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Some number theoretic properties of arbitrary order recursive sequences

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This thesis develops some number theoretic properties associated with sequences, the elements of which satisfy homogeneous linear recurrence relations of arbitrary order. The properties studied are those analogous to the results of Lucas [5] and A.F. Horadam [3] for the second order case, to examine their points of similarity. Their finite field properties have been developed extensively by Selmer [6] and there exists a fairly good knowledge of their global behaviour in terms of existence theorems, but knowledge of local behaviour in terms of reasonably elegant explicit and specific identities is limited.

Two arithmetical functions are defined:

$$\delta(m, s) = \begin{cases} 1 & \text{if } m \mid s , \\ 0 & \text{if } m \nmid s , \end{cases}$$

and

$$\rho_n(m, s) = \begin{cases} 0 & \text{if } \exists\, j : j \mid (n, s) , \quad 1 < j < m , \\ 1 & \text{otherwise;} \end{cases}$$

they are related as Dirichlet inverses under a convolution product $\delta^{-1}(n, n) = \mu(n)\rho_n(n, n)$ where $\mu(n)$ is the Möbius function.

With the aid of these it is possible to establish analogues of the classical number theoretic functions for these sequences of integers, such as

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$$\phi(u_n) = n - 1 - \sum_{m=2}^{n-1} \sum_{s=2}^{n-1} \rho_{u_n}(u_m, u_s)\delta(u_m, u_s)\delta(u_m, u_n) ,$$

and so provide Staudt-Clausen theorems for generalized Bernoulli numbers within the framework of the Jackson calculus of sequences. This solves a problem posed by Ward [7].

The first chapter of the thesis introduces the definitions and notation and relates the sequences to be discussed with other work in this area, especially that of Williams [8]. This leads naturally into the second chapter which looks at general terms and auxiliary polynomials. A discussion of various functional difference equations leads to a definition of a generalized Fibonacci polynomial which generates the elements of these sequences and exhibits many properties similar to those of the classical polynomials of Bernoulli, Euler and Hermite.

A periodic Jacobi-Perron algorithm with suitable convergence criteria and various matrices is used in the third chapter to develop some field properties of the sequences. That the algorithm is a generalization of the ordinary continued fraction algorithm is also illustrated. This algorithm also relates the sequences under discussion to a contraction of Bernoulli's iteration for the solution of polynomial equations and provides specific coefficients for this equation. Some solutions for some linear and Pellian type diophantine equations are also included.

Chapter Four contains multiplication properties of these sequences, including a generalization of Simson's relation, an identity for producing Pythagorean triples, and a lacunary recurrence relation in terms of the δ function which can be used to develop other identities including some in Chapter Six. A restricted solution to an important problem of Gould [1] is developed from the results of this chapter.

Division properties of the sequences are the topics of Chapter Five, and as well as some congruences, a recurrence relation for some Fontené-Ward multinomial coefficients is produced. (These are generalizations of the ordinary multinomial coefficients.) The δ and ρ functions are used to obtain properties for divisibility sequences analogous to the classical number theoretic functions and the relation of these to the generalized integers of E.M. Horadam [4] is explained.

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The sixth chapter studies some combinatorial properties associated with partitions, compositions and intersections of the sequences as well as their asymptotic behaviour and related arbitrary order circular functions. Some of these results generalize Hilton's [2] work on Lucas- and Horadam-type sequences.

The concluding chapter summarizes the principal results and specifies some problems for further research which arise from topics in this thesis, especially some related to pyramidal diophantine equations. Possible approaches to these problems are outlined.

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