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MARCINKIEWICZ MULTIPLIERS ON THE HEISENBERG GROUP

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Let $\mathbf{H}_n$ be the Heisenberg group of dimension $2n+1$. Let $\mathcal{L}_1, \dots, \mathcal{L}_n$ be the partial sub-Laplacians on $\mathbf{H}_n$ and T the central element of the Lie algebra of $\mathbf{H}_n$. We prove that the operator $m(\mathcal{L}_1, \dots, \mathcal{L}_n, -iT)$ is bounded on $L^p(\mathbf{H}_n)$, $1 < p < +\infty$, if the function m satisfies a Marcinkiewicz-type condition in $\mathbf{R}^{n+1}$.

1. INTRODUCTION

This paper deals with spectral multipliers on the Heisenberg group. We denote by $\mathbf{H}_n$ the Heisenberg group of dimension $d = 2n+1$, by $\mathcal{L}_1, \dots, \mathcal{L}_n$ the partial sub-Laplacians and by T the central element of the Lie algebra of $\mathbf{H}_n$. The operators $\mathcal{L}_1, \dots, \mathcal{L}_n, -iT$ form a commutative family of self-adjoint operators, so they admit a joint spectral resolution and it is possible to define the operator $m(\mathcal{L}_1, \dots, \mathcal{L}_n, -iT)$ when m is a bounded Borel function on the joint spectrum of $\{\mathcal{L}_1, \dots, \mathcal{L}_n, -iT\}$. The boundedness on $L^2(\mathbf{H}_n)$ of the operator $m(\mathcal{L}_1, \dots, \mathcal{L}_n, -iT)$ is an immediate consequence of the spectral theorem and the boundedness of the function m. We prove that $m(\mathcal{L}_1, \dots, \mathcal{L}_n, -iT)$ extends to a bounded operator on $L^p(\mathbf{H}_n)$, $1 < p < +\infty$, under suitable Marcinkiewicz-type conditions on the function m.

For the operators of the form $m(\mathcal{L})$, where $\mathcal{L} = \mathcal{L}_1 + \dots + \mathcal{L}_n$ is the sub-Laplacian on $\mathbf{H}_n$, the problem of establishing sufficient conditions on m that make the operator $m(\mathcal{L})$ bounded on $L^p(\mathbf{H}_n)$, $p \neq 2$, has a long history. The first results are due to De Michele and Mauceri [**5**], who have considered a wider class of operators. Later, these results have been extended to stratified groups by Hulanicki and Stein (in [**7**, Chapter 6]), Hulanicki and Jenkins [**10**], Mauceri [**15**], De Michele and Mauceri [**6**]. The best result up to now obtained in this more general context is due to Mauceri and Meda [**16**] and to Christ [**3**]: if the function m satisfies a Hörmander condition of order $\alpha > Q/2$ (where Q is the homogeneous dimension of the stratified group), then the operator $m(\mathcal{L})$ extends to an operator which is bounded on L^p for $1 < p < +\infty$ and of weak type (1,1). More recently, Hebisch [**9**] and Müller and Stein [**19**] have proved that for the Heisenberg group the preceding conclusion is still true if the function m satisfies a Hörmander condition of order $\alpha > d/2$. In the paper of Müller and Stein [**19**] it is also shown that this condition is

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sharp. Operators of the form $m(\mathcal{L}, -iT)$ have been studied by Mauceri [**14**]. In all these works the authors have considered classes of multipliers that satisfy conditions invariant with respect to the natural family of one-parameter dilations on the group. More recently, Müller, Ricci and Stein [**17, 18**] have shown the boundedness on $L^p(\mathbf{H}_n)$, $1 < p < +\infty$, of some classes of operators $m(\mathcal{L}, -iT)$ where m satisfies conditions invariant with respect to a family of multi-parameter dilations, in analogy with the classical Marcinkiewicz theorem on the Euclidean space [**20**, Chapter IV].

Operators of the form $m(\mathcal{L}_1, \dots, \mathcal{L}_n, -iT)$, when m satisfies a Marcinkiewicz-type condition of infinite order in $\mathbf{R}^{n+1}$, have been studied recently by Fraser [**8**], who has characterised their convolution kernels and has shown that these operators are bounded on $L^p(\mathbf{H}_n)$, $1 < p < +\infty$. Our result about the boundedness is stronger, because we only need that m satisfies a condition of finite order. Our techniques, based mainly on Littlewood-Paley decompositions, generalise those of Müller, Ricci and Stein [**18**].

2. Notation and preliminaries

In this paper we set $\mathbf{N} = \{0, 1, 2, \dots\}$, $\mathbf{Z}_+ = \mathbf{N} \setminus \{0\}$, $\mathbf{R}_+ = (0, +\infty)$, $\mathbf{R}^* = \mathbf{R} \setminus \{0\}$.

The $2n+1$-dimensional Heisenberg group $\mathbf{H}_n$ is the nilpotent Lie group whose underlying manifold is $\mathbf{C}^n \times \mathbf{R}$, with multiplication given by

$$(z, t)\,(z', t') = \left(z + z', t + t' + 2\,\mathrm{Im}\,\langle z, z'\rangle\right)$$

where $z = (z_1, \dots, z_n) \in \mathbf{C}^n$, $z' = (z'_1, \dots, z'_n) \in \mathbf{C}^n$, $t, t' \in \mathbf{R}$ and $\langle z, z'\rangle = \sum_{j=1}^{n} z_j \overline{z'_j}$. The Lie algebra of $\mathbf{H}_n$ is generated by the left-invariant vector fields $Z_1, \dots, Z_n, \overline{Z}_1, \dots, \overline{Z}_n, T$, where

$$\begin{aligned} Z_j &= \frac{\partial}{\partial z_j} + i\overline{z_j}\frac{\partial}{\partial t}; \\ \overline{Z}_j &= \frac{\partial}{\partial \overline{z_j}} - iz_j\frac{\partial}{\partial t}; \\ T &= \partial/\partial t. \end{aligned}$$

$\mathbf{H}_n$ is a stratified group endowed with a family of dilations $\{\delta_r : r > 0\}$ defined by

$$\delta_r(z, t) = \left(rz, r^2 t\right).$$

The bi-invariant Haar measure on $\mathbf{H}_n$ coincides with the Lebesgue measure on $\mathbf{R}^{2n+1}$. As usual, we denote by $\mathcal{S}(\mathbf{H}_n)$ the Schwartz space of rapidly decreasing smooth functions on $\mathbf{H}_n$ and by $\mathcal{S}'(\mathbf{H}_n)$ the dual space of $\mathcal{S}(\mathbf{H}_n)$, that is, the space of tempered distributions on $\mathbf{H}_n$. The maximal torus $\mathbf{T}^n$, which we represent by $(-\pi, \pi]^n$, acts by automorphisms on $\mathbf{H}_n$ in the following way:

$$a_\vartheta(z, t) = \left(e^{i\vartheta_1} z_1, \dots, e^{i\vartheta_n} z_n, t\right)$$

where $\vartheta = (\vartheta_1, \dots, \vartheta_n) \in \mathbf{T}^n$. A function f on $\mathbf{H}_n$ is said to be *polyradial* if $f \circ a_\vartheta = f$ for every $\vartheta \in \mathbf{T}^n$, that is, if the value of $f(z,t)$ depends only on $|z_1|, \dots, |z_n|, t$. We denote by $L^p_{\mathbf{T}^n}$ $(1 \leqslant p \leqslant +\infty)$ the space of polyradial functions in $L^p(\mathbf{H}_n)$. The space $L^1_{\mathbf{T}^n}$ is a commutative, closed $*$-subalgebra of $L^1(\mathbf{H}_n)$. A differential operator D on $\mathbf{H}_n$ is said to be $\mathbf{T}^n$*-invariant* if $D(f \circ a_\vartheta) = D(f) \circ a_\vartheta$ for every $f \in C^\infty(\mathbf{H}_n)$ and $\vartheta \in \mathbf{T}^n$. The commutative algebra of $\mathbf{T}^n$-invariant operators is generated by $\mathcal{L}_1, \dots, \mathcal{L}_n, T$, where $\mathcal{L}_1, \dots, \mathcal{L}_n$ are the partial sub-Laplacians on $\mathbf{H}_n$ defined by

$$\mathcal{L}_j = -\frac{1}{2}\left(Z_j \overline{Z}_j + \overline{Z}_j Z_j\right).$$

The sub-Laplacian on $\mathbf{H}_n$ is $\mathcal{L} = \sum_{j=1}^{n} \mathcal{L}_j$. The Gelfand spectrum Δ of $L^1_{\mathbf{T}^n}$ can be identified with $(\mathbf{N}^n \times \mathbf{R}^*) \cup \left([0, +\infty)^n\right)$. The Gelfand transform $\mathcal{G}f$ of a function $f \in L^1_{\mathbf{T}^n}$ is given by

$$\mathcal{G}f(k,\lambda) = \int_{\mathbf{H}_n} f(x)\, \omega_{k,\lambda}(x)\, dx$$

with $(k,\lambda) \in \mathbf{N}^n \times \mathbf{R}^*$ and

$$\omega_{k,\lambda}(z,t) = e^{-i\lambda t} e^{-|\lambda|\cdot|z|^2} \prod_{j=1}^{n} L_{k_j}\left(2|\lambda| \cdot |z_j|^2\right)$$

where L_r $(r \in \mathbf{N})$ is the Laguerre polynomial of type 0 and degree r, defined by

$$L_r(\tau) = \sum_{s=0}^{r} \frac{(-1)^s}{s!} \binom{r}{s} \tau^s.$$

The Godement-Plancherel measure μ on Δ is given by

$$\int_\Delta F(\psi)\, d\mu(\psi) = \frac{2^{n-1}}{\pi^{n+1}} \sum_{k \in \mathbf{N}^n} \int_{\mathbf{R}^*} F(k,\lambda)\, |\lambda|^n\, d\lambda. \tag{2.1}$$

We ignore the remaining part of Δ, because it is of measure zero. By the Godement-Plancherel theory, $\mathcal{G}$ extends uniquely to a unitary operator $\widetilde{\mathcal{G}} : L^2_{\mathbf{T}^n} \longrightarrow L^2(\Delta)$. For the proofs and further information about all these facts, see for instance [**2, 11, 19**].

3. Joint spectral multipliers

The operators $\mathcal{L}_1, \dots, \mathcal{L}_n, -iT$ form a family of commuting self-adjoint operators. Their joint spectrum (see [**2**]) is the subset $\Sigma_1 \cup \Sigma_2$ of $\mathbf{R}^{n+1}$, where

$$\Sigma_1 = \left\{\left((2k_1+1)|\lambda|, \dots, (2k_n+1)|\lambda|, \lambda\right) : k_1, \dots, k_n \in \mathbf{N},\ \lambda \in \mathbf{R}^*\right\}$$

and

$$\Sigma_2 = \left\{(\mu_1, \dots, \mu_n, 0) : \mu_1, \dots, \mu_n \in [0, +\infty)\right\}.$$

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Let us define

$$\Lambda = |T|.$$

Arguing as in [**18**], one shows that also the operators $\Lambda^{-1}\mathcal{L}_1, \dots, \Lambda^{-1}\mathcal{L}_n, -iT$ form a family of commuting self-adjoint operators. Their joint spectrum is $(2\mathbf{N}+1)^n \times \mathbf{R}$. By the spectral theorem, the multiplier operators $m(\mathcal{L}_1, \dots, \mathcal{L}_n, -iT)$ and $m(\Lambda^{-1}\mathcal{L}_1, \dots, \Lambda^{-1}\mathcal{L}_n, -iT)$ are bounded on $L^2(\mathbf{H}_n)$ for all bounded Borel functions m defined on the corresponding joint spectra. Both these operators commute with left translations, so by [**12**] they are given by right convolution with tempered distributions, which we denote by $m(\mathcal{L}_1, \dots, \mathcal{L}_n, -iT)\delta$ and $m(\Lambda^{-1}\mathcal{L}_1, \dots, \Lambda^{-1}\mathcal{L}_n, -iT)\delta$, respectively. We also use the notations

$$M_m = m(\mathcal{L}_1, \dots, \mathcal{L}_n, -iT)\delta;$$
$$N_m = m(\Lambda^{-1}\mathcal{L}_1, \dots, \Lambda^{-1}\mathcal{L}_n, -iT)\delta.$$

By the Godement-Plancherel theory, we have that $M_m \in L^2_{\mathbf{T}^n}$ if and only if the function

$$\tilde{\mathcal{G}}M_m(k,\lambda) = m\Big((2k_1+1)|\lambda|, \dots, (2k_n+1)|\lambda|, \lambda\Big)$$

is in $L^2(\Delta)$. Similarly, we have that $N_m \in L^2_{\mathbf{T}^n}$ if and only if the function

$$\tilde{\mathcal{G}}N_m(k,\lambda) = m(2k_1+1, \dots, 2k_n+1, \lambda)$$

is in $L^2(\Delta)$.

4. Littlewood-Paley decompositions

Fix a function $\chi \in C_c^\infty\big((1/2,2)\big)$ such that $\chi \geqslant 0$ and $\sum_{m\in\mathbf{Z}} \chi(2^{-m}\lambda)^2 = 1$ for $\lambda > 0$. Let $\psi(\lambda) = \chi\big(|\lambda|\big)$ for $\lambda \in \mathbf{R}$. For $j = (j_1, \dots, j_{n+1}) \in \mathbf{Z}^{n+1}$ and $(\mu,\lambda) = (\mu_1, \dots, \mu_n, \lambda) \in \mathbf{R}^{n+1}$ write

$$\chi_j(\mu,\lambda) = \prod_{r=1}^{n} \chi\big(2^{-j_r}\mu_r\big) \cdot \psi\big(2^{-j_{n+1}}\lambda\big).$$

Set

$$\varphi_j = \chi_j(\mathcal{L}_1, \dots, \mathcal{L}_n, -iT)\delta;$$
$$\Phi_j = \chi_j\big(\Lambda^{-1}\mathcal{L}_1, \dots, \Lambda^{-1}\mathcal{L}_n, -iT\big)\delta.$$

The properties of χ imply (see [**1**]) that φ_j and Φ_j are in $\mathcal{S}(\mathbf{H}_n)$ and satisfy

$$\sum_{j\in\mathbf{Z}^{n+1}} \mathcal{G}\varphi_j(k,\lambda)^2 = \sum_{j\in\mathbf{Z}^{n+1}} \mathcal{G}\Phi_j(k,\lambda)^2 = 1.$$

For $u \in \mathcal{S}'(\mathbf{H}_n)$ we define the following Littlewood-Paley functions:

$$g_1(u) = \left(\sum_{j\in\mathbf{Z}^{n+1}} |u * \varphi_j|^2\right)^{1/2};$$
$$g_2(u) = \left(\sum_{j\in\mathbf{Z}^{n+1}} |u * \Phi_j|^2\right)^{1/2}.$$

Arguing as in [**18**], it is easy to prove that g_1 and g_2 are isometries of $L^2(\mathbf{H}_n)$.

PROPOSITION 4.1. *For $1 < p < +\infty$ there exists a constant $C_p \geqslant 1$ such that*

(a) *if $f \in L^p(\mathbf{H}_n)$ then $g_1(f) \in L^p(\mathbf{H}_n)$ and $\left\|g_1(f)\right\|_p \leqslant C_p \|f\|_p$;*

(b) *if $f \in L^2(\mathbf{H}_n)$ and $g_1(f) \in L^p(\mathbf{H}_n)$ then $f \in L^p(\mathbf{H}_n)$ and $\|f\|_p \leqslant C_p \left\|g_1(f)\right\|_p$.*

PROOF: By a standard duality argument (see [**20**, Chapter II]), it suffices to prove (a). Moreover, by some standard randomisation argument based on Khintchin's inequality (see [**21**, Chapter V]), it suffices to prove that there exists $C_p' > 0$ such that

$$\left\|\sum_{j_1=-N}^{N} \cdots \sum_{j_{n+1}=-N}^{N} \varepsilon_{j_1}^{(1)} \cdots \varepsilon_{j_{n+1}}^{(n+1)} (f * \varphi_j)\right\|_p \leqslant C_p' \|f\|_p$$

for every $N \in \mathbf{N}$ and for every choice of the $n+1$ sequences $\{\varepsilon_{j_1}^{(1)}\}_{j_1\in\mathbf{Z}}, \dots, \{\varepsilon_{j_{n+1}}^{(n+1)}\}_{j_{n+1}\in\mathbf{Z}}$ with values in $\{-1, 0, 1\}$. Since $\mathcal{S}(\mathbf{H}_n)$ is dense in $L^p(\mathbf{H}_n)$, a standard approximation argument allows us to assume that $f \in \mathcal{S}(\mathbf{H}_n)$. So

$$\sum_{j_1=-N}^{N} \cdots \sum_{j_{n+1}=-N}^{N} \varepsilon_{j_1}^{(1)} \cdots \varepsilon_{j_{n+1}}^{(n+1)} (f * \varphi_j)$$
$$= \left(\sum_{j_1=-N}^{N} \varepsilon_{j_1}^{(1)} \chi\left(2^{-j_1}\mathcal{L}_1\right)\right) \cdots \left(\sum_{j_n=-N}^{N} \varepsilon_{j_n}^{(n)} \chi\left(2^{-j_n}\mathcal{L}_n\right)\right) \left(\sum_{j_{n+1}=-N}^{N} \varepsilon_{j_{n+1}}^{(n+1)} \psi\left(-2^{-j_{n+1}} iT\right)\right) f.$$

A straight-forward calculation yields

$$\sup_{\lambda>0} \left|\lambda^h \frac{d^h}{d\lambda^h}\left(\sum_{j_r=-N}^{N} \varepsilon_{j_r}^{(r)} \chi\left(2^{-j_r}\lambda\right)\right)\right| \leqslant A_h$$

for $r \in \{1, \dots, n\}$, where the constant A_h is independent of N and of the choice of the sequence $\{\varepsilon_{j_r}^{(r)}\}_{j_r\in\mathbf{Z}}$. Therefore, by a suitable multiplier theorem (see [**7**, Chapter 6]), we have

$$\left\|\sum_{j_r=-N}^{N} \varepsilon_{j_r}^{(r)} \chi\left(2^{-j_r}\mathcal{L}^{\mathbf{H}_1}\right) g\right\|_{L^p(\mathbf{H}_1)} \leqslant M_p \|g\|_{L^p(\mathbf{H}_1)}$$

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for $g \in \mathcal{S}(\mathbf{H}_1)$, where $\mathcal{L}^{\mathbf{H}_1}$ is the sub-Laplacian on $\mathbf{H}_1$ and the constant M_p depends only on p. Applying the transference principle [4] yields

$$\left\| \sum_{j_r=-N}^{N} \varepsilon_{j_r}^{(r)} \chi\left(2^{-j_r}\mathcal{L}_r\right) f \right\|_{L^p(\mathbf{H}_n)} \leqslant M_p \|f\|_{L^p(\mathbf{H}_n)}$$

for $f \in \mathcal{S}(\mathbf{H}_n)$. Similarly we obtain

$$\left\| \sum_{j_{n+1}=-N}^{N} \varepsilon_{j_{n+1}}^{(n+1)} \psi\left(-2^{-j_{n+1}} iT\right) f \right\|_{L^p(\mathbf{H}_n)} \leqslant M_p' \|f\|_{L^p(\mathbf{H}_n)}$$

for $f \in \mathcal{S}(\mathbf{H}_n)$. This gives the conclusion. $\square$

As a corollary of Proposition 4.1, we obtain a weak Marcinkiewicz-type multiplier theorem. For $N \in \mathbf{N}$ and $m \in C^N((\mathbf{R}_+)^n \times \mathbf{R}^*)$ put

$$\|m\|_{(N)} = \sup_{\substack{\alpha \in \mathbf{N}^{n+1} \\ |\alpha| \leqslant N}} \sup_{\substack{\mu \in (\mathbf{R}_+)^n \\ \lambda \in \mathbf{R}^*}} \left| \left(\mu_1 \frac{\partial}{\partial \mu_1}\right)^{\alpha_1} \cdots \left(\mu_n \frac{\partial}{\partial \mu_n}\right)^{\alpha_n} \left(\lambda \frac{\partial}{\partial \lambda}\right)^{\alpha_{n+1}} m(\mu, \lambda) \right|.$$

Corollary 4.2. *There exists $N \in \mathbf{N}$ such that if $m \in C^N\left((\mathbf{R}_+)^n \times \mathbf{R}^*\right)$ and $\|m\|_{(N)} < +\infty$ then the operator $m(\mathcal{L}_1, \dots, \mathcal{L}_n, -iT)$ is bounded on $L^p(\mathbf{H}_n)$, $1 < p < +\infty$, with norm controlled by $\|m\|_{(N)}$.*

We omit the proof of Corollary 4.2, because it is an easy but lengthy adaptment of the proof of Corollary 4.3 in [18], where the operator $m(\mathcal{L}, -iT)$ is considered. The only crucial point is that we apply our Proposition 4.1 instead of the corresponding Proposition 4.1 in [18]. We remark that Corollary 4.2 has also been proved in [8], however by a different method. Once we have Corollary 4.2, arguing again as in [18] we easily obtain the following

Proposition 4.3. *For $1 < p < +\infty$ there exists a constant $C_p \geqslant 1$ such that*

(a) *if $f \in L^p(\mathbf{H}_n)$ then $g_2(f) \in L^p(\mathbf{H}_n)$ and $\|g_2(f)\|_p \leqslant C_p \|f\|_p$;*

(b) *if $f \in L^2(\mathbf{H}_n)$ and $g_2(f) \in L^p(\mathbf{H}_n)$ then $f \in L^p(\mathbf{H}_n)$ and $\|f\|_p \leqslant C_p \|g_2(f)\|_p$.*

5. Functional calculus on the Gelfand spectrum

In Section 2 we have seen that the Gelfand spectrum Δ can be identified, as a measure space, with the space $\mathbf{N}^n \times \mathbf{R}^*$ equipped with the measure μ defined by (2.1). Thus Δ can be considered as a subspace of the measure space $S = \mathbf{Z}^n \times \mathbf{R}$ equipped with the measure $\tilde{\mu}$ defined by

$$\int_S G(\psi)\, d\tilde{\mu}(\psi) = \frac{2^{n-1}}{\pi^{n+1}} \sum_{k \in \mathbf{Z}^n} \int_{\mathbf{R}} G(k, \lambda)\, |\lambda|^n\, d\lambda.$$

We consider the canonical operators $\mathcal{P} : L^2(S) \longrightarrow L^2(\Delta)$ and $\mathcal{Q} : L^2(\Delta) \longrightarrow L^2(S)$ defined by

$$(\mathcal{P}G)(k,\lambda) = G(k,\lambda);$$
$$(\mathcal{Q}F)(k,\lambda) = \begin{cases} F(k,\lambda) & \text{if } k \in \mathbf{N}^n \text{ and } \lambda \in \mathbf{R}^* \\ 0 & \text{otherwise.} \end{cases}$$

Let G be a function on S. For $j \in \{1,\dots,n\}$ and $h \in \mathbf{Z}$ we define the translation operator $\tau_j^{(h)}$ by

$$\left(\tau_j^{(h)}G\right)(k,\lambda) = G(k_1,\dots,k_{j-1},k_j+h,k_{j+1},\dots,k_n,\lambda). \tag{5.1}$$

We also define the difference operator

$$\Delta_j = \tau_j^{(1)} - \tau_j^{(0)}.$$

Finally we define the multiplication operator M_j by

$$(M_jG)(k,\lambda) = k_j\,G(k,\lambda). \tag{5.2}$$

From (5.1) and (5.2) we immediately obtain the following commutation relations between the operators $\tau_j^{(h)}$ and M_j:

$$\begin{aligned} M_iM_j &= M_jM_i; \\ \tau_j^{(h)}M_i &= M_i\tau_j^{(h)} \quad \text{if } i \neq j; \\ \tau_j^{(h)}M_j &= M_j\tau_j^{(h)} + h\tau_j^{(h)}; \\ \tau_j^{(h)}\tau_i^{(l)} &= \tau_i^{(l)}\tau_j^{(h)}; \\ \tau_j^{(h)}\tau_j^{(l)} &= \tau_j^{(h+l)}. \end{aligned}$$

These relations and simple induction arguments lead to the following

Lemma 5.1. *For $\nu,\beta,q \in \mathbf{N}$, $m \in \mathbf{Z}_+$, $h \in \mathbf{Z}$, $j \in \{1,\dots,n\}$ the following identities hold:*

$$M_j^\nu\tau_j^{(h)} = \sum_{r=0}^{\nu}(-h)^{\nu-r}\binom{\nu}{r}\tau_j^{(h)}M_j^r;$$
$$M_j^m\Delta_j = \Delta_jM_j^m + \sum_{r=0}^{m-1}(-1)^{m-r}\binom{m}{r}\tau_j^{(1)}M_j^r;$$
$$M_j^\nu\Delta_j^{\nu+q} = \sum_{r=0}^{\nu}\sum_{s=0}^{q+2^\nu-1}a_{\nu,q,r,s}\,\tau_j^{(s)}M_j^r\Delta_j^r;$$
$$M_j^\nu\Delta_j^{\nu+q}\tau_j^{(h)}M_j^\beta\Delta_j^\beta = \sum_{r=0}^{\nu+\beta}\sum_{s=0}^{2^\beta(q+\nu+2^\nu-1)}b_{\nu,\beta,q,h,r,s}\,\tau_j^{(h+s)}M_j^r\Delta_j^{r+q}.$$

The coefficients $a_{\nu,q,r,s}$ and $b_{\nu,\beta,q,h,r,s}$ in the last two identities are real.

Let p be a polyradial polynomial on $\mathbf{H}_n$. For all $f \in L^2_{\mathbf{T}^n}$ such that $pf \in L^2_{\mathbf{T}^n}$ let us define

$$\partial_p\left(\tilde{\mathcal{G}}f\right) = \tilde{\mathcal{G}}(pf).$$

The operator ∂_p is thus densely defined on $L^2(\Delta)$ and its domain is

$$\operatorname{Dom}\partial_p = \left\{F \in L^2(\Delta) : p \cdot \tilde{\mathcal{G}}^{-1}F \in L^2_{\mathbf{T}^n}\right\}.$$

Straight-forward computations (see [**5, 13, 19**]) yield

$$(5.3)\quad \left(\partial_{|z_j|^2}F\right)(k,\lambda) = \frac{1}{2|\lambda|}\left\{(2k_j+1)F(k,\lambda) - (k_j+1)\left(\tau_j^{(1)}\mathcal{Q}F\right)(k,\lambda) - k_j\left(\tau_j^{(-1)}\mathcal{Q}F\right)(k,\lambda)\right\};$$

$$(5.4)\quad (\partial_{-it}F)(k,\lambda) = \frac{\partial F}{\partial\lambda}(k,\lambda) + \frac{1}{2\lambda}\sum_{j=1}^{n}\left\{F(k,\lambda) - (k_j+1)\left(\tau_j^{(1)}\mathcal{Q}F\right)(k,\lambda) + k_j\left(\tau_j^{(-1)}\mathcal{Q}F\right)(k,\lambda)\right\}.$$

Since every polyradial polynomial on $\mathbf{H}_n$ has the form

$$p(z,t) = \sum_{i_1=0}^{N}\cdots\sum_{i_n=0}^{N}\sum_{l=0}^{N} a_{i_1,\dots,i_n,l}\,|z_1|^{2i_1}\cdots|z_n|^{2i_n}\,(-it)^l$$

with $a_{i_1,\dots,i_n,l} \in \mathbf{C}$, by (5.3) and (5.4) we can extend the operator ∂_p to an operator $\tilde{\partial}_p$ on $L^2(S)$ defined by

$$(5.5)\quad \tilde{\partial}_p = \sum_{i_1=0}^{N}\cdots\sum_{i_n=0}^{N}\sum_{l=0}^{N} a_{i_1,\dots,i_n,l}\,\tilde{\partial}^{i_1}_{|z_1|^2}\cdots\tilde{\partial}^{i_n}_{|z_n|^2}\,\tilde{\partial}^{l}_{-it}$$

where

$$(5.6)\quad \left(\tilde{\partial}_{|z_j|^2}G\right)(k,\lambda) = \frac{1}{2|\lambda|}\left\{(2k_j+1)G(k,\lambda) - (k_j+1)\left(\tau_j^{(1)}G\right)(k,\lambda) - k_j\left(\tau_j^{(-1)}G\right)(k,\lambda)\right\};$$

$$(5.7)\quad \left(\tilde{\partial}_{-it}G\right)(k,\lambda) = \frac{\partial G}{\partial\lambda}(k,\lambda) + \frac{1}{2\lambda}\sum_{j=1}^{n}\left\{G(k,\lambda) - (k_j+1)\left(\tau_j^{(1)}G\right)(k,\lambda) + k_j\left(\tau_j^{(-1)}G\right)(k,\lambda)\right\}.$$

The operator $\tilde{\partial}_p$ is thus densely defined on $L^2(S)$ and its domain is

$$\operatorname{Dom}\tilde{\partial}_p = \left\{G \in L^2(S) : \tilde{\partial}_pG \in L^2(S)\right\}.$$

This domain contains the subspace $\mathcal{Q}(\operatorname{Dom}\partial_p)$. Furthermore, the following identity is valid on $\operatorname{Dom}\partial_p$:

$$(5.8)\quad \partial_p = \mathcal{P}\tilde{\partial}_p\mathcal{Q}.$$

Let us introduce the following notation:

$$(a\cdot\tilde{\tau}_j)=\sum_{h=-H}^{H}a^{(h)}\,\tau_j^{(h)};\tag{5.9}$$

$$(a\cdot\tilde{\tau})=\sum_{h_1=-H}^{H}\cdots\sum_{h_n=-H}^{H}a^{(h_1,\dots,h_n)}\,\tau_1^{(h_1)}\cdots\tau_n^{(h_n)}\tag{5.10}$$

where $H\in\mathbf{N}$ and $a^{(h)},a^{(h_1,\dots,h_n)}\in\mathbf{R}$.

Proposition 5.2.

(a) *For $q\in\mathbf{N}$ and $j\in\{1,\dots,n\}$ we have*

$$\tilde{\partial}^q_{|z_j|^2}=|\lambda|^{-q}\sum_{\nu=0}^{q}(a\cdot\tilde{\tau}_j)\,M_j^{\nu}\Delta_j^{\nu+q}$$

where the integer H and the coefficients $a^{(h)}$ involved in the expression $(a\cdot\tilde{\tau}_j)$ according to (5.9) depend only on q and ν.

(b) *For $q\in\mathbf{N}$ and $T\in\mathbf{R}$ we have*

$$\tilde{\partial}^q_{T^2+t^2}=\sum_{\nu=0}^{2q}\sum_{\substack{\beta\in\mathbf{N}^n\\|\beta|\leqslant 2q-\nu}}\sum_{\gamma=0}^{[q-\nu/2]}(a\cdot\tilde{\tau})\,T^{2\gamma}\cdot|\lambda|^{-(2q-\nu-2\gamma)}\frac{\partial^\nu}{\partial\lambda^\nu}M_1^{\beta_1}\cdots M_n^{\beta_n}\,\Delta_1^{\beta_1}\cdots\Delta_n^{\beta_n}$$

where $[\cdot]$ denotes the greatest integer function and the integer H and the coefficients $a^{(h_1,\dots,h_n)}$ involved in the expression $(a\cdot\tilde{\tau})$ according to (5.10) depend only on $q,\nu,\beta,\gamma,\operatorname{sgn}\lambda$.

Proof: By straight-forward computations, we can rewrite (5.6) and (5.7) as

$$\tilde{\partial}_{|z_j|^2}=-\frac{1}{2|\lambda|}\left\{\tau_j^{(-1)}M_j\Delta_j^2+\left(2\tau_j^{(0)}-\tau_j^{(-1)}\right)\Delta_j\right\};$$

$$\tilde{\partial}_{-it}=\frac{\partial}{\partial\lambda}-\frac{1}{2\lambda}\sum_{j=1}^{n}\left\{\left(\tau_j^{(0)}+\tau_j^{(-1)}\right)M_j\Delta_j+\tau_j^{(1)}-\tau_j^{(-1)}\right\}.$$

Then

$$\begin{aligned}\tilde{\partial}_{T^2+t^2}&=T^2-\tilde{\partial}^2_{-it}\\&=T^2-\frac{\partial^2}{\partial\lambda^2}+\frac{\partial}{\partial\lambda}\left(\frac{1}{2\lambda}\sum_{j=1}^{n}\left\{\left(\tau_j^{(0)}+\tau_j^{(-1)}\right)M_j\Delta_j+\tau_j^{(1)}-\tau_j^{(-1)}\right\}\right)\\&\quad+\frac{1}{2\lambda}\frac{\partial}{\partial\lambda}\left(\sum_{j=1}^{n}\left\{\left(\tau_j^{(0)}+\tau_j^{(-1)}\right)M_j\Delta_j+\tau_j^{(1)}-\tau_j^{(-1)}\right\}\right)\\&\quad-\frac{1}{4\lambda^2}\left(\sum_{i=1}^{n}\left\{\left(\tau_i^{(0)}+\tau_i^{(-1)}\right)M_i\Delta_i+\tau_i^{(1)}-\tau_i^{(-1)}\right\}\right)\\&\qquad\qquad\left(\sum_{j=1}^{n}\left\{\left(\tau_j^{(0)}+\tau_j^{(-1)}\right)M_j\Delta_j+\tau_j^{(1)}-\tau_j^{(-1)}\right\}\right).\end{aligned}$$

Using these expressions for $\widetilde{\partial}_{|z_j|^2}$ and $\widetilde{\partial}_{T^2+t^2}$, we can easily obtain (a) and (b) by induction on q and iterated applications of Lemma 5.1. ∎

The reason why we have considered the space $\mathbf{Z}^n \times \mathbf{R}$ rather than the space $\mathbf{N}^n \times \mathbf{R}^*$ is that $\mathbf{Z}^n \times \mathbf{R}$ has some properties which $\mathbf{N}^n \times \mathbf{R}^*$ does not have: in particular, it is a locally compact Abelian group, so it is possible to define a Fourier transform on it. If f is a function in $L^1(\mathbf{Z}^n \times \mathbf{R})$, the Fourier transform of f is the function $\widehat{f} \in C_0(\mathbf{T}^n \times \mathbf{R})$ defined by

$$\widehat{f}(\vartheta, s) = \sum_{k \in \mathbf{Z}^n} \int_{\mathbf{R}} f(k, \lambda)\, e^{-i(k\cdot\vartheta + \lambda s)}\, d\lambda.$$

The Fourier transform on $\mathbf{Z}^n \times \mathbf{R}$ extends uniquely to a unitary operator (apart from a multiplicative constant) from $L^2(\mathbf{Z}^n \times \mathbf{R})$ to $L^2(\mathbf{T}^n \times \mathbf{R})$.

If f is a suitable function on $\mathbf{Z}^n \times \mathbf{R}$, we have

$$\widehat{\Delta_j f}(\vartheta, s) = \left(e^{i\vartheta_j} - 1\right)\widehat{f}(\vartheta, s);$$
$$\widehat{\frac{\partial f}{\partial \lambda}}(\vartheta, s) = is\widehat{f}(\vartheta, s).$$

Correspondingly, for $\alpha \geqslant 0$ we define fractional powers $|\Delta_j|^\alpha$ and $\left|\frac{\partial}{\partial \lambda}\right|^\alpha$ by

$$\left(|\Delta_j|^\alpha f\right)^\frown(\vartheta, s) = |e^{i\vartheta_j} - 1|^\alpha \widehat{f}(\vartheta, s);$$
$$\left(\left|\frac{\partial}{\partial \lambda}\right|^\alpha f\right)^\frown(\vartheta, s) = |s|^\alpha \widehat{f}(\vartheta, s).$$

Similarly, for $r_1, \dots, r_n, \rho \geqslant 0$, we define the operator $\left(1 + \sum_{j=1}^n |r_j \Delta_j| + |\rho \frac{\partial}{\partial \lambda}|\right)^\alpha I$ by

$$\text{(5.11)} \quad \left(\left(1 + \sum_{j=1}^n |r_j \Delta_j| + \left|\rho \frac{\partial}{\partial \lambda}\right|\right)^\alpha f\right)^\frown(\vartheta, s) = \left(1 + \sum_{j=1}^n r_j |e^{i\vartheta_j} - 1| + \rho|s|\right)^\alpha \widehat{f}(\vartheta, s).$$

We shall use all these notations in Section 6.

6. Multipliers on the joint spectrum

In this section m is a bounded function on $(2\mathbf{N}+1)^n \times \mathbf{R}^*$ such that $m(2k_1+1, \dots, 2k_n+1, \cdot)$ is a Borel function on $\mathbf{R}^*$ for every $k = (k_1, \dots, k_n) \in \mathbf{N}^n$.

Fix a function $\eta \in C_c^\infty\big((1/4, 4)\big)$ such that $\eta \geqslant 0$ and $\eta = 1$ in $[1/2, 2]$. For $j = (j_1, \dots, j_{n+1}) \in \mathbf{Z}^{n+1}$ and $(\mu, \lambda) = (\mu_1, \dots, \mu_n, \lambda) \in \mathbf{R}^{n+1}$ put

$$\text{(6.1)} \qquad \eta_j(\mu, \lambda) = \prod_{r=1}^n \eta\left(2^{-j_r}\mu_r\right) \cdot \eta\left(2^{-j_{n+1}}|\lambda|\right).$$

Set

$$\text{(6.2)} \qquad N_j = (m\eta_j)\left(\Lambda^{-1}\mathcal{L}_1, \dots, \Lambda^{-1}\mathcal{L}_n, -iT\right)\delta.$$

Since the function

$$(k,\lambda) \longmapsto (m\eta_j)(2k_1+1,\dots,2k_n+1,\lambda)$$

is in $L^2(\Delta)$, by (6.2) and the facts established in Section 3 we have that $N_j \in L^2_{\mathbf{T}^n}$ for all $j \in \mathbf{Z}^{n+1}$. We consider the function $m_j \in L^2(S)$ defined by

$$m_j = \mathcal{Q}\tilde{\mathcal{G}}N_j \tag{6.3}$$

where the operators $\mathcal{Q}$ and $\tilde{\mathcal{G}}$ have been introduced in the previous sections. According to (5.11), for $\alpha \geqslant 0$ and $\beta \geqslant 0$ we define the scale-invariant localised Sobolev norm

$$\|m\|_{\ell^2(L^2)_{\alpha,\beta,\mathrm{sloc}}} = \left\{ \sup_{j\in\mathbf{Z}^{n+1}} 2^{-\sum_{r=1}^{n+1} j_r} \sum_{k\in\mathbf{Z}^n} \int_{\mathbf{R}} \left| \left(1+|2^{j_1}\Delta_1|\right)^{\alpha} \cdots \left(1+|2^{j_n}\Delta_n|\right)^{\alpha} \right.\right.$$
$$\left.\left. \left(1+\sum_{r=1}^{n}|2^{j_r}\Delta_r| + \left|2^{j_{n+1}}\frac{\partial}{\partial\lambda}\right|\right)^{\beta} m_j(k,\lambda)\right|^2 d\lambda \right\}^{1/2}. \tag{6.4}$$

We remark that, by standard partition of unity arguments, it can easily be shown that different bump functions η lead to equivalent $\ell^2(L^2)_{\alpha,\beta,\mathrm{sloc}}$ norms.

For $\delta > 0$, $\gamma \geqslant 0$ and $j \in \mathbf{Z}^{n+1}$ let $W_\delta^{(j)}$ and $u_\gamma^{(j)}$ be the weights on $\mathbf{H}_n$ defined by

$$W_\delta^{(j)}(z,t) = 2^{-\sum_{r=1}^{n} j_r - (n+1)j_{n+1}} \cdot \prod_{r=1}^{n}\left(1+2^{(j_r+j_{n+1})/2}|z_r|\right)^{2(1+\delta)} \cdot \left(1+2^{j_{n+1}}|t|\right)^{1+\delta}; \tag{6.5}$$

$$u_\gamma^{(j)}(z,t) = 2^{2\gamma j_{n+1}} \cdot \prod_{r=1}^{n}\left\{1+\left(2^{j_r+j_{n+1}}|z_r|^2\right)^{2\gamma}\right\} \cdot \left(2^{-2j_{n+1}}+t^2\right)^{\gamma}. \tag{6.6}$$

Lemma 6.1. *Suppose $1 < p < +\infty$ and $\delta > 0$. There exists a constant $C = C(p,\delta) > 0$ such that*

$$\left\| m(\Lambda^{-1}\mathcal{L}_1,\dots,\Lambda^{-1}\mathcal{L}_n,-iT)f \right\|_p \leqslant C\,\|f\|_p \cdot \sup_{j\in\mathbf{Z}^{n+1}} \left(\int_{\mathbf{H}_n} \left|N_j(x)\right|^2 W_\delta^{(j)}(x)\,dx\right)^{1/2}$$

for all $f \in L^2(\mathbf{H}_n) \cap L^p(\mathbf{H}_n)$.

The proof of Lemma 6.1 follows strictly the proof of Lemma 5.1 in [**18**], where the operator $m(\Lambda^{-1}\mathcal{L},-iT)$ is considered. The only obvious difference is that we apply our Proposition 4.3 instead of the corresponding Proposition 4.4 in [**18**].

Proposition 6.2. *For every $\gamma \geqslant 0$ there exists a constant $C_\gamma > 0$ such that*

$$\int_{\mathbf{H}_n} \left|N_j(x)\,u_\gamma^{(j)}(x)\right|^2 dx \leqslant C_\gamma \cdot 2^{nj_{n+1}} \cdot \sum_{k\in\mathbf{Z}^n} \int_{\mathbf{R}} \left| \left(1+\sum_{r=1}^{n}|2^{j_r}\Delta_r| + \left|2^{j_{n+1}}\frac{\partial}{\partial\lambda}\right|\right)^{2\gamma} \right.$$
$$\left. \left(1+|2^{j_1}\Delta_1|\right)^{4\gamma}\cdots\left(1+|2^{j_n}\Delta_n|\right)^{4\gamma} m_j(k,\lambda)\right|^2 d\lambda$$

for all $j \in \mathbf{Z}^{n+1}$.

PROOF: By (5.11) it suffices to prove that

$$\int_{\mathbf{H}_n} \left|N_j(x)\, u_\gamma^{(j)}(x)\right|^2 dx \leqslant C_\gamma \cdot 2^{nj_{n+1}} \cdot \int_{\mathbf{T}^n} \int_{\mathbf{R}} \Bigg| \left(1+\sum_{r=1}^{n} 2^{j_r} \left|e^{i\vartheta_r}-1\right| + 2^{j_{n+1}}|\sigma|\right)^{2\gamma} \cdot \prod_{r=1}^{n}\left(1+2^{j_r}\left|e^{i\vartheta_r}-1\right|\right)^{4\gamma} \cdot \widehat{m_j}(\vartheta,\sigma)\Bigg|^2 d\vartheta\, d\sigma. \tag{6.7}$$

Furthermore, it suffices to prove (6.7) if $\gamma \in \mathbf{N}$: the general case will follow by interpolation. In this hypothesis $u_\gamma^{(j)}$ is a polyradial polynomial on $\mathbf{H}_n$. So, by (5.5) and Proposition 5.2, we have

$$\begin{aligned}
\widetilde{\partial}_{u_\gamma^{(j)}} &= 2^{2\gamma j_{n+1}} \sum_{\{r_1,\dots,r_s\}\subset\{1,\dots,n\}} 2^{2\gamma(j_{r_1}+j_{n+1})}\cdots 2^{2\gamma(j_{r_s}+j_{n+1})}\, \widetilde{\partial}^{2\gamma}_{|z_{r_1}|^2}\cdots \widetilde{\partial}^{2\gamma}_{|z_{r_s}|^2}\, \widetilde{\partial}^{\gamma}_{2^{-2j_{n+1}}+t^2} \\
&= 2^{2\gamma j_{n+1}} \Bigg\{ \sum_{\{r_1,\dots,r_s\}\subset\{1,\dots,n\}} 2^{2\gamma\left(sj_{n+1}+\sum_{v=1}^{s} j_{r_v}\right)} |\lambda|^{-2\gamma s} \\
&\quad \left(\sum_{m_1=0}^{2\gamma} (a\cdot\widetilde{\tau}_{r_1})\, M_{r_1}^{m_1}\Delta_{r_1}^{m_1+2\gamma}\right)\cdots\left(\sum_{m_s=0}^{2\gamma}(a\cdot\widetilde{\tau}_{r_s})\, M_{r_s}^{m_s}\Delta_{r_s}^{m_s+2\gamma}\right)\Bigg\} \\
&\quad \left\{\sum_{\nu=0}^{2\gamma} \sum_{\substack{\beta\in\mathbf{N}^n\\ |\beta|\leqslant 2q-\nu}} \sum_{q=0}^{[\gamma-\nu/2]} 2^{-2qj_{n+1}}\,(a\cdot\widetilde{\tau})|\lambda|^{-(2\gamma-\nu-2q)}\frac{\partial^\nu}{\partial\lambda^\nu} M_1^{\beta_1}\cdots M_n^{\beta_n}\Delta_1^{\beta_1}\cdots\Delta_n^{\beta_n}\right\} \\
&= 2^{2\gamma j_{n+1}} \sum_{\{r_1,\dots,r_s\}\subset\{1,\dots,n\}} \sum_{\nu=0}^{2\gamma} \sum_{|\beta|\leqslant 2\gamma-\nu} \sum_{q=0}^{[\gamma-\nu/2]} \sum_{m_1=0}^{2\gamma}\cdots\sum_{m_s=0}^{2\gamma} 2^{-2qj_{n+1}} \\
&\quad \cdot 2^{2\gamma\left(sj_{n+1}+\sum_{v=1}^{s} j_{r_v}\right)} |\lambda|^{-(2\gamma-\nu-2q+2\gamma s)} \frac{\partial^\nu}{\partial\lambda^\nu}(a\cdot\widetilde{\tau}_{r_1})\cdots(a\cdot\widetilde{\tau}_{r_s}) \\
&\quad M_{r_1}^{m_1}\Delta_{r_1}^{m_1+2\gamma}\cdots M_{r_s}^{m_s}\Delta_{r_s}^{m_s+2\gamma}(a\cdot\widetilde{\tau})\, M_1^{\beta_1}\cdots M_n^{\beta_n}\Delta_1^{\beta_1}\cdots\Delta_n^{\beta_n}.
\end{aligned}$$

If we set $\{r_{s+1},\dots,r_n\} = \{1,\dots,n\}\setminus\{r_1,\dots,r_s\}$, by Lemma 5.1 we obtain

$$\begin{aligned}
\widetilde{\partial}_{u_\gamma^{(j)}} = &\sum_{\{r_1,\dots,r_s\}\subset\{1,\dots,n\}} \sum_{\nu=0}^{2\gamma} \sum_{|\beta|\leqslant 2\gamma-\nu} \sum_{q=0}^{[\gamma-\nu/2]} \sum_{m_1=0}^{2\gamma}\cdots\sum_{m_s=0}^{2\gamma} \sum_{h_1=0}^{m_1+\beta_{r_1}}\cdots\sum_{h_s=0}^{m_s+\beta_{r_s}} 2^{2j_{n+1}(\gamma-q)} \\
&\cdot 2^{2\gamma\left(sj_{n+1}+\sum_{v=1}^{s} j_{r_v}\right)} |\lambda|^{-(2\gamma-\nu-2q+2\gamma s)} \frac{\partial^\nu}{\partial\lambda^\nu}(a\cdot\widetilde{\tau}) \\
&M_{r_1}^{h_1}\Delta_{r_1}^{h_1+2\gamma}\cdots M_{r_s}^{h_s}\Delta_{r_s}^{h_s+2\gamma} M_{r_{s+1}}^{\beta_{r_{s+1}}}\Delta_{r_{s+1}}^{\beta_{r_{s+1}}}\cdots M_{r_n}^{\beta_{r_n}}\Delta_{r_n}^{\beta_{r_n}}.
\end{aligned}$$

We observe that in $\operatorname{supp} m_j$ we have $|\lambda| \sim 2^{j_{n+1}}$ and $k_r \sim 2^{j_r}$ for $r \in \{1,\dots,n\}$. So

$$2^{2j_{n+1}(\gamma-q)}\,|\lambda|^{-(2\gamma-\nu-2q)} \sim 2^{\nu j_{n+1}};$$
$$2^{2\gamma s j_{n+1}}\,|\lambda|^{-2\gamma s} \sim 1;$$
$$M_r \sim 2^{j_r}.$$

By these facts and by (5.8) and (6.3) we have

$$\begin{aligned}
&\int_{\mathbf{H}_n} \left|N_j(x)\, u_\gamma^{(j)}(x)\right|^2 dx \\
&= \frac{2^{n-1}}{\pi^{n+1}} \sum_{k\in\mathbf{N}^n} \int_{\mathbf{R}^*} \left|\partial_{u_\gamma^{(j)}}\left(\tilde{\mathcal{G}} N_j\right)(k,\lambda)\right|^2 |\lambda|^n \, d\lambda \\
&\leqslant \frac{2^{n-1}}{\pi^{n+1}} \sum_{k\in\mathbf{Z}^n} \int_{\mathbf{R}} \left|\tilde{\partial}_{u_\gamma^{(j)}} m_j(k,\lambda)\right|^2 |\lambda|^n \, d\lambda \\
&\leqslant C_\gamma \sum_{\{r_1,\dots,r_s\}\subset\{1,\dots,n\}} \sum_{\nu=0}^{2\gamma} \sum_{|\beta|\leqslant 2\gamma-\nu} \sum_{m_1=0}^{2\gamma} \cdots \sum_{m_s=0}^{2\gamma} \sum_{h_1=0}^{m_1+\beta_{r_1}} \cdots \sum_{h_s=0}^{m_s+\beta_{r_s}} \\
&\qquad \sum_{k\in\mathbf{Z}^n} \int_{\mathbf{R}} \left| \left(2^{\nu j_{n+1}} \frac{\partial^\nu}{\partial\lambda^\nu}\right) \left(2^{j_{r_1}} \Delta_{r_1}\right)^{h_1+2\gamma} \cdots \left(2^{j_{r_s}} \Delta_{r_s}\right)^{h_s+2\gamma} \right. \\
&\qquad\qquad \left. \left(2^{j_{r_{s+1}}} \Delta_{r_{s+1}}\right)^{\beta_{r_{s+1}}} \cdots \left(2^{j_{r_n}} \Delta_{r_n}\right)^{\beta_{r_n}} m_j(k,\lambda) \right|^2 2^{n j_{n+1}} \, d\lambda \\
&= C_\gamma \cdot 2^{n j_{n+1}} \cdot \sum_{\{r_1,\dots,r_s\}\subset\{1,\dots,n\}} \sum_{\nu=0}^{2\gamma} \sum_{|\beta|\leqslant 2\gamma-\nu} \sum_{m_1=0}^{2\gamma} \cdots \sum_{m_s=0}^{2\gamma} \sum_{h_1=0}^{m_1+\beta_{r_1}} \cdots \sum_{h_s=0}^{m_s+\beta_{r_s}} \\
&\qquad \int_{\mathbf{T}^n} \int_{\mathbf{R}} \left| 2^{\nu j_{n+1}} \cdot (i\sigma)^\nu \cdot \prod_{v=1}^{s} \left\{2^{j_{r_v}} \left(e^{i\vartheta_{r_v}} - 1\right)\right\}^{h_v+2\gamma} \right. \\
&\qquad\qquad \left. \cdot \prod_{u=s+1}^{n} \left\{2^{j_{r_u}} \left(e^{i\vartheta_{r_u}} - 1\right)\right\}^{\beta_{r_u}} \cdot \widehat{m_j}(\vartheta,\sigma) \right|^2 d\vartheta \, d\sigma \\
&\leqslant C'_\gamma \cdot 2^{n j_{n+1}} \cdot \sum_{\{r_1,\dots,r_s\}\subset\{1,\dots,n\}} \sum_{\nu=0}^{2\gamma} \sum_{|\beta|\leqslant 2\gamma-\nu} \sum_{m_1=0}^{2\gamma} \cdots \sum_{m_s=0}^{2\gamma} \\
&\qquad \int_{\mathbf{T}^n} \int_{\mathbf{R}} \left| \left(2^{j_{n+1}} |\sigma|\right)^\nu \cdot \prod_{v=1}^{s} \left(1 + 2^{j_{r_v}} \left|e^{i\vartheta_{r_v}} - 1\right|\right)^{m_v+\beta_{r_v}+2\gamma} \right. \\
&\qquad\qquad \left. \cdot \prod_{u=s+1}^{n} \left(1 + 2^{j_{r_u}} \left|e^{i\vartheta_{r_u}} - 1\right|\right)^{\beta_{r_u}} \cdot \widehat{m_j}(\vartheta,\sigma) \right|^2 d\vartheta \, d\sigma \\
&\leqslant C'_\gamma \cdot 2^{n j_{n+1}} \cdot \sum_{\{r_1,\dots,r_s\}\subset\{1,\dots,n\}} \sum_{\nu=0}^{2\gamma} \sum_{|\beta|\leqslant 2\gamma-\nu} \sum_{m_1=0}^{2\gamma} \cdots \sum_{m_s=0}^{2\gamma} \\
&\qquad \int_{\mathbf{T}^n} \int_{\mathbf{R}} \left| \left(1 + \sum_{r=1}^{n} 2^{j_r} \left|e^{i\vartheta_r} - 1\right| + 2^{j_{n+1}} |\sigma|\right)^{\nu+|\beta|} \right. \\
&\qquad\qquad \left. \cdot \prod_{v=1}^{s} \left(1 + 2^{j_{r_v}} \left|e^{i\vartheta_{r_v}} - 1\right|\right)^{m_v+2\gamma} \cdot \widehat{m_j}(\vartheta,\sigma) \right|^2 d\vartheta \, d\sigma \\
&\leqslant C''_\gamma \cdot 2^{n j_{n+1}} \cdot \int_{\mathbf{T}^n} \int_{\mathbf{R}} \left| \left(1 + \sum_{r=1}^{n} 2^{j_r} \left|e^{i\vartheta_r} - 1\right| + 2^{j_{n+1}} |\sigma|\right)^{2\gamma} \right. \\
&\qquad \left. \cdot \prod_{r=1}^{n} \left(1 + 2^{j_r} \left|e^{i\vartheta_r} - 1\right|\right)^{4\gamma} \cdot \widehat{m_j}(\vartheta,\sigma) \right|^2 d\vartheta \, d\sigma.
\end{aligned}$$

∎

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Formula (6.4), Lemma 6.1 and Proposition 6.2, by the relation between $W_\delta^{(j)}$ and $u_{1+\delta}^{(j)}$ deducible from (6.5) and (6.6), lead directly to:

Theorem 6.3. *Suppose $\|m\|_{\ell^2(L^2)_{\alpha,\beta,\mathrm{sloc}}} < +\infty$ for some $\alpha > 1$ and $\beta > 1/2$. Then for $1 < p < +\infty$ there exists a constant $C_{\alpha,\beta,p} > 0$, not depending on the function m, such that*

$$\left\|m(\Lambda^{-1}\mathcal{L}_1, \ldots, \Lambda^{-1}\mathcal{L}_n, -iT)f\right\|_p \leqslant C_{\alpha,\beta,p}\,\|m\|_{\ell^2(L^2)_{\alpha,\beta,\mathrm{sloc}}}\,\|f\|_p$$

for all $f \in L^2(\mathbf{H}_n) \cap L^p(\mathbf{H}_n)$.

7. Multipliers on $\mathbf{R}^{n+1}$

We want to prove a weaker but simpler version of Theorem 6.3, where the function m satisfies a Sobolev condition on all $\mathbf{R}^{n+1}$ and not only on the spectrum of the operator. In this context, from the boundedness of the operator $m(\Lambda^{-1}\mathcal{L}_1, \ldots, \Lambda^{-1}\mathcal{L}_n, -iT)$ we shall be able to deduce also the boundedness of the operator $m(\mathcal{L}_1, \ldots, \mathcal{L}_n, -iT)$ under the same hypotheses on m.

In this section m is a bounded Borel function on $(\mathbf{R}_+)^n \times \mathbf{R}^*$. We extend m on all $\mathbf{R}^{n+1}$ by putting $m = 0$ outside $(\mathbf{R}_+)^n \times \mathbf{R}^*$. For $r = (r_1, \ldots, r_{n+1}) \in (\mathbf{R}_+)^{n+1}$ we write

$$m^{(r)}(\mu, \lambda) = m(r_1\mu_1, \ldots, r_n\mu_n, r_{n+1}\lambda).$$

Fix η as in Section 6 and η_0 as in (6.1). For $\alpha \geqslant 0$ and $\beta \geqslant 0$ we define

$$\|m\|_{L^2_{\alpha,\beta,\mathrm{sloc}}} = \sup_{r \in (\mathbf{R}_+)^{n+1}} \|m^{(r)}\eta_0\|_{L^2_{\alpha,\beta}}$$

where the mixed Sobolev norm $\|\cdot\|_{L^2_{\alpha,\beta}}$ is defined by

$$\|g\|^2_{L^2_{\alpha,\beta}} = \int_{\mathbf{R}^{n+1}} \left|\widehat{g}(\xi) \cdot \prod_{j=1}^{n}\left(1 + |\xi_j|\right)^{\alpha} \cdot \left(1 + \sum_{j=1}^{n+1} |\xi_j|\right)^{\beta}\right|^2 d\xi.$$

By applying n times Lemma 2.5 in **[18]**, we have

$$\|m\|_{\ell^2(L^2)_{\alpha,\beta,\mathrm{sloc}}} \leqslant C\,\|m\|_{L^2_{\alpha,\beta,\mathrm{sloc}}}. \tag{7.1}$$

Theorem 7.1. *Suppose $\|m\|_{L^2_{\alpha,\beta,\mathrm{sloc}}} < +\infty$ for some $\alpha > 1$ and $\beta > 1/2$. Then for $1 < p < +\infty$ there exists a constant $C_{\alpha,\beta,p} > 0$, not depending on the function m, such that*

$$\left\|m\left(\Lambda^{-1}\mathcal{L}_1, \ldots, \Lambda^{-1}\mathcal{L}_n, -iT\right)f\right\|_p \leqslant C_{\alpha,\beta,p}\,\|m\|_{L^2_{\alpha,\beta,\mathrm{sloc}}}\|f\|_p$$

and

$$\left\|m(\mathcal{L}_1, \ldots, \mathcal{L}_n, -iT)f\right\|_p \leqslant C_{\alpha,\beta,p}\,\|m\|_{L^2_{\alpha,\beta,\mathrm{sloc}}}\|f\|_p$$

for all $f \in L^2(\mathbf{H}_n) \cap L^p(\mathbf{H}_n)$.

PROOF: The first inequality is a direct consequence of Theorem 6.3 and (7.1). Putting

$$(Sm)(\mu,\lambda) = m\Big(|\lambda|\mu,\lambda\Big)$$

we have that

$$m(\mathcal{L}_1,\dots,\mathcal{L}_n,-iT) = (Sm)\Big(\Lambda^{-1}\mathcal{L}_1,\dots,\Lambda^{-1}\mathcal{L}_n,-iT\Big).$$

Then, in order to prove the second inequality, it suffices to prove that

$$\|Sm\|_{L^2_{\alpha,\beta,\mathrm{sloc}}} \leqslant C_{\alpha,\beta}\,\|m\|_{L^2_{\alpha,\beta,\mathrm{sloc}}}. \tag{7.2}$$

The proof of (7.2) is an easy adaption of the last part of the proof of [**18**, Corollary 2.4]. ∎

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