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LINKAGES FOR THE TRISECTION OF AN ANGLE AND DUPLICATION OF THE CUBE

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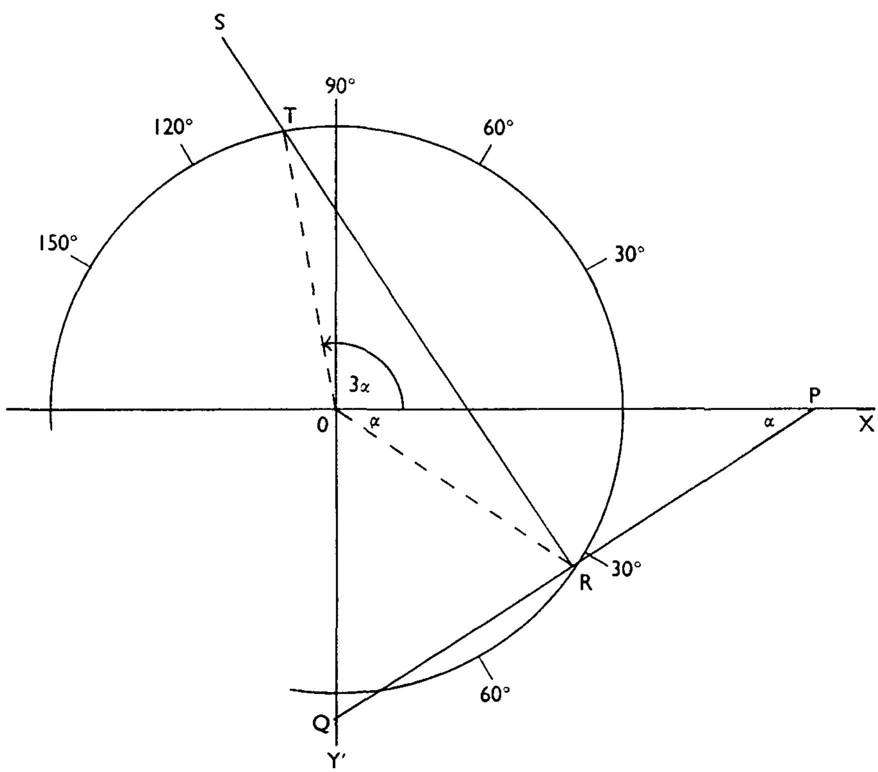
In this note some linkage systems for trisecting an angle and for finding the cube root of a number are described. The models are easily made and are of considerable pedagogic value.



TRISECTION BY *T*-SQUARE

In the notation of the figure, let a *T*-square PQ, with centre at R and of length $2r$, move so that P lies on OX and Q on OY', and let RS meet the circle with centre at O and of radius r in T. Then, if $\angle XOR = \alpha$, $\angle XOT = 3\alpha$; the proof is as follows.

Since $RP = RQ = OT$ and $\angle ROX = \angle RXO = \alpha$, $\angle ORP = 180° - 2\alpha$ and $\angle ORT = 90° - 2\alpha = \angle OTR$. Hence $\angle ROT = 180° - 2\ (90° - 2\alpha) = 4\alpha$ and so $\angle XOT = 3\alpha$.

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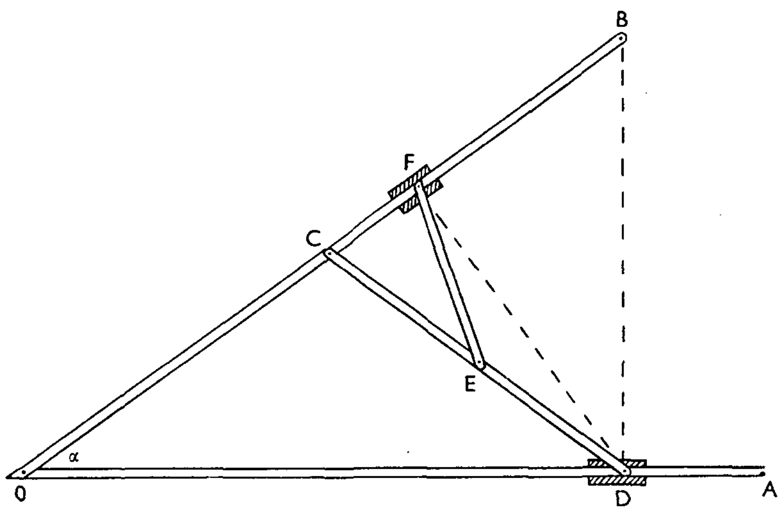
The gadget achieves trisection of an angle by the use of only one moving part; in an actual model P and Q may be constrained to move on the axes by guide pieces, sliding heads or slots. For small values of 3α the accuracy of trisection may be improved by setting T on $3\alpha+90°$ and subtracting $30°$ from the resulting $\alpha+30°$.

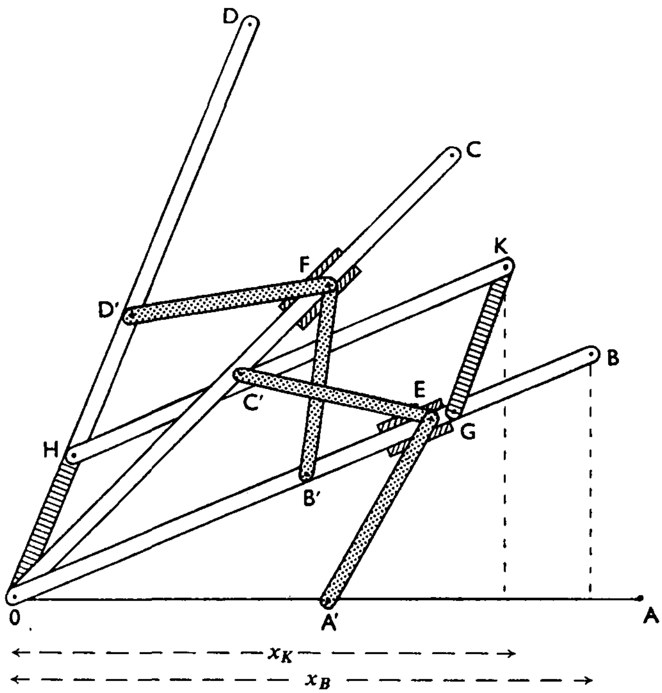


A LINKAGE FOR CUBE DUPLICATION

In the linkage shown, O and C are pivots and D and F are sliding heads. With suitable units, $OB = 1$, $OC = CD = CB = \frac{1}{2}$, $EC = EF = ED = \frac{1}{4}$ and so angles ODB and CFD are angles in semicircles. The linkage is mounted on a base plate having squared paper from which coordinates can be read.

If the $x =$ axis is taken along OA and if $\angle AOB = \alpha$, then $x_F = OF\cos\alpha = OD\cos^2\alpha = OB\cos^3\alpha = OD^3$.

If the linkage is deformed until the abscissa of F has a given value, then the cube root of this number is obtained by reading off the abscissa of D. The case when $x_F = \frac{1}{2}$ is that of cube duplication.



A COMBINED TRISECTOR AND CUBE DUPLICATOR

Overlapping twin kites $OA'EC'$ and $OB'FD'$ are made deformable by sliding heads E, F. $OA = OB = OC = OD = 1$ and $OA' = OB' = OC' = OD' = \frac{1}{2}$, $HK = OG = \frac{3}{4}OB = \frac{3}{4}$, and $GK = OH = \frac{1}{4}$. The construction makes D, K, B collinear and $DK : KB = 3 : 1$.

The linkage functions as a trisector by setting $\angle AOD$ as the angle to be trisected.

Since $OA = 1$, $x_B^3 = \cos^3 \alpha = \frac{1}{4}(\cos 3\alpha + 3 \cos \alpha) = (x_D + 3x_B)/(1+3) = x_\kappa$ since $DK : KB = 3 : 1$. Hence $x_B = \sqrt[3]{x_\kappa}$. When $x_\kappa = \frac{1}{2}$ we get cube duplication. If the linkage is set so that κ falls on the ordinate $x = x_\kappa$, then x_B gives the cube root of x_κ.

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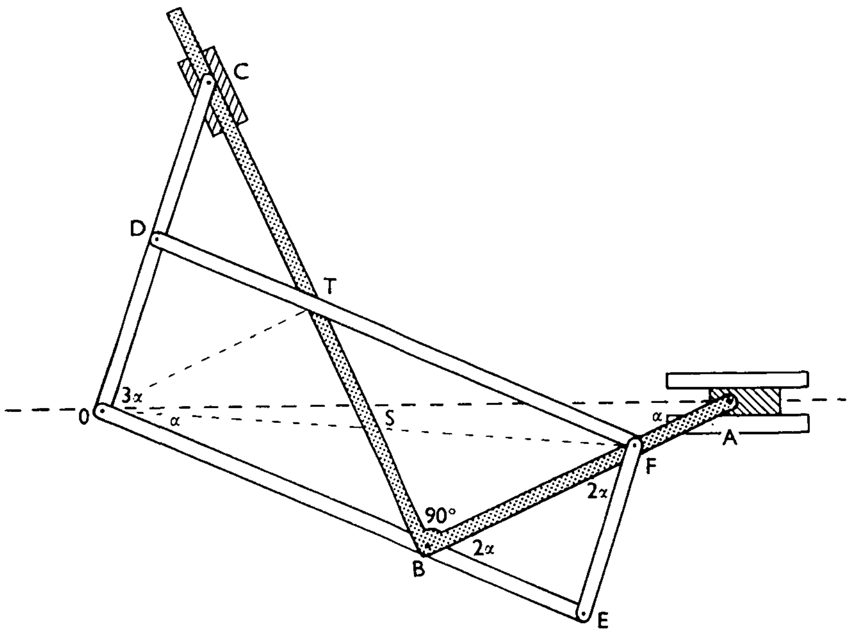

THE CLOCK TRISECTOR AND CUBE DUPLICATOR

In the linkage shown, $OC = OB = BA = 1$, $CD = DO = BE = EF = \frac{1}{2}$, $\angle ABC = 90°$. Hence $\angle OCB = \angle OBC = \angle OBA - 90° = (180° - 2\alpha) - 90° = 90° - 2\alpha$ and $\angle BOC = 180° - 2(90° - 2\alpha) = 4\alpha$. AB and BC form a rigid elbow piece and there are pivots at O, B, E, F, D and sliding heads at C, A. For trisection set $\angle AOC$ to the given angle and read off $\angle AOB$.

Further, $\cos^3 \alpha = \frac{1}{4}(\cos 3\alpha + 3 \cos \alpha) = (x_C + 3x_B)/(1+3) = x_S$, where S is such that $BS = ST = \frac{1}{2}TC$, OT is perpendicular to BC, and $OTFB$ is a parallelogram. Hence $\cos \alpha = \sqrt[3]{x_s} = x_B$.

In an actual model there would be a suitable grid on the base plate to facilitate the reading of abscissae.

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LARGS, AYRSHIRE

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