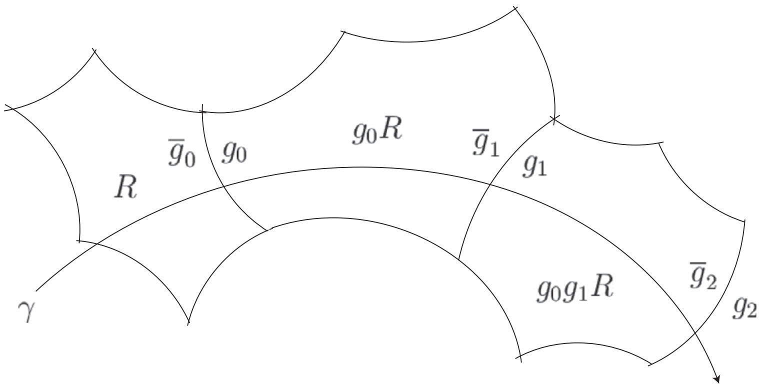

FIGURE 1. An oriented geodesic γ crossing copies of the fundamental domain R.

$d(0, g0) \geq 2 \log |g| - C$ by [10]. The complexity of the action of G is reflected in the fact that the growth rate of $d(0, g0)/|g|$, as $|g| \to \infty$, takes on uncountably many values, and rates of convergence are not uniform. In this paper, we perform a multifractal analysis of this growth rate along oriented geodesics, which are circular arcs orthogonal to the boundary $\mathbb{S}^1$ of $\mathbb{D}$.

Let $R \subset \mathbb{D}$ be a convex, locally finite fundamental domain for G which contains 0 in its interior [3]. The finite set of side-pairings of R is denoted by G_R and defines a symmetric set of generators of G. We call R *admissible* if R has *even corners* [7, 38] and satisfies a technical condition. We refer the reader to §2.1 for the details. Let $\mathscr{R}$ denote the set of oriented complete geodesics γ joining two points in $\mathbb{S}^1$ and intersecting the interior of R. If $\gamma \in \mathscr{R}$ cuts through the copies $R, g_0R, g_0g_1R, \ldots$ of R, with $g_i \in G_R$ and $i = 0, 1, \ldots \in \mathbb{N}$, then $g_0, g_1, g_2, \ldots$ is called the *cutting sequence* of γ (see Figure 1). By slightly perturbing geodesics passing through a vertex of R, we will define for each $\gamma \in \mathscr{R}$ a unique finite or infinite cutting sequence in §2.1. For $\gamma \in \mathscr{R}$ with the cutting sequence $g_0, g_1, \ldots$ of length at least $n \geq 1$, we define

$$t_n(\gamma) = d(0, g_0g_1 \cdots g_{n-1}0),$$

and call $t_n(\gamma)/n$ the homological growth rate of γ [17]. Since R has even corners, $g_0 \ldots g_{n-1}$ has word length n with respect to G_R (see Proposition 2.1). We denote by $\Lambda = \Lambda(G)$ the limit set of G, and by $\Lambda_c = \Lambda_c(G)$ the conical limit set of G. We have $\Lambda_c \subset \Lambda$ and by a result of Beardon and Maskit [4], $\Lambda \setminus \Lambda_c$ is equal to the countable set of parabolic fixed points of elements of G. It turns out in Lemma 2.2 that $\gamma \in \mathscr{R}$ has an infinite cutting sequence if and only if its positive endpoint γ^+ belongs to Λ_c. For $\alpha \geq 0$, we define the *level set*

$$\mathscr{H}(\alpha) = \left\{\xi \in \Lambda_c \colon \text{there exists } \gamma \in \mathscr{R} \text{ such that } \gamma^+ = \xi \text{ and } \lim_{n\to\infty} \frac{t_n(\gamma)}{n} = \alpha\right\}.$$

Since the level sets are pairwise disjoint by Remark 2.11, we have a multifractal decomposition of the conical limit set

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$$\Lambda_c = \left(\bigcup_{\alpha \geq 0} \mathscr{H}(\alpha) \right) \cup \mathscr{H}_{\rm ir},$$

where $\mathscr{H}_{\rm ir}$ denotes the set of $\xi \in \Lambda_c$ for which $t_n(\gamma)/n$ does not converge as $n \to \infty$ for any $\gamma \in \mathscr{R}$ whose positive endpoint is ξ.

If G is of the first kind, that is, $\Lambda = \mathbb{S}^1$, then there exists a constant $\alpha_G \geq 0$ such that $\mathscr{H}(\alpha_G)$ has full Lebesgue measure in $\mathbb{S}^1$. We refer the reader to §A.3 for a proof of this claim and more information on α_G. For a description of the fine structure of Λ, it is necessary to analyse other level sets which are negligible in terms of the Lebesgue measure. Let $\dim_{\rm H}$ denote the Hausdorff dimension on $\mathbb{S}^1$, and for $\alpha \geq 0$, let

$$b(\alpha) = \dim_{\rm H} \mathscr{H}(\alpha).$$

Information on the complexity of the limit set is encoded in the function $\alpha \mapsto b(\alpha)$ called the *spectrum of homological growth rates*, or simply the $\mathscr{H}$-spectrum. Note that the $\mathscr{H}$-spectrum depends on the choice of the fundamental domain R.

The thermodynamic formalism gives an access to the description of the $\mathscr{H}$-spectrum. We define

$$\delta_G = \dim_{\rm H} \Lambda.$$

It is well known [2, 25], [39, Corollary 26] that δ_G is equal to the *Poincaré exponent of G* given by

$$\inf \left\{ \beta \geq 0 \colon \sum_{g \in G} \exp(-\beta d(0, g0)) < +\infty \right\}.$$

Imitating this style, following [21, Theorem 2.1.3], we introduce a *generalized Poincaré exponent* at an inverse temperature $\beta \in \mathbb{R}$ by

$$P(\beta) = \inf \left\{ t \in \mathbb{R} \colon \sum_{g \in G} \exp(-\beta d(0, g0) - t|g|) < +\infty \right\}.$$

We call the function $\beta \in \mathbb{R} \mapsto P(\beta)$ the *geometric pressure function of G with respect to R*, or simply the *pressure function*. The negative convex conjugate of P is for $\alpha \in \mathbb{R}$ given by

$$P^*(-\alpha) = \inf\{\alpha\beta + P(\beta) \colon \beta \in \mathbb{R}\}.$$

We set

$$\alpha_+ = \sup_{\gamma \in \mathscr{R}, \gamma^+ \in \Lambda_c} \limsup_{n \to \infty} \frac{t_n(\gamma)}{n} \quad \text{and} \quad \alpha_- = \inf_{\gamma \in \mathscr{R}, \gamma^+ \in \Lambda_c} \liminf_{n \to \infty} \frac{t_n(\gamma)}{n},$$

and define the *freezing point* by

$$\beta_+ = \sup\{\beta \in \mathbb{R} \colon P(\beta) > -\alpha_- \beta\}.$$

MAIN THEOREM. *Let G be a finitely generated non-elementary Fuchsian group with an admissible fundamental domain R.*

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(a) *We have* $\alpha_- < \alpha_+$, *and the level set* $\mathscr{H}(\alpha)$ *is non-empty if and only if* $\alpha \in [\alpha_-, \alpha_+]$. *The* $\mathscr{H}$*-spectrum is continuous on* $[\alpha_-, \alpha_+]$, *analytic on* (α_-, α_+) *and for each* $\alpha \in [\alpha_-, \alpha_+] \setminus \{0\}$, *we have*

$$b(\alpha) = \frac{P^*(-\alpha)}{\alpha}.$$

Moreover, the $\mathscr{H}$*-spectrum attains its maximum* δ_G *at a unique* $\alpha_G \in [\alpha_-, \alpha_+)$, *is strictly increasing on* $[\alpha_-, \alpha_G]$ *and strictly decreasing on* $[\alpha_G, \alpha_+]$, *and* $\lim_{\alpha \nearrow \alpha_+} b'(\alpha) = -\infty$. *If G has no parabolic element, then* $\alpha_G > \alpha_- > 0$ *and* $\lim_{\alpha \searrow \alpha_-} b'(\alpha) = +\infty$. *If G has a parabolic element, then* $\alpha_G = \alpha_- = 0$.

(b) *The pressure function P is convex and continuously differentiable on* $\mathbb{R}$, *and analytic and strictly convex on* $(-\infty, \beta_+)$. *If G has no parabolic element, then* $\beta_+ = +\infty$. *If G has a parabolic element, then* $\beta_+ = \delta_G$ *and P vanishes on* $[\delta_G, +\infty)$.

For finitely generated, essentially free Kleinian groups in arbitrary dimension, Kesseböhmer and Stratmann [17] analysed homological growth rates along geodesic rays, and analysed the $\mathscr{H}$-spectrum. Our Main Theorem significantly extends [17, Theorem 1.2] to a large class of Fuchsian groups which are not free groups. In particular, the Main Theorem applies to Fuchsian groups uniformizing compact hyperbolic surfaces.

A key ingredient in [17] is that for essentially free Kleinian groups, cutting sequences of geodesic rays directly give a symbolic coding of the limit set by a Markov shift. For Fuchsian groups, essentially free groups are free groups, and hence the Koebe–Morse coding coincides with the Artin coding [38]. For the Fuchsian groups we consider in this paper, the Koebe–Morse and Artin codings do not necessarily coincide [38], namely, cutting sequences do not have a direct link to the dynamics on the limit set. To overcome this difficulty, we use the results for Fuchsian groups with even corners [7, 38].

The Main Theorem is a manifestation of the familiar thermodynamic and multifractal picture for conformal expanding Markov maps possibly with neutral fixed points (see e.g. [11, 12, 18, 23, 26, 28, 29, 31, 32, 36, 42]) in the context of Fuchsian groups. Indeed, one main step in the proof of the Main Theorem is to clarify an elusive coincidence between the $\mathscr{H}$-spectrum and the Lyapunov spectrum of the Bowen–Series (BS) map [7].

Let us compare [17, Theorem 1.2] and the Main Theorem in terms of phase transitions, that is, the loss of analyticity of the pressure function in the case the group has a parabolic element. For essentially free Kleinian groups, two types of phase transitions were detected in [17, Theorem 1.2]: the pressure is not differentiable at the freezing point, or the pressure is continuously differentiable on $\mathbb{R}$ and not analytic at the freezing point. For the Fuchsian groups considered in the Main Theorem, we have shown that only the second type of phase transition occurs.

In fact, the graphs of the $\mathscr{H}$-spectra in Figure 2 are only schematic. If G has no parabolic element, we do not know whether the spectrum is concave on $[\alpha_-, \alpha_+]$ or not (see [13]). Moreover, if G has a parabolic element and the pressure function is C^2, then $P''(\delta_G) = 0$, which implies that the $\mathscr{H}$-spectrum has an inflection point (see Proposition 5.9).

1.1. *Methods of proofs and structure of the paper.* Bowen and Series [7, 38] constructed a piecewise analytic Markov map $f : \Delta \to \mathbb{S}^1$ which is orbit equivalent to the action of G

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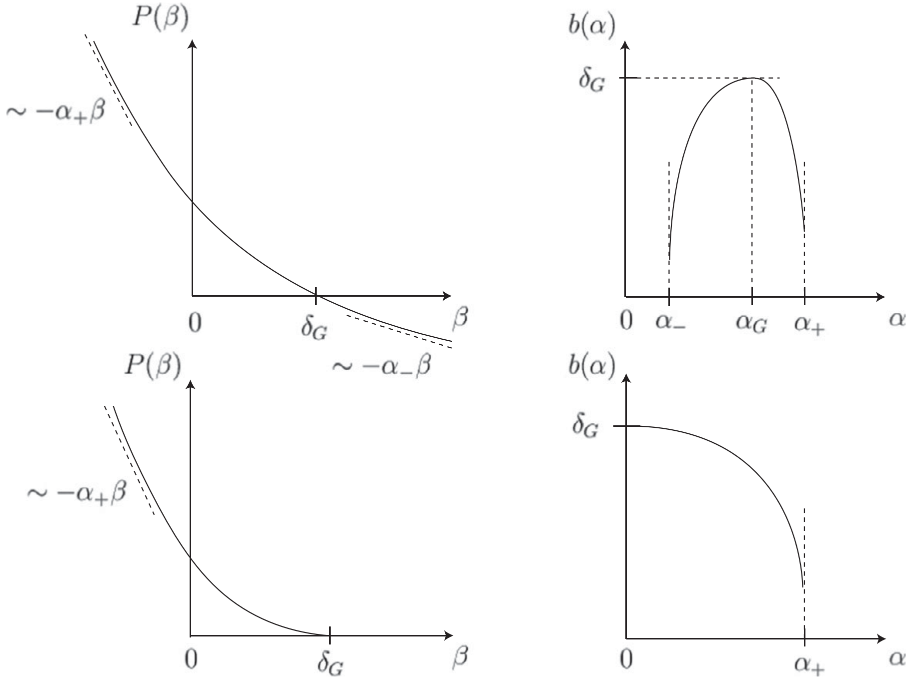

FIGURE 2. The graphs of $\beta \in \mathbb{R} \mapsto P(\beta)$ and $\alpha \in [\alpha_-, \alpha_+] \mapsto b(\alpha)$: G has no parabolic element (upper); G has a parabolic element (lower). We have $\delta_G = \min\{\beta \geq 0 \colon P(\beta) = 0\}$. The constant α_G is the unique maximum point of the $\mathscr{H}$-spectrum, see equation (5.6) for the definition.

on the limit set, now called the *Bowen–Series map*. See equation (2.1) for the definition of Δ. To prove our main results, we use three different symbolic codings (partitions) associated with the limit set Λ and the map f.

In §2, following [**7**, **38**], we introduce the Bowen–Series map and a non-Markov partition well adapted to the group structure, and develop various asymptotic results associated with them. A main conclusion is that (I) the level sets of homological growth rates coincide with the level sets of the pointwise Lyapunov exponents of the map f (Proposition 2.10).

The Markov partition constructed in [**7**] is an infinite partition if and only if G has a parabolic element. In §3, for groups having parabolic elements, we construct a finite Markov partition slightly modifying the construction in [**7**]. Combining this with the non-Markov partition introduced in §2, we show that (II) the generalized Poincaré exponent coincides with the geometric pressure (Proposition 3.8).

By virtue of the identities (I) and (II), the proof of the Main Theorem boils down to implementing the thermodynamic formalism and multifractal analysis for the map f. Series [**37**, Theorem 5.1] showed that some power of f is uniformly expanding if G has no parabolic element. In this case, properties of the pressure function and that of the Lyapunov spectrum of f are well known [**6**, **26**, **28**, **29**, **34**, **42**]. If G has a parabolic element, f has a neutral periodic point and these classical results do not apply. To deal with this case, we take an inducing procedure that is now familiar in the construction of equilibrium states

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(see e.g. [27]). In §4, we construct a uniformly expanding induced Markov map $\tilde{f}$ equipped with an infinite Markov partition that allows us to represent $\tilde{f}$ with a countable Markov shift.

Although the construction of the induced Markov map $\tilde{f}$ essentially follows Bowen and Series [7], one important difference from [7] is that we dispense with the geometric hypothesis (i) of property (*) in [7, p. 406] which states that *each side of the fundamental domain is contained in the isometric circle of the associated side-pairing*. This implies that f is non-contracting, namely

$$\inf_{\Delta} |f'| \geq 1. \tag{1.1}$$

This kind of hypothesis is usually imposed in the thermodynamic formalism as well as the multifractal analysis of pointwise Lyapunov exponents of intermittent Markov maps, to facilitate arguments, see e.g. [11, 14–16, 20, 23, 32, 40, 43], and also [22]. We exploit the discrete group structure and dispense with equation (1.1) altogether. If the Fuchsian group G has no parabolic element, one can apply the Švarc–Milnor lemma to derive that some iterate of f is uniformly expanding [37]. If G has parabolic elements, we use similar ideas to derive uniform expansion of the induced map $\tilde{f}$ (see Lemma 4.4 and Proposition 4.5).

In §5 and Appendix A, we verify several conditions on induced potentials associated with $\tilde{f}$, and apply results of Mauldin and Urbański [21] (see also e.g. [1, 8, 35]) to establish the existence and uniqueness of equilibrium states for the induced map $\tilde{f}$. We then construct equilibrium states for the original map f, and use them to establish the analyticity of the pressure function. Further, we combine results in the previous sections with the dimension formula for level sets of pointwise Lyapunov exponents in [14] to complete the proof of the Main Theorem.

1.2. *Notation.* Throughout, we shall use the notation $a \ll b$ for two positive reals a, b to indicate that a/b is bounded from above by a constant which depends only on G or R. If $a \ll b$ and $b \ll a$, we write $a \asymp b$. For $g \in G$, the inverse of g is denoted by $\bar{g}$, and the word length of g with respect to G_R is denoted by $|g|$. Let $\mathrm{cl}(\cdot)$ and $\mathrm{int}(\cdot)$ denote the closure and interior operations in $\mathbb{S}^1$, respectively. Let $|\cdot|$ denote the Lebesgue measure on $\mathbb{S}^1$, and let $\mathrm{diam}(\cdot)$ denote the Euclidean diameter on $\mathbb{R}^2$. For two distinct points $P, Q \in \mathbb{S}^1$, let $[P, Q]$ denote the closed arc in $\mathbb{S}^1$ that consists of points lying in between P and Q, anticlockwise from P to Q. Similarly, let $[P, Q) = [P, Q] \setminus \{Q\}$, $(P, Q] = [P, Q] \setminus \{P\}$ and $(P, Q) = [P, Q] \setminus \{P, Q\}$.

2. *The Bowen–Series map*

In §2.1, we collect basic facts about cutting sequences and fundamental domains with even corners. In §2.2, we introduce the Bowen–Series map f together with an associated non-Markov symbolic coding called f-expansion. In §2.3, following Series [38], we characterize admissible words for this coding that will be used later. In §2.4, we establish uniform decay of cylinders, and use it in §2.6 to prove a distortion estimate. In §2.5, we describe two orbits in the hyperbolic space, one from the cutting sequence of a geodesic and the other from the f-expansion of the positive endpoint of the same geodesic. In §2.7,

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we relate the size of a cylinder with the corresponding homological growth rate, and use this estimate in §2.8 to show that the level sets of homological growth rates coincide with the level sets of the pointwise Lyapunov exponents of the map f.

2.1. *Cutting sequences for fundamental domains with even corners.* Let $R \subset \mathbb{D}$ be a fundamental domain for G. By a fundamental domain, we always mean a convex and locally finite fundamental domain which contains 0 in its interior [3]. The sides of R are geodesics, or else arcs contained in $\mathbb{S}^1$. The latter sides are called free sides. Note that G is of the first kind if and only if R has no free sides [33, Theorem 12.2.12]. Since G is finitely generated, R has finitely many sides. The sides of R which are not free give rise to a finite set of side-pairing transformations G_R. Recall that G_R is a symmetric set of generators of G.

The copies of R adjacent to R along the sides of R are of the form eR, $e \in G_R$. For every $g \in G$ and $e \in G_R$, we label the side common to gR and geR on the side of geR by e, and on the side of gR by $\bar{e}$. By a side or vertex of $N = G\partial R$, we mean the G-image of a side or vertex of R. We say R has *even corners* if N is a union of complete geodesics ([38], see also [7]). We say R is *admissible* if R has even corners with at least four sides and satisfies the following property: if R has precisely four sides with all vertices in $\mathbb{D}$, then at least three geodesics in N meet at each vertex of R [38, Theorem 3.1]. The even corner assumption is not as restrictive as it appears. In fact, every surface which is uniformized by a finitely generated Fuchsian group has a fundamental domain with this property (see [7, §3] and [38, p. 609, lines 9–10]).

Unless otherwise stated, we assume all geodesics are complete. If γ is an oriented geodesic which passes through a vertex v of N in $\mathbb{D}$, we make the convention that γ is replaced by a curve deformed to the right around v. We shall take as understood that all geodesics have been deformed, where necessary, in this way.

For $\gamma \in \mathscr{R}$, we define a one-sided, finite or infinite sequence $g_0, g_1, g_2, \ldots$ of labels in G_R, called the *cutting sequence* of γ as follows (see Figure 1): g_0 is the exterior label of the side of R across which γ crosses from R to g_0R, and for each $n \geq 1$, we use g_n to denote the exterior label of the side of $g_0 \cdots g_{n-1}R$ across which γ crosses from $g_0 \cdots g_{n-1}R$ to $g_0 \cdots g_nR$.

Given a discrete set S and a set Z of one-sided infinite sequences $(z_n)_{n=0}^{\infty} = z_0z_1 \cdots$ in the Cartesian product topological space $S^{\mathbb{N}}$, let $E(Z)$ denote the set of finite words in S that appear in some element of Z. For an integer $n \geq 1$, let $E^n(Z)$ denote the set of elements of $E(Z)$ with word length n.

A word $w \in E(G_R^{\mathbb{N}})$ represents the group element given by the combination of the symbols under the group operation. From now on, the word length of elements of G is always understood with respect to G_R. We say w is *shortest* if its word length is equal to the word length of the element of G represented by w, and we say w is *reduced* if it does not contain successive letters $e, \bar{e} \in G_R$. Shortest words are reduced. We say $(g_n)_{n=0}^{\infty} \in G_R^{\mathbb{N}}$ is *shortest* if $g_j \cdots g_k$ is shortest, for all $j, k \in \mathbb{N}$ with $j < k$.

PROPOSITION 2.1. [38, Theorem 3.1(ii)] *If R is admissible, then the cutting sequences of $\gamma \in \mathscr{R}$ are shortest.*

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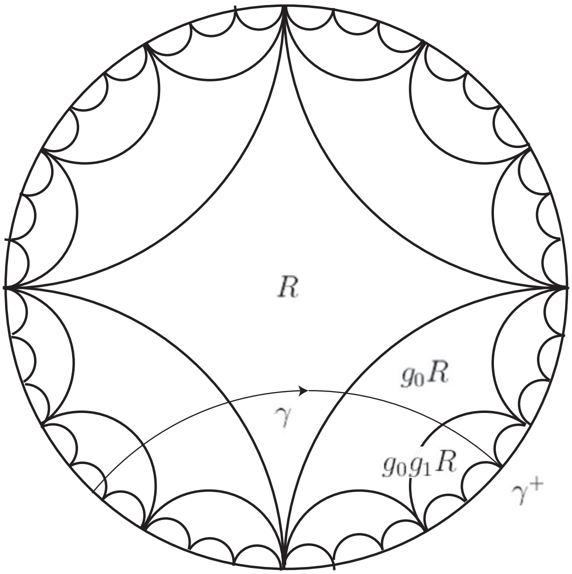

FIGURE 3. An oriented geodesic γ with the finite cutting sequence g_0, g_1 for a free Fuchsian group with two generators.

The cutting sequence of $\gamma \in \mathscr{R}$ may not always be infinite. Figure 3 shows an example with $\gamma^+ \in \Lambda \setminus \Lambda_c$ for a group of the first kind. Note that γ^+ is the image of a cusp of R under G and γ has no infinite cutting sequence. The next lemma characterizes $\gamma \in \mathscr{R}$ with infinite cutting sequence.

LEMMA 2.2. *An element $\gamma \in \mathscr{R}$ has an infinite cutting sequence if and only if $\gamma^+ \in \Lambda_c$. Moreover, for $\gamma \in \mathscr{R}$ with an infinite cutting sequence $(g_n)_{n=0}^{\infty}$, we have*

$$\lim_{n\to\infty} g_0 \cdots g_n 0 = \gamma^+.$$

Proof. First assume that γ has an infinite cutting sequence $(g_n)_{n=0}^{\infty}$. Since cutting sequences are shortest by Proposition 2.1, $(g_0 \cdots g_n)_{n=0}^{\infty} \subset G$ are pairwise distinct. Since R is locally finite, $\operatorname{diam}(g_0 \cdots g_n R) \to 0$ as $n \to \infty$. Hence, $g_0 \cdots g_n 0 \to \gamma^+$ and therefore, $\gamma^+ \in \Lambda$. To prove that $\gamma^+ \in \Lambda_c$, we assume for the sake of contradiction that γ^+ is fixed by some parabolic element of G. By [3, Corollary 9.2.9], γ^+ is the G-image of some cusp of R. This implies that γ has a finite cutting sequence and gives the desired contradiction. Hence, $\gamma^+ \in \Lambda_c$.

Conversely, assume that $\gamma \in \mathscr{R}$ with $\gamma^+ \in \Lambda$ has no infinite cutting sequence. In this case, γ^+ belongs to the Euclidean boundary of some image of R under G. Hence, by [3, Theorem 9.3.8], γ^+ is fixed by some parabolic element of G and therefore, $\gamma^+ \notin \Lambda_c$. The proof is complete. □

2.2. *The definition of the Bowen–Series map.* Let m denote the number of sides of the fundamental domain R, with exterior labels $e_1, \ldots, e_m$ in anticlockwise order. For $1 \le i \le m$, let $C(\bar{e}_i)$ denote the Euclidean closure of the geodesic that contains the side of R with the exterior label e_i. We denote the two endpoints of $C(\bar{e}_i)$ by P_i and Q_{i+1} in

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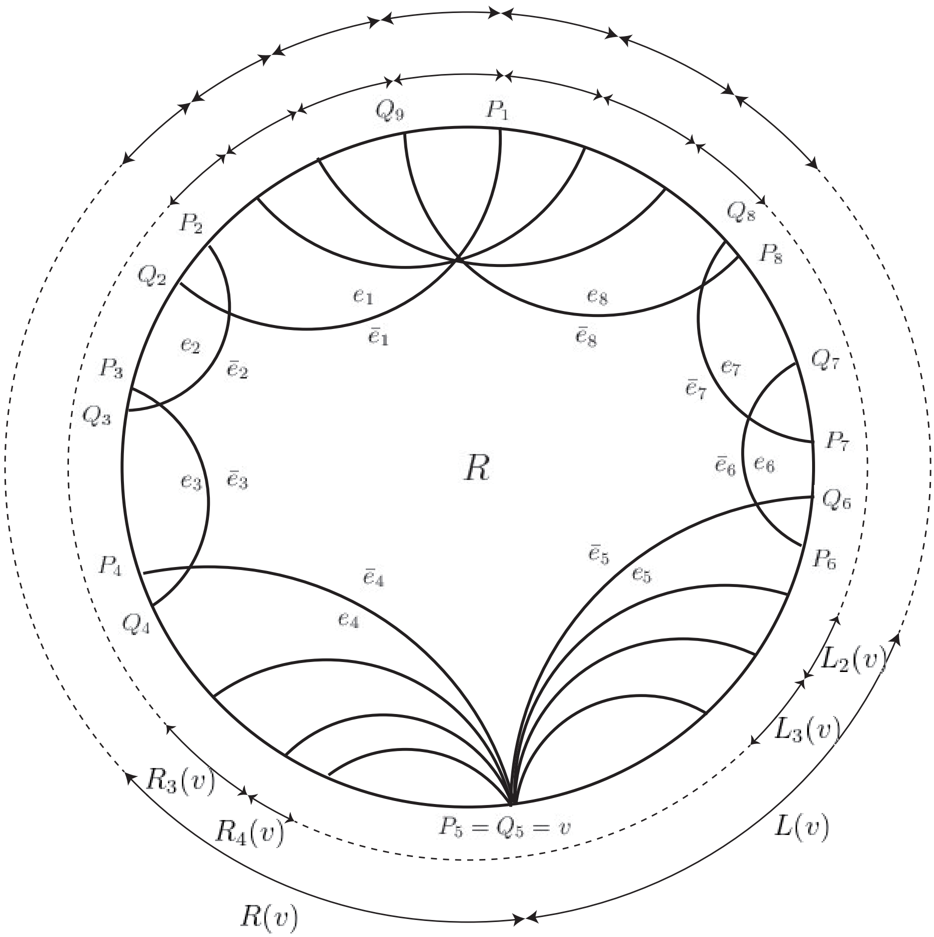

FIGURE 4. A fundamental domain R of a finitely generated Fuchsian group of the first kind with eight sides: v is a cusp, e_1 and e_8, e_2 and e_6, e_3 and e_7, e_4 and e_5 are identified in pairs, which yields a hyperbolic surface of genus 2. The bidirectional arrows in the inner (respectively outer) circle indicate the elements of the partition of $\mathbb{S}^1$ defined by W in equation (3.4) (respectively W' in equation (3.6)). In the coarser partition defined by W', the sets $L_i(v)$ $(i = 2, 3, \ldots)$ are combined into a single set $L(v)$. Similarly, $R_i(v)$ $(i = 3, 4, \ldots)$ are combined into $R(v)$.

anticlockwise order (see Figure 4). If $C(\bar{e}_i) \cap C(\bar{e}_{i+1}) \neq \emptyset$, we put $U_{i+1} = P_{i+1}$, and put $U_{i+1} = Q_{i+1}$ otherwise. For $j \in \mathbb{Z}$ with $i = j \bmod m$, set $e_j = e_i$, $P_j = P_i$, $Q_j = Q_i$ and $U_j = U_i$. We define

$$\Delta = \mathbb{S}^1 \setminus \bigcup_{i=1}^{m} [U_i, P_i). \tag{2.1}$$

Note that $\Delta = \mathbb{S}^1$ if G is of the first kind. According to [7, 38] (in [7], the Bowen–Series map is defined only for groups of the first kind), the Bowen–Series map $f : \Delta \to \mathbb{S}^1$ is given by

$$f|_{[P_i, U_{i+1})}(\xi) = \bar{e}_i \xi. \tag{2.2}$$

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The *f-expansion* of a point $\xi \in \bigcap_{n=0}^{\infty} f^{-n}(\Delta)$ is the one-sided infinite sequence $\xi_f = (e_{i_n})_{n=0}^{\infty} \in G_R^{\mathbb{N}}$ given by

$$f^n(\xi) \in [P_{i_n}, U_{i_n+1}) \quad \text{for } n \geq 0.$$

We set

$$\Sigma^+ = \{\xi_f : \xi \in \Lambda\}.$$

For each $i \in \mathbb{Z}$, the restriction of f to (P_i, U_{i+1}) is analytic and can be extended to a C^∞ map on $[P_i, U_{i+1}]$. The derivatives of f at points P_i (respectively U_{i+1}) are the right-sided (respectively left-sided) derivatives. If P_i (respectively U_{i+1}) is a cusp, it is a neutral periodic point of f.

Standing hypotheses for the rest of the paper. R is an admissible fundamental domain for G, and f is the associated Bowen–Series map.

2.3. *Characterization of admissible BS words.* If v is a vertex of N in $\mathbb{D}$, let $n(v)$ denote the number of sides of N through v. A small circle around v has a cutting sequence $g_1 \cdots g_{2n(v)}$, and $g_1 \cdots g_{2n(v)} = 1$ is one of the defining relations of G. Note that the relator has even word length since R has even corners. A word $w \in E^k(G_R^{\mathbb{N}})$ is a *clockwise* (respectively *anticlockwise*) *cycle* around v if $k \leq 2n(v)$ and there exists a neighbourhood U of v in $\mathbb{D}$ such that w appears in the 'cutting sequence' of any clockwise (respectively anticlockwise) circle around v in U. If moreover $k = n(v)$, we call w a *half-cycle*, and if $k > n(v)$, we call w a *long cycle*.

PROPOSITION 2.3. [38, Theorem 4.2] *A word in $E(G_R^{\mathbb{N}})$ is contained in $E(\Sigma^+)$ if and only if it is shortest and contains no anticlockwise half-cycle.*

2.4. *Uniform decay of BS cylinders.* Let $n \geq 1$ and let $e_{i_0} \cdots e_{i_{n-1}} \in E^n(\Sigma^+)$. We define a *BS cylinder*, or more precisely a BS n-cylinder, by

$$\Theta(e_{i_0} \cdots e_{i_{n-1}}) = \{\xi \in \Delta \colon f^k(\xi) \in [P_{i_k}, U_{i_k+1}) \quad \text{for } 0 \leq k \leq n-1\}.$$

In what follows, we denote elements of $E^n(\Sigma^+)$ by $a_0 \cdots a_{n-1}$, $a_k \in G_R$ for $0 \leq k \leq n-1$, to make a distinction from cutting sequences of geodesics. Put

$$\Theta_{\max,n} = \max_{a_0 \cdots a_{n-1} \in E^n(\Sigma^+)} |\Theta(a_0 \cdots a_{n-1})|.$$

If G has no parabolic element, then $\Theta_{\max,n}$ decays as n increases since some power of f is uniformly expanding [37, Theorem 5.1]. Below we show that this uniform decay of BS cylinders still holds even if G has a parabolic element. Although some results in [37] seem to imply this, we give a self-contained proof for the convenience of the readers. For $e \in G_R$, we denote by $H(\bar{e})$ the open half-space in $\mathbb{D}$ bordered by $C(\bar{e})$ which does not contain R.

LEMMA 2.4. *Let $(a_n)_{n=0}^{\infty} \in \Sigma^+$ and $n \geq 0$. We have $a_0 \cdots a_n 0 \notin a_0 \cdots a_n H(\overline{a}_{n+1})$, and for $k > n$, we have $a_0 \cdots a_k 0 \in a_0 \cdots a_n H(\overline{a}_{n+1})$.*

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Proof. Clearly, $a_0 \cdots a_n 0 \notin a_0 \cdots a_n H(\overline{a}_{n+1})$ and $a_0 \cdots a_{n+1} 0 \in H(\overline{a}_{n+1})$. Since by Proposition 2.3 the elements of $E(\Sigma^+)$ are shortest, $a_0 \cdots a_k 0 \in H(\overline{a}_{n+1})$ for $k > n$. Hence, the lemma follows. □

LEMMA 2.5. *We have*

$$\lim_{n\to\infty} \max_{a_0 \cdots a_{n-1} \in E^n(\Sigma^+)} |\mathbb{S}^1 \cap a_0 \cdots a_{n-1} H(\overline{a}_n)| = 0.$$

In particular, we have $\lim_{n\to\infty} \Theta_{\max,n} = 0$.

Proof. Recall that each $a_0 \cdots a_{n-1} \in E^n(\Sigma^+)$ has word length n by Proposition 2.3. For convenience, we work in the upper half-plane $\mathbb{H}$. We choose a conjugacy which maps a point in the complement of the Euclidean closure of the arc cut off by $H(\overline{a}_0)$ in $\mathbb{S}^1$ to infinity. Put $r_n = \max_{a_0 \cdots a_{n-1} \in E^n(\Sigma^+)} \operatorname{diam}(a_0 \cdots a_{n-1} R)$. Since R is locally finite, we have $r_n \to 0$ as $n \to \infty$.

If $C(\overline{a}_n)$ is a free side of R, then $|\partial\mathbb{H} \cap a_0 \cdots a_{n-1} H(\overline{a}_n)| \leq r_n$. If $C(\overline{a}_n)$ is not a free side, we assume for simplicity that the side s of $a_0 \cdots a_{n-1} R$ contained in $a_0 \cdots a_{n-1} C(\overline{a}_n)$ has one vertex v at infinity, and one vertex v' in $\mathbb{H}$. Denote the other side of $a_0 \cdots a_{n-1} R$ emanating from the vertex v' by s', and the endpoint of s' not equal to v' by v''. Denote by s'' the side of $a_0 \cdots a_{n-1} R$ emanating from the vertex v'' not equal to s'. Since the circular arcs containing s and s'' are disjoint [7, Lemma 2.2], it is easy to see that $|\partial\mathbb{H} \cap a_0 \cdots a_{n-1} H(\overline{a}_n)|$ is bounded from above by the Euclidean distance of v and v''. Since v and v'' are vertices of the fundamental domain $a_0 \cdots a_{n-1} R$, we conclude $|\partial\mathbb{H} \cap a_0 \cdots a_{n-1} H(\overline{a}_n)| \leq r_n$. The remaining cases can be treated in a similar fashion. The proof of the first assertion is complete. The second assertion follows from the first one because $\Theta(a_0 \cdots a_n)$ is contained in the Euclidean closure of $a_0 \cdots a_{n-1} H(\overline{a}_n)$. □

2.5. *Comparison of BS and cutting orbits.* Let $\gamma \in \mathscr{R}$ with $\gamma^+ \in \Lambda$. Since Λ is G-invariant and $\Lambda \subset \Delta$, we obtain $\Lambda \subset \bigcap_{n=0}^{\infty} f^{-n}(\Delta)$. Hence, γ^+ has an infinite f-expansion. Let $(a_n)_{n=0}^{\infty}$ denote the f-expansion of γ^+. We call $(a_0 \cdots a_n 0)_{n=0}^{\infty}$ a *BS orbit* associated with γ. For BS orbits, the convergence is uniform in the following sense.

LEMMA 2.6. *For any* $\varepsilon > 0$, *there exists* $n_0 \geq 1$ *such that if* $\xi \in \Lambda$ *has the* f*-expansion* $(a_n)_{n=0}^{\infty}$, *then for all* $n \geq n_0$, *we have*

$$|a_0 \cdots a_{n-1} 0 - \xi| < \varepsilon.$$

Proof. By Lemma 2.4, we have $a_0 \cdots a_{n-1} 0 \in a_0 \cdots a_{n-2} H(\overline{a}_{n-1})$. By Lemma 2.5, $\operatorname{diam}(a_0 \cdots a_{n-2} H(\overline{a}_{n-1}))$ tends to zero uniformly, as $n \to \infty$. Since ξ has the f-expansion $(a_n)_{n=0}^{\infty}$, it belongs to the Euclidean closure of $a_0 \cdots a_{n-2} H(\overline{a}_{n-1})$ for each n. The lemma follows. □

Let $(g_n)_n$ denote the finite or infinite cutting sequence of γ. We call $(g_0 \cdots g_n 0)_n$ the *cutting orbit* associated with γ. For free groups, the cutting orbit of $\gamma \in \mathscr{R}$ coincides with the f-expansion of γ^+. For non-free groups, this is not always the case. Nevertheless, they differ only slightly in the sense of the next lemma.

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LEMMA 2.7. *Let $\gamma \in \mathscr{R}$ have the infinite cutting sequence $(g_n)_{n=0}^{\infty}$ and let $(a_n)_{n=0}^{\infty}$ be the f-expansion of γ^+. For any $n \geq 0$, $g_0 \cdots g_n R$ and $a_0 \cdots a_n R$ share a common side of N, or else share a common vertex of N in $\mathbb{D}$.*

Proof. By Lemmas 2.2 and 2.6, the cutting orbit $(g_0 \cdots g_n 0)_{n=0}^{\infty}$ and the BS orbit $(a_0 \cdots a_n 0)_{n=0}^{\infty}$ converge to the same point γ^+. Hence, the conclusion is a consequence of [**38**, Proposition 3.2] and [**38**, Theorem 3.1]. □

2.6. *Mild distortion on BS cylinders.* For $n \geq 1$, define

$$D_n = \sup_{a_0 \cdots a_{n-1} \in E^n(\Sigma^+)} \sup_{\xi, \eta \in \Theta(a_0 \cdots a_{n-1})} \frac{|(f^n)'\xi|}{|(f^n)'\eta|}.$$

If G has no parabolic element, some power of f is uniformly expanding [**37**, Theorem 5.1], and so D_n is uniformly bounded. If G has a parabolic element, D_n grows sub-exponentially as n increases, which suffices for all our purposes. We say f has *mild distortion* if $\log D_n = o(n)$ $(n \to \infty)$.

PROPOSITION 2.8. *The Bowen–Series map f has mild distortion.*

Proof. Let $n \geq 2$ and let $a_0 \cdots a_{n-1} \in E^n(\Sigma^+)$. By the chain rule and the mean value theorem for $\log |f'|$, for $\xi, \eta \in \Theta(a_0 \cdots a_{n-1})$, we have

$$\log \frac{|(f^n)'\xi|}{|(f^n)'\eta|} \ll \sum_{j=0}^{n-1} |f^j(\Theta(a_0 \cdots a_{n-1}))| \ll \sum_{j=0}^{n-2} \Theta_{\max, n-j} + 2\pi,$$

which is $o(n)$ by Lemma 2.5. □

2.7. *Decay estimate of BS cylinders.* The next proposition connects the size of a BS n-cylinder with the corresponding growth rate. There exists a constant $\theta_0 > 0$ such that for $n \geq 1$ and $a_0 \cdots a_{n-1} \in E^n(\Sigma^+)$,

$$0 < \theta_0 \leq |\overline{a_0 \cdots a_{n-1}} \Theta(a_0 \cdots a_{n-1})| \leq 2\pi. \tag{2.3}$$

Put

$$n(R) = \max(\{n(v) : \ v \text{ is a vertex of } R \text{ in } \mathbb{D}\} \cup \{1\}).$$

PROPOSITION 2.9. *For any $\gamma \in \mathscr{R}$ with the cutting sequence $(g_n)_{n=0}^{\infty}$ and the f-expansion $(a_n)_{n=0}^{\infty}$ of γ^+, we have*

$$1 \ll \frac{|\Theta(a_0 \cdots a_{n-1})|}{\exp(-t_n(\gamma))} \ll D_n.$$

Proof. By Lemma 2.7, the copies $g_0 \cdots g_{n-1} R$ and $a_0 \cdots a_{n-1} R$ of R share a common side of N, or else share a common vertex of N in $\mathbb{D}$. The triangle inequality yields

$$|t_n(\gamma) - d(0, a_0 \cdots a_{n-1} 0)| \leq n(R) \max\{d(0, g0) : g \in G_R\} \ll 1.$$

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Hence, it suffices to show that for all $n \geq 1$ and $a_0 \cdots a_{n-1} \in E^n(\Sigma^+)$,

$$1 \ll \frac{|\Theta(a_0 \cdots a_{n-1})|}{\exp(-d(0, a_0 \cdots a_{n-1}0))} \ll D_n. \tag{2.4}$$

Let $n \geq 1$ and let $a_0 \cdots a_{n-1} \in E^n(\Sigma^+)$. Let ξ_+ and ξ_- denote the boundary points of $\Theta(a_0 \cdots a_{n-1})$. Let $\theta > 0$ denote the angle between the geodesic arcs joining $a_0 \cdots a_{n-1}0$ to ξ_+ and ξ_-. Since all $a_0, \ldots, a_{n-1}$ are Möbius transformations, equation (2.3) gives

$$\theta_0 \leq \theta \leq 2\pi. \tag{2.5}$$

Split $\Theta(a_0 \cdots a_{n-1})$ into three disjoint arcs $\Theta_+, \Theta_0, \Theta_-$ so that $\xi_+ \in \Theta_+$, $\xi_- \in \Theta_-$ and the $\overline{a_0 \cdots a_{n-1}}$-images of the three arcs have the same Euclidean lengths. We use Θ_+ and Θ_- as a buffer, and estimate $|\Theta_0|$ rather than $|\Theta(a_0 \cdots a_{n-1})|$ itself. The mean value theorem gives

$$\frac{1}{3D_n} \leq \frac{|\Theta_0|}{|\Theta(a_0 \cdots a_{n-1})|} \leq \min\left\{\frac{D_n}{3}, 1\right\}. \tag{2.6}$$

Let $d(0, a_0 \cdots a_{n-1}0) = r$. Rotate the Poincaré disk so that $a_0 \cdots a_{n-1}0$ is placed on the negative part of the real axis. By Lemma 2.6, there exists $n_0 \geq 1$ such that if $n \geq n_0$, then for any $a_0 \cdots a_{n-1} \in E^n(\Sigma^+)$, $\Theta(a_0 \cdots a_{n-1})$ is contained in the Euclidean open ball of radius $1/100$ about -1. In particular, $\Theta(a_0 \cdots a_{n-1})$ does not contain 1. We apply the Möbius transformation $T : \mathbb{P}^1 \to \mathbb{P}^1$ given by

$$T(z) = \frac{\cosh(r/2)z + \sinh(r/2)}{\sinh(r/2)z + \cosh(r/2)}.$$

This carries the four geodesics through $a_0 \cdots a_{n-1}0$ separating $\Theta_+, \Theta_0, \Theta_-$ to rays through 0 at an equal angle $\theta/3$. Since $1 \notin \Theta(a_0 \cdots a_{n-1})$ and $T(1) = 1$, $1 \notin T(\Theta(a_0 \cdots a_{n-1}))$ holds. Therefore, $T(\Theta_0)$ lies in the complement of the domain $\{z \in \mathbb{D}\colon |\arg(z)| \leq \theta_0/3\}$. A calculation shows

$$|(T^{-1})'z| = \left|\cosh\frac{r}{2} - \mathrm{Re}(z)\sinh\frac{r}{2} - \sqrt{-1}\mathrm{Im}(z)\sinh\frac{r}{2}\right|^{-2}.$$

Since $T(\Theta_0)$ is uniformly bounded away from 1 in the Euclidean distance, we have $|(T^{-1})'z| \asymp e^{-r}$. Since $|\Theta_0| = \int_{T(\Theta_0)} |(T^{-1})'z||dz|$, this yields

$$|\Theta_0| \asymp \theta e^{-r}. \tag{2.7}$$

Combining equations (2.5), (2.6) and (2.7), we obtain equation (2.4). □

2.8. *Equality of level sets, boundary of the $\mathscr{H}$-spectrum.* The upper and lower pointwise Lyapunov exponents at a point $\xi \in \Lambda$ are given by

$$\overline{\chi}(\xi) = \limsup_{n\to\infty} \frac{1}{n} \log |(f^n)'\xi| \quad \text{and} \quad \underline{\chi}(\xi) = \liminf_{n\to\infty} \frac{1}{n} \log |(f^n)'\xi|,$$

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respectively. If $\overline{\chi}(\xi) = \underline{\chi}(\xi)$, this common value is called the *pointwise Lyapunov exponent* at η and denoted by $\chi(\xi)$. For each $\alpha \in \mathbb{R}$, define the level set

$$\mathscr{L}(\alpha) = \{\xi \in \Lambda_c : \overline{\chi}(\xi) = \underline{\chi}(\xi) = \alpha\}.$$

The next proposition indicates that the level sets of homological growth rates and that of pointwise Lyapunov exponents coincide.

PROPOSITION 2.10. *For every $\gamma \in \mathscr{R}$ such that $\gamma^+ \in \Lambda_c$ and every $n \geq 1$,*

$$D_n^{-1} \ll \exp(\log |(f^n)'\gamma^+| - t_n(\gamma)) \ll D_n^2.$$

In particular, for every $\alpha \geq 0$, $\mathscr{H}(\alpha) = \mathscr{L}(\alpha)$.

Proof. Let $(a_n)_{n=0}^{\infty}$ denote the f-expansion of γ^+. By the mean value theorem, there exists $\xi \in \Theta(a_0 \cdots a_{n-1})$ such that $|(f^n)'\xi||\Theta(a_0 \cdots a_{n-1})| = |f^n(\Theta(a_0 \cdots a_{n-1}))|$. By equation (2.3), we have $|f^n(\Theta(a_0 \cdots a_{n-1}))| \in [\theta_0, 2\pi]$, and

$$\theta_0 D_n^{-1} |\Theta(a_0 \cdots a_{n-1})|^{-1} \leq |(f^n)'\gamma^+| \leq 2\pi D_n |\Theta(a_0 \cdots a_{n-1})|^{-1}.$$

This together with Proposition 2.9 yields the desired double inequalities. The rest of the assertions follow from Proposition 2.8. □

Remark 2.11. By Proposition 2.10, the level sets $\mathscr{H}(\alpha)$ are pairwise disjoint.

LEMMA 2.12. *We have $\alpha_- = 0$ if and only if G has a parabolic element.*

Proof. If G has no parabolic element, some power of f is uniformly expanding [**37**, Theorem 5.1]. Hence, we have $\alpha_- > 0$ by Proposition 2.10. If G has a parabolic element, then R has a cusp, which is a neutral periodic point of f. By [**14**], the set $\{x \in \Lambda : \overline{\chi}(x) = \underline{\chi}(x) = 0\}$ has positive Hausdorff dimension, while $\Lambda \setminus \Lambda_c$ is a countable set. Hence, we have $\mathscr{L}(0) \neq \emptyset$ and $\mathscr{H}(0) \neq \emptyset$ by Proposition 2.10. Hence, we obtain $\alpha_- = 0$. □

Remark 2.13. In the definition of $t_n(\gamma)$, we may replace the cutting sequence of γ by the f-expansion of the positive endpoint γ^+. By Lemma 2.7, this does not change the level sets $\mathscr{H}(\alpha)$.

3. *Finite Markov structures*

In this section, starting with the definition of Markov maps in §3.1, we construct a finite Markov partition for the Bowen–Series map f in §3.2 by slightly modifying the Markov partition constructed in [**7**]. In §3.3, we use this finite Markov partition to identify the maximal invariant set of the Markov map f as the limit set of G. In §3.4, we introduce a geometric pressure function using the free energy of f-invariant Borel probability measures, and show that the generalized Poincaré exponent coincides with the geometric pressure.

3.1. *Markov maps.* Let S be a discrete set with $\#S \geq 2$. A *Markov map* is a map $F : \Gamma \to \mathbb{S}^1$ such that the following hold.

(M0) There exists a family $(\Gamma(a))_{a\in S}$ of pairwise disjoint arcs in $\mathbb{S}^1$ such that $\Gamma = \bigcup_{a\in S}\Gamma(a)$.

(M1) For each $a \in S$, the restriction $F|_{\Gamma(a)}$ extends to a C^1 diffeomorphism from $\mathrm{cl}(\Gamma(a))$ onto its image.

(M2) If $a, b \in S$ and $F(\Gamma(a)) \cap \Gamma(b)$ has non-empty interior, then $F(\Gamma(a)) \supset \Gamma(b)$.

The family $(\Gamma(a))_{a\in S}$ of arcs is called a *Markov partition* of F.

Condition (M2) determines a transition matrix (M_{ab}) over the countable alphabet S by the rule $M_{ab} = 1$ if $F(\Gamma(a)) \supset \Gamma(b)$ and $M_{ab} = 0$ otherwise. This transition matrix determines a countable topological Markov shift $Y = Y(F, (\Gamma(a))_{a\in S})$ by

$$Y = \{y = (y_n)_{n=0}^{\infty} \in S^{\mathbb{N}} \colon M_{y_n y_{n+1}} = 1 \text{ for } n \geq 0\}. \tag{3.1}$$

We endow Y with the metric $d_Y(y, z) = \exp(-\inf\{n \geq 0 \colon y_n \neq z_n\})$, where we set $\exp(-\infty) = 0$. For $n \geq 1$ and $\omega_0 \cdots \omega_{n-1} \in S^n$, write

$$[\omega_0 \cdots \omega_{n-1}] = \{y \in Y \colon y_k = \omega_k \text{ for } 0 \leq k \leq n-1\}. \tag{3.2}$$

Subsets of Y of this form are called cylinders. The collection of all cylinders forms a base of the topology on Y.

For $\omega \in S^m$ and $\kappa \in S^n$, write $\omega\kappa$ for $\omega_0 \cdots \omega_{m-1}\kappa_0 \cdots \kappa_{n-1} \in S^{m+n}$. For convenience, put $E^0 = \{\emptyset\}$, $|\emptyset| = 0$ and $\omega\emptyset = \omega = \emptyset\omega$ for all $\omega \in E(Y)$. The Markov map F is *finitely irreducible* [21] if there exists a finite subset Ξ of $E(Y) \cup E^0$ such that for all $\omega, \kappa \in E(Y)$, there exists $\lambda \in \Xi$ such that $\omega\lambda\kappa \in E(Y)$.

The symbolic dynamics and the dynamics of F are related by the coding map $\pi_Y : Y \to \mathbb{S}^1$ given by

$$\pi_Y((y_n)_{n=0}^{\infty}) \in \bigcap_{n=1}^{\infty} \mathrm{cl}(\Gamma(y_0 \cdots y_{n-1})), \tag{3.3}$$

where

$$\Gamma(y_0 \cdots y_{n-1}) = \bigcap_{k=0}^{n-1} F^{-k}(\Gamma(y_k)).$$

We shall always assume that the Markov map F has *decay of cylinders* [14], that is, the right-hand side in equation (3.3) is a singleton. We will treat two Markov maps introduced in §§3.2 and 4.1.

3.2. *Construction of a finite Markov partition for the Bowen–Series map.* We recall the construction of a Markov partition for the Bowen–Series map carried out in [7]. Our presentation of this is a slightly expanded version so as to include groups of the second kind. All lemmas quoted from [7] below remain valid for groups of the second kind.

A point $v \in \mathbb{S}^1$ is a *proper vertex* of R at infinity if v is the common endpoint of two sides of R. A point $v \in \mathbb{S}^1$ is called an *improper vertex* of R at infinity if v is the common endpoint of a side and a free side of R. A proper vertex at infinity is also called a *cusp*. The set of all cusps of R is denoted by V_c. Note that each $v \in V_c$ is a fixed point of some

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parabolic element of G. Conversely, if G has a parabolic element, then V_c is non-empty. Let V denote the set of all vertices of R in $\mathbb{D} \cup \mathbb{S}^1$.

For each vertex $v \in V$, we denote by $N(v)$ the set of geodesics in N passing through v, and by $W(v)$ the set of points where the geodesics in $N(v)$ meet $\mathbb{S}^1$. We set

$$W = \bigcup_{v \in V} W(v). \tag{3.4}$$

By [7, Lemma 2.3] and the definition of f in equation (2.2), we have $f(W) \subset W$, and $\lim_{\xi \nearrow U_{i+1}} f(\xi) \in W$ for any $i \in \mathbb{Z}$, where the one-sided limit on $\mathbb{S}^1$ is understood in anticlockwise order. Hence, W induces a Markov partition for f. This partition is an infinite partition if and only if R has a cusp.

If R has a cusp, f is not finitely irreducible with respect to this Markov partition, and so results on the multifractal analysis in [14] are not directly applicable. To make use of the results in [14], we construct a coarser Markov partition below with respect to which f becomes finitely irreducible.

If $v \in V_c$, then we denote the arcs of $\mathbb{S}^1$ cut off by successive points of $W(v)$ in clockwise order from Q_{i+1} to $Q_i = v$ by $L_1(v), L_2(v), \ldots$, and in anticlockwise order from Q_{i+1} to Q_i by $R_1(v), R_2(v), \ldots$, and set

$$L(v) = \overline{\bigcup_{r \geq 2} L_r(v)} \quad \text{and} \quad R(v) = \overline{\bigcup_{r \geq 3} R_r(v)}.$$

By [7, (2.4.1)], we have

$$\begin{aligned} f|_{L_r(v)} &= \bar{e}_i, \ f(L_r(v)) = L_{r-1}(\bar{e}_i(v)) && \text{for } r \geq 2, \text{ and} \\ f|_{R_r(v)} &= \bar{e}_{i-1}, \ f(R_r(v)) = R_{r-1}(\bar{e}_{i-1}(v)) && \text{for } r \geq 2. \end{aligned} \tag{3.5}$$

For each $v \in V$, we define

$$W'(v) = \begin{cases} W(v) & \text{if } v \notin V_c, \\ \partial L_1(v) \cup \{v\} \cup \partial R_2(v) & \text{if } v \in V_c, \end{cases}$$

and set

$$W' = \bigcup_{v \in V} W'(v). \tag{3.6}$$

Note that W' is a finite subset of W. We define a partition of Δ into arcs with endpoints given by two consecutive points in W'. We choose all partition elements to be of the form $[P, Q)$, $P, Q \in \mathbb{S}^1$. We label the partition elements by integers in a finite subset S of $\mathbb{N}$, and denote the element labelled with $a \in S$ by $\Delta(a)$.

PROPOSITION 3.1. *The Bowen–Series map* $f: \Delta \to \mathbb{S}^1$ *defines a finitely irreducible Markov map with a finite Markov partition* $(\Delta(a))_{a \in S}$.

Proof. By [7, Lemmas 2.3 and 2.5], $f|_\Lambda$ is a transitive Markov map with respect to the partition of Δ into arcs with endpoints given by two consecutive points in W. By Lemma 3.2 below, the proof of which is similar to that of [7, Lemma 2.3], f is also a transitive Markov map with respect to the finite Markov partition $(\Delta(a))_{a \in S}$.

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LEMMA 3.2. *We have* $f(W') \subset W'$, *and* $\lim_{\xi \nearrow U_{i+1}} f(\xi) \in W'$ *for any* $i \in \mathbb{Z}$.

Let $P \in W'$. Since f clearly preserves cusps and improper vertices of R, we may assume in the following that P is neither a cusp nor an improper vertex of R.

First suppose that there exists a vertex $v \in V \cap \mathbb{D}$ such that $P \in W'(v)$. Let $i \in \mathbb{Z}$ satisfy $v \in C(\bar{e}_{i-1}) \cap C(\bar{e}_i)$. We distinguish three cases. (i) If $P \in [P_{i-1}, P_i)$, then $f(P) \in \bar{e}_{i-1}(P) \in W(\bar{e}_{i-1}(v)) = W'(\bar{e}_{i-1}(v))$. (ii) If $P \in [P_i, P_{i+1})$, then $f(P) \in \bar{e}_{i+1}(P) \in W(\bar{e}_{i+1}(v)) = W'(\bar{e}_{i+1}(v))$. (iii) If $P \in [P_{i+1}, P_{i-1})$, then, by [7, Lemma 2.2], we have $P = Q_{i+1}$. Since P is neither a cusp nor an improper vertex, it follows that $P \in W'(v')$ for the vertex $v' \in V \cap \mathbb{D}$ of R next to v in anticlockwise order. Hence, by case (i) with v replaced by v', we obtain $f(P) \in W'$.

Next we suppose that there exists a cusp $v' \in V_c$ such that $P \in W'(v')$. Let $j \in \mathbb{Z}$ satisfy $v' \in C(\bar{e}_{j-1}) \cap C(\bar{e}_j)$. Then we have $P \in \partial L_1(v')$ or $P \in \partial R_2(v')$. First suppose that $P \in \partial R_2(v')$. By equation (3.5), we have $f(P) \in \partial R_1(\bar{e}_{j-1}(v')) \subset W'(\bar{e}_{j-1}(v'))$. Now suppose that $P \in \partial L_1(v')$. If $P \in \partial L_1(v') \cap \partial L_2(v)$, then we have $f(P) \in \partial L_1(\bar{e}_j(v')) \subset W'(\bar{e}_j(v'))$ by equation (3.5). Otherwise, we have $P = Q_{j+1} \in \partial L_1(v') \cap \partial R_1(v')$. Since P is neither a cusp nor an improper vertex, it follows that $P \in W'(v'')$ for the vertex $v'' \in V \cap \mathbb{D}$ of R next to v' in anticlockwise order. Hence, by case (i) with v replaced by v'', we obtain $f(P) \in W'$. This completes the proof of $f(W') \subset W'$.

It remains to show $\lim_{\xi \nearrow U_{i+1}} f(\xi) \in W'$ for any $i \in \mathbb{Z}$, where $U_{i+1} = P_{i+1}$ if $C(\bar{e}_i) \cap C(\bar{e}_{i+1}) \neq \emptyset$ and $U_{i+1} = Q_{i+1}$ otherwise (see §2.2). In the former case, with $v \in C(\bar{e}_i) \cap C(\bar{e}_{i+1})$, we have $\lim_{\xi \nearrow U_{i+1}} f(\xi) = \lim_{\xi \nearrow P_{i+1}} \bar{e}_i(\xi) \in W'(\bar{e}_i(v))$. In the latter case, we have $\lim_{\xi \nearrow U_{i+1}} f(\xi) = \bar{e}_i(Q_{i+1}) \in W'$, because $e_i(Q_{i+1})$ is an improper vertex of R. This completes the proof of the lemma and that of Proposition 3.1. □

The Bowen–Series map f determines by equation (3.1) a finitely irreducible Markov shift

$$X = X(f, (\Delta(a))_{a \in S}).$$

The left shift $\sigma : X \to X$ is given by $(\sigma x)_n = x_{n+1}$ for $n \geq 0$. By Lemma 2.5, the coding map $\pi = \pi_X$ given by equation (3.3) is well defined and continuous. We have

$$f \circ \pi = \pi \circ \sigma.$$

BS cylinders and the cylinders in X are related as follows. For each $a_0 \cdots a_{n-1} \in E^n(\Sigma^+)$, the corresponding BS n-cylinder $\Theta(a_0 \cdots a_{n-1})$ is the union of finitely many n-cylinders in X, the number of which is at most $2n(R)$. Conversely, for each $\omega_0 \cdots \omega_{n-1} \in E^n(X)$, there exists a unique element $a_0 \cdots a_{n-1}$ of $E^n(\Sigma^+)$ such that $\Delta(\omega_0 \cdots \omega_{n-1}) \subset \Theta(a_0 \cdots a_{n-1})$. For convenience, we will sometimes identify $\omega_0 \cdots \omega_{n-1}$ with the Möbius transformation $a_0 \cdots a_{n-1}$ in G, and write $\Theta(\omega_0 \cdots \omega_{n-1})$ for $\Theta(a_0 \cdots a_{n-1})$.

3.3. *Identifying the maximal invariant set.* The proposition below asserts that the maximal invariant set of f coincides with the limit set of G. This clearly holds for groups of the first kind, and is known for free groups of the second kind [38, Lemma 2.2].

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PROPOSITION 3.3. *We have*

$$\Lambda = \bigcap_{n=0}^{\infty} f^{-n}(\Delta) = \pi(X).$$

Proof. If G is of the first kind, then clearly all the three sets are equal to $\mathbb{S}^1$. Suppose G is of the second kind. Then Δ is the union of finitely many arcs. Points in $\partial\Delta \setminus \Delta$ are improper vertices of R. Since each improper vertex of R is paired with another, it is easy to see that improper vertices of R are not limit points and, in particular, $\partial\Delta \setminus \Delta$ is not contained in Λ. Interior points of the complement of Δ are not limit points, because no copy of R can accumulate at such a point. We have verified that $\Lambda \subset \Delta$.

Since Λ is G-invariant and $\Lambda \subset \Delta$, we obtain $\Lambda \subset \bigcap_{n=0}^{\infty} f^{-n}(\Delta)$. To prove the equalities in the proposition, we first show the next lemma.

LEMMA 3.4. *We have* $\bigcap_{n=0}^{\infty} f^{-n}(\Delta) \subset \pi(X)$.

Proof. Let $\xi \in \bigcap_{n=0}^{\infty} f^{-n}(\Delta)$. Define $x = (x_n)_{n=0}^{\infty} \in S^{\mathbb{N}}$ by $f^n(\xi) \in \Delta(x_n)$. This is well defined since the elements $\Delta(a)$, $a \in S$ of the Markov partition are pairwise disjoint. Since f preserves orientation and the elements of the Markov partition are arcs of the form $[P, Q)$, $P, Q \in \mathbb{S}^1$, x belongs to X. Clearly, we have $\xi \in \pi(x)$. □

To complete the proof of Proposition 3.3, it remains to show $\pi(X) \subset \Lambda$. Since $f|_{\Lambda}$ is transitive by Proposition 3.1, the periodic points of σ are dense in X. Since π is continuous, it suffices to show that for any $k \geq 1$ and any fixed point $x = (x_n)_{n=0}^{\infty} \in X$ of σ^k, $\pi(x) \in \Lambda$ holds. Observe that the Möbius transformation $\overline{x_0 \cdots x_{k-1}} \in G$ satisfies $\overline{x_0 \cdots x_{k-1}}(\pi(x)) = \pi(x)$. Since $x_0 \cdots x_{k-1}$ is not the identity in G by Proposition 2.3, and since Λ contains all fixed points of elements of $G \setminus \{1\}$ in $\mathbb{S}^1$, we obtain $\pi(x) \in \Lambda$. □

Let Y be a topological space, $Y_0 \subset Y$ and let $F\colon Y_0 \to Y$ be a Borel map. Let $\mathcal{M}(Y_0, F)$ denote the set of Borel probability measures on $\bigcap_{n=0}^{\infty} F^{-n}(Y_0)$ which are invariant under the restriction of F to this set. For each $\mu \in \mathcal{M}(Y_0, F)$, let $h(\mu)$ denote the measure-theoretic entropy of μ with respect to F.

We will use the following correspondence of invariant measures on X and Λ.

LEMMA 3.5. *For any $\mu \in \mathcal{M}(\Lambda, f)$, there exists $\nu \in \mathcal{M}(X, \sigma)$ such that $\mu = \nu \circ \pi^{-1}$ and $h(\mu) = h(\nu)$. Conversely, for any $\nu \in \mathcal{M}(X, \sigma)$, the measure $\mu = \nu \circ \pi^{-1}$ belongs to $\mathcal{M}(\Lambda, f)$ and satisfies $h(\mu) = h(\nu)$.*

Proof. The coding map π is one-to-one except on the preimage of the countable set $B = \bigcup_{n=0}^{\infty} f^{-n}(\bigcup_{a \in S} \partial\Delta(a))$. Since $\mathbb{S}^1$ is one-dimensional, π is at most two-to-one on B. Since f preserves boundary points of the elements of the Markov partition, $f^{-1}(B) = B$ and so $\sigma^{-1}(\pi^{-1}(B)) = \pi^{-1}(B)$.

We have $f \circ \pi = \pi \circ \sigma$, and the restriction of π to $X \setminus \pi^{-1}(B)$ has a continuous inverse. Hence, π induces a measurable bijection between $X \setminus \pi^{-1}(B)$ and $\pi(X) \setminus B$. This and $\Lambda \subset \pi(X)$ in Proposition 3.3 imply that for any $\mu \in \mathcal{M}(\Lambda, f)$ with $\mu(B) = 0$, there exists $\nu \in \mathcal{M}(X, \sigma)$ such that $\mu = \nu \circ \pi^{-1}$ and $h(\mu) = h(\nu)$.

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If $\mu \in \mathcal{M}(\Lambda, f)$ and $\mu(B) > 0$, there exist $\rho \in (0, 1]$ and $\mu_1, \mu_2 \in \mathcal{M}(\Lambda, f)$ such that $\mu_1(B) = 0$, $\mu_2(B) = 1$ and $\mu = (1-\rho)\mu_1 + \rho\mu_2$. Since B is a countable set, μ_2 is supported on a periodic orbit of f. By Proposition 3.3, there exists $\nu_2 \in \mathcal{M}(X, \sigma)$ that is supported on a periodic orbit of σ and satisfies $\mu_2 = \nu_2 \circ \pi^{-1}$. By the previous paragraph, there exists $\nu_1 \in \mathcal{M}(X, \sigma)$ with $\mu_1 = \nu_1 \circ \pi^{-1}$. Set $\nu = (1-\rho)\nu_1 + \rho\nu_2$. Then $\mu = \nu \circ \pi^{-1}$ and $h(\mu) = (1-\rho)h(\mu_1) = (1-\rho)h(\nu_2) = h(\nu)$, as required in the first assertion of the lemma. A proof of the second one is analogous. □

3.4. *Equality of pressure and generalized Poincaré exponent.* The piecewise analytic function $\phi\colon \Lambda \to \mathbb{R}$ given by

$$\phi = -\log|f'|$$

plays an important role. For $\mu \in \mathcal{M}(\Lambda, f)$, define the *Lyapunov exponent* of μ by $\chi(\mu) = -\int \phi \, d\mu$. The *geometric pressure function*, or simply the pressure, is the function $\beta \in \mathbb{R} \mapsto P(\beta\phi, f)$ given by

$$P(\beta\phi, f) = \sup\{h(\mu) - \beta\chi(\mu) \colon \mu \in \mathcal{M}(\Lambda, f)\}.$$

A measure in $\mathcal{M}(\Lambda, f)$ which attains this supremum is called an *equilibrium state* for the potential $\beta\phi$. By the affinity of entropy and Lyapunov exponent on measures in $\mathcal{M}(\Lambda, f)$, the geometric pressure function is convex. It is non-increasing since any measure in $\mathcal{M}(\Lambda, f)$ has a non-negative Lyapunov exponent as in Lemma 3.7 below.

LEMMA 3.6. *We have*

$$\alpha_+ = \sup\{\chi(\mu) \colon \mu \in \mathcal{M}(\Lambda, f)\} \quad \textit{and} \quad \alpha_- = \inf\{\chi(\mu) \colon \mu \in \mathcal{M}(\Lambda, f)\}.$$

Proof. Using Proposition 2.8 and the irreducibility of the finite Markov shift X in Proposition 3.1, we can construct a measure supported on periodic points whose Lyapunov exponent is arbitrarily close to α_+. Hence, $\sup\{\overline{\chi}(\xi) \colon \xi \in \Lambda_c\} \leq \sup\{\chi(\mu) \colon \mu \in \mathcal{M}(\Lambda, f)\}$ holds. The reverse inequality follows from Birkhoff's ergodic theorem. Combining this equality with $\alpha_+ = \sup\{\overline{\chi}(\xi) \colon \xi \in \Lambda_c\}$ which follows from Proposition 2.10, we obtain the first equality in the lemma. A proof of the second one is analogous. □

LEMMA 3.7. *For any $\mu \in \mathcal{M}(\Lambda, f)$, we have $\chi(\mu) \geq 0$.*

Proof. From Lemma 3.6 and $\alpha_- \geq 0$. □

Although ϕ may have discontinuities, the function $\varphi\colon X \to \mathbb{R}$ given by

$$\varphi = \phi \circ \pi \tag{3.7}$$

is continuous. For $\beta \in \mathbb{R}$, the topological pressure of the potential $\beta\varphi\colon X \to \mathbb{R}$ with respect to σ is given by

$$P(\beta\varphi, \sigma) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega \in E^n(X)} \sup_{[\omega]} \exp\left(\beta \sum_{k=0}^{n-1} \varphi \circ \sigma^k\right),$$

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under the cylinder notation in equation (3.2). Since φ is continuous, the variational principle holds:

$$P(\beta\varphi, \sigma) = \sup \left\{ h(\nu) + \beta \int \varphi \, d\nu \colon \nu \in \mathcal{M}(X, \sigma) \right\}.$$

Since σ is expansive and X is a subshift over the finite set S, the entropy function is upper semicontinuous on $\mathcal{M}(X, \sigma)$. Since φ is continuous and $\mathcal{M}(X, \sigma)$ is compact with respect to the weak* topology, this supremum is attained. By Lemma 3.5, there is an equilibrium state for the potential $\beta\phi$.

PROPOSITION 3.8. *For all $\beta \in \mathbb{R}$, we have*

$$P(\beta) = P(\beta\varphi, \sigma) = P(\beta\phi, f).$$

Let $g \in G$. A *shortest representation* of g is a representation of g that contains exactly $|g|$ generators in G_R. A shortest representation of g is *admissible* if it is contained in $E(\Sigma^+)$.

LEMMA 3.9. *Every $g \in G \setminus \{1\}$ has a unique admissible shortest representation.*

Proof. Let $g = e_{i_1} \cdots e_{i_{|g|}}$ be a shortest representation of g. We replace all anticlockwise half-cycles in this representation by the corresponding clockwise half-cycles, and obtain (possibly) another shortest representation $g = e_{j_1} \cdots e_{j_{|g|}}$ that contains no anticlockwise half-cycle. By Proposition 2.3, $e_{j_1} \cdots e_{j_{|g|}} \in E(\Sigma^+)$ holds.

Let $g = e_{j_1} \cdots e_{j_{|g|}}$, $g = e_{k_1} \cdots e_{k_{|g|}}$ be two admissible shortest representations of g. Suppose $e_{j_1} \neq e_{k_1}$. Then we have a relation $\bar{e}_{k_{|g|}} \cdots \bar{e}_{k_1} e_{j_1} \cdots e_{j_{|g|}} = 1$. Since the vertex cycles give a complete set of the relations of G and both representations of g are shortest, $\bar{e}_{k_{|g|}} \cdots \bar{e}_{k_1} e_{j_1} \cdots e_{j_{|g|}}$ contains a cycle that contains $\bar{e}_{k_1} e_{j_1}$. It follows that one of the two representations of g contains an anticlockwise half cycle, and this yields a contradiction since both representations are admissible. Hence, we obtain $e_{j_1} = e_{k_1}$. Repeating this argument, we obtain $e_{j_i} \neq e_{k_i}$ for $1 \leq i \leq |g|$. □

Proof of Proposition 3.8. For $n \geq 1$ and $a_0 \cdots a_{n-1} \in E^n(\Sigma^+)$, let $E^n(X, a_0 \cdots a_{n-1})$ denote the set of elements of ω in $E^n(X)$ such that $\Delta(\omega) \subset \Theta(a_0 \cdots a_{n-1})$. Clearly,

$$1 \leq \# E^n(X, a_0 \cdots a_{n-1}) \leq 2n(R). \tag{3.8}$$

By equation (2.3), for $x \in \Theta(a_0 \cdots a_{n-1})$,

$$\theta_0 D_n^{-1} \leq \frac{|\Theta(a_0 \cdots a_{n-1})|}{|(f^n)'x|^{-1}} \leq 2\pi D_n. \tag{3.9}$$

By Proposition 2.9, and equations (3.8) and (3.9), there exists a constant $C \geq 1$ such that for all $\beta, t \in \mathbb{R}$, we have

$$\begin{aligned} C^{-\beta} D_n^{-2\beta} (2\pi)^{-\beta} &\leq \frac{\sum_{\omega \in E^n(X, a_0 \cdots a_{n-1})} \sup_{[\omega]} \exp(\beta \sum_{k=0}^{n-1} \varphi \circ \sigma^k) e^{-tn}}{\exp(-\beta d(0, a_0 \cdots a_{n-1} 0)) e^{-tn}} \\ &\leq 2n(R) \cdot C^{\beta} D_n^{2\beta} \theta_0^{-\beta}. \end{aligned} \tag{3.10}$$

By Lemma 3.9 and Proposition 2.3, there is a one-to-one correspondence between $G \setminus \{1\}$ and $E(\Sigma^+)$. Therefore, rearranging the double inequalities in equation (3.10), summing the result over all words in $E^n(\Sigma^+)$, then summing the result over all $n \geq 1$ and then using [21, Theorem 2.1.3], we obtain

$$P(\beta\varphi, \sigma) \leq \inf\left\{t \in \mathbb{R} \colon \sum_{n=1}^{\infty} D_n^{2\beta} \sum_{g \in G, |g|=n} \exp(-\beta d(0, g0) - t|g|) < +\infty\right\},$$

and $P(\beta\varphi, \sigma) \leq P(\beta)$. A similar reasoning shows the reverse inequality. Lemma 3.5 implies $P(\beta\varphi, \sigma) = P(\beta\phi, f)$. This completes the proof of Proposition 3.8. □

4. *Building-induced expansion*

The aim of this section is to construct from the Bowen–Series map f a uniformly expanding induced Markov map $\tilde{f}$. We construct the induced Markov map $\tilde{f}$ in §4.1 as a first return map to a large subset of Δ which misses small neighbourhoods of the cusps. Although this construction is essentially the same as in [7], to build a uniform expansion without assuming the non-contracting condition in equation (1.1), we use a linear growth lemma on induced scale (Lemma 4.3) that relies on a geometric ingredient developed in §4.2. Finally, in §4.3, we verify the uniform expansion of $\tilde{f}$.

4.1. *Construction of an induced Markov map.*

Let f be the Bowen–Series map with the finite Markov partition $(\Delta(a))_{a\in S}$ constructed in §3.2. Note that $\Delta(a) \cap \Lambda \neq \emptyset$ for $a \in S$. Define the inducing domain

$$\Delta_0 = \Delta \setminus \left(\bigcup_{v \in V_c} L(v) \cup R(v)\right)$$

and the first return time $t \colon \Delta_0 \to \mathbb{N} \cup \{\infty\}$ to Δ_0 by

$$t(\xi) = \inf\{n \geq 1 \colon f^n(\xi) \in \Delta_0\}.$$

Define

$$\tilde{\Delta} = \{\xi \in \Delta_0 \colon t(\xi) < +\infty\}.$$

Note that both Δ_0 and $\tilde{\Delta}$ are non-empty sets, which are illustrated in Figure 5. We now define the induced map

$$\tilde{f} \colon \tilde{\Delta} \to \mathbb{S}^1, \quad \xi \mapsto f^{t(\xi)}(\xi),$$

and set

$$\tilde{\Lambda} = \bigcap_{n=0}^{\infty} \tilde{f}^{-n}(\tilde{\Delta}).$$

Replacing each $\Delta(a)$, $a \in S$, by the countably many cylinders on which t is finite and constant, we obtain a Markov partition for $\tilde{f}$ given by the sets $(\tilde{\Delta}(\tilde{a}))_{\tilde{a}\in\tilde{S}}$, where $\tilde{S}$ is a countably infinite subset of $E(X)$ and each $\tilde{\Delta}(\tilde{a})$ has the form $\tilde{\Delta}(\tilde{a}) = \Delta(a_1) \cap \{t = n\} \cap f^{-n}(\Delta(a_2))$ for some $n \geq 1$ and $a_1, a_2 \in S$. This determines by equation (3.1) a countable Markov shift

$$\tilde{X} = \tilde{X}(\tilde{f}, (\tilde{\Delta}(\tilde{a}))_{\tilde{a}\in\tilde{S}}).$$

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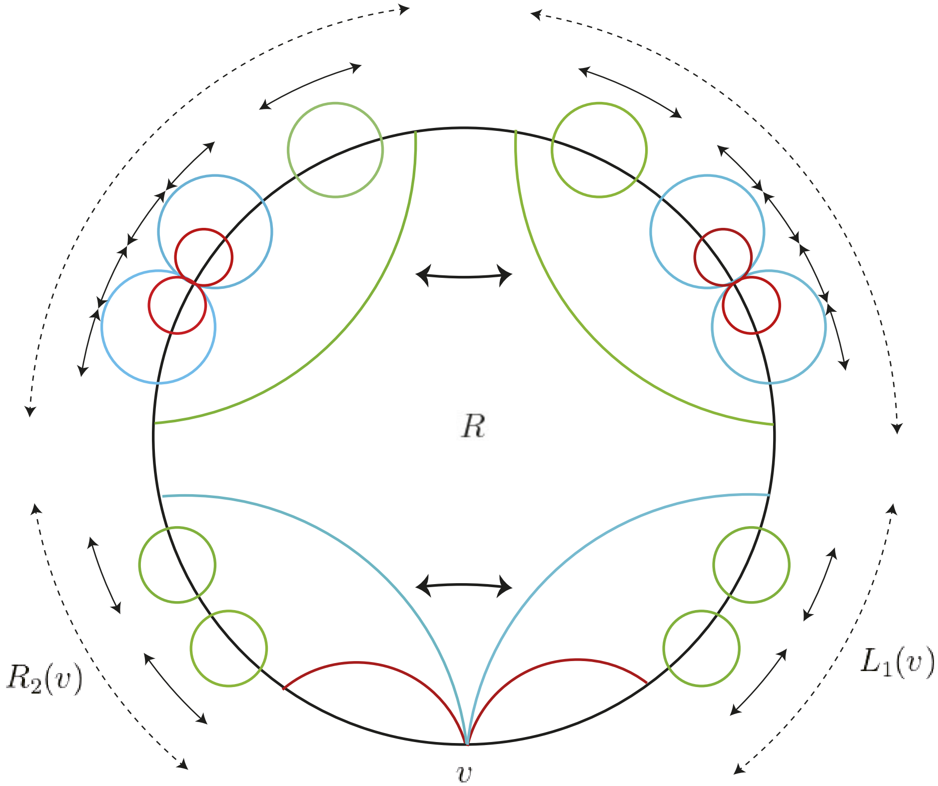

FIGURE 5. A free Fuchsian group with two generators and one cusp v in its fundamental domain R. The sides of R with the same colour are identified. The coloured complete circles are mapped to the bigger ones with the same colour (partially drawn) by the corresponding Möbius transformations defining f. The outer dotted bidirectional arrows altogether indicate the inducing domain $\Delta_0 \subset \mathbb{S}^1$. The inner bidirectional arrows altogether indicate the domain $\tilde{\Delta}$ that is contained in Δ_0. The first return time t to Δ_0 is 1 except at points in the four red circles where it is 2, 3, 4, ...

4.2. *Control of deviations of cutting orbits.* Let $\gamma \in \mathscr{R}$ with the infinite cutting sequence $(g_n)_{n=0}^{\infty}$. If G has a parabolic element, R has a cusp and the cutting orbit $(g_0 \cdots g_n 0)_{n=0}^{\infty}$ may deviate from γ. In this subsection, we elaborate on uniform bounds on this deviation using $\tilde{f}$.

For $\xi \in \mathbb{D}$ and $A \subset \mathbb{D}$, we denote $d(\xi, A) = \inf\{d(\xi, \eta) \colon \eta \in A\}$.

LEMMA 4.1. *There exist $C_0 > 0$ and an integer $M_0 \geq 1$ such that if $\gamma \in \mathscr{R}$ satisfies $\gamma^+ \in \Lambda_c$ and $f^k(\gamma^+) \in \tilde{\Delta}$ for some $k \geq M_0$, then there exists $n_* \in \{k - M_0, \dots, k + M_0\}$ such that*

$$d(g_0 \cdots g_{n_*} 0, \gamma \cap g_0 \cdots g_{n_*} R) \leq C_0,$$

where $(g_n)_{n=0}^{\infty}$ denotes the cutting sequence of γ.

Proof. Let $C_0 > 0$ be so large that the hyperbolic disk around 0 of radius C_0 covers all of the intersection of R and the Nielsen region of G [3, §8.5] except small neighbourhoods of the cusps. Let $M_0 \geq 1$ be a large number to be determined later.

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Suppose for a contradiction the assertion of the lemma fails. Then there exists $\gamma \in \mathscr{R}$ with infinite cutting sequence $(g_n)_{n=0}^{\infty}$ (see Lemma 2.2) and there exists $k \geq M_0$ such that $f^k(\gamma^+) \in \tilde{\Delta}$ and, for every $n \in \{k - M_0, \ldots, k + M_0\}$,

$$d(g_0 \cdots g_n 0, \gamma \cap g_0 \cdots g_n R) > C_0.$$

This means that γ performs a deep cusp excursion between the $(k - M_0)$th and the $(k + M_0)$th crossing of fundamental domains and therefore, the cutting symbols $g_{k-M_0}, \ldots, g_{k+M_0}$ of γ are given by the periodic sequence of labels of sides ending at one of the cusps of R, say $v_0 \in V_c$. We conclude by Lemmas 2.4 and 2.7 that the partial BS orbit $(a_0 \cdots a_n 0)_{n:\, |n-k| \leq M_0 - 2}$ appears in the same order in the partial cutting orbit $(g_0 \cdots g_n 0)_{n:\, |n-k| \leq M_0}$. This implies that the first $M_0 - 2$ symbols of the f-expansion of $f^k(\gamma^+)$ are given by the periodic sequence of sides ending at the cusp v_0. If M_0 is large enough depending on the prime periods of the cusps, this implies $f^k(\gamma^+) \in L(v_0) \cup R(v_0)$ contradicting $f^k(\gamma^+) \in \tilde{\Delta}$. □

PROPOSITION 4.2. *There exists $C > 0$ such that for all $n \geq 1$ sufficiently large and $\tilde{\omega}_0 \cdots \tilde{\omega}_{n-1} \in E^n(\tilde{X})$, and for all $\gamma \in \mathscr{R}$ with $\gamma^+ \in \tilde{\Delta}(\tilde{\omega}_0 \cdots \tilde{\omega}_{n-1}) \cap \Lambda_c$,*

$$d(\tilde{\omega}_0 \cdots \tilde{\omega}_{n-1} 0, \gamma) \leq C.$$

Proof. Let C_0 and M_0 denote the constants in Lemma 4.1. Let $n > M_0$. There exists $k \geq n$ and $a_1 \cdots a_k \in E^k(\Sigma^+)$ such that $\tilde{\omega}_0 \cdots \tilde{\omega}_{n-1} = a_1 \cdots a_k$ and $f^k(\gamma^+) = (\tilde{f})^n(\gamma^+) \in \tilde{\Delta}$. By Lemma 4.1, there exists $n_* \in \{k - M_0, \ldots, k + M_0\}$ such that $d(g_0 \cdots g_{n_*} 0, \gamma \cap g_0 \cdots g_{n_*} R) \leq C_0$. By the triangle inequality, we have

$$d(a_1 \cdots a_k 0, \gamma) \leq 2M_0 \max_{g \in G_R} d(0, g0) + d(a_1 \cdots a_{n_*} 0, g_0 \cdots g_{n_*} 0) + C_0.$$

Since the second term of the right-hand side does not exceed $n(R) \max_{g \in G_R}\{d(0, g0)\}$ by Lemma 2.7, the proposition follows. □

4.3. *Uniform expansion of the induced map.* If the Fuchsian group G has no parabolic element, that is, G is convex cocompact, then the next lemma follows from the Švarc–Milnor lemma.

LEMMA 4.3. (Linear growth on induced scale) *There exists $\alpha_0 > 0$ such that for all sufficiently large $n \geq 1$ and every $\tilde{\omega}_0 \cdots \tilde{\omega}_{n-1} \in E^n(\tilde{X})$, we have*

$$d(0, \tilde{\omega}_0 \cdots \tilde{\omega}_{n-1} 0) \geq \alpha_0 n.$$

Proof. Let C_0 and M_0 denote the constants in Lemma 4.1. Let $n > M_0$ and $\tilde{\omega}_0 \cdots \tilde{\omega}_{n-1} \in E^n(\tilde{X})$. Let $\gamma \in \mathscr{R}$ such that $\gamma^+ \in \Lambda_c \cap \tilde{\Delta}(\tilde{\omega}_0 \cdots \tilde{\omega}_{n-1})$ with the cutting sequence $(g_j)_{j=0}^{\infty}$. By Lemma 4.1, for each $k \in \{M_0, \ldots, n-1\}$, we fix an integer $j(k)$ such that $|j(k) - |\tilde{\omega}_0 \cdots \tilde{\omega}_k|| \leq M_0$ and

$$d(g_0 \cdots g_{j(k)} 0, \gamma \cap g_0 \cdots g_{j(k)} R) \leq C_0. \tag{4.1}$$

We write all the distinct elements of the sequence $j(M_0), \ldots, j(n-1)$ as $j_1, j_2, \ldots, j_q$ in the increasing order with some $q \geq (n - 1 - M_0)/(2M_0)$. For each $1 \leq k \leq q$, there

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exists $p_k \in \gamma \cap g_0 \cdots g_{j_k} R$ such that $d(g_0 \cdots g_{j_k} 0, p_k) \leq C_0$. Using equation (4.1) and Lemma 2.7, we derive the existence of a uniform constant $C' > 0$ such that

$$d(0, \tilde{\omega}_0 \cdots \tilde{\omega}_{n-1} 0) \geq d(p_1, p_q) - C'.$$

Divide the geodesic segment from p_1 to p_q into segments of hyperbolic length C_0, with one shorter segment, say $\gamma_1, \ldots, \gamma_N$, for some $N \geq 1$. By equation (4.1), for each $1 \leq \ell \leq q$, the orbit point $g_0 \cdots g_{j_\ell} 0$ is within the hyperbolic distance C_0 of one of the geodesic segments $\gamma_1, \ldots, \gamma_N$.

Partition the set $\{g_0 \cdots g_{j_\ell} 0 \colon 1 \leq \ell \leq q\}$ into subsets $O_1, \ldots, O_N$ so that $d(O_k, \gamma_k) \leq C_0$ for $1 \leq k \leq N$. Since G acts properly discontinuously on $\mathbb{D}$, there exists an integer $M \geq 1$ such that $\# O_k \leq M$ for $1 \leq k \leq N$. Hence, $N \geq q/M$. Combining this with equation (4.1) yields

$$d(0, \tilde{\omega}_0 \cdots \tilde{\omega}_{n-1} 0) \geq C_0 \bigg(\frac{q}{M} - 1 \bigg) - C' \geq C_0 \bigg(\frac{n - 1 - M_0}{2 M_0 M} - 1 \bigg) - C'.$$

Hence, the lemma follows for $\alpha_0 = C_0/(3 M_0 M)$ and sufficiently large n. □

PROPOSITION 4.4. *There exists $\alpha_0 > 0$ such that for all sufficiently large $n \geq 1$, we have*

$$\inf_{\tilde{\omega}_0 \cdots \tilde{\omega}_{n-1} \in E^n(\tilde{X})} \inf_{\xi \in \tilde{\Delta}(\tilde{\omega}_0 \cdots \tilde{\omega}_{n-1}) \cap \Lambda_c} |(\tilde{f}^n)' \xi| \gg e^{\alpha_0 n}$$

and

$$\operatorname{diam}(\tilde{\Delta}(\tilde{\omega}_0 \cdots \tilde{\omega}_{n-1}) \cap \Lambda_c) \ll e^{-\alpha_0 n}.$$

Proof. Let $C > 0$ denote the constant in Proposition 4.2. Let $n \geq 1$ and $\tilde{\omega}_0 \cdots \tilde{\omega}_{n-1} \in E^n(\tilde{X})$ and let $\xi \in \tilde{\Delta}(\tilde{\omega}_0 \cdots \tilde{\omega}_{n-1}) \cap \Lambda_c$. Let $\gamma \in \mathscr{R}$ be the ray through zero with $\gamma^+ = \xi$. By Proposition 4.2, for all sufficiently large $n \geq 1$, we have

$$d(\tilde{\omega}_0 \cdots \tilde{\omega}_{n-1} 0, \gamma) \leq C. \tag{4.2}$$

Since $\tilde{f}^n(\gamma^+) = (\tilde{\omega}_0 \cdots \tilde{\omega}_{n-1})^{-1} \gamma^+$, it follows from the well-known properties of the Poisson kernel [3] that

$$\log |(\tilde{f}^n)' \gamma^+| = d(0, p),$$

where $p \in \mathbb{D}$ denotes the point of intersection between γ and the horocircle at γ^+ through $\tilde{\omega}_0 \cdots \tilde{\omega}_{n-1} 0$. By equation (4.2), we have $d(p, \tilde{\omega}_0 \cdots \tilde{\omega}_{n-1} 0) \leq 2C$ and thus,

$$|\log |(\tilde{f}^n)' \gamma^+| - d(0, \tilde{\omega}_0 \cdots \tilde{\omega}_{n-1} 0)| \leq 2C.$$

The first assertion of the proposition now follows from Lemma 4.3.

To prove the second assertion, first note that the estimate in equation (4.2) remains intact if $\gamma \in \mathscr{R}$ is a ray through zero whose endpoint γ^+ is in between two points in $\tilde{\Delta}(\tilde{\omega}_0 \cdots \tilde{\omega}_{n-1}) \cap \Lambda_c$. Consequently, the first assertion of the proposition also holds if ξ is taken from the smallest arc in $\mathbb{S}^1$ containing $\tilde{\Delta}(\tilde{\omega}_0 \cdots \tilde{\omega}_{n-1}) \cap \Lambda_c$. From this and the mean value theorem, the second assertion of the proposition follows. □

LEMMA 4.5. *We have $\alpha_- < \alpha_+$.*

Proof. If G has a parabolic element, then $\alpha_- = 0$ by Lemma 2.12, and $\alpha_+ > 0$ since the induced Markov interval map $\tilde{f}$ is uniformly expanding by Proposition 4.4. If G has no parabolic element, it follows from [19, Corollary 11.3] that the function φ in equation (3.7) is not cohomologous to a constant. Since f is piecewise C^2 and some iterate of f is uniformly expanding, φ is Hölder continuous. By a standard argument [6, Proposition 4.5], we conclude that $\alpha_- < \alpha_+$. □

5. *Thermodynamic formalism and multifractal analysis*

In this section, we implement the thermodynamic formalism and the multifractal analysis for the Bowen–Series map. In §5.1, we establish the uniqueness of equilibrium states and the analyticity of the geometric pressure function. In §§5.2 and 5.3, we apply results in [14] to obtain formulae for the Hausdorff dimension of level sets and the limit set. In §5.4, we derive formulae for the $\mathscr{H}$-spectrum and its first-order derivative in terms of the pressure. In §5.5, we complete the proof of the Main Theorem.

5.1. *Uniqueness of equilibrium states, regularity of pressure.*

The next proposition is a key ingredient for the proofs of our main results. The proof relies heavily on the existence of an induced system which is uniformly expanding (see Proposition 4.4). Except for this geometrical fact, the arguments are well known, and can be found in [17], [21, §8] and [27], for example. For the convenience of the reader, we include a proof in Appendix A.

PROPOSITION 5.1. *The Bowen–Series map f satisfies all of the following.*

(a) *For any $\beta \in (-\infty, \beta_+)$, there exists a unique equilibrium state for the potential $-\beta \log |f'|$, denoted by μ_β. We have $\beta_+ = +\infty$ if and only if G has no parabolic element.*

(b) *The geometric pressure function P is analytic on $(-\infty, \beta_+)$.*

(c) *For all $\beta \in (-\infty, \beta_+)$, $P'(\beta) = -\chi(\mu_\beta)$. In particular, the function $\beta \in (-\infty, \beta_+) \mapsto \chi(\mu_\beta)$ is analytic.*

5.2. *Dimension formula for level sets.*

We recall a few relevant definitions in [14]. A measure $\mu \in \mathcal{M}(\Lambda, f)$ is *expanding* if $\chi(\mu) > 0$. The *dimension* of a measure $\mu \in \mathcal{M}(\Lambda, f)$ is defined by

$$\dim(\mu) = \begin{cases} \dfrac{h(\mu)}{\chi(\mu)} & \text{if } \mu \text{ is expanding,} \\ 0 & \text{otherwise.} \end{cases}$$

For an ergodic expanding measure μ, the dimension $\dim(\mu)$ is equal to the infimum of the Hausdorff dimensions of sets with full μ-measure (see e.g. [21, Theorem 4.4.2]). In particular, $\delta_G \geq \dim(\mu)$ holds for any $\mu \in \mathcal{M}(\Lambda, f)$.

We say f is *saturated* if

$$\delta_G = \sup\{\dim(\mu) \colon \mu \in \mathcal{M}(\Lambda, f)\}. \tag{5.1}$$

If G has no parabolic element, it is known [5, 37] that the supremum in equation (5.1) is attained by a unique element and, in particular, f is saturated. This unique measure

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is equivalent to the normalized δ_G-dimensional Hausdorff measure on Λ [25, 39]. The saturation is important because it ensures that the the dimension formula in [14, Main Theorem] accounts for any level set of positive Hausdorff dimension. Even in the case where G has a parabolic element, the saturation still holds, although there is no measure which attains the supremum in equation (5.1).

PROPOSITION 5.2. *The Bowen–Series map f is saturated.*

Proof. The case where G has no parabolic element has already been explained. Suppose G has a parabolic element. If equation (1.1) holds, then f is a non-uniformly expanding, finitely irreducible Markov map in the sense of [14]. Since $\Lambda \setminus \bigcup_{n=0}^{\infty} f^{-n}(V_c)$ is contained in $\bigcup_{n=0}^{\infty} f^{-n}(\tilde{\Lambda})$ and V_c is countable, we have $\dim_{\mathrm{H}}(\Lambda) = \dim_{\mathrm{H}}(\tilde{\Lambda})$. Hence, f is saturated by [14, Proposition 5.2(c)]. Even if equation (1.1) does not hold, we have shown in Proposition 4.4 that some power of the induced Markov map $\tilde{f}$ is uniformly expanding. Hence, the argument in the proof of [14, Proposition 5.2(c)] works almost verbatim to conclude that f is saturated. □

PROPOSITION 5.3. *The Bowen–Series map f satisfies all of the following.*

(a) *We have $\mathscr{H}(\alpha) \neq \emptyset$ if and only if $\alpha \in [\alpha_-, \alpha_+]$.*

(b) *For all $\alpha \in [\alpha_-, \alpha_+]$, we have*

$$b(\alpha) = \lim_{\varepsilon \to 0} \sup\{\dim(\mu) \colon \mu \in \mathcal{M}(\Lambda, f),\ |\chi(\mu) - \alpha| < \varepsilon\}. \tag{5.2}$$

(c) *For all $\alpha \in [\alpha_-, \alpha_+] \setminus \{0\}$, we have*

$$b(\alpha) = \max\{\dim(\mu) \colon \mu \in \mathcal{M}(\Lambda, f),\ \chi(\mu) = \alpha\}.$$

Proof. Proposition 2.10 gives $\mathscr{H}(\alpha) = \mathscr{L}(\alpha)$, and so $b(\alpha) = \dim_{\mathrm{H}} \mathscr{L}(\alpha)$. If G has no parabolic element, then some power of f is uniformly expanding [37, Theorem 5.1], and so the result is well known, see for example [24, 26, 28, 29, 42].

Suppose G has a parabolic element. The assertion in item (a) follows from [14, Main Theorem(a)]. To derive the desired formula in item (b), we aim to apply [14, Main Theorem(b)]. By Proposition 3.1, f is a finitely irreducible Markov map. By Proposition 2.8, f has mild distortion, and by Proposition 5.2, f is saturated. In addition to these conditions, in [14, Main Theorem(b)], it is assumed that the map satisfies a non-contracting condition as in equation (1.1). However, the non-contracting condition was used in [14] only to ensure the non-existence of points with negative pointwise Lyapunov exponent. Although we do not assume the Bowen–Series map f satisfies equation (1.1), the formulae in [14, Main Theorem(b)] remain intact for the level sets $\mathscr{L}(\alpha)$ since all points in these sets have non-negative pointwise Lyapunov exponents. This proves the desired formula in item (b).

Let $\alpha \in [\alpha_-, \alpha_+] \setminus \{0\}$. To remove the limit $\varepsilon \to 0$ in equation (5.2), we use Lemma 3.5 to transfer the problem to $\mathcal{M}(X, \sigma)$. Since the function $\varphi : X \to \mathbb{R}$ is continuous and the entropy function is upper semicontinuous on the compact space $\mathcal{M}(X, \sigma)$, we can choose a convergent sequence $\{\mu_n\}_{n=1}^{\infty}$ in $\mathcal{M}(X, \sigma)$ with positive entropy such that its weak* limit point μ_∞ is an expanding measure satisfying $\dim(\mu_\infty \circ \pi^{-1}) = b(\alpha)$. This yields the desired formula in item (c). □

5.3. *Bowen's formula.* The next type of formula, first established in [**5**] for Fuchsian groups without parabolic elements, is called Bowen's formula. It is known for conformal graph directed Markov systems [**21**, Theorem 4.2.13] which, in dimension one, correspond to uniformly expanding finitely irreducible Markov maps. Bowen's formula is also known for parabolic iterated function systems [**21**, Theorem 8.3.6] and essentially free Kleinian groups with parabolic elements [**17**].

PROPOSITION 5.4. (Bowen's formula) *We have*

$$\delta_G = \min\{\beta \geq 0\colon P(\beta) = 0\}.$$

Proof. Put $\delta_0 = \sup\{\dim(\mu)\colon \mu \in \mathcal{M}(\Lambda, f)\}$. Since f is saturated by Proposition 5.2, we have $\delta_0 = \delta_G$. Set $\delta_1 = \min\{\beta \geq 0\colon P(\beta) = 0\}$. It suffices to show $\delta_0 = \delta_1$. By definition, $\delta_0 \geq \dim(\mu)$ holds for any expanding measure $\mu \in \mathcal{M}(\Lambda, f)$. Hence, $P(\delta_0) \leq 0$ and so $\delta_0 \geq \delta_1$. Suppose for a contradiction that $\delta_0 > \delta_1$. Then there exists $\varepsilon > 0$ such that $\delta_0 > \delta_1 + \varepsilon$ and an expanding measure μ such that $\dim(\mu) > \delta_1 + \varepsilon$, and so $P(\delta_1 + \varepsilon) > 0$. However, by the definition of δ_1 and the monotonicity of pressure, we have $P(\delta_1 + \varepsilon) \leq P(\delta_1 + \varepsilon/2) \leq 0$, and a contradiction arises. Therefore, $\delta_0 = \delta_1$ holds. □

5.4. *Dimension formula for level sets in terms of pressure.* We call $\mu \in \mathcal{M}(\Lambda, f)$ satisfying $\dim(\mu) = \delta_G$ a *measure of maximal dimension for G*. For the proof of the next proposition, we refer the reader to Appendix A.2.

PROPOSITION 5.5. *There exists a measure of maximal dimension for G if and only if G has no parabolic element.*

LEMMA 5.6. *If G has a parabolic element, then* $\lim_{\beta \nearrow \beta_+} P'(\beta) \geq 0$.

Proof. By Proposition 5.1(a), we have $\beta_+ < \infty$. Suppose for a contradiction that $\lim_{\beta \nearrow \beta_+} P'(\beta) < 0$. Take a sequence $\{\beta_n\}_{n=1}^{\infty}$ with $\beta_n \nearrow \beta_+$ as $n \to \infty$ and $\lim_{n\to\infty} P'(\beta_n) < 0$. Let μ_{β_n} be the equilibrium state for the potential $-\beta_n \log |f'|$ and let μ be an weak* accumulation point of $\{\mu_{\beta_n}\}_{n=1}^{\infty}$. Recall that X is a finite Markov shift where the entropy function is upper semicontinuous, and the function $\varphi\colon X \to \mathbb{R}$ in equation (3.7) is continuous. Hence, μ is an equilibrium state for the potential $-\beta_+ \log |f'|$, namely, $h(\mu) - \beta_+\chi(\mu) = 0$. Since $P'(\beta_n) = -\chi(\mu_{\beta_n})$ by Proposition 5.1(c), we have $\chi(\mu) = \lim_{n\to\infty} \chi(\mu_{\beta_n}) = -\lim_{n\to\infty} P'(\beta_n) > 0$. By Proposition 5.4, μ is a measure of maximal dimension for G, which is a contradiction to Proposition 5.5. □

PROPOSITION 5.7. *If G has a parabolic element, then the pressure is equal to zero on* $[\beta_+, +\infty)$*,* $\beta_+ = \delta_G$ *and the pressure function P is* C^1 *on* $\mathbb{R}$*. Moreover, P is strictly convex on* $(-\infty, \delta_G)$*.*

Proof. Lemma 2.12 gives $\alpha_- = 0$. By the definition of β_+, the pressure is equal to zero on $[\beta_+, +\infty)$. By Proposition 5.4, we have $\beta_+ = \delta_G$. By Proposition 5.1(b), P is analytic on $(-\infty, \beta_+)$. The continuous differentiability of P at $\beta = \delta_G$ is a consequence of Lemma 5.6 and the convexity of P. This shows that P is C^1 on $\mathbb{R}$. Since P is analytic, convex and non-increasing on $(-\infty, \delta_G)$, an elementary inductive argument on the power series expansion of P shows that either P is affine on $(-\infty, \delta_G)$ or strictly convex on $(-\infty, \delta_G)$. The first case is ruled out by Lemma 5.6 and the assumption that G is non-elementary, which gives $\delta_G > 0$. □

From Lemma 3.6, Proposition 3.8 and Proposition 5.7, we have

$$\alpha_+ = -\lim_{\beta \searrow -\infty} P'(\beta) \quad \text{and} \quad \alpha_- = -\lim_{\beta \nearrow \beta_+} P'(\beta).$$

By Proposition 5.1(c), Proposition 5.7 and the implicit function theorem, there exists a strictly decreasing analytic function $\beta\colon (\alpha_-, \alpha_+) \to (-\infty, \beta_+)$ satisfying $-P'(\beta(\alpha)) = \chi(\mu_{\beta(\alpha)}) = \alpha$. We have

$$\lim_{\alpha \searrow \alpha_-} \beta(\alpha) = \beta_+ \quad \text{and} \quad \lim_{\alpha \nearrow \alpha_+} \beta(\alpha) = -\infty. \tag{5.3}$$

PROPOSITION 5.8. *For all $\alpha \in (\alpha_-, \alpha_+)$, we have*

$$\alpha b(\alpha) = P(\beta(\alpha)) + \alpha\beta(\alpha) \quad \text{and} \quad b(\alpha) = \frac{P^*(-\alpha)}{\alpha}. \tag{5.4}$$

Moreover, the $\mathscr{H}$-spectrum is analytic on (α_-, α_+) and satisfies

$$b'(\alpha) = \frac{-P(\beta(\alpha))}{\alpha^2}. \tag{5.5}$$

Proof. We have $P(\beta(\alpha)) + \alpha\beta(\alpha) = h(\mu_{\beta(\alpha)}) - \beta(\alpha)\alpha + \alpha\beta(\alpha) = h(\mu_{\beta(\alpha)}) \leq \alpha b(\alpha)$, where the last inequality follows from Proposition 5.3(c). Again by Proposition 5.3(c), there exists an expanding measure $\mu \in \mathcal{M}(\Lambda, f)$ such that $\chi(\mu) = \alpha$ and $\dim(\mu) = b(\alpha)$. Then, $\alpha b(\alpha) = h(\mu) = h(\mu) - \beta(\alpha)\alpha + \alpha\beta(\alpha) \leq P(\beta(\alpha)) + \alpha\beta(\alpha)$, and so the first equality in equation (5.4) holds.

The second equality in equation (5.4) follows from the first. Since P and β are analytic, the $\mathscr{H}$-spectrum is analytic on (α_-, α_+) by the first formula in equation (5.4). Differentiating the first equality in equation (5.4), and combining with the first equality in equation (5.4) and the fact that $P'(\beta(\alpha)) = -\alpha$, yields the equality in equation (5.5). □

5.5. *Proof of the Main Theorem.* Lemma 4.5 gives $\alpha_- < \alpha_+$. By Proposition 5.3(a), we have $\mathscr{H}(\alpha) \neq \emptyset$ if and only if $\alpha \in [\alpha_-, \alpha_+]$. The analyticity of the $\mathscr{H}$-spectrum on (α_-, α_+) is due to Proposition 5.8. Equation (5.2) implies that the $\mathscr{H}$-spectrum is upper semicontinuous on $[\alpha_-, \alpha_+]$. The lower semicontinuity of the spectrum at $\alpha \in \{\alpha_-, \alpha_+\} \setminus \{0\}$ can be derived from Proposition 5.3(c). To prove this, we may assume that $b(\alpha) > 0$ and denote by $\mu \in \mathcal{M}(\Lambda, f)$ an expanding measure with $\dim(\mu) = b(\alpha)$. Let $\alpha' \in (\alpha_-, \alpha_+)$ with $\alpha' \neq \alpha$ and $\mu' \in \mathcal{M}(\Lambda, f)$ such that $\chi(\mu') = \alpha'$. Applying Proposition 5.3(c) to convex combinations $p\mu + (1-p)\mu'$ and letting $p \to 1$ shows that the spectrum is lower semicontinuous at α. The remaining case $\alpha_- = 0$ is covered by [**14**, Main Theorem(b)(ii)]. We have thus shown that b is continuous on $[\alpha_-, \alpha_+]$. By the second formula in equation (5.4), we have $b(\alpha) = P^*(-\alpha)/\alpha$ for $\alpha \in (\alpha_-, \alpha_+)$. Since P^* is continuous on $[\alpha_-, \alpha_+]$, this formula extends to $[\alpha_-, \alpha_+] \setminus \{0\}$.

To complete the proof of item (a), we define

$$\alpha_G = -P'(\delta_G). \tag{5.6}$$

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If G has no parabolic element, we have $\beta(\alpha_G) = \delta_G$, and so $b'(\alpha_G) = 0$, and $b'(\alpha)(\alpha - \alpha_G) < 0$ for $\alpha \in (\alpha_-, \alpha_+) \setminus \{\alpha_G\}$ by equation (5.5). If G has a parabolic element, then Proposition 5.7 gives $\alpha_G = 0$, and [**14**, Main Theorem(b)(ii)] gives $\lim_{\alpha \searrow \alpha_-} b(\alpha) = b(\alpha_-) = \delta_G$. Moreover, equation (5.5) implies $b'(\alpha) < 0$ for $\alpha \in (\alpha_-, \alpha_+)$, and so the $\mathscr{H}$-spectrum is strictly monotone decreasing on $[\alpha_-, \alpha_+]$.

Finally, it follows from equations (5.3) and (5.5) that $\lim_{\alpha \nearrow \alpha_+} b'(\alpha) = -\infty$. Similarly, if G has no parabolic element, then equations (5.3) and (5.5) give $\lim_{\alpha \searrow \alpha_-} b'(\alpha) = +\infty$. The proof of item (a) is complete. The assertions in item (b) follow from Proposition 5.7.

Finally, we show that the regularity of the pressure is related to the existence of an inflection point in the spectrum.

PROPOSITION 5.9. *If G has a parabolic element and the geometric pressure function is C^2, then $P''(\delta_G) = 0$ and the $\mathscr{H}$-spectrum has an inflection point.*

Proof. Recall that $\beta(\alpha)$ is the unique solution of the equation $P'(\beta) + \alpha = 0$. By the implicit function theorem, $\alpha \in (\alpha_-, \alpha_+) \mapsto \beta(\alpha)$ is differentiable and

$$\beta'(\alpha) = -\frac{1}{P''(\beta(\alpha))}. \tag{5.7}$$

We apply l'Hôpital's rule to equation (5.5) together with equation (5.7) to obtain $\lim_{\alpha \searrow \alpha_-} b'(\alpha) = -\lim_{\beta \nearrow \delta_G} 1/(2P''(\beta))$ when one of the two limits exists. Since $P''(\beta) > 0$ for $\beta < \delta_G$, we obtain $\lim_{\alpha \searrow \alpha_-} b'(\alpha) = -\infty$. Since $\lim_{\alpha \nearrow \alpha_+} b'(\alpha) = -\infty$ by the Main Theorem, we must have $b''(\alpha_*) = 0$ for some $\alpha_* \in (\alpha_-, \alpha_+)$. □

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A. *Appendix. Supplementary proofs*

A.1. *Proof of Proposition 5.1.* All the statements for G without parabolic elements are well known [**6**, **34**], since some power of f is uniformly expanding [**37**, Theorem 5.1] in this case. Hence, we assume G has a parabolic element. Our strategy is to apply to $\tilde{X}$ the results in [**21**] on the thermodynamic formalism for countable Markov shifts.

Let $\tilde{\sigma}\colon \tilde{X} \to \tilde{X}$ denote the left shift. We write $\tilde{\pi}$ for $\pi_{\tilde{X}}$ and $\tilde{t}$ for $t \circ \tilde{\pi}$. Note that $\tilde{t}$ is constant on each 1-cylinder $\tilde{\Delta}(\tilde{a})$ in $\tilde{X}$. Let $\tilde{t}(\tilde{a})$ denote the constant value of $\tilde{t}$ on each partition element $\tilde{\Delta}(\tilde{a})$. For $n \geq 1$, we set

$$\tilde{S}(n) = \{\tilde{a} \in \tilde{S} \colon \tilde{t}(\tilde{a}) = n\}.$$

LEMMA A.1. *There exists $n_0 \geq 2$ such that for all $n \geq n_0$, we have $1 \leq \#\tilde{S}(n) \leq (\#S)^3$.*

Proof. From the definition of the Markov map $\tilde{f}$ in §4.1, for each $\tilde{a} \in \tilde{S}$ with $\tilde{t}(\tilde{a}) = n \geq 2$, there exists a unique element $\omega_0\omega_1 \cdots \omega_n$ of $E^{n+1}(X)$ with $\Delta(\omega_0\omega_1 \cdots \omega_n) = \tilde{\Delta}(\tilde{a})$. Let v denote the cusp that is contained in $\mathrm{cl}(\Delta(\omega_1))$. Since $f^i(v) \in \mathrm{cl}(\Delta(\omega_{i+1}))$ for $0 \leq i \leq n-1$, the sequence $\omega_0\omega_1 \cdots \omega_n$ is determined by the three symbols ω_0, ω_1 and ω_n in S. Hence, the upper bound follows. The lower bound immediately follows from equation (3.5). $\square$

LEMMA A.2. *For any $\tilde{a} \in \tilde{S}$ and $\xi \in \tilde{\Delta}(\tilde{a})$, we have $|(\tilde{f})'\xi| \asymp \tilde{t}(\tilde{a})^2$.*

Proof. Follows from [7, Lemma 2.8]. $\square$

For $(\beta, \zeta) \in \mathbb{R}^2$, we define an induced potential $\Phi_{\beta,\zeta} : \tilde{X} \to \mathbb{R}$ by

$$\Phi_{\beta,\zeta}(\tilde{x}) = -\beta \log |(\tilde{f})'\tilde{\pi}(\tilde{x})| - \zeta \tilde{t}(\tilde{x}),$$

and an induced pressure

$$\mathscr{P}(\beta, \zeta) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\tilde{\omega}_0\cdots\tilde{\omega}_{n-1}\in E^n(\tilde{X})} \sup_{[\tilde{\omega}_0\cdots\tilde{\omega}_{n-1}]} \exp\bigg(\sum_{k=0}^{n-1} \Phi_{\beta,\zeta} \circ \tilde{\sigma}^k\bigg),$$

under the cylinder notation in equation (3.2). Since logarithm of the series is sub-additive in n, this limit exists and is never $-\infty$. By [21, Theorem 2.1.8], the variational principle holds:

$$\mathscr{P}(\beta, \zeta) = \sup\bigg\{h(\tilde{\mu}) + \int \Phi_{\beta,\zeta}\, d\tilde{\mu} : \tilde{\mu} \in \mathcal{M}(\tilde{X}, \tilde{\sigma}), \int \Phi_{\beta,\zeta}\, d\tilde{\mu} > -\infty\bigg\}.$$

In the case $\mathscr{P}(\beta, \zeta) < +\infty$, measures which attain this supremum are called *equilibrium states* for the potential $\Phi_{\beta,\zeta}$.

We aim to verify sufficient conditions in [21, Theorem 2.2.9, Corollary 2.7.5] for the existence and uniqueness of a shift-invariant Gibbs state and the equilibrium state for the potential $\Phi_{\beta,\zeta}$. (The term 'bounded function' in [21, Corollary 2.7.5] should be 'function bounded from above'.) We say $\Phi_{\beta,\zeta}$ is *summable* if

$$\sum_{\tilde{a}\in\tilde{S}} \sup_{[\tilde{a}]} \exp \Phi_{\beta,\zeta} < +\infty.$$

It is easy to see that the summability of $\Phi_{\beta,\zeta}$ implies the finiteness of $\mathscr{P}(\beta, \zeta)$.

LEMMA A.3. *The potential $\Phi_{\beta,\zeta}$ is summable if and only if $\zeta > 0$ or $(\beta, \zeta) = (\beta, 0)$ with $\beta > 1/2$.*

Proof. By combining Lemmas A.1 and A.2. $\square$

Recall that $d_{\tilde{X}}$ denotes the metric on $\tilde{X}$. A function $\Psi : \tilde{X} \to \mathbb{R}$ is *locally Hölder continuous* if there exist constants $C > 0$ and $\theta \in (0, 1]$ such that for any $\tilde{a} \in \tilde{S}$ and all $\tilde{x}, \tilde{y} \in [\tilde{a}]$, we have

$$|\Psi(\tilde{x}) - \Psi(\tilde{y})| \le C(d_{\tilde{X}}(\tilde{x}, \tilde{y}))^\theta.$$

LEMMA A.4. *For any $(\beta, \zeta) \in \mathbb{R}^2$, $\Phi_{\beta,\zeta}$ is locally Hölder continuous.*

Proof. Let $\tilde{a} \in \tilde{S}$ and let $\tilde{x}, \tilde{y} \in [\tilde{a}]$. Let $n \ge 1$ be such that $d_{\tilde{X}}(\tilde{x}, \tilde{y}) = e^{-n}$. By the mean value theorem and Proposition 4.4 there exists $\alpha_0 > 0$ such that $|\tilde{\pi}(\tilde{\sigma}\tilde{x}) - \tilde{\pi}(\tilde{\sigma}\tilde{y})| \ll e^{-\alpha_0(n-1)}$. Combining this with the Rényi condition in [7, Lemma 2.8] we obtain

$$|\log |(\tilde{f})'\tilde{\pi}(\tilde{x})| - \log |(\tilde{f})'\tilde{\pi}(\tilde{y})|| \ll e^{-\alpha_0(n-1)}.$$

This implies that $\log |(\tilde{f})'| \circ \tilde{\pi}$ is locally Hölder continuous with $\theta = \min\{\alpha_0/2, 1\}$. Moreover, $\tilde{t}$ is locally Hölder continuous since it is constant on each induced 1-cylinder. Hence, $\Phi_{\beta,\zeta}$ is locally Hölder continuous. □

LEMMA A.5. *The Markov map $\tilde{f}$ with the Markov partition $(\tilde{\Delta}(\tilde{a}))_{\tilde{a}\in\tilde{S}}$ is finitely irreducible.*

Proof. By the definition of $\tilde{f}$ and the transitivity of the Markov map *f*, the proof is straightforward. □

By [21, Corollary 2.7.5] together with Lemmas A.3 and A.4 and Proposition 5.7, for any $\beta \in \mathbb{R}$, there exists a unique $\tilde{\sigma}$-invariant Gibbs state $\tilde{\mu}_\beta$ for $\Phi_{\beta,P(\beta)}$, namely, there exists a constant $C \geq 1$ such that for any $\tilde{x} \in \tilde{X}$ and $n \geq 1$,

$$C^{-1} \leq \frac{\tilde{\mu}_\beta[\tilde{x}_0 \cdots \tilde{x}_{n-1}]}{\exp(-\mathscr{P}(\beta, P(\beta))n + \sum_{k=0}^{n-1} \Phi_{\beta,P(\beta)}(\tilde{\sigma}^k \tilde{x}))} \leq C. \tag{A.1}$$

LEMMA A.6. *If $\beta < \beta_+$, then $\int \tilde{t}\, d\tilde{\mu}_\beta < +\infty$ and $\int \Phi_{\beta,P(\beta)}\, d\tilde{\mu}_\beta > -\infty$. If $\beta \geq \beta_+$, then $\int \tilde{t}\, d\tilde{\mu}_\beta = +\infty$ if and only if $\beta \geq 1$.*

Proof. Let $\beta < \beta_+$. Let C denote the constant given by equation (A.1). By Lemma A.2, we have

$$\sum_{n=1}^{\infty} n \sum_{\tilde{a}\in\tilde{S}(n)} \tilde{\mu}_\beta[\tilde{a}] \asymp C e^{-\mathscr{P}(\beta,P(\beta))} \sum_{n=1}^{\infty} n^{1-2\beta} e^{-P(\beta)n}, \tag{A.2}$$

which is finite since $P(\beta) > 0$. From equation (A.2) and Lemma A.1, we obtain $\int \tilde{t}\, d\tilde{\mu}_\beta = \sum_{\tilde{a}\in\tilde{S}} \tilde{t}(\tilde{a})\tilde{\mu}_\beta[\tilde{a}] < +\infty$, and also $\int \log |(\tilde{f})'| \circ \tilde{\pi}\, d\tilde{\mu}_\beta < +\infty$. Therefore, $\int \Phi_{\beta,P(\beta)}\, d\tilde{\mu}_\beta > -\infty$ holds. If $\beta \geq \beta_+$, then $P(\beta) = 0$ by Proposition 5.7. By Lemma A.1, we obtain $\int \tilde{t}\, d\tilde{\mu}_\beta = \sum_{\tilde{a}\in\tilde{S}} \tilde{t}(\tilde{a})\tilde{\mu}_\beta[\tilde{a}] \asymp \sum_{n=1}^{\infty} n^{1-2\beta} < +\infty$ if $\beta > 1$ and $\int \tilde{t}\, d\tilde{\mu}_\beta = +\infty$ if $\beta \leq 1$. □

Now let $\beta < \beta_+$. By [21, Theorem 2.2.9] together with Lemma A.6, $\tilde{\mu}_\beta$ is the unique equilibrium state for the potential $\Phi_{\beta,P(\beta)}$, namely

$$\mathscr{P}(\beta, P(\beta)) = h(\tilde{\mu}_\beta) + \int -\beta \log |(\tilde{f})'| \circ \tilde{\pi} - P(\beta)\tilde{t}\, d\tilde{\mu}_\beta. \tag{A.3}$$

The measure given by

$$\mu_\beta = \frac{1}{\int \tilde{t}\, d\tilde{\mu}_\beta} \sum_{n=0}^{\infty} \tilde{\mu}_\beta|_{\{\tilde{t}>n\}} \circ (f^n \circ \tilde{\pi})^{-1}$$

belongs to $\mathcal{M}(\Lambda, f)$ and by the Abramov–Kac formula [27, Theorem 2.3] satisfies

$$\mathscr{P}(\beta, P(\beta)) = (h(\mu_\beta) - \beta\chi(\mu_\beta) - P(\beta)) \int \tilde{t}\, d\tilde{\mu}_\beta. \tag{A.4}$$

LEMMA A.7. *If $\beta < \beta_+$, then $\mathscr{P}(\beta, P(\beta)) = 0$.*

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Proof. Let $\varepsilon > 0$. From Lemma 3.5 and the fact that any measure in $\mathcal{M}(X, \sigma)$ is approximated in the weak* topology by ergodic ones with similar entropy [9, Theorem B], it follows that there exists an ergodic $\nu \in \mathcal{M}(\Lambda, f)$ with $h(\nu) > 0$ and $h(\nu) - \beta\chi(\nu) > P(\beta) - \varepsilon$. Since $\Lambda \setminus \bigcup_{n=0}^{\infty} f^{-n}(V_c) \subset \bigcup_{n=0}^{\infty} f^{-n}(\tilde{\Lambda})$, measures in $\mathcal{M}(\Lambda, f)$ supported in the complement of $\tilde{\Lambda}$ have zero entropy, and so $\nu(\tilde{\Lambda}) > 0$. The normalized restriction $\bar{\nu}$ of ν to $\tilde{\Lambda}$ is $\tilde{f}$-invariant and so the measure $\tilde{\nu} = \bar{\nu} \circ \tilde{\pi}^{-1}$ belongs to $\mathcal{M}(\tilde{X}, \tilde{\sigma})$. The variational principle for the potential $\Phi_{\beta, P(\beta)}$ yields

$$\begin{aligned} \mathscr{P}(\beta, P(\beta) - \varepsilon) &\geq h(\tilde{\nu}) + \int -\beta \log |(\tilde{f})'| \circ \tilde{\pi} - (P(\beta) - \varepsilon)\tilde{t}\, d\tilde{\nu} \\ &= (h(\nu) - \beta\chi(\nu) - P(\beta) + \varepsilon) \int \tilde{t}\, d\tilde{\nu} \geq 0. \end{aligned}$$

Since $\beta < \beta_+$, we have $P(\beta) > 0$. By Lemma A.3, $\mathscr{P}(\beta, P(\beta) - \varepsilon)$ is finite for sufficiently small $\varepsilon \geq 0$. The variational principle [21, Theorem 2.1.8] implies that the non-negative function $\varepsilon \mapsto \mathscr{P}(\beta, P(\beta) - \varepsilon)$ is convex on a neighbourhood of $\varepsilon = 0$. Hence, we obtain $\mathscr{P}(\beta, P(\beta)) \geq 0$. Combining this with equation (A.4), we conclude that $\mathscr{P}(\beta, P(\beta)) = 0$. □

From equation (A.4) and Lemma A.7, μ_β is an equilibrium state of f for the potential $\beta\phi$. From the uniqueness of $\tilde{\mu}_\beta$, such an equilibrium state is unique, namely μ_β is the unique equilibrium state of f for this potential. Since $P(0) > 0$, $\alpha_- = 0$ and $P(1) = 0$, we have $0 < \beta_+ \leq 1$. This completes the proof of item (a).

Next we show the analyticity of the pressure. Let $\beta_0 \in (-\infty, \beta_+)$. By [21, Theorem 2.6.12] together with Lemma A.3, $\mathscr{P}(\beta, \zeta)$ is analytic at $(\beta_0, P(\beta_0))$, and so can be extended to a holomorphic function in a complex neighbourhood of $(\beta_0, P(\beta_0))$. Note that the analyticity results in [21] continue to hold for finitely irreducible shift spaces, see also [30]. Lemma A.7 gives $\mathscr{P}(\beta_0, P(\beta_0)) = 0$, and equation (A.3) shows $\partial\mathscr{P}(\beta, P(\beta))/\partial\zeta = -\int \tilde{t}\, d\tilde{\mu}_\beta \neq 0$. By the implicit function theorem for holomorphic functions, the pressure is analytic at $\beta = \beta_0$. This completes the proof of item (b).

Finally, we verify item (c). By [21, Theorem 2.6.13], we have that $\partial\mathscr{P}(\beta, P(\beta))/\partial\beta = -\int \log |\tilde{f}| d\tilde{\mu}_\beta$. The implicit function theorem gives

$$P'(\beta) = -\frac{\partial\mathscr{P}(\beta, P(\beta))/\partial\beta}{\partial\mathscr{P}(\beta, P(\beta))/\partial\zeta} = -\frac{\int \log |\tilde{f}|\, d\tilde{\mu}_\beta}{\int \tilde{t}\, d\tilde{\mu}_\beta} = -\chi(\mu_\beta),$$

as required. The proof of Proposition 5.1 is complete.

A.2. *Proof of Proposition 5.5.* It is well known that G has a measure of maximal dimension if G has no parabolic element. Now assume that G has a parabolic element and assume for a contradiction that there exists a measure of maximal dimension μ. From Proposition 5.4, $\mu \circ \pi$ is an equilibrium state for the potential $-\delta_G \log |f'| \circ \pi$. Since $\delta_G > 0$ by Proposition 5.4, we have $h(\mu \circ \pi) > 0$, and hence, $\mu(\tilde{\Lambda}) > 0$. The normalized restriction $\tilde{\mu}$ of $\mu \circ \pi$ to $\pi^{-1}(\tilde{\Lambda})$ is $\tilde{\sigma}$-invariant and satisfies $\int \tilde{t}\, d\tilde{\mu} < +\infty$ by Kac's formula. Moreover,

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$$h(\tilde{\mu}) + \int \Phi_{\delta_G,0}\, d\tilde{\mu} = (h(\mu) - \delta_G \chi(\mu)) \int \tilde{t}\, d\tilde{\mu} = 0.$$

Proposition 5.4 and approximations by finite subsystems together imply $\mathscr{P}(\delta_G, 0) \leq 0$. Hence, $\mathscr{P}(\delta_G, 0) = 0$ and by [**21**, Theorem 2.1.9], $\Phi_{\delta_G,0}$ is summable. By [**21**, Corollary 2.7.5], $\tilde{\mu}$ is the unique Gibbs-equilibrium state for the potential $\Phi_{\delta_G,0}$. By Lemma A.2, we obtain a constant $C > 0$ with $\int \tilde{t}\, d\tilde{\mu} \geq C \sum_{n=1}^{\infty} n \cdot n^{-2\delta_G} = +\infty$, which is a contradiction.

A.3. *Typical homological growth rates.* Recall that α_G is the unique maximal point of the $\mathscr{H}$-spectrum: $b(\alpha_G) = \delta_G$ by the Main Theorem. It is well known [**2**, Theorem 1.2] that G is of the second kind if and only if $\delta(G) < 1$. Hence, $b(\alpha_G) = 1$ if and only if G is of the first kind. Even more, the following holds.

PROPOSITION A.8. *If G is of the first kind, then $|\Lambda \setminus \mathscr{H}(\alpha_G)| = 0$.*

Proof. By Proposition 2.10, it is enough to show that $(1/n) \log |(f^n)'|$ converges Lebesgue almost everywhere (a.e.) to the constant α_G in equation (5.6) as $n \to \infty$. If G has no parabolic element, there exists an ergodic f-invariant probability measure μ_{ac} that is absolutely continuous with respect to the Lebesgue measure. Since $\sigma : X \to X$ is transitive, the support of μ_{ac} is equal to Λ. By Birkhoff's ergodic theorem, $(1/n) \log |(f^n)'|$ converges Lebesgue a.e. to the Lyapunov exponent of μ_{ac}, namely, the Lebesgue measure of the set $\Lambda \setminus \mathscr{H}(\chi(\mu_{\mathrm{ac}}))$ is 0 and $\chi(\mu_{\mathrm{ac}}) = \alpha_G$ holds.

If G has a parabolic element, then $\alpha_G = 0$ by the Main Theorem. It suffices to show that for any open set U containing all neutral periodic points of f,

$$\lim_{n\to\infty} \frac{1}{n} \#\{0 \leq i \leq n-1 : f^i(\xi) \in U\} = 1 \tag{A.5}$$

for Lebesgue almost every $\xi \in \Lambda$. From the ergodicity and the Gibbs property of the $\tilde{\sigma}$-invariant measure $\tilde{\mu}_1$, it follows that the f-invariant measure $\mu_{\mathrm{ac}} = \sum_{n=0}^{\infty} \tilde{\mu}_1|_{\{\tilde{t}>n\}} \circ (f^n \circ \tilde{\pi})^{-1}$ is ergodic and absolutely continuous with respect to the Lebesgue measure. Moreover, the density of μ_{ac} is positive everywhere and infinite only at the neutral periodic points of f. Since $\beta_+ = \delta_G = 1$, by Lemma A.6, we have $\mu_{\mathrm{ac}}(\Lambda) = \int \tilde{t}\, d\tilde{\mu}_1 = +\infty$. By [**41**, Theorem 1.14], for $h \in L^1(\mu_{\mathrm{ac}})$, we have $\lim_{n\to\infty} (1/n) \sum_{k=0}^{n-1} h \circ f^k = 0$ μ_{ac}-a.e. Since $\Lambda \setminus U$ has finite μ_{ac} measure, taking h as the indicator of $\Lambda \setminus U$ proves equation (A.5) for Lebesgue almost every $\eta \in \Lambda$. □

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