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ASPERÓ–MOTA ITERATION AND THE SIZE OF THE CONTINUUM

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Abstract. In this paper we build an Asperó–Mota iteration of length ω_2 that adds a family of $\aleph_2$ many club subsets of ω_1 which cannot be diagonalized while preserving $\aleph_2$. This result discloses a technical limitation of some types of Asperó–Mota iterations.

§1. Introduction. Shelah introduced countable support iterations of proper forcing notions, which enable us to obtain a large number of consistency results. But by technical limitations, the size of the continuum cannot be larger than $\aleph_2$ in such consistency results. Asperó and Mota introduced a new iteration technique for proper forcing notions that enables us to obtain some consistency results with the continuum larger than $\aleph_2$. Asperó–Mota iterations are equipped with symmetric systems of models as side conditions, idea due to Todorčević (see, e.g., [12, Section 4]). The main ingredient is not only the use of symmetric systems of models but also symmetric systems of models with markers. Asperó–Mota iterations are used in the papers [2–5, 10, 13]. In [5, 10], the iterations require that markers of models in symmetric systems also have symmetry in a suitable sense (Definition 4.1(el), (ho), (up), and (down)).

The Asperó–Mota iterations used in [10] have length ω_2 and are proper. The Asperó–Mota iterations used in [5] have length beyond ω_2 and are claimed to be proper. The proof, however, contains a flaw, which has been acknowledged in personal communication. The problems of the proof from [5] are generated by the fact that the iteration in the paper is greater than ω_2.[1]

The forcing iteration in this paper deals with Asperó–Mota iterations with symmetric markers, like in [5, 10]. We disclose a technical limitation of this type of iterations; in fact our results show that the length of Asperó–Mota iterations with symmetric markers in the style of [5, 10] must be at most ω_2 in order to ensure their properness.

To achieve our goal, we consider a family of club subsets of ω_1. Specifically, it is proved that Asperó–Mota iteration can force a certain assertion, which is called **(c)** in this paper, concerning the existence of a family of club subsets of ω_1 which cannot be diagonalized while preserving $\aleph_2$. The assertion was introduced by Justin Tatch

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[1]This paper was in fact retracted in 2022 [6].

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Moore in personal communication and was inspired by the results in [1, Section 2]. Moreover, the natural proof of properness does not work when the length of the iteration is greater than ω_2.

This paper is organized as follows. Section 2 is devoted to the basic facts of the assertion **(c)**: In Section 2.1, the assertion **(c)** is introduced; in Section 2.2, we introduce forcing notions to force the assertion **(c)**. Forcing notions stated in Section 2.2 consist of finite objects. This idea is applied to the Asperó–Mota iterations presented in this paper. The rest of the sections are devoted to our main goal, that is, to prove that the assertion **(c)** can be forced by Asperó–Mota iterations. In Section 3, we introduce relational structures with which we will equip our Asperó–Mota style iterations. This notion is necessary for symmetric systems with symmetric markers. We define our iteration in Section 4, and in Section 5 we prove that it forces **(c)**. As part of this proof, we show that the iteration is proper whenever its length is at most ω_2. In the last section we explain why the proof of properness breaks down when the length of our the iteration is greater than ω_2.

§2. The assertion (c) and forcing (c) by finite approximations.

2.1. The assertion (c). Galvin showed that, if the Continuum Hypothesis holds, then for any family of $\aleph_2$ many club subsets of ω_1, there exists a subfamily of size $\aleph_1$ whose intersection is a club [8, Section 3.2]. Abraham and Shelah showed that the assumption of CH in this theorem of Galvin is necessary. More precisely, they showed that it is consistent that there exists a family of $\aleph_2$ many club subsets of ω_1 such that the intersection of any uncountable subfamily is finite [1, Section 2]. (Notice that such a family cannot be diagonalized without collapsing $\aleph_1$.) They proved this consistency result by an involved forcing construction using countable objects and ccc forcing notions. Justin Tatch Moore introduced the assertion **(c)**, which is inspired by this result of Abraham–Shelah. His assertion **(c)** can be forced by a countable support iteration of proper forcing notions.

All definitions, propositions, and remarks in the subsection are due to Moore. Throughout the article, we assume the following.

Assumptions throughout the paper 2.1.

- $\mathcal{C} = \langle C_\delta : \delta \in \omega_1 \cap \mathsf{Lim}\rangle$ is a ladder system on ω_1, that is, each C_δ is a cofinal subset of δ of order type ω; moreover, we suppose that each C_δ consists of successor ordinals (hence, for any limit ordinals δ and γ in ω_1 with $\delta < \gamma$, $C_\delta \cap (\gamma+1) = C_\delta \cap \gamma$),
- the set $2^{<\omega}$ is equipped with the discrete topology, and $(2^{<\omega})^\omega$ is considered as the product space of copies of the discrete space $2^{<\omega}$,
- for each $v \in (2^{<\omega})^{<\omega}$, we denote

$$[v] := \left\{ g \in \left(2^{<\omega}\right)^\omega : v \subseteq g \right\},$$

which is a basic open subset of the space $(2^{<\omega})^\omega$; here we recall that $v \subseteq g$ means that, for every $n \in \mathrm{dom}(v)$, $v(n) = g(n)$.

Definition 2.2. For a set X of injective functions from ω into $2^{<\omega}$ and $r \in 2^\omega$, we say that a club subset E of ω_1 captures r relative to ($\mathcal{C}$ and) X if, for any limit point δ of E, there are $f \in X$ and $\varepsilon \in \delta$ such that, for any $\xi \in (E \cap \delta) \setminus \varepsilon$, $f(|C_\delta \cap \xi|) \subseteq r$.

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PROPOSITION 2.3. *Suppose that R is a set of reals and X is a set of injective functions from ω into $2^{<\omega}$ of size $\aleph_1$. If, for each $r \in R$, there exists a club subset E_r of ω_1 which captures r relative to X and if the set $\{E_r : r \in R\}$ can be diagonalized, then the size of R is not larger than $\aleph_1$.*

PROOF. Suppose that a club subset E of ω_1 diagonalizes all club sets E_r, $r \in R$, (i.e., for each $r \in R$, $E \setminus E_r$ is bounded in ω_1) and R is of size $\geq \aleph_2$. Then there exists $\eta \in \omega_1$ such that the set

$$R' := \{r \in R : E \setminus \eta \subseteq E_r\}$$

is of size $\geq \aleph_2$. Let δ be a limit point of the set $E \setminus \eta$. Since X is of size $\aleph_1$, there are an injective function f in X and $\varepsilon \in \delta$ such that $\varepsilon \geq \eta$ and the set

$$R'' := \{r \in R' : \forall \xi \in (E_r \cap \delta) \setminus \varepsilon, f(|C_\delta \cap \xi|) \subseteq r\}$$

is of size $\geq \aleph_2$. But then, for any $r \in R''$,

$$\forall \xi \in (E \cap \delta) \setminus \varepsilon, f(|C_\delta \cap \xi|) \subseteq r.$$

This contradicts the fact that R'' has at least two different reals. ⊣

DEFINITION 2.4. Define the assertion **(c)** to be the statement that there are a set X of injective functions from ω into $2^{<\omega}$ of size $\aleph_1$ and a collection of $\aleph_2$-many reals each one of which can be captured by a club of ω_1 relative to X.

REMARK 2.5. By Proposition 2.3, any collection of $\aleph_2$ many club subsets each of which captures a real r relative to X cannot be diagonalized in any outer model with the same $\aleph_2$ if all the r's are distinct.

REMARK 2.6. Moore pointed out that, if X is a non-meager subset of injective functions from ω into $2^{<\omega}$ and $r \in 2^\omega$, the forcing notion of all countable approximations to a club subset of ω_1 that captures r is proper and adds no new reals (however it may not be σ-closed). Moreover, if CH holds, then it satisfies the $\aleph_2$-proper isomorphic condition ($\aleph_2$-pic). Therefore, the assertion **(c)** can be forced by a countable support iteration.

2.2. Forcing (c) by finite approximations. In this subsection, we deal with a forcing notion to force the assertion **(c)** different from the one referred to in Remark 2.6. Our forcing notion is equipped with models as side conditions [12, Section 4]. The proofs of the basic facts of this forcing notion should help the reader understand the machinery of the proofs dealing with our Asperó–Mota iteration in Section 5.

Suppose that X is a non-meager subset of injective functions from ω into $2^{<\omega}$, and $r \in 2^\omega$. Let $\kappa := \left(2^{\aleph_0}\right)^+$. Define $\mathfrak{M}(X,r)$ to be the set of countable elementary submodels of H_κ which contain the set $\{\mathcal{C}, X, r\}$. Each member of $\mathfrak{M}(X,r)$ is considered as a substructure of the structure $\langle H_\kappa, \in, \omega_1, \mathcal{C}, X, r\rangle$. For each $M \in \mathfrak{M}(X,r)$, the transitive collapse of M is considered as the structure $\langle \mathrm{trcl}(M), \in, \omega_1 \cap M, \mathcal{C} \restriction M, X \cap M, r\rangle$, which is denoted by $\overline{M}$. Ψ_M denotes the transitive collapsing map from M onto $\overline{M}$. For each $M \in \mathfrak{M}(X,r)$, since M is countable and ω_1 is of uncountable cofinality, $\omega_1 \cap M$ is a countable ordinal. And if M and M' in $\mathfrak{M}(X,r)$ have the same transitive collapse, then $\omega_1 \cap M =$

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$\omega_1 \cap M'$, and the composition $\Psi_{M'}{}^{-1} \circ \Psi_M$ is an isomorphism from the structure $\langle M, \in, \omega_1, \mathcal{C}, X, r\rangle$ onto the structure $\langle M', \in, \omega_1, \mathcal{C}, X, r\rangle$.

In this subsection, we suppose $2^{\aleph_0} = \aleph_1$. If M and M' in $\mathfrak{M}(X, r)$ satisfy $\omega_1 \cap M = \omega_1 \cap M'$, then $\mathbb{R} \cap M = \mathbb{R} \cap M'$. So the set of Borel codes in M coincides with those in M'. Therefore, for any $f \in (2^{<\omega})^\omega$, f is Cohen over M iff f is Cohen over M'. In this subsection, we identify a non-meager set X (which is of size $\aleph_1$) with some fixed enumeration of X of length ω_1. Then, if M and M' in $\mathfrak{M}(X, r)$ satisfies $\omega_1 \cap M = \omega_1 \cap M'$, then $X \cap M = X \cap M'$. We notice that, for any $M \in \mathfrak{M}(X, r)$ and any $f \in (2^{<\omega})^\omega$ which is Cohen over M, the set $\{n \in \omega : f(n) \subseteq r\}$ is infinite.

In Section 4, we will define an Asperó–Mota iteration of forcing notions playing the same role as the following forcing notions.

Definition 2.7. Define the forcing notion $\mathbb{Q}(X, r)$ consisting of the triples $p = \langle \mathcal{N}_p^0, \mathcal{N}_p^1, A_p\rangle$ such that

(sym) $\mathcal{N}_p^0 \cup \mathcal{N}_p^1$ is a finite subset of $\mathfrak{M}(X, r)$ such that
- for each $M, M' \in \mathcal{N}_p^0 \cup \mathcal{N}_p^1$, if $\omega_1 \cap M = \omega_1 \cap M'$, then $\overline{M} = \overline{M'}$,
- for each $M, M' \in \mathcal{N}_p^0 \cup \mathcal{N}_p^1$, if $\omega_1 \cap M' < \omega_1 \cap M$, then there exists $M'' \in \mathcal{N}_p^0 \cup \mathcal{N}_p^1$ such that $\overline{M''} = \overline{M}$ and $M' \in M''$,

(ob) A_p is a finite set of tuples of the form $\sigma = \langle \varepsilon_\sigma, \delta_\sigma, \gamma_\sigma, f_\sigma\rangle$ such that $\varepsilon_\sigma, \delta_\sigma, \gamma_\sigma \in \omega_1$, $\varepsilon_\sigma < \delta_\sigma < \gamma_\sigma$, and $f_\sigma \in X$,

(ob-2)
- the set $\{\delta_\sigma : \sigma \in A_p\}$ includes the set $\{\omega_1 \cap N : N \in \mathcal{N}_p^0\}$, and the set $\{\gamma_\sigma : \sigma \in A_p\}$ includes the set $\{\omega_1 \cap M : M \in \mathcal{N}_p^1\}$,
- for any $\sigma \in A_p$ and any $N \in \mathcal{N}_p^0$, if $\omega_1 \cap N = \delta_\sigma$, then there exists $M \in \mathcal{N}_p^1$ such that $N \in M$ and $\omega_1 \cap M = \gamma_\sigma$,
- for any $\sigma \in A_p$ and any $M \in \mathcal{N}_p^1$, if $\omega_1 \cap M = \gamma_\sigma$, then f_σ is Cohen over M,
- for any $\sigma \in A_p$ and $N \in \mathcal{N}_p^0$, if $\delta_\sigma < \omega_1 \cap N$, then $\sigma \in N$,

(cl) for any $\{\sigma, \tau\} \in [A_p]^2$, either $\gamma_\sigma < \delta_\tau$ or $\gamma_\tau < \delta_\sigma$, and

(w) for any $\sigma \in A_p$, if δ_σ is a limit ordinal, then the set $\{n \in \omega : f_\sigma(n) \subseteq r\}$ is infinite, and for any $\tau \in A_p \setminus \{\sigma\}$ with $\varepsilon_\sigma < \delta_\tau < \delta_\sigma$, $f_\sigma(|C_{\delta_\sigma} \cap \delta_\tau|) \subseteq r$ and $f_\sigma(|C_{\delta_\sigma} \cap (\gamma_\tau + 1)|) \subseteq r$,

for each $p, q \in \mathbb{Q}(X, r)$, $q \leq_{\mathbb{Q}(X,r)} p$ if $\mathcal{N}_q^0 \supseteq \mathcal{N}_p^0$, $\mathcal{N}_q^1 \supseteq \mathcal{N}_p^1$, $A_q \supseteq A_p$.

We notice that, for any $p \in \mathbb{Q}(X, r)$, any $N \in \mathcal{N}_p^0$ and any $M \in \mathcal{N}_p^1$, $\omega_1 \cap N \neq \omega_1 \cap M$.

Lemma 2.8. *For any non-meager set X of injective functions from ω into $2^{<\omega}$ and any $r \in 2^\omega$, $\mathbb{Q}(X, r)$ is proper.*

Proof. Let θ be a large enough regular cardinal, N^* a countable elementary submodel of H_θ which contains $\{\mathcal{C}, X, r, H_\kappa\}$, and $p \in \mathbb{Q}(X, r) \cap N^*$. Let M^* be a countable elementary submodel of H_θ which has the set $\{N^*\}$, and let $\varepsilon_* \in \omega_1 \cap N^*$ such that $\delta_\sigma < \varepsilon_*$ for every $\sigma \in A_p$. We denote $N_* := N^* \cap H_\kappa$ and $M_* := M^* \cap H_\kappa$. Since X is non-meager and M^* is countable, there exists a function f_* in X which is Cohen over M^*. Define $\sigma_* := \langle \varepsilon_*, \omega_1 \cap N_*, \omega_1 \cap M_*, f_*\rangle$ and

$p^+ := \langle \mathcal{N}_p^0 \cup \{N_*\}, \mathcal{N}_p^1 \cup \{M_*\}, A_p \cup \{\sigma_*\}\rangle$. p^+ is a condition of $\mathbb{Q}(X, r)$, and hence is an extension of p.

Let us show that p^+ is an N^*-generic condition of $\mathbb{Q}(X, r)$. Let $\mathcal{D}$ be a predense subset of $\mathbb{Q}(X, r)$ which belongs to N^*, and q an extension of p^+ in $\mathbb{Q}(X, r)$. By extending q if necessary, we may assume that q is an extension of some member of $\mathcal{D}$. Moreover, by extending q if necessary again, we may assume that the set $(\mathcal{N}_q^0 \cup \mathcal{N}_q^1) \cap N_*$ includes the set

$$\left\{ \left((\Psi_{N_*})^{-1} \circ \Psi_N\right)(K) : \ N \in \mathcal{N}_q^0 \text{ with } \omega_1 \cap N = \omega_1 \cap N_*, \ K \in (\mathcal{N}_q^0 \cup \mathcal{N}_q^1) \cap N \right\}.$$

Define $\mathcal{E}$ to be the set of the conditions u of $\mathbb{Q}(X, r)$ such that

- u is an extension of some member of $\mathcal{D}$ in $\mathbb{Q}(X, r)$,
- $\mathcal{N}_u^0 \cap N = \mathcal{N}_q^0 \cap N_*, \mathcal{N}_u^1 \cap M = \mathcal{N}_q^1 \cap M_*$ $(= \mathcal{N}_q^1 \cap N_*)$, and $N \in M$ for some $N \in \mathcal{N}_u^0$ and some $M \in \mathcal{N}_u^1$,
- $A_u \supseteq A_q \cap N_*$, and, for any $\sigma \in A_u \setminus (A_q \cap N_*)$ and any $\tau \in A_q \setminus (N_* \cup \{\sigma_*\})$,

$$\max\left(C_{\delta_\tau} \cap N_*\right) < \delta_\sigma.$$

Then $p^+ \in \mathcal{E}$. Since the set

$$\left\{\mathbb{Q}(X, r), \mathcal{D}, \mathcal{N}_q^0 \cap N_*, \mathcal{N}_q^1 \cap M_*, A_q \cap N_*, \langle C_{\delta_\tau} \cap N_* : \tau \in A_q \setminus (N_* \cup \{\sigma_*\})\rangle\right\}$$

belongs to N^*, by elementarity of N^*, $\mathcal{E}$ belongs to N^*. We note that

♠ if $u \in \mathcal{E} \cap N_*$, then for any $\sigma \in A_u \setminus (A_q \cap N_*)$ and any $\tau \in A_q \setminus (N_* \cup \{\sigma_*\})$,

$$C_{\delta_\tau} \cap \delta_\sigma = C_{\delta_\tau} \cap \gamma_\sigma = C_{\delta_\tau} \cap N_* = C_{\delta_\tau} \cap \delta_{\sigma_*}.$$

It follows from elementarity of N^* that

♦ for any $\eta \in \omega_1 \cap N_*$, there exists $u \in \mathcal{E} \cap N^*$ such that, for every $\tau \in A_u \setminus (A_q \cap N_*)$, $\eta < \delta_\tau$, hence, for any $n \in \omega$, there exists $u \in \mathcal{E} \cap N^*$ such that, for every $\tau \in A_u \setminus (A_q \cap N_*)$, $\left|C_{\omega_1 \cap N_*} \cap \gamma_\tau\right| \geq \left|C_{\omega_1 \cap N_*} \cap \delta_\tau\right| \geq n$.

Define Z to be the set of the functions g from ω into $2^{<\omega}$ such that there exists $u \in \mathcal{E} \cap N^*$ which satisfies that for any $\tau \in A_u \setminus (A_q \cap N_*)$, $g(\left|C_{\omega_1 \cap N_*} \cap \delta_\tau\right|) \subseteq r$ and $g(\left|C_{\omega_1 \cap N_*} \cap (\gamma_\tau + 1)\right|) \subseteq r$. Since Z is defined from $\mathcal{E}$ and $C_{\omega_1 \cap N_*}$ $(= C_{\delta_{\sigma_*}})$, and the set $\left\{\mathcal{E}, N^*, C_{\omega_1 \cap N_*}\right\}$ is in M^*, Z belongs to M^*. By the property ♦ of $\mathcal{E} \cap N^*$, Z is a dense open subset of the space $(2^{<\omega})^\omega$. Since f_* is Cohen over M^*, f_* belongs to Z. Let u be a witness that f_* belongs to Z.

Define $u' = \langle \mathcal{N}_{u'}^0, \mathcal{N}_{u'}^1, A_{u'}\rangle$ such that

$$\mathcal{N}_{u'}^0 := \mathcal{N}_q^0 \cup \mathcal{N}_u^0 \cup \left\{ \left(\Psi_N^{-1} \circ \Psi_{N_*}\right)(N') : N \in \mathcal{N}_q^0 \text{ with } \omega_1 \cap N = \omega_1 \cap N_*, N' \in \mathcal{N}_u^0 \right\},$$

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$$\mathcal{N}^1_{u'} := \mathcal{N}^1_q \cup \mathcal{N}^1_u \cup \Big\{ \left(\Psi_M{}^{-1} \circ \Psi_{N_*}\right)(M') : M \in \mathcal{N}^1_q \text{ with } \omega_1 \cap M = \omega_1 \cap M_*, M' \in \mathcal{N}^1_u \Big\},$$

$$A_{u'} := A_q \cup A_u.$$

Let us show that u' is a condition of $\mathbb{Q}(X, r)$. Then it follows that u' is a common extension of q and u, which completes the proof. Since u and q satisfies (ob) and (ob-2), so does u'. Since $A_{u'}$ is an end-extension of $A_q \cap N_*$ and $u' \in N^*$, $A_{u'}$ satisfies (cl). We will check two non-trivial cases of (w) for u'. Suppose that $\sigma \in A_u \setminus (A_q \cap N_*)$ and $\tau \in A_q \setminus (N_* \cup \{\sigma_*\})$. Then $\delta_\sigma < \delta_\tau$. If $\varepsilon_\tau < \delta_\sigma$, then $\varepsilon_\tau < \omega_1 \cap N_*$, and so, by ♠,

$$f_\tau(|C_{\delta_\tau} \cap \delta_\sigma|) = f_\tau(|C_{\delta_\tau} \cap (\gamma_\sigma + 1)|) = f_\tau(|C_{\delta_\tau} \cap \delta_{\sigma_*}|) \subseteq r.$$

This takes care of one non-trivial case. Since u is a witness that $f_* \in Z$,

$$f_{\sigma_*}(\left|C_{\delta_{\sigma_*}} \cap \delta_\sigma\right|) = f_*(\left|C_{\omega_1 \cap N_*} \cap \delta_\sigma\right|) \subseteq r$$

and

$$f_{\sigma_*}(\left|C_{\delta_{\sigma_*}} \cap (\gamma_\sigma + 1)\right|) = f_*(\left|C_{\omega_1 \cap N_*} \cap (\gamma_\sigma + 1)\right|) \subseteq r.$$

This takes care of the other non-trivial case. Therefore, $A_{u'}$ satisfies (w). ⊣

PROPOSITION 2.9. *For any non-meager set X of injective functions from ω into $2^{<\omega}$ and any $r \in 2^\omega$,*

$$\Vdash_{\mathbb{Q}(X,r)} \text{“}\bigcup\left\{[\delta_\sigma, \gamma_\sigma] : p \in \dot{G}, \sigma \in A_p\right\} \supseteq (\omega_1 \setminus \delta) \cap \mathsf{Lim} \textit{ for some } \delta \in \omega_1\text{”},$$

where $[\delta, \gamma] := \{\xi \in \gamma + 1 : \delta \leq \xi\}$, and Lim *denotes the class of the limit ordinals.*

PROOF. Let $p \in \mathbb{Q}(X, r)$ and $\xi \in \omega_1 \cap \mathsf{Lim}$. Suppose that ξ is not in the set $\bigcup_{\sigma \in A_p} [\delta_\sigma, \gamma_\sigma]$ and that there exists $\sigma_0 \in A_p$ such that $\gamma_{\sigma_0} < \xi$. Since ξ is a limit ordinal, $\gamma_{\sigma_0} + 1 < \xi$. We may assume that σ_0 is a largest tuple of A_p with this property. Take $f_0 \in X \cap \bigcap \mathcal{N}^0_p$.

If there are no $\tau \in A_p$ such that $\xi < \delta_\tau$, then let $\delta \in \xi$ be a successor ordinal such that $\gamma_\sigma < \delta < \xi$ for every $\sigma \in A_p$, and define $q := \left\langle \mathcal{N}^0_p, \mathcal{N}^1_p, A_p \cup \left\{\left\langle \gamma_{\sigma_0}, \delta, \xi + 1, f_0 \right\rangle\right\}\right\rangle$. Suppose that $\xi < \gamma_\tau$ for some $\tau \in A_p$. Let $\sigma_1 \in A_p$ be the smallest tuple with the property that $\xi < \gamma_{\sigma_1}$. Then by our assumption, $\xi < \delta_{\sigma_1}$. If δ_{σ_1} is a successor ordinal, then define $q := \left\langle \mathcal{N}^0_p, \mathcal{N}^1_p, A_p \cup \left\{\left\langle \gamma_{\sigma_0}, \gamma_{\sigma_0} + 1, \delta_{\sigma_1} - 1, f_0 \right\rangle\right\}\right\rangle$. If δ_{σ_1} is a limit ordinal, then take $\gamma \in \delta_{\sigma_1}$ such that, for any $\tau \in A_p$ with $\delta_\tau > \delta_{\sigma_1}$, if δ_τ is a limit ordinal, then

$$C_{\delta_\tau} \cap (\gamma + 1) = C_{\delta_\tau} \cap \delta_{\sigma_1}.$$

It follows that, if $\varepsilon_\tau < \delta_{\sigma_1}$, then $f_\tau(|C_{\delta_\tau} \cap (\gamma + 1)|) \subseteq r$. By extending γ if necessary, we may assume that

$$f_{\gamma_{\sigma_1}}(|C_{\delta_{\sigma_1}} \cap (\gamma + 1)|) \subseteq r.$$

This can be done because the set $\{n \in \omega : f_{\gamma_{\sigma_1}}(n) \subseteq r\}$ is infinite. Define $q := \langle \mathcal{N}_p^0, \mathcal{N}_p^1, A_p \cup \{\langle \gamma_{\sigma_0}, \gamma_{\sigma_0}+1, \gamma, f_0\rangle\}\rangle$.

In each case, q is a condition of $\mathbb{Q}(X, r)$, and hence it is an extension of p in $\mathbb{Q}(X, r)$ and

$$q \Vdash_{\mathbb{Q}(X,r)} \text{“}\xi \in \bigcup\{[\delta_\sigma, \gamma_\sigma] : u \in \dot{G}, \sigma \in A_u\}\text{”},$$

which finishes the proof. ⊣

It follows from this proposition that

$\Vdash_{\mathbb{Q}(X,r)}$ “the set $\{\delta_\sigma : p \in \dot{G}, \sigma \in A_p\}$ is a club subset of ω_1, and captures r relative to X”.

The following lemma shows that $\mathbb{Q}(X, r)$ almost preserves $\sqsubseteq^{\text{Cohen}}$ in the sense of Goldstern [9, Section 6, Application 3] (see also [7, Section 6.3.C]). It follows that $\mathbb{Q}(X, r)$ preserves non-meager sets of reals (from [7, Lemmas 6.3.16 and 6.3.17]). This is necessary to guarantee that a countable support iteration of forcing notions of the form $\mathbb{Q}(X, r)$ is still proper because the non-meagerness of X is used to prove properness of $\mathbb{Q}(X, r)$. The preservation of $\sqsubseteq^{\text{Cohen}}$ is closed under countable support iterations [7, Theorems 6.1.13 and 6.3.20]. Therefore, for any non-meager set X of injective functions from ω into $2^{<\omega}$, a countable support iteration of forcing notion of the form $\mathbb{Q}(X, r)$ is still proper and forces X to be non-meager.

LEMMA 2.10. *Let X be a non-meager set of injective functions from ω into $2^{<\omega}$, $r \in 2^\omega$, θ a regular cardinal such that $H_\kappa \in H_\theta$, and λ a regular cardinal such that $H_\theta \in H_\lambda$. Then, for any countable elementary submodel N^* of H_λ which contains the set $\{\mathcal{C}, X, r, H_\kappa, H_\theta\}$, any $c \in 2^\omega$ which is Cohen over N^*, and any $p \in \mathbb{Q}(X, r) \cap N^*$, there exists an extension p^+ of p in $\mathbb{Q}(X, r)$ such that p^+ is an $(N^*, \mathbb{Q}(X, r))$-generic condition and*

$$p^+ \Vdash_{\mathbb{Q}(X,r)} \text{“}c \text{ is Cohen over } N^*[\dot{G}_{\mathbb{Q}(X,r)}]\text{”}.$$

PROOF. As in the previous proof, let N^* be a countable elementary submodel of H_λ which contains $\{\mathcal{C}, X, r, H_\kappa, H_\theta, p\}$, M^* a countable elementary submodel of H_λ which has the set $\{N^*\}$, $\varepsilon_* \in \omega_1 \cap N^*$ such that $\delta_\sigma^p < \varepsilon_*$ for every $\sigma \in A_p$, and f_* a member of X which is Cohen over M^*. Define $\sigma_* := \langle \varepsilon_*, \omega_1 \cap N^*, \omega_1 \cap M^*, f_*\rangle$ and

$$p^+ := \langle \mathcal{N}_p^0 \cup \{N^* \cap H_\kappa\}, \mathcal{N}_p^1 \cup \{M^* \cap H_\kappa\}, A_p \cup \{\sigma_*\}\rangle.$$

We have shown in Lemma 2.8 that p^+ is $(N^*, \mathbb{Q}(X, r))$-generic. Let us show that

$$p^+ \Vdash_{\mathbb{Q}(X,r)} \text{“}c \text{ is Cohen over } N^*[\dot{G}_{\mathbb{Q}(X,r)}]\text{”}.$$

Suppose not. Then there are a $\mathbb{Q}(X, r)$-name $\dot{F}$ for a nowhere dense subset of 2^ω and $q \leq_{\mathbb{Q}(X,r)} p$ such that $\dot{F} \in N^*$ and

$$q \Vdash_{\mathbb{Q}(X,r)} \text{“}c \in \dot{F}\text{”}.$$

Define $\mathcal{D}$ to be the set of the conditions u of $\mathbb{Q}(X, r)$ such that there are countable elementary submodels N' and M' of H_θ such that

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- $(\mathcal{N}_u^0 \cup \mathcal{N}_u^1 \cup A_u) \cap N' = (\mathcal{N}_q^0 \cup \mathcal{N}_q^1 \cup A_q) \cap N^*$,
- $\Big\{\mathcal{C}, X, r, H_\kappa, \dot{F}, \{C_{\delta_\sigma} \cap N^* : \sigma \in A_q \setminus (N^* \cup \{\sigma_*\})\}, \{\varepsilon_\sigma : \sigma \in A_q\} \cap N^*\Big\} \in N' \in M'$,
- $N' \cap H_\kappa \in \mathcal{N}_u^0$ and $M' \cap H_\kappa \in \mathcal{N}_u^1$, and
- there is a $\sigma \in A_u$ such that $\omega_1 \cap N' = \delta_\sigma$ and $\omega_1 \cap M' = \gamma_\sigma$.

Then $q \in \mathcal{D} \in N^*$. Since q is $(N^*, \mathbb{Q}(X, r))$-generic, there exists $u \in \mathcal{D} \cap N^*$ that is compatible with q in $\mathbb{Q}(X, r)$. Let q^+ be a common extension of q and u in $\mathbb{Q}(X, r)$ such that there are countable elementary submodels N_0^* and M_0^* of H_θ such that

- $(\mathcal{N}_u^0 \cup \mathcal{N}_u^1 \cup A_u) \cap N_0^* = (\mathcal{N}_q^0 \cup \mathcal{N}_q^1 \cup A_q) \cap N^*$,
- $\Big\{\mathcal{C}, X, r, H_\kappa, \dot{F}, \{C_{\delta_\sigma} \cap N^* : \sigma \in A_q \setminus (N^* \cup \{\sigma_*\})\}, \{\varepsilon_\sigma : \sigma \in A_q\} \cap N^*\Big\} \in N_0^* \in M_0^* \in N^*$,
- $N_0 := N_0^* \cap H_\kappa \in \mathcal{N}_{q^+}^0$ and $M_0 := M_0^* \cap H_\kappa \in \mathcal{N}_{q^+}^1$, and
- there is a unique $\sigma_0 \in A_{q^+}$ such that $\omega_1 \cap N_0 = \delta_{\sigma_0}$ and $\omega_1 \cap M_0 = \gamma_{\sigma_0}$.

As seen in the previous lemma, q^+ is $(N_0^*, \mathbb{Q}(X, r))$-generic. By extending q^+ if necessary, we may assume that the set $\left(\mathcal{N}_{q^+}^0 \cup \mathcal{N}_{q^+}^1\right) \cap N^*$ contains the set

$$\Big\{ \left((\Psi_{N^* \cap H_\kappa})^{-1} \circ \Psi_N \right)(K) : N \in \mathcal{N}_{q^+}^0 \text{ with } \omega_1 \cap N = \omega_1 \cap N^*, \\ K \in \left(\mathcal{N}_{q^+}^0 \cup \mathcal{N}_{q^+}^1\right) \cap N \Big\}.$$

Let $\zeta_0 \in \omega_1 \cap N_0$ be such that

- for every $\sigma \in A_q \cap N_0$ $(= A_q \cap N^* = A_{q^+} \cap N_0)$, $\gamma_\sigma < \zeta_0$, and
- for every $\sigma \in A_q \setminus N_0$ (then $\delta_\sigma \geq \omega_1 \cap N^* > \omega_1 \cap N_0 = \delta_{\sigma_0}$),
  - if $C_{\delta_\sigma} \cap N_0 \neq \emptyset$, then $\max\left(C_{\delta_\sigma} \cap N_0\right) < \zeta_0$, and
  - if $\varepsilon_\sigma < \omega_1 \cap N_0$, then $\varepsilon_\sigma < \zeta_0$.

For each $v \in 2^{<\omega}$, each $\eta \in \omega_1$, and each $x \in [\mathfrak{M}(X, r)]^{<\aleph_0}$, define $\mathcal{E}(v, \eta, x)$ to be the set of the conditions u of $\mathbb{Q}(X, r)$ such that

- $A_u \cap (\eta^3 \times X) = A_q \cap N_0$ and, for any $\tau \in A_u \setminus (A_q \cap N_0)$, $\delta_\tau \geq \eta$,

♦ the set

$$\left\{\left\langle \varepsilon_\sigma, C_{\delta_\sigma} \cap \eta, f_\sigma \restriction \left|C_{\delta_\sigma} \cap \eta\right| \right\rangle : \sigma \in A_u \setminus \left(A_q \cap N_0\right) \text{ with } \varepsilon_\sigma < \eta\right\}$$

is equal to the set

$$\left\{\left\langle \varepsilon_\sigma, C_{\delta_\sigma} \cap N_0, f_\sigma \restriction \left|C_{\delta_\sigma} \cap N_0\right| \right\rangle : \sigma \in A_q \setminus N_0 \text{ with } \varepsilon_\sigma < \zeta_0\right\},$$

- there are $N \in \mathcal{N}_u^0$ and $M \in \mathcal{N}_u^1$ such that $x \in N$, $\mathcal{N}_u^0 \cap N = \mathcal{N}_q^0 \cap N_0$, $\omega_1 \cap N = \min\left\{\delta_\sigma : \sigma \in A_u \setminus (A_q \cap N_0)\right\}$, $\mathcal{N}_u^1 \cap M = \mathcal{N}_q^1 \cap M_0$ $(= \mathcal{N}_q^1 \cap N_0)$, $\omega_1 \cap M = \min\left\{\gamma_\sigma : \sigma \in A_u \setminus (A_q \cap N_0)\right\}$, and $N \in M$,
- $u \Vdash_{\mathbb{Q}(X,r)}$" $\dot{F} \cap [v] \neq \emptyset$".

By the choice of N_0^*, the set

$$\left\{\mathcal{E}(v, \eta, x) : v \in 2^{<\omega}, \eta \in \omega_1, x \in \left[\mathfrak{M}(X, r)\right]^{<\aleph_0}\right\}$$

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belongs to N_0. Define

$$Y := \Big\{a \in 2^\omega : \text{ for any } k \in \omega, \text{ any } \eta \in \omega_1 \setminus \zeta_0, \text{ and any } x \in [\mathfrak{M}(X,r)]^{<\aleph_0}, \mathcal{E}(a \restriction k, \eta, x) \neq \emptyset\Big\}.$$

Y also belongs to N_0.

We claim that Y is nowhere dense in 2^ω. Let v be in $2^{<\omega}$. Take an extension q' of q^+ in $\mathbb{Q}(X,r)$ and an end extension v' of v in $2^{<\omega}$ such that

$$q' \Vdash_{\mathbb{Q}(X,r)} \text{“}\dot{F} \cap [v'] = \emptyset\text{”}.$$

By extending q' if necessary, we may assume that the set $\left(\mathcal{N}^0_{q'} \cup \mathcal{N}^1_{q'}\right) \cap N_0$ includes the set

$$\Big\{ \left(\left(\Psi_{N_0}\right)^{-1} \circ \Psi_N\right)(K) : N \in \mathcal{N}^0_{q'} \text{ with } \omega_1 \cap N = \omega_1 \cap N_0, K \in \left(\mathcal{N}^0_{q'} \cup \mathcal{N}^1_{q'}\right) \cap N\Big\}.$$

Let us show that $Y \cap [v'] = \emptyset$. Suppose not. Then there exists $a \in Y \cap [v']$. Let $k \in \omega$ be such that $v' \subseteq a \restriction k$, and let $\zeta_1 \in \omega_1 \cap N_0$ be such that

- for every $\sigma \in A_{q'} \cap N_0$, $\gamma_\sigma < \zeta_1$, and
- for every $\sigma \in A_{q'} \setminus (N_0 \cup \{\sigma_0\})$, if $C_{\delta_\sigma} \cap N_0 \neq \emptyset$, then $\max\left(C_{\delta_\sigma} \cap N_0\right) < \zeta_1$.

Define Z to be the set of the functions h from ω into $2^{<\omega}$ such that there exists $u \in \mathcal{E}(a \restriction k, \zeta_1, \left(\mathcal{N}^0_{q'} \cup \mathcal{N}^1_{q'}\right) \cap N_0) \cap N_0^*$ which satisfies that, for any $\sigma \in A_u \setminus (A_q \cap N_0)$, $h(\left|C_{\omega_1 \cap N_0} \cap \delta_\sigma\right|) \subseteq r$ and $h(\left|C_{\omega_1 \cap N_0} \cap (\gamma_\sigma + 1)\right|) \subseteq r$. Since $a \in Y$, it follows from elementarity of N_0^* again that Z is a dense open subset of $(2^{<\omega})^\omega$. We note that Z is in M_0^*. Since f_{σ_0} is Cohen over M_0^*, f_{σ_0} is in Z. Take $u \in \mathcal{E}(a \restriction k, \zeta_1, \left(\mathcal{N}^0_{q'} \cup \mathcal{N}^1_{q'}\right) \cap N_0) \cap N_0^*$ which witnesses $f_{\sigma_0} \in Z$. So there are $N \in \mathcal{N}^0_u$ and $M \in \mathcal{N}^1_u$ such that $\omega_1 \cap N = \min\{\delta_\sigma : \sigma \in A_u \setminus (A_q \cap N_0)\}$, $\omega_1 \cap M = \min\{\gamma_\sigma : \sigma \in A_u \setminus (A_q \cap N_0)\}$, $\left(\mathcal{N}^0_{q'} \cup \mathcal{N}^1_{q'}\right) \cap N_0 \in N \in M$, $\mathcal{N}^0_u \cap N = \mathcal{N}^0_q \cap N_0$, and $\mathcal{N}^1_u \cap M = \mathcal{N}^1_q \cap M_0$. Then

$$\mathcal{N}^0_u \cap N = \mathcal{N}^0_q \cap N_0 \subseteq \mathcal{N}^0_{q'} \cap N_0 \in N \in \mathcal{N}^0_u$$

and

$$\mathcal{N}^1_u \cap M = \mathcal{N}^1_u \cap N = \mathcal{N}^1_q \cap N_0 \subseteq \mathcal{N}^1_{q'} \cap N_0 \in N \in M \in \mathcal{N}^1_u.$$

Therefore, if $A_u \cup A_{q'}$ satisfies (w), as in the proof of properness of $\mathbb{Q}(X,r)$, u and q' are compatible in $\mathbb{Q}(X,r)$. However, a common extension of u and q' in $\mathbb{Q}(X,r)$ forces both $\dot{F} \cap [v'] = \emptyset$ and $\dot{F} \cap [a \restriction k] \neq \emptyset$, which contradicts $v' \subseteq a \restriction k$. Therefore, if $A_u \cup A_{q'}$ satisfies (w), then $Y \cap [v'] = \emptyset$.

We will show that $A_u \cup A_{q'}$ satisfies (w). The non-trivial case is that $\sigma \in A_u \setminus (A_q \cap N_0)$ such that $\varepsilon_\sigma < \zeta_1$, and $\tau \in \left(A_{q'} \cap N_0\right) \setminus A_q$ such that $\varepsilon_\sigma < \delta_\tau$. Then $\delta_\tau < \zeta_1 < \delta_\sigma$. Moreover by ♦, there exists $\sigma' \in A_q \setminus N_0$ such that

$$\langle \varepsilon_\sigma, C_{\delta_\sigma} \cap \zeta_1, f_\sigma \restriction |C_{\delta_\sigma} \cap \zeta_1| \rangle = \left\langle \varepsilon_{\sigma'}, C_{\delta_{\sigma'}} \cap N_0, f_{\sigma'} \restriction \left|C_{\delta_{\sigma'}} \cap N_0\right|\right\rangle.$$

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Then $\sigma' \in A_q \subseteq A_{q^+} \subseteq A_{q'}$. Since $\tau \in A_{q'} \cap N_0$, $\delta_\tau < \gamma_\tau < \zeta_1 < \omega_1 \cap N_0$. Thus $C_{\delta_\sigma} \cap \delta_\tau = C_{\delta_{\sigma'}} \cap \delta_\tau$ and $C_{\delta_\sigma} \cap (\gamma_\tau + 1) = C_{\delta_{\sigma'}} \cap (\gamma_\tau + 1)$. Thus, since $\{\sigma', \tau\} \subseteq A_{q'}$ and $\varepsilon_{\sigma'} = \varepsilon_\sigma < \delta_\tau < \omega_1 \cap N_0 < \delta_{\sigma'}$,

$$f_\sigma(|C_{\delta_\sigma} \cap \delta_\tau|) = f_{\sigma'}\left(\left|C_{\delta_{\sigma'}} \cap \delta_\tau\right|\right) \subseteq r$$

and

$$f_\sigma(|C_{\delta_\sigma} \cap (\gamma_\tau + 1)|) = f_{\sigma'}\left(\left|C_{\delta_{\sigma'}} \cap (\gamma_\tau + 1)\right|\right) \subseteq r.$$

Since c is Cohen over N^*, c is not in Y. Thus, there are $k \in \omega$, $\eta \in \omega_1 \setminus \zeta_0$, and $x \in [\mathfrak{M}(X, r)]^{<\aleph_0}$ such that $\mathcal{E}(c \restriction k, \eta, x)$ is empty. Since $c \restriction k \in N_0^*$, by elementarity of N_0^*, we may assume that $\eta \in \left(\omega_1 \cap N_0^*\right) \setminus \zeta_0$, and $x \in [\mathfrak{M}(X, r)]^{<\aleph_0} \cap N_0^*$. But then q belongs to $\mathcal{E}(c \restriction k, \eta, x)$, which is a contradiction. ⊣

As in the proof of Proposition 5.1 that we will see later on, we can show that $\mathbb{Q}(X, r)$ has the $\aleph_2$-chain condition ($\aleph_2$-cc). Moreover, as in [12, Section 4], we can show that $\mathbb{Q}(X, r)$ has the $\aleph_2$-pic, which is defined by Shelah [11, Chapter VIII, Section 2] (see also [12, Section 4]). The $\aleph_2$-pic is a stronger condition than the $\aleph_2$-cc, and is closed under countable support iterations. Therefore, the following theorem is a consequence of the lemmas and observations in the present subsections.

Theorem 2.11. *Suppose that* $2^{\aleph_0} = \aleph_1$, *X is a non-meager subset of injective functions from ω into $2^{<\omega}$, and some diamond principle* (*which is used in the book-keeping argument of a countable support iteration*) *holds. Then a countable support iteration of forcing notions of the form $\mathbb{Q}(X, r)$ with some booking argument forces the assertion* (**c**).

§3. Symmetric systems of relational structures. This section is similar to Section 4 of the paper [10]. The idea of this section is due to Tadatoshi Miyamoto. The notion in this section will be used in the definition of our forcing notion which forces the assertion (**c**).

Assumptions throughout the paper 3.1. Throughout the rest of the paper, suppose that

- $2^{\aleph_0} = \aleph_1$ holds,
- $\mathbb{R}$ stands for the set of real numbers, and $\vec{\mathbb{R}}$ is a fixed enumeration of $\mathbb{R}$,
- X is a non-meager set of injective functions from ω into $2^{<\omega}$ (of size $\aleph_1$),
- κ is an uncountable regular cardinal such that $\kappa \geq \aleph_2$ and $2^{<\kappa} = \kappa$,
- Φ is a surjection from κ to H_κ such that for every $x \in H_\kappa$, $\Phi^{-1}[\{x\}]$ is unbounded in κ.

If M and M' are countable elementary submodels of H_κ with the set $\left\{\vec{\mathbb{R}}\right\}$ such that $\omega_1 \cap M = \omega_1 \cap M'$, then $\mathbb{R} \cap M = \mathbb{R} \cap M'$. So, as in Section 2.2, the set of Borel codes in M coincides with the one in M'. Therefore, for any $f \in (2^{<\omega})^\omega$, f is Cohen over M iff f is Cohen over M'.

Assumptions throughout the paper 3.2.

- $\vec{X} = \langle f_\delta : \delta \in \omega_1 \rangle$ is an enumeration of X such that, for any countable elementary submodel M of H_κ which contains the set $\left\{\vec{\mathbb{R}}\right\}$, $f_{\omega_1 \cap M}$ is Cohen over M.

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Definition 3.3. $\mathfrak{M}_0$ is the set of countable elementary submodels N of H_κ such that $\left\{\vec{\mathbb{R}}, \vec{X}\right\} \in N$ and the structure $\langle N, \in, \Phi \cap N\rangle$ is an elementary substructure of the structure $\langle H_\kappa, \in, \Phi\rangle$.

As in Section 2.2, we always consider the members of $\mathfrak{M}_0$ as substructures of the structure $\left\langle H_\kappa, \in, \omega_1, \vec{\mathbb{R}}, \vec{X}, \Phi\right\rangle$. For each $M \in \mathfrak{M}_0$, the transitive collapse of M is considered as the structure $\left\langle \mathrm{trcl}(M), \in, \omega_1 \cap M, \vec{\mathbb{R}} \cap M, \vec{X} \cap M, \overline{\Phi \cap M}\right\rangle$, which is denoted by $\overline{M}$, where $\overline{\Phi \cap M}$ is considered as the image, under the collapsing function of M, of $\Phi \cap M$. As in Section 2.2, Ψ_M denotes the transitive collapsing map from M onto $\overline{M}$. So when M and M' in $\mathfrak{M}_0$ are isomorphic, the composition ${\Psi_{M'}}^{-1} \circ \Psi_M$ is an isomorphism from the structure $\left\langle M, \in, \omega_1, \vec{\mathbb{R}}, \vec{X}, \Phi \cap M\right\rangle$ onto the structure $\left\langle M', \in, \omega_1, \vec{\mathbb{R}}, \vec{X}, \Phi \cap M'\right\rangle$.

Definition 3.4. A finite subset $\mathcal{M}$ of $\mathfrak{M}_0$ is called a *symmetric system* if

(ho) for each $M, M' \in \mathcal{M}$, if $\omega_1 \cap M = \omega_1 \cap M'$, then $\overline{M} = \overline{M'}$,

(up) for each $M, M' \in \mathcal{M}$, if $\omega_1 \cap M' < \omega_1 \cap M$, then there exists $M'' \in \mathcal{M}$ such that $\overline{M''} = \overline{M}$ and $M' \in M''$,

(down) for each $M_0, M_1 \in \mathcal{M}$ and each $M' \in \mathcal{M} \cap M_0$, if $\overline{M_0} = \overline{M_1}$, then $\left({\Psi_{M_1}}^{-1} \circ \Psi_{M_0}\right)(M')$ belongs to $\mathcal{M}$, and

(id) for each $M, M' \in \mathcal{M}$, if $\omega_1 \cap M = \omega_1 \cap M'$, then the function

$$\left({\Psi_{M'}}^{-1} \circ \Psi_M\right) \restriction (M \cap M')$$

is the identity.

The requirement (id) comes from the Asperó–Mota iteration [3]. This was used to show properness whenever the length of the iteration has uncountable cofinality. In this paper, the requirement (id) will be used in other places, for example, Propositions 3.10 and 3.11.

We will deal with symmetric systems of relational structures. To introduce such systems, we define the following notions, and mention some necessary propositions.

Definition 3.5. Let $(\mathbb{P}, \leq_{\mathbb{P}})$ be a forcing notion with *the κ-chain condition* such that $\mathbb{P} \subseteq H_\kappa$. We define *the expanded relational structure by* $\mathbb{P}$ to the relational structure

$$\left\langle H_\kappa, \in, \mathbb{P}, \leq_{\mathbb{P}}, H^{\mathbb{P}}_\kappa, R^{\mathbb{P}}_=, R^{\mathbb{P}}_\in, \vec{\mathbb{R}}, \vec{X}, \Phi\right\rangle,$$

where

- $H^{\mathbb{P}}_\kappa := V^{\mathbb{P}} \cap H_\kappa$, where $V^{\mathbb{P}}$ denotes the class of all $\mathbb{P}$-names,
- $R^{\mathbb{P}}_= := \left\{(p, \tau, \pi) \in \left(\mathbb{P} \times V^{\mathbb{P}} \times V^{\mathbb{P}}\right) \cap H_\kappa : p \Vdash_{\mathbb{P}} \text{“}\tau = \pi\text{”}\right\}$, and
- $R^{\mathbb{P}}_\in := \left\{(p, \tau, \pi) \in \left(\mathbb{P} \times V^{\mathbb{P}} \times V^{\mathbb{P}}\right) \cap H_\kappa : p \Vdash_{\mathbb{P}} \text{“}\tau \in \pi\text{”}\right\}$.

For each $M \in \mathfrak{M}_0$, M is also considered as the substructure

$$\left\langle M, \in \cap M^2, \mathbb{P} \cap M, \leq_{\mathbb{P}} \cap M^2, H^{\mathbb{P}}_\kappa \cap M, R^{\mathbb{P}}_= \cap M^3, R^{\mathbb{P}}_\in \cap M^3, \vec{\mathbb{R}}, \vec{X}, \Phi \cap M\right\rangle$$

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of the expanded relational structure by $\mathbb{P}$, and we write $M \prec \mathbb{P}$ when the structure M is an elementary substructure of the expanded relational structure by $\mathbb{P}$.

The forcing notions we will define in Section 4 are *not* members of H_κ, but subsets of H_κ. The following can be proved in a similar way as in [11, Chapter III, Section 2]. Here, $p \in \mathbb{P}$ is called $(M, \mathbb{P})$-generic if, for any predense subset $\mathcal{D} \in M$ of $\mathbb{P}$, $\mathcal{D} \cap M$ is predense below p in $\mathbb{P}$. For the proof of the following proposition, see, e.g., [11, Chapter III, Theorem 2.11].

Proposition 3.6. *Suppose that $\mathbb{P}$ is a forcing notion with the κ-chain condition such that $\mathbb{P} \subseteq H_\kappa$.*

1. *If θ is a large enough regular cardinal for $\mathbb{P}$ and M^* is a countable elementary submodel of H_θ which contains the set*

$$\left\{ H_\kappa, \in, \mathbb{P}, \leq_{\mathbb{P}}, \vec{\mathbb{R}}, \vec{X}, \Phi \right\}$$

   as a member, then $M^* \cap H_\kappa \in \mathfrak{M}_0$ and $M^* \cap H_\kappa \prec \mathbb{P}$.
2. *For any $M \in \mathfrak{M}_0$ with $M \prec \mathbb{P}$, and any $p \in \mathbb{P}$, the followings are equivalent:*
   - *p is $(M, \mathbb{P})$-generic,*
   - *$p \Vdash_{\mathbb{P}}$ "$M[\dot{G}] \cap {H_\kappa}^V = M$", where ${H_\kappa}^V$ denotes H_κ in the ground model, and*
   - *$p \Vdash_{\mathbb{P}}$ "$M[\dot{G}] \cap \kappa = M \cap \kappa$".*
3. *For any $M \in \mathfrak{M}_0$ with $M \prec \mathbb{P}$,*

   $\Vdash_{\mathbb{P}}$ "*the structure*
$$\Big\langle M[\dot{G}], \in \cap M[\dot{G}]^2, H_\kappa^V \cap M[\dot{G}], \mathbb{P} \cap M[\dot{G}], \leq_{\mathbb{P}} \cap M[\dot{G}]^2, \dot{G} \cap M[\dot{G}], H_\kappa^{\mathbb{P}} \cap M[\dot{G}], R_=^{\mathbb{P}} \cap M[\dot{G}]^3, R_\in^{\mathbb{P}} \cap M[\dot{G}]^3, \vec{\mathbb{R}}, \vec{X}, \Phi \cap M[\dot{G}] \Big\rangle$$
   is an elementary substructure of the structure
$$\Big\langle H_\kappa^{V[\dot{G}]}, \in, H_\kappa^V, \mathbb{P}, \leq_{\mathbb{P}}, \dot{G}, H_\kappa^{\mathbb{P}}, R_=^{\mathbb{P}}, R_\in^{\mathbb{P}}, \vec{\mathbb{R}}, \vec{X}, \Phi \Big\rangle\text{"}.$$

Notation 3.7. For $\alpha \in \kappa + 1$, $n \in \omega$, and a sequence $\left\langle X_\xi^i : i \in n, \xi \in \alpha \right\rangle$ of subsets of H_κ, we denote

$$\langle\!\langle X_\xi^i : i \in n, \xi \in \alpha \rangle\!\rangle := \left\{ \langle i, \xi, x \rangle : i \in n, \xi \in \alpha, x \in X_\xi \right\}.$$

Then the tuple $\langle\!\langle X_\xi^i : i \in n, \xi \in \alpha \rangle\!\rangle$ is also a subset of H_κ.

Definition 3.8. Let $\alpha \in \kappa + 1$, and $\langle \mathbb{P}_\xi : \xi \leq \alpha \rangle$ a sequence of forcing notions such that $\mathbb{P}_\xi \subseteq H_\kappa$ and $\mathbb{P}_\xi$ has the κ-chain condition for each $\xi \leq \alpha$. We define the *expanded relational structure by $\langle \mathbb{P}_\xi : \xi \leq \alpha \rangle$* to the structure

$$\Big\langle H_\kappa, \in, \mathbb{P}_\alpha, \leq_{\mathbb{P}_\alpha}, H_\kappa^{\mathbb{P}_\alpha}, R_=^{\mathbb{P}_\alpha}, R_\in^{\mathbb{P}_\alpha}, \langle\!\langle H_\kappa^{\mathbb{P}_\xi}, R_=^{\mathbb{P}_\xi}, R_\in^{\mathbb{P}_\xi} : \xi \in \alpha \rangle\!\rangle \Big\rangle.$$

For each $M \in \mathfrak{M}_0$, M is also considered as the substructure

$$\Big\langle M, \in \cap M^2, \mathbb{P}_\alpha \cap M, \leq_{\mathbb{P}_\alpha} \cap M^2, H_\kappa^{\mathbb{P}_\alpha} \cap M, R_=^{\mathbb{P}_\alpha} \cap M^3, R_\in^{\mathbb{P}_\alpha} \cap M^3, \langle\!\langle H_\kappa^{\mathbb{P}_\xi} \cap M, R_=^{\mathbb{P}_\xi} \cap M^3, R_\in^{\mathbb{P}_\xi} \cap M^3 : \xi \in \alpha \cap M \rangle\!\rangle \Big\rangle$$

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of the expanded relational structure by $\langle \mathbb{P}_\xi : \xi \leq \alpha \rangle$, and we write $M \prec \langle \mathbb{P}_\xi : \xi \leq \alpha \rangle$ when the structure M is an elementary substructure of the expanded relational structure by $\langle \mathbb{P}_\xi : \xi \leq \alpha \rangle$.

The following is a variation of Proposition 3.6 for iterated forcing.

PROPOSITION 3.9. *Suppose that $\alpha \in \kappa + 1$, and $\langle \mathbb{P}_\xi : \xi \leq \alpha \rangle$ is a sequence of forcing notions such that $\mathbb{P}_\xi \subseteq H_\kappa$ and $\mathbb{P}_\xi$ has the κ-chain condition for each $\xi \leq \alpha$.*

1. *If θ is a large enough regular cardinal for the iteration $\langle \mathbb{P}_\xi : \xi \leq \alpha \rangle$ and M^* is a countable elementary submodel of H_θ which contains the set*
$$\left\{ H_\kappa, \in, \langle \mathbb{P}_\xi : \xi \leq \alpha \rangle, \vec{\mathbb{R}}, \vec{X}, \Phi \right\}$$
as a member, then $M^* \cap H_\kappa \in \mathfrak{M}_0$ and $M^* \cap H_\kappa \prec \langle \mathbb{P}_\xi : \xi \leq \alpha \rangle$.
2. *If $\alpha < \kappa$, then for any $M \in \mathfrak{M}_0$ with $M \prec \langle \mathbb{P}_\xi : \xi \leq \alpha \rangle$, α belongs to M.*
3. *For any $M \in \mathfrak{M}_0$ with $M \prec \langle \mathbb{P}_\xi : \xi \leq \alpha \rangle$ and any $\beta \in \alpha$, if $\beta \in M$, then $M \prec \langle \mathbb{P}_\xi : \xi \leq \beta \rangle$.*

The following is necessary for our symmetric systems of relational structures. This is the reason why we introduce the relational structures equipped with forcing notions that are subsets of H_κ.

PROPOSITION 3.10. *Suppose that $M, N_0, N_1 \in \mathfrak{M}_0$ are elementary substructures of the expanded relational structure by $\langle \mathbb{P}_\xi : \xi \leq \alpha \rangle$, N_0 and N_1 are isomorphic as substructures of the expanded relational structure by the sequence $\langle \mathbb{P}_\xi : \xi \leq \alpha \rangle$ (then the map $\Psi = {\Psi_{N_1}}^{-1} \circ \Psi_{N_0}$ is the isomorphism from N_0 onto N_1 as substructures of the expanded relational structure by $\langle \mathbb{P}_\xi : \xi \leq \alpha \rangle$), $\beta \leq \alpha$ is such that $\Psi(\beta) = \beta$, and $M \in \mathfrak{M}_0 \cap N_0$. Then*

- *if $M \prec \left\langle \mathbb{P}_\xi : \xi \leq \alpha \right\rangle$, then the structure*
$$\Big\langle M, \in \cap M^2, \mathbb{P}_\beta \cap M, \leq_{\mathbb{P}_\beta} \cap M^2, H_\kappa^{\mathbb{P}_\beta} \cap M, R_=^{\mathbb{P}_\beta} \cap M^3, R_\in^{\mathbb{P}_\beta} \cap M^3, \\ \langle\langle H_\kappa^{\mathbb{P}_\xi} \cap M, R_=^{\mathbb{P}_\xi} \cap M^3, R_\in^{\mathbb{P}_\xi} \cap M^3 : \xi \in \beta \cap M \rangle\rangle \Big\rangle$$
is an elementary substructure of the structure
$$\Big\langle N_0, \in \cap {N_0}^2, \mathbb{P}_\beta \cap N_0, \leq_{\mathbb{P}_\beta} \cap {N_0}^2, H_\kappa^{\mathbb{P}_\beta} \cap N_0, R_=^{\mathbb{P}_\beta} \cap {N_0}^3, R_\in^{\mathbb{P}_\beta} \cap {N_0}^3, \\ \langle\langle H_\kappa^{\mathbb{P}_\xi} \cap N_0, R_=^{\mathbb{P}_\xi} \cap {N_0}^3, R_\in^{\mathbb{P}_\xi} \cap {N_0}^3 : \xi \in \beta \cap N_0 \rangle\rangle \Big\rangle,$$
and
- *$\Psi(M) \prec \left\langle \mathbb{P}_\xi : \xi \leq \beta \right\rangle$, and the structure*
$$\Big\langle \Psi(M), \in \cap \Psi(M)^2, \mathbb{P}_\beta \cap \Psi(M), \leq_{\mathbb{P}_\beta} \cap \Psi(M)^2, H_\kappa^{\mathbb{P}_\beta} \cap \Psi(M), R_=^{\mathbb{P}_\beta} \cap \Psi(M)^3, \\ R_\in^{\mathbb{P}_\beta} \cap \Psi(M)^3, \langle\langle H_\kappa^{\mathbb{P}_\xi} \cap \Psi(M), R_=^{\mathbb{P}_\xi} \cap \Psi(M)^3, R_\in^{\mathbb{P}_\xi} \cap \Psi(M)^3 : \xi \in \beta \cap \Psi(M) \rangle\rangle \Big\rangle$$
is an elementary substructure of the structure
$$\Big\langle N_1, \in \cap {N_1}^2, \mathbb{P}_\beta \cap N_1, \leq_{\mathbb{P}_\beta} \cap {N_1}^2, H_\kappa^{\mathbb{P}_\beta} \cap N_1, R_=^{\mathbb{P}_\beta} \cap {N_1}^3, R_\in^{\mathbb{P}_\beta} \cap {N_1}^3, \\ \langle\langle H_\kappa^{\mathbb{P}_\xi} \cap N_1, R_=^{\mathbb{P}_\xi} \cap {N_1}^3, R_\in^{\mathbb{P}_\xi} \cap {N_1}^3 : \xi \in \beta \cap N_1 \rangle\rangle \Big\rangle.$$

The following is a key point of the proof of properness of our forcing notions.

PROPOSITION 3.11. *Suppose that* $\alpha \in \omega_2 \leq \kappa$, $N_0, N_1 \in \mathfrak{M}_0$, *and* N_0 *and* N_1 *are isomorphic as the substructures of the expanded relational structure by* $\langle \mathbb{P}_\xi : \xi \leq \alpha \rangle$. *Then* $N_0 \cap \alpha = N_1 \cap \alpha$.

PROOF. Let us only show the case that α is uncountable. By our assumption, $\alpha \in N_0 \cap N_1$ and $N_0 \cap \omega_1 = N_1 \cap \omega_1$. Then there exists a bijection $f : \omega_1 \to \alpha$ which is in both N_0 and N_1. Then

$$N_0 \cap \alpha = f[N_0 \cap \omega_1] = f[N_1 \cap \omega_1] = N_1 \cap \alpha. \qquad \dashv$$

§4. Definition of Asperó–Mota iteration to force (c). In this section, we define our forcing notion $\mathbb{P}_\kappa$ that forces the assertion **(c)**. $\mathbb{P}_\kappa$ is defined by an Asperó–Mota iteration of forcing notions playing the same role as $\mathbb{Q}(X, r)$ in Section 2.2.

We notice that, for each $M \in \mathfrak{M}_0$ and $\alpha \in \kappa + 1$, any initial segment of $\alpha \cap M$ is of the form $\beta \cap M$ for some $\beta \in \alpha + 1$ (which is not necessary unique). For each $\alpha \in \kappa + 1$, we will define the forcing notion $\mathbb{P}_\alpha$ to be a subset of the set

$$\begin{aligned} U_\alpha := \; & [\mathfrak{M}_0]^{<\aleph_0} \\ & \times \left\{ \bigcup_{\langle M, \beta \rangle \in Z} \{M\} \times (\beta \cap M) : Z \in [\mathfrak{M}_0 \times (\alpha + 1)]^{<\aleph_0} \right\} \\ & \times \left(\bigcup_{D \in [\alpha]^{<\aleph_0}} \left([\omega_1 \times \omega_1 \times \omega_1]^{<\omega} \right)^D \right). \end{aligned}$$

Since $\mathfrak{M}_0$ is a subset of H_κ, for each $\alpha \in \kappa + 1$, the forcing notion $\mathbb{P}_\alpha$ is a subset of H_κ.

To define $\mathbb{P}_\alpha$, we introduce the following notation. For each $\alpha \in \kappa + 1$ and $p = (\mathcal{N}_p, R_p, A_p) \in U_\alpha$,

- $\operatorname{dom}(R_p) := \{M : \text{ there is } \zeta \in \alpha \text{ so that } \langle M, \zeta \rangle \in R_p\}$,
- $\operatorname{ran}(R_p) := \{\zeta : \text{ there is } M \in \mathfrak{M}_0 \text{ so that } \langle M, \zeta \rangle \in R_p\}$,
- for each $I \subseteq \alpha$,
$$R_p{}^{-1}[I] := \{M : \text{ there is } \zeta \in I \text{ so that } \langle M, \zeta \rangle \in R_p\},$$
- for each $M \in \operatorname{dom}(R_p)$,
$$R_p(M) := \{\zeta \in \operatorname{ran}(R_p) : \langle M, \zeta \rangle \in R_p\},$$
- for each $\beta \in \alpha$, define $p \restriction \beta = (\mathcal{N}_{p \restriction \beta}, R_{p \restriction \beta}, A_{p \restriction \beta})$ to be the member of U_β such that
  - $\mathcal{N}_{p \restriction \beta} := \mathcal{N}_p$,
  - $R_{p \restriction \beta} := R_p \cap (\mathfrak{M}_0 \times \beta)$, and
  - $A_{p \restriction \beta} := A_p \restriction \beta$, the restriction of the function A_p to the set β.

For $p \in U_\alpha$ and $M \in \mathcal{N}_p$, members of the set $R_p(M)$ are called *markers of* M.

We define a forcing notion $\mathbb{P}_\alpha$ satisfying the $\aleph_2$-cc (under CH), by recursion on $\alpha \in \kappa + 1$. When we have defined the sequence $\langle \mathbb{P}_\xi : \xi \leq \alpha \rangle$ of forcing notions, we

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will define the subset $\mathfrak{M}^P_\alpha$ of $\mathfrak{M}_0$ by

$$\mathfrak{M}^P_\alpha := \{M \in \mathfrak{M}_0 : M \prec \langle \mathbb{P}_\xi : \xi \leq \alpha \rangle\}.$$

As in Proposition 3.9(2), if $\alpha \in \kappa$ and $M \in \mathfrak{M}^P_\alpha$, then $\alpha \in M$. As seen below, for each $\alpha \in \kappa$, $\mathbb{P}_\alpha$ will be defined from the set

$$\left\{\omega_1, \vec{\mathbb{R}}, \vec{X}, H_\kappa, \Phi, \langle\!\langle \mathfrak{M}^P_\xi : \xi \in \alpha \rangle\!\rangle \right\}.$$

$\mathbb{P}_\kappa$ can be considered as the direct limit of $\langle \mathbb{P}_\alpha : \alpha \in \kappa \rangle$.

Definition 4.1. The forcing notion $\mathbb{P}_\alpha$ is defined by recursion on $\alpha \in \kappa + 1$. However, each $\mathbb{P}_\alpha$ is defined uniformly. $\mathbb{P}_\alpha$ consists of the members $p = (\mathcal{N}_p, R_p, A_p)$ of U_α satisfying the following conditions:

(ob) • $\mathcal{N}_p$ is finite and forms a symmetric system, and
- $\operatorname{dom}(R_p) \subseteq \mathcal{N}_p$, and, for each $M \in \operatorname{dom}(R_p)$, $R_p(M)$ is an initial segment of the set $\alpha \cap M$.

(el) For each $\zeta \in \alpha$, ${R_p}^{-1}[\{\zeta\}] \subseteq \mathfrak{M}^P_\zeta$.

(ho) For each $\zeta \in \alpha$ and each $M_0, M_1 \in {R_p}^{-1}[\{\zeta\}]$, if $\omega_1 \cap M_0 = \omega_1 \cap M_1$, then the structure $\left\langle M_0, \in, \vec{\mathbb{R}}, \vec{X}, \Phi \restriction M_0, \langle\!\langle \mathbb{P}_\xi : \xi \in (\zeta+1) \cap M_0 \rangle\!\rangle \right\rangle$ is isomorphic to the structure $\left\langle M_1, \in, \vec{\mathbb{R}}, \vec{X}, \Phi \restriction M_1, \langle\!\langle \mathbb{P}_\xi : \xi \in (\zeta+1) \cap M_1 \rangle\!\rangle \right\rangle$.

(up) For each $\zeta \in \alpha$ and each $M, N_0 \in {R_p}^{-1}[\{\zeta\}]$, if $\omega_1 \cap M < \omega_1 \cap N_0$, then there exists $N_1 \in {R_p}^{-1}[\{\zeta\}]$ such that $M \in N_1$ and $\omega_1 \cap N_1 = \omega_1 \cap N_0$.

(down) For each $\zeta \in \alpha$ and each $M, N_0, N_1 \in {R_p}^{-1}[\{\zeta\}]$, if $M \in N_0$ and $\omega_1 \cap N_0 = \omega_1 \cap N_1$, then $\left({\Psi_{N_1}}^{-1} \circ \Psi_{N_0}\right)(M) \in {R_p}^{-1}[\{\zeta\}]$.

(g) If $\xi \in \operatorname{dom}(A_p)$ and $p \restriction \xi$ belongs to $\mathbb{P}_\xi$, then $\Phi(\xi) = \{\dot{r}_\xi\}$ such that $\dot{r}_\xi$ is a $\mathbb{P}_\xi$-name for a function from ω into 2. Moreover,

(g-ob) $A_p(\xi)$ is a finite set of triples of the form $\sigma = \langle \varepsilon_\sigma, \delta_\sigma, \gamma_\sigma \rangle$ such that $\varepsilon_\sigma \in \delta_\sigma \in \gamma_\sigma \in \omega_1$,

(g-ob-2) the set $\{\delta_\sigma : \sigma \in A_p\}$ includes the set $\{\omega_1 \cap N : N \in {R_p}^{-1}[\{\xi\}]\}$,

(g-cl) for each $\{\sigma, \tau\} \in [A_p(\xi)]^2$, either $\gamma_\sigma < \delta_\tau$ or $\gamma_\tau < \delta_\sigma$,

(g-w) for any $\sigma \in A_p(\xi)$, if δ_σ is a limit ordinal, then

$$p \restriction \xi \Vdash_{\mathbb{P}_\xi} \text{“}\{n \in \omega : f_{\gamma_\sigma}(n) \subseteq \dot{r}_\xi\} \text{ is infinite, and, for any } \tau \in A_p(\xi) \setminus \{\sigma\} \text{ with } \varepsilon_\sigma < \delta_\tau < \delta_\sigma,\ f_{\gamma_\sigma}(|C_{\delta_\sigma} \cap \delta_\tau|) \subseteq \dot{r}_\xi \text{ and } f_{\gamma_\sigma}(|C_{\delta_\sigma} \cap (\gamma_\tau + 1)|) \subseteq \dot{r}_\xi\text{”},$$

(g-m) for each $\sigma \in A_p(\xi)$ and each $N \in {R_p}^{-1}[\{\xi\}]$ (that is, $\{N\} \times ((\xi+1) \cap N) \subseteq R_p$) with $\omega_1 \cap N = \delta_\sigma$, there exists $M \in \mathcal{N}_p \cap \mathfrak{M}^P_\xi$ such that
- $N \in M$,
- $\omega_1 \cap M = \gamma_\sigma$, and
- $\{M\} \times (\xi \cap M) \subseteq R_p$ (then, by (g-ob-2) and (g-cl), $M \notin {R_p}^{-1}[\{\xi\}]$).

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By definition, we can check that, for each $p \in \mathbb{P}_\alpha$ and $\zeta \in \alpha$, $p \restriction \zeta$ is a condition of $\mathbb{P}_\zeta$. The order of $\mathbb{P}_\alpha$ is defined as follows: For each $p, q \in \mathbb{P}_\alpha$, $q \leq_{\mathbb{P}_\alpha} p$ iff

- $\mathcal{N}_q \supseteq \mathcal{N}_p$,
- $R_q \supseteq R_p$,
- $\operatorname{dom}(A_q) \supseteq \operatorname{dom}(A_p)$,
- for each $\xi \in \operatorname{dom}(A_p)$, $A_q(\xi) \supseteq A_p(\xi)$.

By definition, we can check that, for each $p, q \in \mathbb{P}_\alpha$ with $q \leq_{\mathbb{P}_\alpha} p$, and each $\zeta \in \alpha$, $q \restriction \zeta \leq_{\mathbb{P}_\zeta} p \restriction \zeta$. By definition, $\mathbb{P}_\kappa$ is equivalent to the direct limit of $\langle \mathbb{P}_\alpha : \alpha \in \kappa \rangle$.

LEMMA 4.2. *Suppose that $\alpha, \beta \in \kappa + 1$ with $\beta < \alpha$. Then $\mathbb{P}_\beta$ can be completely embeddable into $\mathbb{P}_\alpha$.*

PROOF. By definition, any condition of $\mathbb{P}_\beta$ is also a condition of $\mathbb{P}_\alpha$. Suppose that $q \in \mathbb{P}_\beta$, $p \in \mathbb{P}_\alpha$, and $q \leq_{\mathbb{P}_\beta} p \restriction \beta$. Then define

$$r := \left\langle \mathcal{N}_q \cup \mathcal{N}_p, R_q \cup R_p, A_q \cup \left(A_p \restriction [\beta, \alpha)\right)\right\rangle.$$

We can check that r is a condition of $\mathbb{P}_\zeta$. So r is an extension of p in $\mathbb{P}_\alpha$. Such an r is a canonical common extension of q and p in $\mathbb{P}_\alpha$. Hence the identity map from $\mathbb{P}_\beta$ into $\mathbb{P}_\alpha$ is a complete embedding. ⊣

DEFINITION 4.3. For each $\xi \in \kappa$, define the $\mathbb{P}_{\xi+1}$-name $\dot{E}_\xi$ by

$$\Vdash_{\mathbb{P}_{\xi+1}} \text{“}\dot{E}_\xi := \left\{\delta_\sigma : q \in \dot{G}_{\mathbb{P}_{\xi+1}}, \sigma \in A_p(\xi)\right\}\text{”}.$$

OBSERVATION 4.4. It is proved in the next section that for each $\xi \in \omega_2$, $\mathbb{P}_\xi$ is proper. Then, as in the case of $\mathbb{Q}(X, r)$ in Section 2.2, $\dot{E}_\xi$ is a $\mathbb{P}_{\xi+1}$-name for a club subset of ω_1. By the definition, for each $\xi \in \omega_2$, if $\Phi(\xi) = \{\dot{r}_\xi\}$ and $\dot{r}_\xi$ is a $\mathbb{P}_\xi$-name for a function from ω into 2, then $\mathbb{P}_{\xi+1}$ forces that $\dot{E}_\xi$ captures $\dot{r}_\xi$ relative to X.

OBSERVATION 4.5. The requirement (g-cl) is necessary in the definition of $\mathbb{P}_\kappa$ to show Lemma 5.2. To show properness of $\mathbb{P}_\alpha$ (for each $\alpha \leq \omega_2$) equipped with (g-cl), we want the requirements (el), (ho), (up), and (down) in Definition 4.1. This is the reason why we introduced a symmetric system of relational structures.

§5. Forcing (c) by $\mathbb{P}_{\omega_2}$.

PROPOSITION 5.1. *For every $\alpha \in \kappa + 1$, $\mathbb{P}_\alpha$ has the $\aleph_2$-chain condition. In fact, every subset of $\mathbb{P}_\alpha$ of size $\aleph_2$ has a pairwise compatible subset of size $\aleph_2$.*

PROOF. Suppose that $\alpha \in \kappa + 1$ and $\{p_\zeta : \zeta \in \omega_2\}$ is a set of $\aleph_2$-many conditions in $\mathbb{P}_\alpha$. Recall that CH holds (Assumption 3.1). By shrinking the set if necessary, we may assume that

- the set $\{\mathcal{N}_{p_\zeta} : \zeta \in \omega_2\}$ forms a Δ-system,
- the set $\{\operatorname{dom}(A_{p_\zeta}) : \zeta \in \omega_2\}$ forms a Δ-system with root D,

(•) the set $\left\{\left(\bigcup \mathcal{N}_{p_\zeta}\right) \cap \kappa : \zeta \in \omega_2\right\}$ forms a Δ-system with root K (which is a countable subset of κ),

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($\bullet$) for each $\xi \in K$, the set

$$\left\{\overline{M} : M \in {R_{p_\zeta}}^{-1}\big[\,[\xi,\kappa)\,\big] \cap \mathfrak{M}^P_\xi\right\}$$

does not depend on $\zeta \in \omega_2$,

($\bullet$) for each $\zeta \in \omega_2$, $\big(\mathrm{dom}(A_{p_\zeta}) \setminus D\big) \cap K = \emptyset$,

($\bullet$) for each $\zeta, \zeta' \in \omega_2$, each $M \in \mathcal{N}_{p_\zeta}$ and each $M' \in \mathcal{N}_{p_{\zeta'}}$, if $\overline{M} = \overline{M'}$, then $M \cap \kappa$ and $M' \cap \kappa$ are order isomorphic and the corresponding isomorphism fixes $\kappa \cap M \cap M'$ (which is a subset of K)[2], and

- for each $\xi \in D$, the coordinate $A_{p_\zeta}(\xi) \in [\omega_1 \times \omega_1 \times \omega_1]^{<\aleph_0}$ does not depend on $\zeta \in \omega_2$.

Then we claim that for each distinct ζ and ζ', p_ζ and $p_{\zeta'}$ are compatible in $\mathbb{P}_\alpha$. To see this, let $q \in U_\alpha$ such that

- $\mathcal{N}_q := \mathcal{N}_{p_\zeta} \cup \mathcal{N}_{p_{\zeta'}}$,
- $R_q := R_{p_\zeta} \cup R_{p_{\zeta'}}$, and
- A_q is the function with domain $\mathrm{dom}(A_{p_\zeta}) \cup \mathrm{dom}(A_{p_{\zeta'}})$ such that, for each $\xi \in \mathrm{dom}(A_{p_\zeta}) \cup \mathrm{dom}(A_{p_{\zeta'}})$,

$$A_q(\xi) := A_{p_\zeta}(\xi) \cup A_{p_{\zeta'}}(\xi)$$

(which is equal to $A_{p_\zeta}(\xi)$ or $A_{p_{\zeta'}}(\xi)$).

Such a q is a canonical amalgamation of p_ζ and $p_{\zeta'}$. Then by the above items ($\bullet$), $\mathcal{N}_q$ and R_q satisfy Definition 4.1(ob), (el), (ho), (up), and (down). Recall that for each $M \in \mathfrak{M}_0$ and each $\alpha \in \kappa$, if $\alpha \notin M$, then $M \notin \mathfrak{M}^P_\alpha$. So for any $\{\zeta, \zeta'\} \in [\omega_2]^2$ and any $\alpha \in \mathrm{dom}(A_{p_\zeta}) \setminus D$, $\mathrm{dom}(R_{p_{\zeta'}}) \cap \mathfrak{M}^P_\alpha = \emptyset$, and hence ${R_q}^{-1}[\{\alpha\}] = {R_{p_\zeta}}^{-1}[\{\alpha\}]$. Therefore q satisfies (g). Thus q is a condition of $\mathbb{P}_\alpha$, and is a common extension of p_ζ and $p_{\zeta'}$. ⊣

LEMMA 5.2. *Each α in $\omega_2 + 1$ satisfies the following assertions.*

$(\mathsf{p})_\alpha$: *For any $p \in \mathbb{P}_\alpha$ and any $N \in \mathcal{N}_p \cap \mathfrak{M}^P_\alpha$ such that $\{N\} \times (\alpha \cap N) \subseteq R_p$, p is $(N, \mathbb{P}_\alpha)$-generic.*

$(\mathsf{C})_\alpha$: *For any $p \in \mathbb{P}_\alpha$ and any $N \in \mathcal{N}_p \cap \mathfrak{M}^P_\alpha$ such that $\{N\} \times (\alpha \cap N) \subseteq R_p$,*

$$p \Vdash_{\mathbb{P}_\alpha} \text{“} f_{\omega_1 \cap N} \text{ is Cohen over } N[\dot{G}_{\mathbb{P}_\alpha}]\text{”}.$$

This is proved by induction on $\alpha \in \omega_2 + 1$. A point is that, if $\alpha \in \omega_2 + 1$, $p \in \mathbb{P}$, and $N \in \mathcal{N}_p$, then $R_p(N) \subseteq \omega_2$. In the following proof, we will use Propositions 3.10 and 3.11 frequently.

PROOF OF $(\mathsf{p})_0$. This proof is a standard proof in the context of the side condition method (see, e.g., [12, Lemma 4]), and similar to the proof in [10]. Suppose that $p \in \mathbb{P}_0$ (then $R_p = A_p = \emptyset$), $N \in \mathcal{N}_p \cap \mathfrak{M}^P_0$, $\mathcal{D} \in N$ is a predense subset of $\mathbb{P}_0$, and $q \leq_{\mathbb{P}_0} p$. We notice that q is of the form $(\mathcal{N}_q, \emptyset, \emptyset)$. It suffices to find $u' \in \mathcal{D} \cap N$ which is compatible with q in $\mathbb{P}_0$.

[2] In [3], Asperó and Mota point out that the corresponding isomorphism between M and M' fixes $\kappa \cap M \cap M'$ iff for every two consecutive ordinals ξ_0 and ξ_1, the order types of the sets $\{\mu \in \kappa \cap M : \xi_0 < \mu < \xi_1\}$ and $\{\mu \in \kappa \cap M' : \xi_0 < \mu < \xi_1\}$ are the same (these order types are countable ordinals).

By extending q if necessary, we may assume that there exists $u \in \mathcal{D}$ such that $q \leq_{\mathbb{P}_0} u$. Define

$$\mathcal{E} := \Big\{ r \in \mathbb{P}_0 : \text{ there are } u' \in \mathcal{D} \text{ and } M \in \mathcal{N}_r \text{ such that } r \leq_{\mathbb{P}_0} u' \text{ and } \mathcal{N}_r \cap M = \mathcal{N}_q \cap N \Big\}.$$

Since the set $\mathcal{N}_q \cap N$ belongs to N, $\mathcal{E}$ is a definable class in the expanded relational structure by $\mathbb{P}_0$ with parameters in N. Moreover, we note that $q \in \mathcal{E}$. So by elementarity of N, there exist $r \in \mathcal{E} \cap N$ and $u' \in \mathcal{D} \cap N$ such that $r \leq_{\mathbb{P}_0} u'$. Define $q' \in U_0$ such that

$$\mathcal{N}_{q'} := \mathcal{N}_q \cup \mathcal{N}_r \cup \Big\{ \left({\Psi_{M'}}^{-1} \circ \Psi_N \right)(M) : M' \in \mathcal{N}_q \text{ with } \omega_1 \cap M' = \omega_1 \cap N, M \in \mathcal{N}_r \Big\}.$$

$\mathcal{N}_{q'}$ forms a symmetric system. q' is the canonical amalgamation of q and r. Therefore, q' is a condition of $\mathbb{P}_0$ and a common extension of q and r in $\mathbb{P}_0$. ⊣

Proof of $(\mathsf{C})_0$. Suppose that $p \in \mathbb{P}_0$, $N \in \mathcal{N}_p \cap \mathfrak{M}_0^P$, and $\{\dot{F}_n : n \in \omega\}$ is a set of $\mathbb{P}_0$-names for nowhere dense subsets of $(2^{<\omega})^\omega$ such that the sequence $\langle \dot{F}_n : n \in \omega \rangle$ belongs to N. Let us show that

$$p \Vdash_{\mathbb{P}_0} \text{“} f_{\omega_1 \cap N} \notin \bigcup_{n \in \omega} \dot{F}_n \text{”}.$$

Suppose not, and let q be an extension of p in $\mathbb{P}_0$ such that, for some $n \in \omega$,

$$q \Vdash_{\mathbb{P}_0} \text{“} f_{\omega_1 \cap N} \in \dot{F}_n \text{”}.$$

For each $v \in (2^{<\omega})^{<\omega}$, each $x \in \left[\mathfrak{M}_0^P\right]^{<\aleph_0}$, and each $r \in \mathbb{P}_0$, define $\varphi_0(v, x, r)$ to be the assertion that there exists $K \in \mathcal{N}_r$ such that

- $\{\omega_1 \cap M : M \in \mathcal{N}_r \cap K\} = \{\omega_1 \cap M : M \in \mathcal{N}_q \cap N\}$,
- $\mathcal{N}_q \cap N \subseteq \mathcal{N}_r \cap K$,
- $x \in K$, and
- $r \Vdash_{\mathbb{P}_0}$“ $\dot{F}_n \cap [v] \neq \emptyset$ ”,

and define

$$Y := \Big\{ g \in (2^{<\omega})^\omega : \text{ for any } k \in \omega \text{ and any } x \in \left[\mathfrak{M}_0^P\right]^{<\aleph_0}, \text{ there is } r \in \mathbb{P}_0 \text{ which satisfies } \varphi_0(g \restriction k, x, r) \Big\}.$$

Since $\{\dot{F}_n, \mathcal{N}_p \cap N\} \in N \in \mathfrak{M}_0^P$, we have $Y \in N$.

We claim that Y is nowhere dense. To show this, let $v \in (2^{<\omega})^{<\omega}$. Take $v' \in (2^{<\omega})^{<\omega}$ and $s \leq_{\mathbb{P}_0} q$ such that $v \subseteq v'$, and

$$s \Vdash_{\mathbb{P}_0} \text{“}\dot{F}_n \cap [v'] = \emptyset\text{”}.$$

Let us show that $Y \cap [v'] = \emptyset$. If not, there exists $g \in Y \cap [v']$. Let $k \in \omega$ be such that $v' \subseteq g \restriction k$. Then there exists $r \in \mathbb{P}_0$ which satisfies $\varphi_0(g \restriction k, \mathcal{N}_s, r)$. Let us fix

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$K \in \mathcal{N}_r$ witness to $\varphi_0(g \restriction k, \mathcal{N}_s, r)$. Define $r' \in U_0$ such that

$$\mathcal{N}_{r'} := \mathcal{N}_s \cup \mathcal{N}_r \cup \left\{ \left(\Psi_{M'}{}^{-1} \circ \Psi_K\right)(M) : M' \in \mathcal{N}_r \text{ with } \omega_1 \cap M' = \omega_1 \cap K, M \in \mathcal{N}_s \right\}.$$

Then r' is a common extension of s and r in $\mathbb{P}_0$, and hence

$$r' \Vdash_{\mathbb{P}_0} \text{“}\dot{F}_n \cap [v'] = \emptyset \text{ and } \dot{F}_n \cap [g \restriction k] \neq \emptyset\text{”},$$

which is a contradiction.

We claim that $f_{\omega_1 \cap N}$ belongs to Y. This contradicts the fact that $f_{\omega_1 \cap N}$ is Cohen over N. To show that $f_{\omega_1 \cap N} \in Y$, assume that $f_{\omega_1 \cap N} \notin Y$. Then there are $k \in \omega$ and $x \in \left[\mathfrak{M}_0^P\right]^{<\aleph_0}$ such that there are no $r \in \mathbb{P}_0$ which satisfies $\varphi_0(f_{\omega_1 \cap N} \restriction k, x, r)$. Since $f_{\omega_1 \cap N} \restriction k \in N$ and N is an elementary substructure of the expanded relational structure by $\mathbb{P}_0$, there exists $x' \in \left[\mathfrak{M}_0^P\right]^{<\aleph_0} \cap N$ such that there are no $r \in \mathbb{P}_0$ which satisfies $\varphi_0(f_{\omega_1 \cap N} \restriction k, x', r)$. However, q satisfies $\varphi_0(f_{\omega_1 \cap N} \restriction k, x', q)$, which is a contradiction. ⊣

Proof of $(\mathsf{p})_{\alpha+1}$. Suppose that $\alpha \in \omega_2$, $p \in \mathbb{P}_{\alpha+1}$, $N \in \mathcal{N}_p \cap \mathfrak{M}_{\alpha+1}^P$ which satisfies that $\{N\} \times ((\alpha+1) \cap N) \subseteq R_p$, $\mathcal{D} \in N$ is a predense subset of $\mathbb{P}_{\alpha+1}$, and $q \leq_{\mathbb{P}_{\alpha+1}} p$. By extending q if necessary, we may assume that q is an extension of some member of $\mathcal{D}$. Since $N \in \mathfrak{M}_{\alpha+1}^P$, by Proposition 3.9(3) and the fact that $\alpha + 1 \in N$ (hence $\alpha \in N$), $N \in \mathfrak{M}_\alpha^P$. So by the induction hypothesis $(\mathsf{p})_\alpha$, $q \restriction \alpha$ is $(N, \mathbb{P}_\alpha)$-generic. It suffices to find $u \in \mathbb{P}_{\alpha+1} \cap N$ which is compatible with q in $\mathbb{P}_{\alpha+1}$ such that u is an extension of some member of $\mathcal{D} \cap N$. When $\alpha \notin \mathrm{dom}(A_q)$, the argument is similar to the proof of $(\mathsf{p})_0$. So we suppose that $\alpha \in \mathrm{dom}(A_q)$.

Define $\mathcal{E}$ to be the set of the conditions u of $\mathbb{P}_{\alpha+1}$ such that

- u is an extension of some member of $\mathcal{D}$ in $\mathbb{P}_{\alpha+1}$,
- $\mathcal{N}_u \cap M = \mathcal{N}_q \cap N$ and $A_u(\alpha) \cap M = A_q(\alpha) \cap N$ for some $M \in R_u{}^{-1}[\{\alpha\}]$, and
- for any $\sigma \in A_q(\alpha) \setminus N$ with $\delta_\sigma > \omega_1 \cap N$ and $C_{\delta_\sigma} \cap N \neq \emptyset$,
$$\max\left(C_{\delta_\sigma} \cap N\right) < \min\left\{\delta_\tau : \tau \in A_u(\alpha) \setminus \left(A_q(\alpha) \cap N\right)\right\}.$$

Then $q \in \mathcal{E}$, and $\mathcal{E}$ is a definable class in the expanded relational structure by $\mathbb{P}_{\alpha+1}$ with parameters $\mathcal{D}, \mathcal{N}_q \cap N, A_q(\alpha) \cap N, \{C_{\delta_\sigma} \cap N : \sigma \in A_q(\alpha) \setminus N, \delta_\sigma > \omega_1 \cap N\}$, all of which are in N. Since $q \restriction \alpha$ is $(N, \mathbb{P}_\alpha)$-generic, for any $\eta \in \omega_1 \cap N$,

$$\star\ q \restriction \alpha \Vdash_{\mathbb{P}_\alpha} \text{“there exists } u \in \mathcal{E} \cap N \text{ such that } u \restriction \alpha \in \dot{G}_{\mathbb{P}_\alpha} \text{ and } \eta < \min\left\{\delta_\tau : \tau \in A_u(\alpha) \setminus \left(A_q(\alpha) \cap N\right)\right\}\text{”}.$$

Define $\dot{Z}$ to be a $\mathbb{P}_\alpha$-name such that

$$\Vdash_{\mathbb{P}_\alpha} \text{“}\dot{Z} \text{ is the set of the functions } g \text{ from } \omega \text{ into } 2^{<\omega} \text{ such that there exists } u \in \mathcal{E} \cap N \text{ such that } u \restriction \alpha \in \dot{G}_{\mathbb{P}_\alpha} \text{ and, for any } \tau \in A_u(\alpha) \setminus \left(A_q(\alpha) \cap N\right), g(\left|C_{\omega_1 \cap N} \cap \delta_\tau\right|) \subseteq \dot{r}_\alpha \text{ and } g(\left|C_{\omega_1 \cap N} \cap (\gamma_\tau + 1)\right|) \subseteq \dot{r}_\alpha\text{”}.$$

By (g-ob-2), (g-cl), and $N \in R_q{}^{-1}[\{\alpha\}]$, there exists the unique $\sigma_0 \in A_q(\alpha)$ such that $\delta_{\sigma_0} = \omega_1 \cap N$. By (g-m), we can take $M \in \mathcal{N}_q \cap \mathfrak{M}_\alpha^P$ such that $N \in M$, $\omega_1 \cap M = \gamma_{\sigma_0}$, and $\{M\} \times (\alpha \cap M) \subseteq R_q$. By $\star$ above, $q \restriction \alpha \Vdash_{\mathbb{P}_\alpha}$“$\dot{Z}$ is dense open in $(2^{<\omega})^\omega$”.

$\dot{Z}$ is defined from the set $\{N, A_q(\alpha) \cap N, C_{\omega_1 \cap N}, \dot{r}_\alpha\}$ (which is in M), $\mathcal{E}$ and $\dot{G}_{\mathbb{P}_\alpha}$, and is forced to be an open subset of $(2^{<\omega})^\omega$. Since M is an elementary substructure of the expanded relational structure by $\mathbb{P}_\alpha$, by Proposition 5.1, $\dot{Z}$ can be considered as an element of M. By the induction hypothesis $(\mathsf{C})_\alpha$, $q \restriction \alpha \Vdash_{\mathbb{P}_\alpha}$" $f_{\omega_1 \cap M}$ is Cohen over $M[\dot{G}_{\mathbb{P}_\alpha}]$". It follows that $q \restriction \alpha \Vdash_{\mathbb{P}_\alpha}$" $f_{\omega_1 \cap M} \in \dot{Z}$". Take $r \in \mathbb{P}_\alpha$ and $u \in \mathcal{E} \cap N$ such that $r \leq_{\mathbb{P}_\alpha} q \restriction \alpha$ and

$$r \Vdash_{\mathbb{P}_\alpha} \text{“}u \text{ is a witness that } f_{\omega_1 \cap M} \in \dot{Z}\text{”}.$$

Let r' be a common extension of r and $u \restriction \alpha$. Define $q' \in U_{\alpha+1}$ such that

- $\mathcal{N}_{q'} := \mathcal{N}_{r'}$,
- $R_{q'} := R_{r'}$
$$\cup \Big\{ \big\langle \big({\Psi_{M'}}^{-1} \circ \Psi_N\big)(K), \alpha \big\rangle : M' \in {R_q}^{-1}[\{\alpha\}] \text{ with } \omega_1 \cap M' = \omega_1 \cap N, \\ K \in {R_u}^{-1}[\{\alpha\}] \setminus {R_q}^{-1}[\{\alpha\}] \Big\},$$
- $A_{q'} \restriction \alpha := A_{r'}$, and
- $A_{q'}(\alpha) = A_u(\alpha) \cup A_q(\alpha)$.

We claim that q' is a condition of $\mathbb{P}_{\alpha+1}$. Since $q' \restriction \alpha = r'$, $q' \restriction \alpha$ is a condition of $\mathbb{P}_\alpha$. Since $q \in \mathbb{P}_{\alpha+1}$, $u \in \mathbb{P}_{\alpha+1} \cap N$, $N \in \mathfrak{M}^P_{\alpha+1}$, and $\alpha \in \omega_2$, by Propositions 3.10 and 3.11, ${R_{q'}}^{-1}[\{\alpha\}]$ satisfies (el), (ho), (up), and (down). Since q and u are conditions of $\mathbb{P}_{\alpha+1}$, $A_{q'}(\alpha)$ satisfies (g-ob), (g-ob-2), and (g-cl). It follows from the choice of r and u that $A_{q'}(\alpha)$ satisfies (g-w). Moreover, by $\alpha \in \omega_2$ and Proposition 3.10, $A_{q'}(\alpha)$ satisfies (g-m). Therefore q' is a condition of $\mathbb{P}_{\alpha+1}$. So q' is a common extension of q and u in $\mathbb{P}_{\alpha+1}$. By elementarity of N and the fact that $u \in \mathcal{E} \cap N$, u is an extension of some member of $\mathcal{D} \cap N$ in $\mathbb{P}_{\alpha+1}$. ⊣

The following proposition will be used in the rest of the proof.

Proposition 5.3. *Suppose that $\alpha \in \omega_2$, $(\mathsf{p})_{\alpha+1}$ holds, $p \in \mathbb{P}_{\alpha+1}$, $N \in \mathcal{N}_p \cap \mathfrak{M}^P_{\alpha+1}$ such that $\{N\} \times ((\alpha+1) \cap N) \subseteq R_p$, and $\mathcal{D}$ is a definable class in the expanded relational structure by $\mathbb{P}_{\alpha+1}$ with parameters in N such that $p \in \mathcal{D}$. Then there exists $q \in \mathcal{D} \cap N$ which is compatible with p in $\mathbb{P}_{\alpha+1}$.*

Proof of Proposition 5.3. Define $\mathcal{D}'$ to be the set of the conditions u of $\mathbb{P}_{\alpha+1}$ such that either $u \in \mathcal{D}$ or u is incompatible with any element of $\mathcal{D}$ in $\mathbb{P}_{\alpha+1}$. Then $\mathcal{D}'$ is a predense subset of $\mathbb{P}_{\alpha+1}$. Since N is an elementary substructure of the expanded relational structure by $\mathbb{P}_{\alpha+1}$, by Proposition 5.1, there exists a maximal antichain $\mathcal{A}$ in N that is a subset of $\mathcal{D}'$. By $(\mathsf{p})_{\alpha+1}$, p is $(N, \mathbb{P}_{\alpha+1})$-generic. So there exists $q \in \mathcal{A} \cap N$ such that q is compatible with p in $\mathbb{P}_{\alpha+1}$. Since $p \in \mathcal{D}$, q has to be in $\mathcal{D}$. ⊣

The following proof has similarities with the proof of Lemma 2.10 (although it is not identical to it).

Proof of $(\mathsf{C})_{\alpha+1}$. Suppose that $p \in \mathbb{P}_{\alpha+1}$, $N \in \mathcal{N}_p \cap \mathfrak{M}^P_{\alpha+1}$ which satisfies that $\{N\} \times ((\alpha+1) \cap N) \subseteq R_p$, and $\{\dot{F}_n : n \in \omega\}$ is a set of $\mathbb{P}_{\alpha+1}$-names for nowhere dense subsets of $(2^{<\omega})^\omega$ such that $\{\dot{F}_n : n \in \omega\} \in N$. Let us show that $p \Vdash_{\mathbb{P}_{\alpha+1}}$ " $f_{\omega_1 \cap N} \notin \bigcup_{n \in \omega} \dot{F}_n$".

Suppose not, and let $q \leq_{\mathbb{P}_{\alpha+1}} p$ and $n \in \omega$ be such that

$$q \Vdash_{\mathbb{P}_{\alpha+1}} \text{“} f_{\omega_1 \cap N} \in \dot{F}_n \text{”}.$$

Let $\sigma_N \in A_q(\alpha)$ be such that $\delta_{\sigma_N} = \omega_1 \cap N$.

Only in this paragraph, for each set x (in the ground model), we denote by $\check{x}$ the canonical $\mathbb{P}_\alpha$-name which represents the set x in the forcing extension. For each $v \in (2^{<\omega})^{<\omega}$ and each $A \in [\omega_1 \times \omega_1 \times \omega_1]^{<\aleph_0}$, define $\dot{\Sigma}'(v, A)$ to be the $\mathbb{P}_\alpha$-name which consists of all the pairs $\left\langle u, \check{B} \right\rangle$ such that

- $u \in \mathbb{P}_\alpha$ and $B \in [\omega_1 \times \omega_1 \times \omega_1]^{<\aleph_0}$,
- there exists $u' \in \mathbb{P}_{\alpha+1}$ such that
  - $u \leq_{\mathbb{P}_\alpha} u' \restriction \alpha$,
  - $R_q{}^{-1}[\{\alpha\}] \cap N \subseteq R_{u'}{}^{-1}[\{\alpha\}]$,
  - $A_{u'}(\alpha) = A \cup B$,
  - for any $\sigma \in A$ and any $\tau \in B$, $\delta_\sigma < \delta_\tau$, and
  - $u' \Vdash_{\mathbb{P}_{\alpha+1}}$“ $\dot{F}_n \cap [v] = \emptyset$ ” (here we omit the check-notation for $\mathbb{P}_{\alpha+1}$).

For each $v \in (2^{<\omega})^{<\omega}$ and each $A \in [\omega_1 \times \omega_1 \times \omega_1]^{<\aleph_0}$, $\dot{\Sigma}'(v, A)$ is a definable class in the expanded relational structure by $\mathbb{P}_{\alpha+1}$ with parameters in $H(\kappa)$. Moreover, if $A \in N$, then $\dot{\Sigma}'(v, A)$ is a definable class in the expanded relational structure by $\mathbb{P}_{\alpha+1}$ with parameters in N. By Proposition 5.1, there exists $\Sigma \in H(\kappa)$ such that

- $\Sigma \subseteq (2^{<\omega})^{<\omega} \times [\omega_1 \times \omega_1 \times \omega_1]^{<\aleph_0} \times \mathbb{P}_\alpha \times [\omega_1 \times \omega_1 \times \omega_1]^{<\aleph_0}$,
- for each $v \in (2^{<\omega})^{<\omega}$ and each $A, B \in [\omega_1 \times \omega_1 \times \omega_1]^{<\aleph_0}$, the set $\mathcal{A}(v, A, B) := \{u \in \mathbb{P}_\alpha : \langle v, A, u, B \rangle \in \Sigma\}$ is a maximal antichain (of size $\aleph_1$), and, for each $u \in \mathcal{A}(v, A, B)$, either $\left\langle u, \check{B} \right\rangle \in \dot{\Sigma}'(v, A)$, or no extension v of u in $\mathbb{P}_\alpha$ satisfies $\left\langle v, \check{B} \right\rangle \in \dot{\Sigma}'(v, A)$.

Since N is an elementary substructure of the expanded relational structure by $\mathbb{P}_{\alpha+1}$, we may assume that $\Sigma \in N$. For each $v \in (2^{<\omega})^{<\omega}$ and each $A \in [\omega_1 \times \omega_1 \times \omega_1]^{<\aleph_0}$, define $\dot{\Sigma}(v, A)$ to be the $\mathbb{P}_\alpha$-name such that

$$\dot{\Sigma}(v, A) := \bigcup_{B \in [\omega_1 \times \omega_1 \times \omega_1]^{<\aleph_0}} \left\{ \left\langle u, \check{B} \right\rangle : u \in \mathcal{A}(v, A, B) \text{ and } \left\langle u, \check{B} \right\rangle \in \dot{\Sigma}'(v, A) \right\}.$$

Then, the sequence $\left\langle \dot{\Sigma}(v, A) : v \in (2^{<\omega})^{<\omega}, A \in [\omega_1 \times \omega_1 \times \omega_1]^{<\aleph_0} \right\rangle$ belongs to N, and, for each $v \in (2^{<\omega})^{<\omega}$ and each $A \in [\omega_1 \times \omega_1 \times \omega_1]^{<\aleph_0}$, $\Vdash_{\mathbb{P}_\alpha}$“$\dot{\Sigma}(v, A) = \dot{\Sigma}'(v, A)$ ”. From now on, we omit the check-notation in the forcing language.

Since $\{N\} \times ((\alpha + 1) \cap N) \subseteq R_q$, by $(\mathsf{p})_{\alpha+1}$ and Proposition 5.3, there exists $q' \in \mathbb{P}_{\alpha+1} \cap N$ such that

- q' is compatible with q in $\mathbb{P}_{\alpha+1}$,
- $R_q{}^{-1}[\{\alpha\}] \cap N \subseteq R_{q'}{}^{-1}[\{\alpha\}]$,
- there exists $N_0 \in R_{q'}{}^{-1}[\{\alpha\}]$ such that
  - $A_{q'}(\alpha) \cap N_0 = A_q(\alpha) \cap N$,
  - the set

$$\left\{ \left\langle \varepsilon_\sigma, C_{\delta_\sigma} \cap N_0, f_{\gamma_\sigma} \restriction \left| C_{\delta_\sigma} \cap N_0 \right| \right\rangle : \sigma \in A_{q'}(\alpha) \setminus N_0 \text{ with } \varepsilon_\sigma \in N_0 \;\&\; \delta_\sigma \neq \omega_1 \cap N_0 \right\}$$

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is equal to the set

$$\left\{\left\langle \varepsilon_\sigma, C_{\delta_\sigma}\cap N, f_{\gamma_\sigma}\restriction\left|C_{\delta_\sigma}\cap N\right|\right\rangle : \sigma\in A_q(\alpha)\setminus(N\cup\{\sigma_N\}) \text{ with } \varepsilon_\sigma\in N\right\},$$

and

– N_0 contains the sets $\left\{\varepsilon_\sigma : \sigma\in A_q(\alpha)\setminus N\right\}\cap N$, $\{C_{\gamma_\sigma}\cap N : \sigma\in A_q(\alpha)\setminus(N\cup\{\sigma_N\})\}$ and $\left\langle\dot\Sigma(v,A) : v\in(2^{<\omega})^{<\omega}, A\in[\omega_1\times\omega_1\times\omega_1]^{<\aleph_0}\right\rangle$ as members.

Let q^+ be a common extension of q' and q in $\mathbb{P}_{\alpha+1}$. We notice that N_0 contains the set $\left\{C_{\omega_1\cap N}\cap N_0, f_{\gamma_{\sigma_N}}\restriction\left(\left|C_{\omega_1\cap N}\cap N_0\right|+1\right)\right\}$ and

$$q^+\restriction\alpha\Vdash_{\mathbb{P}_\alpha} \text{“}f_{\gamma_{\sigma_N}}(\left|C_{\omega_1\cap N}\cap N_0\right|)\subseteq\dot r_\alpha\text{”}.$$

In a similar way as in the definition of $\dot\Sigma'(v,A)$ before, we have a $\mathbb{P}_\alpha$-name $\dot Z$ for a subset of $(2^{<\omega})^\omega$ such that $\dot Z$ is a definable class in the expanded relational structure by $\mathbb{P}_{\alpha+1}$ with parameters in N and, for any $u\in\mathbb{P}_\alpha$ and any $v\in(2^{<\omega})^{<\omega}$, if $u\Vdash_{\mathbb{P}_\alpha}$“$[v]\subseteq\dot Z$”, then there exists $u'\in\mathbb{P}_{\alpha+1}$ such that

- $u\leq_{\mathbb{P}_\alpha}u'\restriction\alpha$,
- $u'\leq_{\mathbb{P}_{\alpha+1}}q'$,
- for each $\tau\in A_{u'}(\alpha)\cap N_0$, if $\varepsilon_{\sigma_N}<\delta_\tau$, then $\gamma_\tau+1<\omega_1\cap N_0$ and

$$u'\restriction\alpha\Vdash_{\mathbb{P}_\alpha}\text{“}f_{\gamma_{\sigma_N}}(\left|C_{\omega_1\cap N}\cap\delta_\tau\right|)\subseteq\dot r_\alpha \text{ and } f_{\gamma_{\sigma_N}}(\left|C_{\omega_1\cap N}\cap(\gamma_\tau+1)\right|)\subseteq\dot r_\alpha\text{”},$$

- $A_{u'}(\alpha)$ has σ such that $\delta_\sigma=\omega_1\cap N_0$, and
- $u'\Vdash_{\mathbb{P}_{\alpha+1}}$“$\dot F_n\cap[v]=\emptyset$”.

Now $\dot F_n$ is a $\mathbb{P}_{\alpha+1}$-name for a nowhere dense subset of $(2^{<\omega})^\omega$, $\sigma_N\in A_{q^+}(\alpha)$, $A_{q^+}(\alpha)$ has σ such that $\delta_\sigma=\omega_1\cap N_0$, and $q^+\leq_{\mathbb{P}_{\alpha+1}}q'$, so

$$q^+\restriction\alpha\Vdash_{\mathbb{P}_\alpha}\text{“}\dot Z \text{ is a dense open subset of } (2^{<\omega})^\omega\text{”}.$$

Since $\dot Z$ is a definable class in the expanded relational structure by $\mathbb{P}_{\alpha+1}$ with parameters in N, by Proposition 5.1, $\dot Z$ can be considered as an element of N. So by $(\mathsf{C})_\alpha$,

$$q^+\restriction\alpha\Vdash_{\mathbb{P}_\alpha}\text{“}f_{\omega_1\cap N}\in\dot Z\text{”}.$$

Thus, there are $r\leq_{\mathbb{P}_\alpha}q^+\restriction\alpha$ and $k\in\omega$ such that

$$r\Vdash_{\mathbb{P}_\alpha}\text{“}[f_{\omega_1\cap N}\restriction k]\subseteq\dot Z\text{”}.$$

Let $u_0\in\mathbb{P}_{\alpha+1}$ be a witness to $r\Vdash_{\mathbb{P}_\alpha}$“$[f_{\omega_1\cap N}\restriction k]\subseteq\dot Z$”. Then

$$r\Vdash_{\mathbb{P}_\alpha}\text{“}A_{u_0}\setminus N_0\in\dot\Sigma(f_{\omega_1\cap N}\restriction k, A_{u_0}(\alpha)\cap N_0)\text{”}.$$

Let $\zeta_0\in\omega_1\cap N_0$ be such that

- for any $\tau\in A_{u_0}(\alpha)\cap N_0$, $\gamma_\tau<\zeta_0$,
- for any $\sigma\in A_q(\alpha)\setminus N$,
  – if $\varepsilon_\sigma\in N$, then $\varepsilon_\sigma<\zeta_0$, and
  – $C_{\delta_\sigma}\cap N_0\subseteq\zeta_0$.

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Since $\{N_0\} \times (\alpha \cap N_0) \subseteq R_{q^+} \subseteq R_r$, by $(\mathsf{p})_\alpha$, r is $(N_0, \mathbb{P}_\alpha)$-generic. Since the set $\left\{f_{\omega_1\cap N} \restriction k, A_{u_0}(\alpha) \cap N_0\right\}$ is in N_0, $\dot{\Sigma}(f_{\omega_1\cap N} \restriction k, A_{u_0}(\alpha) \cap N_0)$ is also in N_0. Moreover, $A_{u_0}(\alpha) \setminus N_0$ has σ such that $\delta_\sigma = \omega_1 \cap N_0$, which is larger than ζ_0. Hence, by Proposition 5.3, there are $r' \leq_{\mathbb{P}_\alpha} r$ and $B \in [\omega_1 \times \omega_1 \times \omega_1]^{<\aleph_0} \cap N_0$ such that

- for every $\tau \in B$, $\zeta_0 < \delta_\tau$, and
- $r' \Vdash_{\mathbb{P}_\alpha}$"$B \in \dot{\Sigma}(f_{\omega_1\cap N} \restriction k, A_{u_0}(\alpha) \cap N_0)$".

Since $\{N\} \times (\alpha \cap N) \subseteq R_{q^+} \subseteq R_{r'}$, by $(\mathsf{p})_\alpha$, r' is $(N, \mathbb{P}_\alpha)$-generic. Since N is an elementary substructure of the expanded relational structure by $\mathbb{P}_{\alpha+1}$, there exists $u_1 \in \mathbb{P}_{\alpha+1} \cap N$ which witnesses $r' \Vdash_{\mathbb{P}_\alpha}$"$B \in \dot{\Sigma}(f_{\omega_1\cap N} \restriction k, A_{u_0}(\alpha) \cap N_0)$". Then r' is a common extension of u_1 and $q \restriction \alpha$ in $\mathbb{P}_\alpha$. Define $s \in U_{\alpha+1}$ such that

- $\mathcal{N}_s := \mathcal{N}_{r'}$,
- $R_s := R_{u_1} \cup R_q$
  $\cup \left\{ \left\langle \left({\Psi_{M'}}^{-1} \circ \Psi_N\right)(K), \alpha \right\rangle : M' \in {R_q}^{-1}[\{\alpha\}] \text{ with } \omega_1 \cap M' = \omega_1 \cap N, \; K \in {R_{u_1}}^{-1}[\{\alpha\}] \right\}$,
- $A_s \restriction \alpha := A_{r'}$, and
- $A_s(\alpha) := A_{u_1}(\alpha) \cup A_q(\alpha)$.

Now $s \restriction \alpha = r' \in \mathbb{P}_\alpha$. Since $u_1 \in N$ and

$$ {R_q}^{-1}[\{\alpha\}] \cap N \subseteq {R_{u_1}}^{-1}[\{\alpha\}] \in N \in {R_q}^{-1}[\{\alpha\}], $$

${R_s}^{-1}[\{\alpha\}]$ satisfies (el), (ho), (up), and (down). $A_s(\alpha)$ satisfies (g-ob), (g-ob-2), (g-cl), and (g-m). We will check that $A_s(\alpha)$ satisfies (g-w). Since $u_1 \leq_{\mathbb{P}_{\alpha+1}} q'$,

$$ A_q(\alpha) \cap N = A_{q'}(\alpha) \cap N_0 \subseteq A_{u_1}(\alpha) \cap N_0. $$

By the choice of u_1,

$$ A_{u_1}(\alpha) = \left(A_{u_0}(\alpha) \cap N_0\right) \cup B \subseteq N_0 \subseteq N. $$

Let $\tau \in A_{u_1}(\alpha)$ and $\sigma \in A_q(\alpha) \setminus (N \cup \{\sigma_N\})$. Then there exists $\sigma' \in A_{q'}(\alpha) \setminus N_0$ such that

$$ \langle \varepsilon_\sigma, C_{\delta_\sigma} \cap N_0, f_{\gamma_\sigma} \restriction |C_{\delta_\sigma} \cap N_0| \rangle = \left\langle \varepsilon_{\sigma'}, C_{\delta_{\sigma'}} \cap N, f_{\gamma_{\sigma'}} \restriction \left| C_{\delta_{\sigma'}} \cap N \right| \right\rangle. $$

So if $\varepsilon_\sigma < \delta_\tau$, then, since $u_1 \leq_{\mathbb{P}_{\alpha+1}} q'$,

$$ \begin{aligned} u_1 \restriction \alpha \Vdash_{\mathbb{P}_\alpha} \text{“} & f_{\gamma_\sigma}(|C_{\delta_\sigma} \cap \delta_\tau|) = f_{\gamma_{\sigma'}}\left(\left|C_{\delta_{\sigma'}} \cap \delta_\tau\right|\right) \subseteq \dot{r}_\alpha \text{ and} \\ & f_{\gamma_\sigma}(|C_{\delta_\sigma} \cap (\gamma_\tau + 1)|) = f_{\gamma_{\sigma'}}\left(\left|C_{\delta_{\sigma'}} \cap (\gamma_\tau + 1)\right|\right) \subseteq \dot{r}_\alpha \text{”}. \end{aligned} $$

Let $\tau' \in A_{u_1}(\alpha) \cap N_0$. By the choice of u_1, if $\varepsilon_{\sigma_N} < \delta_\tau$, then

$$ u_1 \restriction \alpha \Vdash_{\mathbb{P}_\alpha} \text{“} f_{\gamma_{\sigma_N}}\left(\left|C_{\omega_1\cap N} \cap \delta_\tau\right|\right) \subseteq \dot{r}_\alpha \text{ and } f_{\gamma_{\sigma_N}}\left(\left|C_{\omega_1\cap N} \cap (\gamma_\tau + 1)\right|\right) \subseteq \dot{r}_\alpha \text{”}. $$

Let $\tau'' \in A_{u_1}(\alpha) \setminus N_0$. Then $\zeta_0 < \delta_{\tau''}$, and hence $C_{\omega_1\cap N} \cap \delta_{\tau''} = C_{\omega_1\cap N} \cap (\gamma_{\tau''} + 1) = C_{\omega_1\cap N} \cap N_0$. Since $r' \leq_{\mathbb{P}_\alpha} q^+ \restriction \alpha$,

$$ \begin{aligned} r' \Vdash_{\mathbb{P}_\alpha} \text{“} & f_{\gamma_{\sigma_N}}\left(\left|C_{\omega_1\cap N} \cap \delta_{\tau''}\right|\right) = f_{\gamma_{\sigma_N}}\left(\left|C_{\omega_1\cap N} \cap (\gamma_{\tau''} + 1)\right|\right) \\ & = f_{\gamma_{\sigma_N}}\left(\left|C_{\omega_1\cap N} \cap N_0\right|\right) \subseteq \dot{r}_\alpha \text{”}. \end{aligned} $$

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Thus $A_s(\alpha)$ satisfies (g-w). Therefore s is a condition of $\mathbb{P}_{\alpha+1}$. It follows that s is a common extension of u_1 and q. However, then

$$s \Vdash_{\mathbb{P}_{\alpha+1}} \text{“}\dot{F}_n \cap [f_{\omega_1 \cap N} \restriction k] = \emptyset \text{ and } f_{\omega_1 \cap N} \in \dot{F}_n\text{”},$$

which is a contradiction, and finishes the proof of $(\mathsf{C})_{\alpha+1}$. ⊣

We will use Proposition 5.5 in the proof of $(\mathsf{p})_\alpha$ and $(\mathsf{C})_\alpha$ for a limit ordinal α.

PROPOSITION 5.4. *Suppose that* $\alpha \in \omega_2 + 1$, $(\mathsf{p})_\beta$ *and* $(\mathsf{C})_\beta$ *hold for every* $\beta < \alpha$, $p \in \mathbb{P}_\alpha$, $\xi \in \operatorname{dom}(A_p)$, $M \in {R_p}^{-1}[\{\xi\}] \cap \mathfrak{M}^P_{\xi+1}$, *and* $N \in \mathfrak{M}^P_\xi \cap M$ *contains the set* $\{\mathcal{N}_p \cap M, \langle A_p(\zeta) \cap M : \zeta \in \operatorname{dom}(A_p) \cap (\xi+1) \cap M \rangle\}$ *and is such that, for every* $\zeta \in \operatorname{dom}(A_p) \cap (\xi+1) \cap N$ *and every* $\sigma \in A_p(\zeta)$ *with* $\varepsilon_\sigma < \omega_1 \cap N < \delta_\sigma$,

$$p \restriction \zeta \Vdash_{\mathbb{P}_\zeta} \text{“}f_{\gamma_\sigma}(|C_{\delta_\sigma} \cap \omega_1 \cap N|) \subseteq \dot{r}_\zeta\text{”}.$$

Then there is some $q \leq_{\mathbb{P}_{\xi+1}} p \restriction (\xi+1)$ *such that* $\langle N, \xi \rangle \in R_q$ *and* $A_q = A_p \cup \{\sigma\}$ *for some* σ *with* $\delta_\sigma = \omega_1 \cap N$.

PROOF OF PROPOSITION 5.4. We will prove this by induction on ξ. Let $\xi_0 := \max\big(\operatorname{dom}(A_p) \cap \xi \cap M\big)$. Note that $\xi_0 \in N$, and hence $N \in \mathfrak{M}^P_{\xi_0} \cap M$. By the induction hypothesis, there exists $p_0 \leq_{\mathbb{P}_{\xi_0+1}} p \restriction (\xi_0+1)$ such that $N \in {R_{p_0}}^{-1}[\{\xi_0\}]$. Let $p_0' = \big\langle \mathcal{N}_{p_0}, R_{p_0} \cup R_p, A_{p_0} \cup \big(A_p \restriction [\xi_0+1, \xi)\big)\big\rangle$, which is a canonical common extension of p_0 and p in $\mathbb{P}_\xi$. Since p_0' is $(M, \mathbb{P}_{\xi+1})$-generic, by Proposition 5.3, we can take an extension $p_1 \leq_{\mathbb{P}_{\xi+1}} p_0'$ and $N_1 \in {R_{p_1}}^{-1}[\{\xi\}] \cap M$ which contains N as a member. Then, by (g-ob-2) and (g-w), for every $\zeta \in \operatorname{dom}(A_{p_1}) \cap N$ and every $\sigma \in A_{p_1}(\zeta)$ with $\varepsilon_\sigma < \omega_1 \cap N_1 < \delta_\sigma$,

$$p_1 \restriction \zeta \Vdash_{\mathbb{P}_\zeta} \text{“}f_{\gamma_\sigma}(|C_{\delta_\sigma} \cap \omega_1 \cap N_1|) \subseteq \dot{r}_\zeta\text{”}.$$

Recall that, for each limit ordinal δ, $C_\delta \cap \omega_1 \cap N_1 = C_\delta \cap (\omega_1 \cap N_1 + 1)$. Take $\varepsilon \in \omega_1 \cap N$ such that

$$\max\{\gamma_\sigma : \sigma \in A_p(\xi) \cap M\} < \varepsilon.$$

Define $q \in U_{\xi+1}$ such that $\mathcal{N}_q := \mathcal{N}_{p_1}$,

$$R_q := R_{p_1 \restriction \xi} \cup \Big\{ \Big\langle \Big({\Psi_K}^{-1} \circ \Psi_M\Big)(N), \xi \Big\rangle : K \in {R_p}^{-1}[\{\xi\}] \text{ with } \omega_1 \cap K = \omega_1 \cap M \Big\},$$

$A_q \restriction \xi := A_{p_1} \restriction \xi$, and

$$A_q(\xi) := A_p(\xi) \cup \{\langle \varepsilon, \omega_1 \cap N, \omega_1 \cap N_1 \rangle\}.$$

By Propositions 3.10 and 3.11, R_q satisfies (el), (ho), (up), and (down) in Definition 4.1. By $(\mathsf{C})_\xi$ and the fact that $\{N_1\} \times (\xi \cap N_1) \subseteq R_{p_1 \restriction \xi} \subseteq R_q$,

$$q \restriction \xi \Vdash_{\mathbb{P}_\xi} \text{“}f_{\omega_1 \cap N_1} \text{ is Cohen over } N_1[\dot{G}_{\mathbb{P}_\xi}] \text{ and } \dot{r}_\xi \in N_1, \text{ hence } \big\{n \in \omega : f_{\omega_1 \cap N_1}(n) \subseteq \dot{r}_\xi\big\} \text{ is infinite”}.$$

Moreover, by the roles of ε and N_1, $A_q(\xi)$ satisfies (g-ob), (g-ob-2) (g-cl), and (g-w). By the role of the set N_1, $A_q(\xi)$ satisfies (g-m) for R_q. Therefore, q is a condition of

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$\mathbb{P}_{\xi+1}$. This q is what we want. (We notice that $q \restriction \xi$ is an extension of $p_1 \restriction \xi$ in $\mathbb{P}_\xi$; however, q may *not* be an extension of p_1 in $\mathbb{P}_{\xi+1}$.) ⊣

PROPOSITION 5.5. *Suppose that $\alpha \in \omega_2 + 1$, $(\mathsf{p})_\beta$ and $(\mathsf{C})_\beta$ hold for every $\beta < \alpha$, $q, r \in \mathbb{P}_\alpha$, and $\xi < \alpha$ is such that*

- *$q \restriction \xi$ and $r \restriction \xi$ are compatible in $\mathbb{P}_\xi$,*
- *$\xi \notin \mathrm{dom}(A_q)$ and $\xi \in \mathrm{dom}(A_r)$,*
- *there exists $N \in {R_q}^{-1}[\{\xi\}]$ such that $r \in N$ and, for any $M \in {R_q}^{-1}[\{\xi\}]$, $\omega_1 \cap M \geq \omega_1 \cap N$ and $M \in \mathfrak{M}^P_{\xi+1}$.*

Then there exists a common extension $q' \in \mathbb{P}_{\xi+1}$ of $q \restriction (\xi+1)$ and $r \restriction (\xi+1)$ in $\mathbb{P}_{\xi+1}$ such that

$$\left\{\delta_\sigma : \sigma \in A_{q'}(\xi) \setminus A_r(\xi)\right\} = \left\{\omega_1 \cap M : M \in {R_q}^{-1}[\{\xi\}]\right\}.$$

PROOF OF PROPOSITION 5.5. Let $p_{-1} \in \mathbb{P}_{\xi+1}$ be such that

- $p_{-1} \restriction \xi$ is a common extension of $q \restriction \xi$ and $r \restriction \xi$ in $\mathbb{P}_\xi$,
- p_{-1} is an extension of r in $\mathbb{P}_{\xi+1}$,
- ${R_{p_{-1}}}^{-1}[\{\xi\}] = {R_r}^{-1}[\{\xi\}]$, and
- $A_{p_{-1}}(\xi) = A_r(\xi)$.

Take a maximal $\in$-chain $\{M_i : i \leq n\}$ of ${R_q}^{-1}[\{\xi\}]$ such that $M_0 = N$. For each $i \leq n$, since $\{M_i\} \times (\xi \cap M_i) \subseteq R_{p_{-1}}$, for every $\zeta \in \mathrm{dom}(A_{p_{-1}}) \cap \xi \cap M_i$ and every $\sigma \in A_{p_{-1}}(\zeta)$ with $\varepsilon_\sigma < \omega_1 \cap M_i < \delta_\sigma$,

$$p \restriction \zeta \Vdash_{\mathbb{P}_\zeta} \text{“} f_{\gamma_\sigma}(|C_{\delta_\sigma} \cap \omega_1 \cap M_i|) \subseteq \dot{r}_\zeta \text{”}.$$

By using Proposition 5.4 $(n+1)$ times repeatedly, for each $i \leq n$, we can construct an extension $p_i \in \mathbb{P}_{\xi+1}$ such that

- $p_i \restriction \xi$ is an extension of p_{i-1},
- $\langle M_i, \xi \rangle \in R_{p_i}$, and
- $A_{p_i}(\xi) = A_{p_{i-1}}(\xi) \cup \{\sigma_i\}$ for some σ_i such that $\delta_{\sigma_i} = \omega_1 \cap M_i$.

Then p_n is what we want. ⊣

PROOF OF $(\mathsf{p})_\alpha$ FOR A LIMIT ORDINAL α. Suppose that $\alpha \in \omega_2 + 1$ is a limit ordinal, $p \in \mathbb{P}_\alpha$, $N \in \mathcal{N}_p \cap \mathfrak{M}^P_\alpha$ satisfies that $\{N\} \times (\alpha \cap N) \subseteq R_p$, $\mathcal{D} \in N$ is a predense subset of $\mathbb{P}_\alpha$, and $q \leq_{\mathbb{P}_\alpha} p$ is an extension of some member of $\mathcal{D}$. By extending q if necessary, we may assume that there exists $u \in \mathcal{D}$ such that $q \leq_{\mathbb{P}_\alpha} u$. It suffices to find $u' \in \mathcal{D} \cap N$ which is compatible with q in $\mathbb{P}_\alpha$. We have the case when α has uncountable cofinality and the case when it has countable cofinality. In the latter case, $\alpha \cap N$ is cofinal in α and hence we can take $\beta \in \alpha \cap N$ such that $\max(\mathrm{dom}(A_q)) < \beta$. But we may not be able to take such a β in the former case, that is, it may happen that $\mathrm{dom}(A_q)$ is not bounded by $\sup(\alpha \cap N)$. So we need more argument for the former case than for the latter case.

Suppose that α is of uncountable cofinality and $\mathrm{dom}(A_q) \not\subseteq \sup(\alpha \cap N)$. (If $\mathrm{dom}(A_q) \subseteq \sup(\alpha \cap N)$, then the proof will be simpler, same to the proof of the case that α is of countable cofinality.) Since $\mathcal{N}_q$ forms a symmetric system, for each $M' \in \mathcal{N}_q$ with $\omega_1 \cap M' < \omega_1 \cap N$, there exists $M \in \mathcal{N}_q$ such that $\omega_1 \cap M = \omega_1 \cap N$

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and $M' \in M$, and then, by the requirement (id) in Definition 3.4,

$$\begin{aligned}\sup(M' \cap N \cap \alpha) &= \sup(\left(\Psi_{N\cap H_\kappa}{}^{-1} \circ \Psi_M\right)(M') \cap M \cap \alpha) \\ &\leq \sup(\left(\Psi_{N\cap H_\kappa}{}^{-1} \circ \Psi_M\right)(M') \cap \alpha).\end{aligned}$$

Since N thinks that the set $\left(\Psi_{N\cap H_\kappa}{}^{-1} \circ \Psi_M\right)(M')$ is countable and α is of uncountable cofinality,

$$\sup(\left(\Psi_{N\cap H_\kappa}{}^{-1} \circ \Psi_M\right)(M') \cap \alpha) \in N \cap \alpha.$$

So there are large enough $\beta \in \alpha \cap N$ and $\zeta \in \omega_1 \cap N$, which means that

- $\max(\mathrm{dom}(A_q) \cap \sup(\alpha \cap N)) < \beta$,
- $\max(\{\sup(R_q(M)) : M \in \mathrm{dom}(R_q)\} \cap N) < \beta$,
- for every $M' \in \mathcal{N}_q$ with $\omega_1 \cap M' < \omega_1 \cap N$,
$$\sup(M' \cap N \cap \alpha) < \beta,$$
- $\{\omega_1 \cap M : M \in \mathcal{N}_q \cap N\} \subseteq \zeta$,
- for any $\xi \in \mathrm{dom}(A_q) \cap N$ and any $\sigma \in A_q(\xi) \cap N$, $\gamma_\sigma < \zeta$.

By the second and the third requirements on β, we observe that

$\langle 1 \rangle$ for every $\xi \in [\beta, \alpha) \cap N$ and every $K \in \mathcal{N}_q$ with $\omega_1 \cap K < \omega_1 \cap N$, $K \notin \mathfrak{M}^P_\xi$ (because if K was in $\mathfrak{M}^P_\xi$, ξ would be in K).

Define

$$\mathcal{E} := \Big\{ r \restriction \beta : \ r \in \mathbb{P}_\alpha \text{ such that} \quad \bullet \text{ there exists } u \in \mathcal{D} \text{ so that } r \leq_{\mathbb{P}_\alpha} u, \text{ and} \quad \bullet\ \{\omega_1 \cap K : K \in \mathcal{N}_r\} \cap \zeta = \{\omega_1 \cap K : K \in \mathcal{N}_q\} \cap N \Big\}.$$

We notice that $q \restriction \beta \in \mathcal{E}$ and $\mathcal{E}$ is a definable class in the expanded relational structure by $\mathbb{P}_\alpha$ with parameters in N. By Proposition 3.9, since $\beta \in N \in \mathfrak{M}^P_\alpha$, $N \in \mathfrak{M}^P_\beta$. Moreover, it follows that

$$\{N\} \times (\beta \cap N) \subseteq R_{p\restriction\beta} \subseteq R_{q\restriction\beta}.$$

So, by the induction hypothesis $(\mathsf{p})_\beta$, $q \restriction \beta$ is $(N, \mathbb{P}_\beta)$-generic. Hence there exists p_1 in the set $\mathcal{E} \cap N$ which is compatible with the condition $q \restriction \beta$ in $\mathbb{P}_\beta$. Let $r \in \mathbb{P}_\alpha \cap N$ and $u \in \mathcal{D} \cap N$ witness that $p_1 \in \mathcal{E}$.

Let us show that q and r are compatible in $\mathbb{P}_\alpha$, which finishes the proof of this case. Let $p_2 \in \mathbb{P}_\beta$ be a common extension of $q \restriction \beta$ and p_1 $(= r \restriction \beta)$. We note that $\mathrm{dom}(A_{p_2}) \subseteq \beta$ and

$\langle 2 \rangle$ $\mathrm{dom}(A_q) \cap \mathrm{dom}(A_r) \cap [\beta, \alpha) = \emptyset$, more precisely,

$$\begin{aligned}\beta < \min\left(\mathrm{dom}(A_r) \setminus \beta\right) &\leq \max\left(\mathrm{dom}(A_r)\right) \\ &< \sup(\alpha \cap N) < \min\left(\mathrm{dom}(A_q) \setminus \beta\right),\end{aligned}$$

because $\mathrm{dom}(A_r) \subseteq N$ and $\max(\mathrm{dom}(A_q) \cap \sup(\alpha \cap N)) < \beta$. Since $r \in N$,

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⟨3⟩ $\mathcal{N}_r \subseteq N$,
⟨4⟩ for each $\xi \in [\beta, \alpha) \cap N$,
- if $\xi \in \mathrm{dom}(A_r)$, then $A_r(\xi) \subseteq N$,
- $\left\{K \in {R_q}^{-1}\left[\,[\xi, \alpha)\,\right] \cap \mathfrak{M}^P_\xi : \omega_1 \cap K < \omega_1 \cap N\right\} = \emptyset$ (which follows from ⟨1⟩), and
- if $M \in {R_q}^{-1}[\{\xi\}]$, then $\omega_1 \cap M \geq \omega_1 \cap N$ and $\sup(R_q(M)) \geq \sup(\alpha \cap N)$ (by the role of β), and hence $M \in {R_q}^{-1}[\{\xi+1\}] \subseteq \mathfrak{M}^P_{\xi+1}$,

  and

⟨5⟩ for each $\xi \in [\beta, \alpha) \setminus N$, $\mathcal{N}_r \cap \mathfrak{M}^P_\xi = \emptyset$, in fact, no element of $\mathcal{N}_r$ contains ξ as a member.

Let $\xi_0 := \min\left(\mathrm{dom}(A_r) \setminus \beta\right)$. Define $p_2' \in U_{\xi_0}$ such that

- $$R_{p_2'} := R_{p_2} \cup R_{q \restriction \xi_0} \cup \Big\{ \big\langle \left({\Psi_M}^{-1} \circ \Psi_N\right)(K), \xi \big\rangle : \langle K, \xi \rangle \in R_{r \restriction \xi_0},\ M \in {R_q}^{-1}[\{\xi\}] \text{ with } \omega_1 \cap M = \omega_1 \cap N \Big\},$$
- $\mathcal{N}_{p_2'} := \mathrm{dom}(R_{p_2'})$, and
- $A_{p_2'} := A_{p_2}$.

Now $\mathrm{dom}(A_r) \cap \xi_0 = \mathrm{dom}(A_r) \cap \beta$ and $\mathrm{dom}(A_q) \cap \xi_0 = \mathrm{dom}(A_q) \cap \beta$. By Propositions 3.10 and 3.11 and the fact that $\alpha \in \omega_2$, $R_{p_2'}$ satisfies (el), (ho), (up), and (down) in Definition 4.1, and so p_2' is a condition of $\mathbb{P}_{\xi_0}$. Hence p_2' is a common extension of $r \restriction \xi_0$ and $q \restriction \xi_0$.

By ⟨4⟩, we can apply Proposition 5.5 to find a common extension q'_{ξ_0} of p_2', $r \restriction (\xi_0 + 1)$, and $q \restriction (\xi_0 + 1)$ in $\mathbb{P}_{\xi_0+1}$. Let $\{\xi_i : i \leq m\}$ be the increasing enumeration of the set $\mathrm{dom}(A_r) \setminus \beta$. By ⟨4⟩ again, for each $i \leq m$ with $i \neq 0$, we can apply Proposition 5.5 to find a common extension q'_{ξ_i} of $q'_{\xi_{i-1}}$, $r \restriction (\xi_i + 1)$, and $q \restriction (\xi_i + 1)$ in $\mathbb{P}_{\xi_i+1}$.

Define $q'_\alpha \in U_\alpha$ such that

- $$R_{q'_\alpha} := R_{q'_{\xi_m}} \cup R_q \cup \bigcup \Big\{ \left\{\left({\Psi_M}^{-1} \circ \Psi_N\right)(K)\right\} \times \left((\zeta+1) \cap \left({\Psi_M}^{-1} \circ \Psi_N\right)(K)\right) : \zeta \geq \beta, \langle K, \zeta \rangle \in R_r, \langle M, \zeta \rangle \in R_q \text{ with } \omega_1 \cap M = \omega_1 \cap N \Big\},$$
- $\mathcal{N}_{q'_\alpha} := \mathrm{dom}(R_{q'_\alpha})$, and
- $A_{q'_\alpha} := A_{q'_{\xi_m}} \cup \left(A_q \restriction [\beta, \alpha)\right)$.

By Propositions 3.10 and 3.11 and the fact that $\alpha \in \omega_2$ again, $R_{q'_\alpha}$ satisfies (el), (ho), (up), and (down) in Definition 4.1. By ⟨5⟩, for each $\xi \in \mathrm{dom}(A_q) \cap [\beta, \alpha)$, $A_{p'_\alpha}(\xi)$ satisfies (g) for $R_{q'_\alpha}$. Therefore, q'_α is a condition of $\mathbb{P}_\alpha$, and so is a common extension of r and q in $\mathbb{P}_\alpha$. This finishes the proof of this case.

Suppose that α is of countable cofinality. Take a large enough ordinal $\beta \in \alpha \cap N$, which means that

- $\max(\mathrm{dom}(A_q)) < \beta$, and
- for each $M \in \mathrm{dom}(R_q)$, either $R_q(M) \subseteq \beta$ or $R_q(M)$ is cofinal in α,

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and define

$$\mathcal{E} := \Big\{ r \restriction \beta : \ r \in \mathbb{P}_\alpha \text{ such that} \begin{array}{l} \bullet \text{ there exists } u' \in \mathcal{D} \text{ so that } r \leq_{\mathbb{P}_\alpha} u', \\ \bullet \ \mathrm{dom}(A_r) \subseteq \beta, \text{ and} \\ \bullet \text{ for each } M \in \mathcal{N}_r, \text{ either } R_r(M) \subseteq \beta \text{ or } R_r(M) \text{ is cofinal in } \alpha \end{array} \Big\}.$$

We note that $q \in \mathcal{E}$ and $\mathcal{E}$ is a definable class in the expanded relational structure by $\mathbb{P}_\alpha$ with parameters in N. By the induction hypothesis $(\mathsf{p})_\beta$ and the fact that $\beta \in N \in \mathfrak{M}^P_\alpha$, $q \restriction \beta$ is $(N, \mathbb{P}_\beta)$-generic. So there exists $p_1 \in \mathcal{E} \cap N$ which is compatible with the condition $q \restriction \beta$ in $\mathbb{P}_\beta$. Let $r \in \mathbb{P}_\alpha \cap N$ and $u' \in \mathcal{D} \cap N$ witness that $p_1 \in \mathcal{E}$, and let $p_2 \in \mathbb{P}_\beta$ be a common extension of $q \restriction \beta$ and p_1 $(= r \restriction \beta)$.

Define $q' \in U_\alpha$ such that

- $R_{q'} := R_{p_2} \cup R_r \cup R_q$
  $\cup \bigcup \Big\{ \{ (\Psi_M{}^{-1} \circ \Psi_N)(K) \} \times (\alpha \cap (\Psi_M{}^{-1} \circ \Psi_N)(K)) :$
  $\xi \geq \beta, \langle K, \xi \rangle \in R_r, \langle M, \xi \rangle \in R_q \text{ with } \omega_1 \cap M = \omega_1 \cap N \Big\}$,
- $\mathcal{N}_{q'} := \mathrm{dom}(R_{q'})$, and
- $A_{q'} := A_{p_2}$.

Then q' is a condition of $\mathbb{P}_\alpha$, and is a common extension of q, r, and u' in $\mathbb{P}_\alpha$. ⊣

The following proof is similar to one of Lemma 2.10.

Proof of $(\mathsf{C})_\alpha$ for a limit ordinal α. Suppose that $\alpha \in \omega_2 + 1$ is a limit ordinal, $p \in \mathbb{P}_\alpha$, $N \in \mathcal{N}_p \cap \mathfrak{M}^P_\alpha$ satisfies that $\{N\} \times (\alpha \cap N) \subseteq R_p$, and $\{\dot{F}_n : n \in \omega\}$ is a set of $\mathbb{P}_\alpha$-names for nowhere dense subsets of $(2^{<\omega})^\omega$ such that $\{\dot{F}_n : n \in \omega\} \in N$. Let us show that $p \Vdash_{\mathbb{P}_\alpha}$"$f_{\omega_1 \cap N} \notin \bigcup_{n \in \omega} \dot{F}_n$".

Suppose not, and let $q \leq_{\mathbb{P}_\alpha} p$ and $n \in \omega$ be such that

$$q \Vdash_{\mathbb{P}_\alpha} \text{“} f_{\omega_1 \cap N} \in \dot{F}_n \text{”}.$$

As in the proof of $(\mathsf{p})_\alpha$ when α is a limit ordinal, we need to separate two cases.

Suppose that α is of uncountable cofinality and $\mathrm{dom}(A_q) \not\subseteq \sup(\alpha \cap N)$. Let $\beta \in \alpha \cap N$ and $\zeta \in \omega_1 \cap N$ be large enough ordinals for the condition q as in the proof of $(\mathsf{p})_\alpha$. By the induction hypothesis $(\mathsf{p})_\beta$ and the fact that $\beta \in N \in \mathfrak{M}^P_\alpha$, $q \restriction \beta$ is $(N, \mathbb{P}_\beta)$-generic. For each $v \in (2^{<\omega})^{<\omega}$, each $\beta' \in \alpha \setminus \beta$, each $\eta \in \omega_1$ and each $u \in \mathbb{P}_\alpha$, define $\varphi_\alpha(v, \beta', \eta, u)$ to be the assertion that

- $\mathrm{dom}(A_u) \cap \beta' = \mathrm{dom}(A_p) \cap \beta$,
- $\{\omega_1 \cap K : K \in \mathcal{N}_u\} \cap \eta = \{\omega_1 \cap K : K \in \mathcal{N}_q\} \cap N$, and
- $u \Vdash_{\mathbb{P}_\alpha}$"$\dot{F}_n \cap [v] \neq \emptyset$".

Define a $\mathbb{P}_\beta$-name $\dot{Y}$ such that

$$\Vdash_{\mathbb{P}_\beta} \text{“}\dot{Y} = \Big\{ g \in (2^{<\omega})^\omega : \begin{array}{l} \text{for any } k \in \omega, \text{ any } \beta' \in \alpha \setminus \beta, \text{ and any } \eta \in \omega_1 \setminus \zeta, \\ \text{there exists } u \in \mathbb{P}_\alpha \text{ such that } u \restriction \beta \in \dot{G}_{\mathbb{P}_\beta} \text{ and} \\ \varphi_\alpha(g \restriction k, \beta', \eta, u) \end{array} \Big\}\text{”}.$$

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Then $\dot{Y}$ is a definable class in the expanded relational structure by $\mathbb{P}_\alpha$ with parameters in N. $\dot{Y}$ is forced to be a closed subset of $(2^{<\omega})^\omega$. So by Proposition 5.1, $\dot{Y}$ can be considered as an element of N.

We claim that

$$q \restriction \beta \Vdash_{\mathbb{P}_\beta} \text{“}\dot{Y} \text{ is nowhere dense”}.$$

Let $r \leq_{\mathbb{P}_\beta} q \restriction \beta$ and $v \in (2^{<\omega})^{<\omega}$, and let q' be a common extension of both r and q in $\mathbb{P}_\alpha$. Then there are $q'' \leq_{\mathbb{P}_\alpha} q'$ and an end-extension v' of v in $(2^{<\omega})^{<\omega}$ such that

$$q'' \Vdash_{\mathbb{P}_\alpha} \text{“}\dot{F}_n \cap [v'] = \emptyset\text{”}.$$

$q'' \restriction \beta$ is an extension of r in $\mathbb{P}_\beta$. Let us show that

$$q'' \restriction \beta \nVdash_{\mathbb{P}_\beta} \text{“}\dot{Y} \cap [v'] \neq \emptyset\text{”}.$$

Assume not. Let $\dot{g}$ a $\mathbb{P}_\beta$-name such that

$$q'' \restriction \beta \Vdash_{\mathbb{P}_\beta} \text{“}\dot{g} \in \dot{Y} \cap [v']\text{”}.$$

Take $s \leq_{\mathbb{P}_\beta} q'' \restriction \beta$, $k \in \omega$, and $v'' \in (2^{<\omega})^{<\omega}$ such that

$$s \Vdash_{\mathbb{P}_\beta} \text{“}v' \subseteq \dot{g} \restriction k = v''\text{”}.$$

Let $\beta' \in (\alpha \cap N) \setminus \beta$ and let $\eta \in (\omega_1 \cap N) \setminus \zeta$ be large enough ordinals for q'' as in the proof of $(\mathsf{p})_\alpha$. By the definition of $\dot{Y}$, $\{\dot{Y}, \beta', \eta, v''\} \in N$, and the fact that $q'' \restriction \beta$ is also $(N, \mathbb{P}_\beta)$-generic and that N is an elementary substructure of the expanded relational structure by $\mathbb{P}_\alpha$, there exists an extension s' of s in $\mathbb{P}_\beta$ and $u \in \mathbb{P}_\alpha \cap N$ such that $s' \leq_{\mathbb{P}_\beta} u \restriction \beta$ and $\varphi_\alpha(v'', \beta', \eta, u)$. As in the proof of $(\mathsf{p})_\alpha$, by the roles of β' and η, we can build a common extension t of s', q'', and u in $\mathbb{P}_\alpha$. (To build t, for each coordinate ξ in $(\mathrm{dom}(A_{q''}) \cup \mathrm{dom}(A_u)) \cap [\beta, \sup(\alpha \cap N))$, we construct a preparatory condition $t_\xi \in \mathbb{P}_{\xi+1}$ like q'_{ξ_i} as in the proof of $(\mathsf{p})_\alpha$.) Then

$$t \Vdash_{\mathbb{P}_\alpha} \text{“}\dot{F}_n \cap [v'] = \emptyset \text{ and } \dot{F}_n \cap [v''] \neq \emptyset\text{”},$$

which is a contradiction.

We claim that

$$q \restriction \beta \Vdash_{\mathbb{P}_\beta} \text{“}f_{\omega_1 \cap N} \in \dot{Y}\text{”}.$$

This contradicts the induction hypothesis $(\mathsf{C})_\beta$, and finishes the proof of this case.

Assume not. Then there exists $r \leq_{\mathbb{P}_\beta} q \restriction \beta$ such that

$$r \Vdash_{\mathbb{P}_\beta} \text{“}f_{\omega_1 \cap N} \notin \dot{Y}\text{”}.$$

By extending r if necessary, we may assume that there are $k \in \omega$, $\beta' \in \alpha \setminus \beta$, and $\eta \in \omega_1 \setminus \zeta$ such that

$$r \Vdash_{\mathbb{P}_\beta} \text{“there are no } u \in \mathbb{P}_\alpha \text{ such that } u \restriction \beta \in \dot{G}_{\mathbb{P}_\beta} \text{ and } \varphi_\alpha(f_{\omega_1 \cap N} \restriction k, \beta', \eta, u)\text{”}.$$

By the induction hypothesis $(\mathsf{p})_\beta$, r is $(N, \mathbb{P}_\beta)$-generic. So by extending r again if necessary, we may assume that $\beta' \in (\alpha \cap N) \setminus \beta$ and $\eta \in (\omega_1 \cap N) \setminus \zeta$. However,

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then

$$r \Vdash_{\mathbb{P}_\beta} \text{“} q \restriction \beta \in \dot{G}_{\mathbb{P}_\beta} \text{ and } \varphi_\alpha(f_{\omega_1 \cap N} \restriction k, \beta', \eta, q)\text{”},$$

which is a contradiction.

Suppose that α is of countable cofinality. The proof below is similar to the one in the case of uncountable cofinality. The differences are the necessary property of β and the definition of $\dot{Y}$. Let $\beta \in \alpha \cap N$ be a large enough ordinal as in the proof of $(\mathsf{p})_\alpha$. Let $\zeta \in \omega_1 \cap N$ be such that

$$\{\omega_1 \cap K : K \in \mathcal{N}_q\} \cap \zeta = \{\omega_1 \cap K : K \in \mathcal{N}_q\} \cap N.$$

For each $v \in (2^{<\omega})^{<\omega}$, each $\eta \in \omega_1$, and each $u \in \mathbb{P}_\alpha$, define $\varphi_\alpha(v, \eta, u)$ to be the assertion that

- $\mathrm{dom}(A_u) \subseteq \beta$,
- for each $M \in \mathcal{N}_u$, either $R_u(M) \subseteq \beta$ or $R_u(M)$ is cofinal in α,
- $\{\omega_1 \cap K : K \in \mathcal{N}_u\} \cap \eta = \left\{\omega_1 \cap K : K \in \mathcal{N}_q\right\} \cap N$, and
- $u \Vdash_{\mathbb{P}_\alpha}$“$\dot{F}_n \cap [v] \neq \emptyset$”,

and define $\dot{Y}$ to be a $\mathbb{P}_\beta$-name such that

$$\Vdash_{\mathbb{P}_\beta} \text{“}\dot{Y} = \Big\{g \in (2^{<\omega})^\omega : \text{ for any } k \in \omega \text{ and any } \eta \in \omega_1 \setminus \zeta, \text{ there exists } u \in \mathbb{P}_\alpha \text{ such that } u \restriction \beta \in \dot{G}_{\mathbb{P}_\beta} \text{ and } \varphi_\alpha(g \restriction k, \eta, u)\Big\}\text{”}.$$

Then $\dot{Y}$ is a definable class in the expanded relational structure by $\mathbb{P}_\alpha$ with parameters in N, and is forced to be a closed subset of $(2^{<\omega})^\omega$. So by Proposition 5.1, $\dot{Y}$ can be considered as an element of N.

We claim that

$$q \restriction \beta \Vdash_{\mathbb{P}_\beta} \text{“}\dot{Y} \text{ is nowhere dense”}.$$

Let $r \leq_{\mathbb{P}_\beta} q \restriction \beta$ and $v \in (2^{<\omega})^{<\omega}$, and let q' be a common extension of both r and q in $\mathbb{P}_\alpha$. Then there are $q'' \leq_{\mathbb{P}_\alpha} q'$ and an end-extension v' of v in $(2^{<\omega})^{<\omega}$ such that

$$q'' \Vdash_{\mathbb{P}_\alpha} \text{“}\dot{F}_n \cap [v'] = \emptyset\text{”}.$$

$q'' \restriction \beta$ is an extension of r in $\mathbb{P}_\beta$. Let us show that

$$q'' \restriction \beta \nVdash_{\mathbb{P}_\beta} \text{“}\dot{Y} \cap [v'] \neq \emptyset\text{”}.$$

Assume not, and let $\dot{g}$ be a $\mathbb{P}_\beta$-name such that

$$q'' \restriction \beta \Vdash_{\mathbb{P}_\beta} \text{“}\dot{g} \in \dot{Y} \cap [v']\text{”}.$$

Take $s \leq_{\mathbb{P}_\beta} q'' \restriction \beta$, $k \in \omega$ and $v'' \in (2^{<\omega})^{<\omega}$ such that

$$s \Vdash_{\mathbb{P}_\beta} \text{“}v' \subseteq \dot{g} \restriction k = v''\text{”}.$$

Take $\eta \in \omega_1 \cap N$ such that $\eta \geq \zeta$ and, for any $\xi \in \mathrm{dom}(A_{q''}) \cap [\beta, \alpha) \cap N$ (then $N \in R_{q''}[\{\xi\}]$) and any $\sigma \in A_{q''}(\xi) \cap N$, $\gamma_\sigma < \eta$. As in the previous case, we take an extension s of $q'' \restriction \beta$ in $\mathbb{P}_\beta$ and $u \in \mathbb{P}_\alpha \cap N$ such that $s \leq_{\mathbb{P}_\beta} u \restriction \beta$ and $\varphi_\alpha(v'', \eta, u)$. For any $\xi \in \mathrm{dom}(A_{q''}) \cap [\beta, \alpha)$,

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- if $\xi \in N$, then
  - $N \in {R_{q''}}^{-1}[\{\xi\}] \cap \mathfrak{M}^P_{\xi+1}$,
  - ${R_u}^{-1}[\{\xi\}] = {R_u}^{-1}[\{\xi+1\}] \subseteq \mathfrak{M}^P_{\xi+1}$ (because u satisfies $\varphi_\alpha(v'', \eta, u)$),
  - $\{\omega_1 \cap M : M \in {R_u}^{-1}[\{\xi\}]\} \cap \eta \subseteq \{\delta_\sigma : \sigma \in A_{q''}(\xi)\}$,
  - for any $\sigma \in A_q(\xi) \cap N$, $\gamma_\sigma < \eta$, and
  - $\max\{\omega_1 \cap M : M \in {R_u}^{-1}[\{\xi\}]\} < \omega_1 \cap N$,
- if $\xi \notin N$, by Proposition 3.11 and the fact that $u \in N$, ${R_u}^{-1}[\{\xi\}] = \emptyset$.

Hence, as the construction of q'_{ξ_m} in the proof of $(\mathsf{p})_\alpha$ before, we can find a common extension t of s, u, and q''. But then t forces a contradiction.

We claim that

$$q \restriction \beta \Vdash_{\mathbb{P}_\beta} \text{“} f_{\omega_1 \cap N} \in \dot{Y} \text{”}.$$

This contradicts the induction hypothesis $(\mathsf{C})_\beta$, and finishes the proof of this case.

Assume not, then there exists $r \leq_{\mathbb{P}_\beta} q \restriction \beta$ such that

$$r \Vdash_{\mathbb{P}_\beta} \text{“} f_{\omega_1 \cap N} \notin \dot{Y} \text{”}.$$

By extending r if necessary, we may assume that there are $k \in \omega$ and $\eta \in \omega_1 \setminus \zeta$ such that

$$r \Vdash_{\mathbb{P}_\beta} \text{“there are no } u \in \mathbb{P}_\alpha \text{ such that } u \restriction \beta \in \dot{G}_{\mathbb{P}_\beta} \text{ and } \varphi_\alpha(f_{\omega_1 \cap N} \restriction k, \eta, u)\text{”}.$$

By the induction hypothesis $(\mathsf{p})_\beta$, r is $(N, \mathbb{P}_\beta)$-generic. So by extending r again if necessary, we may assume that $\eta \in (\omega_1 \setminus \zeta) \cap N$. However then

$$r \Vdash_{\mathbb{P}_\beta} \text{“} q \restriction \beta \in \dot{G}_{\mathbb{P}_\beta} \text{ and } \varphi_\alpha(f_{\omega_1 \cap N} \restriction k, \eta, q)\text{”},$$

which is a contradiction. ⊣

Lemma 5.6. *For any $\mathbb{P}_{\omega_2}$-name $\dot{r}$ for a member of 2^ω,*

$$\Vdash_{\mathbb{P}_{\omega_2}} \text{“there is } \alpha \in \omega \text{ such that } \dot{E}_\alpha \text{ captures } \dot{r} \text{ relative to the set } X\text{”}.$$

Proof. $\dot{E}_\xi$ is defined in Definition 4.3. By Proposition 5.1, we may assume that $\dot{r}$ belongs to $H(\kappa)$. Let $p \in \mathbb{P}_{\omega_2}$. Take $\alpha \in \omega_2$ such that $\Phi(\alpha) = \{\dot{r}_\alpha\} = \{\dot{r}\}$, and $\mathrm{ran}(R_p) \subseteq \alpha$. Then $\mathrm{dom}(A_p) \subseteq \alpha$.

Let θ be a large enough regular cardinal for the forcing notion $\mathbb{P}_{\alpha+1}$. Take any $\varepsilon \in \omega_1$, and take countable elementary submodels N_0^* and N_1^* of H_θ such that $\{\vec{\mathbb{R}}, \vec{X}, H_\kappa, \mathbb{P}_\alpha, p, \alpha, \varepsilon\} \in N_0^* \in N_1^*$. Then both $N_0^* \cap H_\kappa$ and $N_1^* \cap H_\kappa$ are in $\mathfrak{M}^P_\alpha \cap \mathfrak{M}^P_{\alpha+1}$. By Proposition 5.4, there exists an extension q of p in $\mathbb{P}_{\alpha+1}$ such that ${R_q}^{-1}[\{\alpha\}] = \{N_0^* \cap H_\kappa\}$ and $A_q(\alpha) = \{\langle \varepsilon, \omega_1 \cap N_0^*, \omega_1 \cap N_1^* \rangle\}$. Then, by Lemma 5.2, $\mathbb{P}_{\alpha+1}$ is proper and q is $(N_0^* \cap H_\kappa, \mathbb{P}_{\alpha+1})$-generic. So as seen in Observation 4.4,

$$q \Vdash_{\mathbb{P}_{\omega_2}} \text{“} \dot{E}_\alpha \text{ is a club subset of } \omega_1 \text{ and captures } \dot{r}\ (= \dot{r}_\alpha) \text{ relative to } X\text{”}. \quad ⊣$$

Since $\mathbb{P}_{\omega_2}$ forces that $2^{\aleph_0} = \aleph_2$, we conclude the following.

Theorem 5.7. $\mathbb{P}_{\omega_2}$ *forces the assertion* **(c)**.

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§6. Properness and the length of the iteration. For each $\xi \in \kappa$, define the $\mathbb{P}_{\xi+1}$-name $\dot{S}_\xi$ by

$$\Vdash_{\mathbb{P}_{\xi+1}} \text{“}\dot{S}_\xi := \left\{\omega_1 \cap N : p \in \dot{G}_{\mathbb{P}_{\xi+1}}, N \in {R_p}^{-1}[\{\xi\}]\right\}\text{”}.$$

Let $p \in \mathbb{P}_{\omega_2}$ and $\xi \in \mathrm{dom}(A_p)$ (then $\xi \in \omega_2$). If M belongs to ${R_p}^{-1}[\{\xi\}]$, that is, $\{M\} \times \big((\xi+1) \cap M\big) \subseteq R_p$, then

$$p \restriction (\xi+1) \Vdash_{\mathbb{P}_{\xi+1}} \text{“}\omega_1 \cap M \in \dot{S}_\xi\text{”}.$$

Moreover, if M also belongs to $\mathfrak{M}^p_{\xi+1}$, then p is $(M, \mathbb{P}_{\xi+1})$-generic by Lemma 5.2, and therefore

$$p \restriction (\xi+1) \Vdash_{\mathbb{P}_{\xi+1}} \text{“}\omega_1 \cap M \text{ is a limit point of } \dot{S}_\xi\text{”}.$$

We notice that

$$p \restriction (\xi+1) \Vdash_{\mathbb{P}_{\xi+1}} \text{“}\dot{S}_\xi \text{ is a stationary subset of } \omega_1\text{”}.$$

If $\Phi(\xi) = \{\dot{r}_\xi\}$ and $\dot{r}_\xi$ is a $\mathbb{P}_\xi$-name for a function from ω into 2, then

$$\begin{aligned}\Vdash_{\mathbb{P}_{\xi+1}} \text{“ }&\text{the set } \dot{S}_\xi \text{ is a stationary subset of } \omega_1, \text{ and captures } r \text{ relative to } X,\\ &\text{that is, for any limit point } \delta \text{ of } \dot{S}_\xi \text{ (which means that } \delta \in \dot{S}_\xi \text{ and}\\ &\dot{S}_\xi \cap \delta \text{ is cofinal in } \delta), \text{ there exists } f \in X \text{ and } \varepsilon \in \delta \text{ such that, for any}\\ &\xi \in (\dot{S}_\xi \cap \delta) \setminus \varepsilon, f(|C_\delta \cap \xi|) \subseteq r\text{”}.\end{aligned}$$

Suppose that $p \in \mathbb{P}_{\omega_2}$, $\xi, \zeta \in \mathrm{dom}(A_p)$ with $\xi < \zeta$, and $M \in {R_p}^{-1}[\{\zeta\}]$ (then p forces $\omega_1 \cap M$ to be in $\dot{S}_\zeta$). If $\xi \in M$, then

$$p \Vdash_{\mathbb{P}_\zeta} \text{“}\omega_1 \cap M \in \dot{S}_\xi\text{”}.$$

Therefore, by Proposition 3.10 and the requirements (ho) and (up) in the definition of $\mathbb{P}_\alpha$,

$$p \Vdash_{\mathbb{P}_\zeta} \text{“}\dot{S}_\zeta \setminus M \subseteq \dot{S}_\xi\text{”}.$$

Therefore, $\mathbb{P}_{\omega_2}$ forces that there are $R \in [2^\omega]^{\aleph_2}$ and a sequence $\langle S_\xi : \xi \in \omega_2\rangle$ of stationary subsets of ω_1 such that

- for each $r \in R$, there exists $\xi \in \omega_2$ such that S_ξ captures r relative to X.
- for each $\xi, \zeta \in \omega_2$, if $\xi < \zeta$, then $S_\zeta \setminus S_\xi$ is bounded in ω_1.

As in the proof of Proposition 2.3, the set $\{S_\xi : \xi \in \omega_2\}$ cannot be diagonalized by any stationary subset of ω_1 without collapsing $\aleph_2$. This observation leads to the following conclusion.

LEMMA 6.1. *If $\kappa > \omega_2$, then $(\mathsf{p})_{\omega_2+1}$ in Lemma 5.2 fails.*

This suggests that $\mathbb{P}_{\omega_2+1}$ should not be proper.

PROOF. Suppose that $(\mathsf{p})_{\omega_2+1}$ in Lemma 5.2 holds. Then, as we have already proved, $\dot{S}_{\omega_2}$ is a $\mathbb{P}_{\omega_2+1}$-name for a stationary subset of ω_1. For each $\eta \in \omega_2$, the

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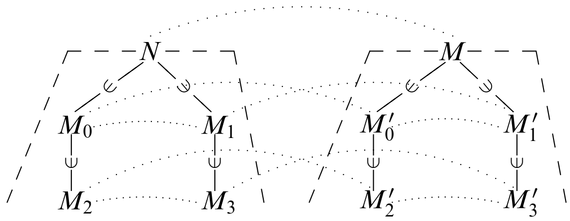
set of conditions p of $\mathbb{P}_{\omega_2+1}$ such that there are $M \in \mathcal{N}_p$ and $\xi \in \omega_2 \setminus \eta$ such that $\{\xi, \omega_2\} \in M$, $\{M\} \times \big((\omega_2+1) \cap M\big) \subseteq R_p$, and $\{\xi, \omega_2\} \subseteq \mathrm{dom}(A_p)$ is dense in $\mathbb{P}_{\omega_2+1}$. Therefore

$$\Vdash_{\mathbb{P}_{\omega_2+1}} \text{“}\Big\{\xi : \exists p \in \dot{G}_{\mathbb{P}_{\omega_2+1}} \left(\xi \in \mathrm{dom}(A_p) \cap \omega_2\right)\Big\} \text{ is of size } \aleph_2,$$
$$\dot{S}_{\omega_2} \text{ is a stationary subset of } \omega_1 \text{ and diagonalizes the set}$$
$$\Big\{\dot{S}_\xi : \exists p \in \dot{G}_{\mathbb{P}_{\omega_2+1}} \left(\xi \in \mathrm{dom}(A_p) \cap \omega_2\right)\Big\}\text{”}.$$

However, this is a contradiction. ⊣

REMARK 6.2. There are reasons to suspect that $\mathbb{P}_\alpha$, for $\alpha > \omega_2$, may fail to be proper, even disregarding the working parts of the forcing. Suppose that the length of the iteration is $\omega_2 + 1$, $q \in \mathbb{P}_{\omega_2+1}$, $\{M, N\} \subseteq {R_q}^{-1}[\{\omega_2\}]$, $r \in \mathbb{P}_{\omega_2+1} \cap N$ which is a nice copy of q inside N as in the proof of $(\mathsf{p})_\alpha$ for a limit ordinal α, $\{M_0, M_1, M_2, M_3\} \subseteq {R_r}^{-1}[\{\omega_2\}] \setminus M$ such that $M_2 \in M_0$, $M_3 \in M_1$, $M_0 \cap \omega_1 = M_1 \cap \omega_1$, $M_2 \cap \omega_1 = M_3 \cap \omega_1$, and $M_i' := \left({\Psi_M}^{-1} \circ \Psi_N\right)(M_i) \in \mathcal{N}_{q'} \setminus (\mathcal{N}_q \cup \mathcal{N}_r)$ for each $i \in \{0, 1, 2, 3\}$, as in the following figure.



Then $\{M_0', M_1', M_2', M_3'\} \subseteq {R_{q'}}^{-1}[\{\omega_2\}]$, and ${R_{q'}}^{-1}[\{\omega_2\}]$ forms a symmetric system. Let $\zeta \in \alpha$. It should be satisfied that ${R_{q'}}^{-1}[\{\zeta\}] \subseteq \mathfrak{M}_\zeta^P$ and ${R_{q'}}^{-1}[\{\zeta\}]$ forms a symmetric system. Now we have no guarantees that the assertion ${R_{q'}}^{-1}[\{\zeta\}] \subseteq \mathfrak{M}_\zeta^P$ is true. And even if this is true, since Proposition 3.11 may fail for $\alpha = \omega_2$, ${R_{q'}}^{-1}[\{\zeta\}]$ may fail to form a symmetric system. For example, it may happen that $\zeta \in M_1'$, $\zeta \in M_2'$ (hence $\zeta \in M_0'$), and $\zeta \notin M_3'$. Then ${R_{q'}}^{-1}[\{\zeta\}]$ does not satisfy (down). Therefore, in this case, q and r may fail to be compatible in $\mathbb{P}_{\omega_2+1}$.

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