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OPERATORS WHICH FACTOR THROUGH CONVEX BANACH LATTICES

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Introduction and notations. We investigate here classes of operators T between Banach spaces E and F, which have factorization of the form

$$\begin{tikzcd} E \arrow[r, "T"] \arrow[dr, "V"'] & F \arrow[r, "j_F"] & F'' \\ & L \arrow[ur, "U"'] & \end{tikzcd}$$

where L is a Banach lattice, V is a p-convex operator, U is a q-concave operator (definitions below) and j_F is the cannonical embedding of F in F''. We show that for fixed p, q this class forms a perfect normed ideal of operators $M_{p,q}$, generalizing the ideal $I_{p,q}$ of [**5**]. We prove (Proposition 5) that $M_{p,q}$ may be characterized by factorization through p-convex and q-concave Banach lattices. We use this fact together with a variant of the complex interpolation method introduced in [**1**], to show that an operator which belongs to $M_{p,q}$ may be factored through a Banach lattice with modulus of uniform convexity (uniform smoothness) of power type arbitrarily close to q (to p). This last result yields similar geometric properties in subspaces of spaces having G.L. − l.u.st.

This is a revised version of a previous work under the same title. After completing that work we received T. Figiel's paper [**2**] and learned that, using the Lions-Peetre's interpolation method he gets the main results (Proposition 4) of § 3 here.

We use here standard notations of Banach space theory. Banach spaces are considered over the field of real numbers (the results are true, with appropriate definitions, in the complex case as well).

If E is a Banach space, E' is its dual space, for $x \in E$, $x' \in E'$, we use alternatively the notations $x'(x)$, $\langle x, x' \rangle$, $\langle x', x \rangle$. We denote

$$B(E) = \{x \in E | \ \|x\| \leqq 1\} \quad S(E) = \{x \in E| \ \|x\| = 1\}.$$

An "Operator" between Banach spaces is a bounded linear operator, $L(E, F)$ is the space of all operators between E and F.

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A standard reference to ideals of operators is [**5**]; specifically we use the ideals π_p of p-absolutely summing operators, I_p of p-integral operators and Γ_p of L_p-factorizable operators. Let α be an ideal norm on tensor products $E \otimes F$ (considered as subspaces of $L(E', F)$). Then α is a $\otimes$-norm if for all $u \in E \otimes F, \alpha(u) = \inf \alpha(u, M, N)$ where the inf is taken over all finite dimensional subspaces $M \subset E$ and $N \subset F$ such that $u \in M \otimes N, \alpha(u, M, N)$ is the α-norm of u as an element of $M \otimes N$.

$[A^*, \alpha^*]$ is the adjoint ideal of the ideal of finite rank operators with the norm α (if α is a $\otimes$-norm then $[A^*(F, E'), \alpha^*] = (E \otimes_\alpha F)')$. As a standard reference to Banach lattices we use [**9**]; in particular, if L is a Banach lattice, $1 \leqq p, q < \infty$ and $T \in L(E, L)$ (resp. $T \in L(L, E)$) then T is p-convex (q-concave) if there exists $K > 0$ such that for all $x_1, \ldots, x_n \in E$,

$$\|(\sum|Tx_i|^p)^{1/p}\| \leqq K(\sum\|x_i\|^p)^{1/p}$$

$$(\text{for all } f_1, \ldots, f_n \in L,\ (\sum\|Tf_i\|^q)^{1/q} \leqq K\|(\sum|f_i|^q)^{1/q}\|)$$

we denote $\inf K = K^{(p)}(T) (= K_{(q)}(T))$. If the identity I of L is p-convex (q-concave) we say that L is a p-convex (q-concave) lattice and denote $K^{(p)}(L) = K^{(p)}(I)$ $(K_{(q)}(L) = K_{(q)}(I))$. We say that L has an *upper-p-estimate* $M^{(p)}(L)$ (a *lower q-estimate* $M_{(q)}(L)$) if the inequalities of p-convexity (q-concavity) are valid for disjoint elements in L, $M^{(p)}(L)$ $(M_{(q)}(L))$ is then, the infimum of the appropriate constants.

A basis $(e_i)_{i \in \mathbf{N}}$ of a Banach space E is called a *monotone unconditional basis* (monotone u.c. basis) if for all $\{\alpha_1, \ldots, \alpha_n) \in \mathbf{R}^n$,

$$\left\|\sum_{i=1}^n \alpha_i e_i\right\| = \left\|\sum_{i=1}^n |\alpha_i| e_i\right\|.$$

The concept of local unconditional structure in the sense of Gordon and Lewis (G.L. – l.u.st) is defined in [**3**]. It is well known that E has G.L. – l.u.st if and only if the cannonical embedding $j: E \to E''$ has a factorization $J = VU$ where $U \in L(E, L)$, $V \in L(L, E'')$ and L is a Banach lattice.

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1. The $\otimes$-norm $\eta_{p,q}$.

Definition 1. For $u \in E \otimes F$ we define

$$(1) \qquad \eta_{p,q}(u) = \inf (\theta_{p,q}(\{x_i \otimes y_i\}_{i=1}^n)).$$

The inf is taken over all representations of u of the form

$$u = \sum_{i=1}^n x_i \otimes y_i$$

and $\theta_{p,q}$ is defined by

$$\theta_{p,q}\left(\{x_i \otimes y_i\}_{i=1}^n\right) = \sup\left\{\sum_{i=1}^n \left(\sum_{k=1}^\infty |x_k'(x_i)|^p\right)^{1/p}\left(\sum_{l=1}^\infty |y_l'(y_i)|^{q'}\right)^{1/q'} \;\middle|\; ||(x_k')_{k=1}^\infty||_{l_p(E')} \leqq 1; ||(y_l')_{l=1}^\infty||_{l_{q'}(F')} \leqq 1; \frac{1}{q} + \frac{1}{q'} = 1\right\}.$$

We omit the proof of the next proposition since it is just a simple verification.

PROPOSITION 2. *$\eta_{p,q}$ is a $\otimes$-norm.*

PROPOSITION 3. *For $u \in E' \otimes F$*

$$\eta_{p,q}(u) = \inf K^{(p)}(\alpha)K_{(q)}(\beta).$$

The inf *is taken over all finite dimensional spaces U with a monotone* u.c. *basis, and factorizations of u (considered as an operator u: $E \to F$) of the form*

(2)
$$\begin{tikzcd} E \arrow[rr, "u"] \arrow[dr, "\alpha"'] & & F \\ & U \arrow[ur, "\beta"'] & \end{tikzcd}$$

Proof. Suppose u has a factorization of the form (2). Let $\{e_i, e_i'\}$ be a monotone u.c. basis in U. Define $x_i' = \alpha'(e_i')$ and $y_i = \beta(e_i)$. We get

$$u = \beta \circ \alpha = \sum_{i=1}^n x_i' \otimes y_i.$$

Moreover, by definition we have:

$$K^{(p)}(\alpha) = \sup\left\{\left\|\sum_{i=1}^n \left(\sum_k |x_i'(x_k)|^p\right)^{1/p} e_i\right\| \;\middle|\; ||(x_k)||_{l_p(E)} \leqq 1\right\}$$

$$K_{(q)}(\beta) = K^{(q')}(\beta') = \sup\left\{\left\|\sum_{i=1}^n \left(\sum_l |(y_l'(y_i)|^{q'}\right)^{1/q'} e_i'\right\| \;\middle|\; ||(y_l')||_{l_{q'}(E')} \leqq 1\right\}.$$

Therefore for an appropriate choice of (x_k) and (y_p') we have:

$$\begin{aligned}\theta_{p,q}(\{x_i' \otimes y_i\}) &\leqq \sum_{i=1}^n \left(\sum_k |x_i'(x_k)|^p\right)^{1/p}\left(\sum_l |y_l'(y_i)|^{q'}\right)^{1/q'} + \epsilon \\ &= \left[\sum_{i=1}^n \left(\sum_l |y_l'(y_i)|^{q'}\right)^{1/q'} e_i'\right]\left(\sum_{i=1}^n \left(\sum_k |x_i'(x_k)|^p\right)^{1/p} e_i\right) + \epsilon \\ &\leqq K^{(p)}(\alpha)K_{(q)}(\beta) + \epsilon.\end{aligned}$$

Hence $\{\eta_{p,q}(u) \leqq K^{(p)}(\alpha)K_{(q)}(\beta)$.

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To prove the other inequality, suppose $u = \sum_{i=1}^{n} x_i \otimes y_i$ is a representation which satisfies

$$\theta_{p,q}(\{x_i' \otimes y_i\}_{i=1}^{n}) \leqq \eta_{p,q}(u) + \epsilon.$$

We define the space U to be $\mathbf{R}^n$ with the norm:

$$(3) \qquad !a! = \sup \left\{ \sum_{i=1}^{n} \left| a_i \left(\sum_{l} |y_l'(y_i)|^{q'} \right)^{1/q'} \right| \,\Big|\, ||(y_l')||_{l_{q'}(F')} \leqq 1 \right\}$$

for $a = (a_i)_{i=1}^{n} \in \mathbf{R}^n$.

The unit vector in U has u.c. constant 1 ($!a!$ is determined by $|a|$ alone).

Define α: $E \rightarrow U$ by

$$\alpha = \sum_{i=1}^{n} x_i' \otimes e_i,$$

β: $U \rightarrow F$ by

$$\beta(a) = \sum_{i=1}^{n} a_i y_i.$$

Then clearly $u = \beta \circ \alpha$ and

$$K^{(p)}(\alpha) = \sup \left\{ \left\| \sum_{i=1}^{n} \left(\sum_{k} |x_i'(x_k)|^p \right)^{1/p} e_i \right\| \,\Big|\, ||(x_k)||_{l_p(E)} \leqq 1 \right\}$$
$$= \theta_{p,q}(\{x_1' \otimes y_i\}_{i=1}^{n}) \leqq \eta_{p,q}(u) + \epsilon;$$
$$K_{(q)}(\beta) = \sup \left\{ \left\| \sum_{i=1}^{n} \left(\sum_{l} |y_l'(y_i)|^{q'} \right)^{1/q'} e_i' \right\| \,\Big|\, ||(y_l')||_{l_{q'}(F')} \leqq 1 \right\}$$
$$= \sup \left\{ \sum_{i=1}^{n} |a_i| \left(\sum_{l} |y_l'(y_i)|^{q'} \right)^{1/q'} \,\Big|\, ||(y_l')||_{l_{q'}(F')} \leqq 1; !a! = 1 \right\}$$
$$= 1.$$

Therefore

$$K^{(p)}(\alpha) K_{(q)}(\beta) = \theta_{p,q}(\{x_i \otimes y_i\}_{i=1}^{n}) \leqq \eta_{p,q}(u) + \epsilon.$$

2. Operators factoring through a Banach lattice. We say that $T \in L(E, F)$ factors p, q through a Banach lattice ($T \in M_{p,q}(E, F)$) if $j_F T$ has the factorization

$$(1) \qquad \begin{tikzcd} E \arrow[r, "T"] \arrow[dr, "U"'] & F \arrow[r, "j_F"] & F'' \\ & L \arrow[ur, "V"'] & \end{tikzcd}$$

where j_F is the canonical embedding, L is a Banach lattice, V a q-concave

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operator and U a p-convex operator. We define

$$\mu_{p,q}(T) = \inf K^{(p)}(U)K_{(q)}(V),$$

the inf being taken over all factorizations of the form (1).

PROPOSITION 1. a) $[M_{p,q}, \mu_{p,q}]$ *is a perfect normed ideal of operators;* b) $[M_{p,q}, \mu_{p,q}] = \eta_{p,q}{}^{**}$.

Proof. It is clear that b) implies a). We prove b).
(i) $[M_{p,q}, \mu_{p,q}] \subset \eta_{p,q}{}^{**}$.

LEMMA 2. *Let G be a finite dimensional subspace of an order complete Banach lattice L. Given $\delta > 0$ there are $x_1, \ldots, x_n$ in L, $x_i \perp x_j$ for $i \neq j$, $x_i \geqq 0$ for all i, such that there is an operator*

$$S: G \rightarrow \operatorname{span} \{x_i\}_{i=1}^{n}$$

which satisfies for all $y_1, \ldots, y_m$ in G and all $1 \leqq p < \infty$:

$$(2) \qquad \left\| \left(\sum_{j=1}^{m} |y_j|^p \right)^{1/p} - \left(\sum_{j=1}^{m} |Sy_j|^p \right)^{1/p} \right\|_L \leqq \delta \left(\sum_{j=1}^{m} ||y_j||_L{}^p \right)^{1/p}.$$

Proof of Lemma 2. Let $0 \leqq a \in L$ be such that $B(G) \subset [-a, a]$. $\hat{I}(a)$ (the completion of span $[-a, a]$ with respect to the norm for which $[-a, a]$ is the unit ball) is isometric and order isomorphic to a $C(K)$ space, which is order complete since L is order complete. Therefore K is Stonian. The extension $j: C(K) \rightarrow L$ of the inclusion $I(a) \subset L$ is of norm $||a||$. An element $x \in I(a)$ will be considered alternatively as an element of L or of $C(K)$. The subspace G_1 spanned by G in $C(K)$ is isomorphic to G. Let $d > 0$ be such that for all $x \in G$,

$$||x||_{C(K)} \leqq d||x||_L.$$

Since K is Stonian we have, for given $\eta > 0$, clopen sets $A_1, \ldots, A_n \subset K$, disjoint, and an operator $U: G_1 \rightarrow \operatorname{span} \{\chi_{A_i}\}_{i=1}{}^n$ such that for all $x \in G$,

$$||x - Ux||_{C(K)} \leqq \eta ||x||_{C(K)}.$$

Then, for all $t \in K$ we have for $w_1, \ldots, w_m \in G_1$:

$$\left| \left(\sum_j |w_j(t)|^p \right)^{1/p} - \left(\sum_j |(Uw_j)(t)|^p \right)^{1/p} \right|$$
$$\leqq \left(\sum |w_j(t) - (Uw_j)(t)|^p \right)^{1/p} \leqq \left(\sum ||w_j - Uw_j||_{C(K)}^p \right)^{1/p}$$
$$\leqq \eta \left(\sum ||w_j||_{C(K)}^p \right)^{1/p}.$$

We now define $S: G \rightarrow L$ by $Sx = jUx$, and $x_i = j\chi_{A_i}$:

Then, for $y_1, \ldots, y_m \in G$ we have (since j is a homomorphism of lattices)

$$\left\| \left(\sum_j |y_j|^p \right)^{1/p} - \left(\sum_j |Sy_j|^p \right)^{1/p} \right\|_L \leqq \|a\| \left\| \left(\sum_j |y_j|^p \right)^{1/p} \right.$$
$$\left. - \left(\sum_j |Uy_j|^p \right)^{1/p} \right\|_{C(K)} \leqq \|a\| \eta \left(\sum_j \|y_j\|_{C(K)}^p \right)^{1/p}$$
$$\leqq \|a\| d\eta \left(\sum_j \|y_i\|_L^p \right)^{1/p}$$

so that a choice of $\eta < \delta/\|a\|d$ will give (2).

Using Lemma 2 and an argument which is dual to it, combined with a perturbation argument and local reflexivity, one can prove the following lemma, whose standard proof we omit.

LEMMA 3. *If at least one of E or F is finite dimensional and $T \in L(E, F)$ has a factorization of $j_F T$ of the form*

(3)
$$\begin{tikzcd} E \arrow[r, "T"] \arrow[dr, "A"'] & F \arrow[r, "j_F"] & F'' \\ & L \arrow[ur, "B"'] & \end{tikzcd}$$

with L a Banach lattice, A a p-convex operator and B a q-concave operator then for every $\epsilon > 0$, there is a finite dimensional U with a monotone u.c. *basis $x_1, \ldots, x_m$ and a factorization*

(4)
$$\begin{tikzcd} E \arrow[rr, "T"] \arrow[dr, "\alpha"'] & & F \\ & U \arrow[ur, "\beta"'] & \end{tikzcd}$$

with $K^{(p)}(\alpha)K_{(q)}(\beta) \leqq (1 + \epsilon)K^{(p)}(A)K_{(q)}(B)$.

We are ready now to prove (i). If $j_F T$ factors in the form (1), then by Lemma 3, for every finite dimensional $E_1 \subset E$ and finite dimensional $F^1 \subset F'$ the operator $j'Ti$ (i: $E_1 \hookrightarrow E$, j: $F^1 \hookrightarrow F'$) has a factorization

$$\begin{tikzcd} E_1 \arrow[rr, "j'Ti"] \arrow[dr, "\alpha"'] & & F^{1\prime} \\ & U \arrow[ur, "\beta"'] & \end{tikzcd}$$

with U finite dimensional with a monotone u.c. basis and such that

$$K^{(p)}(\alpha)K_{(q)}(\beta) \leqq (1 + \epsilon)K^{(p)}(U)K_{(q)}(V).$$

It follows that $T \in \eta_{p,q}^{**}$ and $\eta_{p,q}^{**}(T) \leqq \mu_{p,q}(T)$.

ii) $\eta_{p,q}^{**} \subset [M_{p,q}, \mu_{p,q}]$. This is proved by standard ultra-product methods.

From Proposition 1 it follows that the adjoint ideal $[M_{p,q}^*, \mu_{p,q}^*]$ is the adjoint ideal of $\eta_{p,q}$. Let $T \in L(E, F)$. Denote by K_1 the unit ball of $l_{q'}(E')$ with the relative w^* topology in it with respect to $l_q(E)$. K_2 is the unit ball of $l_p(F'')$ with the analogous topology. The following result is proved by the same method as that of [**4**].

PROPOSITION 4. $\mu_{p,q}^*(T) = \inf b$. *The* inf *is taken over all* $b > 0$ *such that there is a Radon probability measure* μ *on* $K_1 \times K_2$ *such that for all* $x \in E$ *and* $y' \in F'$ *holds:*

$$|\langle Tx, y'\rangle| \leqq b \int_{K_1 \times K_2} ||(x_k'(x))_{k=1}^{\infty}||_{l_{q'}} ||(y_l''(y'))_{l=1}^{\infty}||_{l_p} d\mu((x_k')(y_l'')).$$

We now refer to the following concepts:

An operator $h: L \rightarrow M$ between two Banach lattices is called a *homomorphism* if it is positive and $h(x^+) = (h(x))^+$, $h(x^-) = (h(x))^-$ for every $x \in L$. ($x = x^+ - x^-$, $x^\pm \geqq 0$, $x^+ \perp x^-$, is the canonical representation of $x \in L$.) We call h a *strong homomorphism* if $\overline{h(L)}$ is an ideal of M. We call h a *very strong homomorphism* if $h(L)$ is an ideal of M (not necessarily closed).

PROPOSITION 5. *Let* $T \in L(E, F)$. $T \in M_{p,q}(E, F)$ *if and only if* $j_F T$ *has a factorization of the form:*

$$\begin{array}{c} E \xrightarrow{T} F \xrightarrow{j_F} F'' \\ {\scriptstyle U}\searrow \qquad \nearrow{\scriptstyle V} \\ L \xrightarrow{Q} M \end{array} \tag{5}$$

where L *is a* p-*convex Banach lattice*, $K^{(p)}(L) = 1$, M *is a* q-*concave Banach lattice*, $K_{(q)}(M) = 1$ *and* Q *is a very strong lattice homomorphism. Also* $\|U\|, \|V\| \leqq 1$. *Moreover*,

$$\mu_{p,q}(T) = \inf\{\|Q\|; Q \textit{ as in } (5)\}.$$

The proof of Proposition 5 will be done in a number of steps.

Let L be a Banach lattice and $B: L \rightarrow F$ a q-concave operator we define on L:

$$|||x||| = \sup\left\{\left(\sum_{i=1}^{m} ||Bx_i||^q\right)^{1/q} \middle| \left(\sum |x_i|^q\right)^{1/q} \leqq |x|\right\}.$$

LEMMA 6. $|||\cdot|||$ *is a lattice semi-norm on* L *and is continuous with respect to the norm in* L. (*In fact* $|||\cdot||| \leqq K_{(q)}(B)\|\cdot\|$.)

Proof. a) $|||\cdot|||$ is finite and continuous. Let $x \in L$ and

$$\left(\sum_{i=1}^{m} |x_i|^q\right)^{1/q} \leqq |x|.$$

Then

$$\left(\sum_{i=1}^{m} ||Bx_i||^q\right)^{1/q} \leqq K_{(q)}(B)\left\|\left(\sum_{i=1}^{m} |x_i|^q\right)^{1/q}\right\| \leqq K_{(q)}(B)||x||.$$

Therefore for all $x \in L$,

$$|||x||| \leqq K_{(q)}(B)||x||.$$

b) Positive homogeneity of $|||\cdot|||$ is clear.

c) It is clear from the definition that

$$|||x||| = |||\,|x|\,||| \text{ and } |x| \leqq |y| \Rightarrow |||x||| \leqq |||y|||.$$

d) The triangle inequality. Let $x, y \in L$ and $z = |x| + |y|$. If $I(z) =$ span $[-z, z]$ then $\hat{I}(z) = C(K)$ for some compact K. We have the following diagram:

$$\begin{tikzcd} I(z) \arrow[r, "i"] & C(K) \arrow[r, "j"] \arrow[d, hook] & L \arrow[r, "B"] & F \\ & B(K) \arrow[r, "j''|B(K)"] & L'' \arrow[r, "B''"] & F'' \end{tikzcd} \tag{6}$$

where i is the inclusion, j is the extension of the inclusion $I(z) \subset L$, $B(K)$ is the space of bounded Borel functions on K, $C(K) \subset B(K)$ in a natural way and $B(K)$ is considered as a subspace of $C(K)''$ by the identification of $h \in B(K)$ with $h \in C(K)''$:

$$h(\mu) = \int_K h d\mu \quad (\mu \in C(K)').$$

Let $F = (f_1, \ldots, f_m) \in B(K)^m$ with $(\sum|f_i|^q)^{1/q} = 1$. We put

$$||g||_F = \left(\sum_{i=1}^{m} ||B''j''(f_i g)||^q\right)^{1/q}.$$

It is easy to check that

$$\sup_F ||g||_F = \sup\{(\sum||B''j''(g_i)||^q)^{1/q}; \quad (\sum|g_i|^q)^{1/q} \leqq |g|\}. \tag{7}$$

The term on the right-hand side of (7) will be denoted by $P(g)$. For each F, $||\cdot||_F$ is clearly a semi-norm on $B(K)$, therefore $P(g)$ is a semi-norm on $B(K)$. It is possible now to verify that for $w \in I(z)$, $|||w||| = P(iw)$; therefore $|||\cdot|||$ is a semi-norm on $I(z)$, and in particular,

$$|||x + y||| \leqq |||x||| + |||y|||.$$

This proves Lemma 6.

Let $\mathcal{N}$ be the closed subspace of L:

$$\mathcal{N} = \{x \in L \mid |||x||| = 0\}.$$

Let M be the completion of $L/\mathcal{N}$ with respect to the norm $\|Cx\| = |||x|||$ where C is the quotient map. Then M is a Banach lattice under the natural order. Since for all $x \in L$, $\|Bx\| \leqq |||x|||$, B induces in a natural way an operator B_1: $M \to F$, with $\|B_1\| \leqq 1$.

We have also that C is a strong homomorphism and from (a) in the proof of Lemma 6, we conclude that $\|C\| \leqq K_{(q)}(B)$. It is easily verified that M is a q-concave Banach lattice and $K_{(q)}(M) = 1$.

Proof of Proposition 5. Suppose $j_F T$ has a factorization of the form (1). The operator U': $L' \to E'$ is p'-concave. By the preceding lemmas we have a factorization

$$\begin{tikzcd} L' \arrow[rr, "U'"] \arrow[dr, "C_1"'] & & E' \\ & M_1 \arrow[ur, "U^1"'] & \end{tikzcd}$$

where M_1 is p'-concave, $K_{(p')}(M_1) = 1$, C_1 is a strong homomorphism, $\|C_1\| \leqq K^{(p)}(U)$ and $\|U^1\| \leqq 1$.

By passing to the dual diagram and repeating the argument, we get the factorization (5) (we can always pass from Q to Q'', thus, by [**7**] we may assume Q is a very strong homomorphism).

The ideal $[I_{p,q}, i_{p,q}]$ was defined in [**5**].

COROLLARY 7. *For* $p > q$, $[M_{p,q}, \mu_{p,q}] = [I_{p,q}, i_{p,q}]$. (*We remark that for* $p = q$, $[M_{p,q}, \mu_{p,q}] = [\Gamma_p, \gamma_p]$, *this was proved by Krivine* [**6**].)

Proof. Due to Proposition 5, it is enough to consider a lattice homomorphism Q: $L \to M$ where L is p-convex and M is q-concave and to show that Q has a factorization

$$\begin{tikzcd} L \arrow[r, "Q"] \arrow[d, "A"'] & M \\ L_p(\mu) \arrow[r, "I_g"] & L_q(\mu) \arrow[u, "B"'] \end{tikzcd}$$

where (Ω, μ) is some measure space, $\|A\|$, $\|B\| \leqq 1$ and I_g is the operator of multiplication by a function $0 \leqq g \in L_r(\mu)$ $(q^{-1} = p^{-1} + r^{-1})$ with $\|g\|_r = \|Q\|$. M is q-concave. $K^{(p)}(Q) = \|Q\|$ (L is p-convex and Q a homomorphism), and since $p > q$ we have also $K^{(q)}(Q) \leqq \|Q\|$. Then, by [**7**] there is a $L_q(\mu)$ and a factorization

$$\begin{tikzcd} L \arrow[rr, dash, "Q"] \arrow[drr, "C"'] & & M \\ & & L_q(\mu) \arrow[u, "B"'] \end{tikzcd}$$

with $\|B\| \leqq 1$, $\|C\| \leqq \|Q\|$, C is a positive operator. Hence by [**7**], it is a p-convex operator (L being p-convex). By [**13**, Theorem 8] C has a factorization

$$\begin{tikzcd} L \arrow[rr, "C"] \arrow[d, "A"'] & & L_q(\mu) \\ L_p(\mu) \arrow[urr, "I_g"'] & & \end{tikzcd}$$

for an appropriate $g \in L_p(\mu)$.

COROLLARY 8. a) *Let E be a space of cotype q $(2 \leqq q < \infty)$ and L a Banach lattice which is p-convex for some $p > q$. Then for all r, s with $q < r < s \leqq p$ we have $T \in I_{s,r}(L, E)$ for all $T \in L(L, E)$.*

b) *Let E' be of cotype p' $(1 < p \leqq 2)$ L a q-concave Banach lattice $(q < p)$. Then for all r, s; $q \leqq r < s < p$, we have $T \in I_{s,r}(E, L)$ for all $T \in L(E, L)$.*

If $q = 2$ in *a*) or $p = 2$ in *b*), we may put $r = 2$ in *a*), $s = 2$ in *b*).

Proof. b) follows from a) by duality. To prove a), from [**12**] it follows that for all $r > q$ and $T \in L(L, E)$, T is r-concave. Since L is s-convex for all $s \leqq p$, we have $T \in M_{s,r}(L, E)$ and Corollary 8 completes the proof.

A consequence of Corollary 7 and the results of [**14**] and [**8**] is

COROLLARY 9. *Let L be a q-concave Banach lattice with $q < 2$ and E a subspace of L. Then one of the following mutually exclusive possibilities holds:*

a) *There exists p, $1 \leqq p \leqq q$ such that E contains uniformly l_p^n.*

b) *There exists $r, q < r \leqq 2$ and there is a probability measure space (Ω, Σ, μ) such that E is isomorphic to a subspace of $L_r(\mu)$, and on E all the $L_s(\mu)$ norms with $0 < s \leqq r$ are equivalent.*

PROPOSITION 10. *Let L be a p-convex Banach lattice, and $T \in L(E, L)$ with $T' \in \Pi_p(L', E')$. Then T is lattice bounded in L''; that is, there exists f, $0 \leqq f \in L''$ such that*

$$jT(B(E)) \subset [-f, f]$$

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(j: $L \to L''$ *the cannonical injection*). *Moreover,*

$$\|f\| \leqq K^{(p)}(L)\pi_p(T').$$

Proof. We use the following construction. Let M be a Banach lattice with M' p-convex ($1 \leqq p < \infty$). (We assume, for simplicity, $K^{(p)}(M') = 1$.) Let (Ω, Σ, μ) be a measure space and let $L_*^p(\Omega, \Sigma, \mu, M')$ (in short, $L_*^p(\mu, M')$) be the Banach space of w^*-scalarly measurable functions ϕ from Ω into M', such that

$$\Phi = \sup\{|\langle\phi, x\rangle|, \|x\|_L \leqq 1\}$$

exists and belongs to $L_p(\mu)$. The norm in $L_*^p(\mu, M')$ is $\|\phi\| = \|\Phi\|_{L_p(\mu)}$ (see [**17**]). Let $\phi \in L_*^p(\mu, M')$ and let $A = (A_j)_{j=1}^n$ be a finite collection of disjoint measurable subsets of Ω with $\mu(A_j) < \infty$ ($j = 1, \ldots, n$). We define $f_A \in M'$ by

$$(8) \quad f_A = \left(\sum_{j=1}^n \mu(A_j)^{1-p}\left|\int_{A_j} \phi(\omega)d\mu(\omega)\right|^p\right)^{1/p}.$$

$$\left(\int_{A_j} \phi(\omega)d\mu \text{ is the element of } M' \text{ defined by } \left(\int_{A_j} \phi(\omega)d\mu(x) = \int_{A_j} \langle\phi(\omega), x\rangle\, d\mu.\right)\right).$$

Since $K^{(p)}(M') = 1$, we have

$$\begin{aligned}||f_A|| &\leqq \left(\sum_{j=1}^n \mu(A_j)^{1-p}\left\|\int_{A_j} \phi(\omega)d\mu\right\|^p\right)^{1/p} \\ &\leqq \left[\sum_{j=1}^n \mu(A_j)^{1-p}\left(\int_{A_j} \Phi(\omega)d\mu\right)^p\right]^{1/p} \leqq \left(\sum_{j=1}^n \int_{A_j} \Phi(\omega)^p d\mu\right)^{1/p} \\ &\leqq ||\phi||_{L_*^p(\mu, M')}.\end{aligned}$$

There is therefore a subnet (A^α) of the net of such finite collections with the order induced by refinement, such that

$$f_{A^\alpha} \xrightarrow{w^*} f \in M''.$$

Of course, $f \geqq 0$ and

$$||f|| \leqq ||\phi||_{L_*^p(\mu, M')}.$$

Let $0 \leqq x \in M$ and $A = (A_j)_{j=1}{}^n$ the same collection of sets as before. Then

$$\begin{aligned}&\left(\sum_{j=1}^n \mu(A_j)^{1-p}\left|\int_A \langle\phi(\omega), x\rangle\, d\mu\right|^p\right)^{1/p} \\ &\qquad \leqq \left[\sum_{j=1}^n \mu(A_j)^{1-p}\left(\left|\int_{A_j} \phi(\omega)d\mu\right|(x)\right)^p\right]^{1/p}.\end{aligned}$$

The last expression is equal, for some n-tuple $(\alpha_j)_{n=1}{}^n$ with $\sum|\alpha_j|^{p'} = 1$, to

$$\sum_{=1}^{n} \alpha_j\mu(A_j)^{(1-p)/p}\left|\int_{A_j} \phi(\omega)d\mu\right|(x)$$

$$= \left[\sum_{j=1}^{n} \alpha_j\mu(A_j)^{(1-p)/p}\left|\int_{A_j} \phi(\omega)d\mu\right|\right](x)$$

$$\leqq \left[\sum_{j=1}^{n} \mu(A_j)^{1-p}\left|\int_{A_j} \phi(\omega)d\mu\right|^p\right]^{1/p}(x) = f_A(x).$$

Now, if for all α, $A^\alpha = (A_j{}^\alpha)_{j=1}{}^{n(\alpha)}$ then

$$(9) \quad \left(\int_\Omega |\langle\phi(\omega), x\rangle|^p d\mu(\omega)\right)^{1/p} =$$

$$\lim_\alpha \left(\sum_{j=1}^{n(\alpha)} \mu(A_j{}^\alpha)^{1-p}\left|\int_{A_j} \langle\phi(\omega), x\rangle d\mu(\omega)\right|^p\right)^{1/p} \leqq \lim_\alpha f_{A^\alpha}(x) = f(x).$$

We now use (9) to prove the proposition; we assume $K^{(p)}(L) = 1$. Since $T' \in \pi_p(E', L')$, there is a positive Radon measure μ on $\Omega = S(L'')$ such that

$$(10) \quad \mu(\Omega)^{1/p} = \pi_p(T')$$

and for all $y' \in L'$

$$(11) \quad ||T'y'|| \leqq \left(\int_\Omega |y''(y')|^p d\mu(y'')\right)^{1/p}.$$

We define $f \in L''$ as in the above construction, with $\phi \in L_*{}^p(L'')$, $\phi(y'') = y''$. We have

$$\|f\| \leqq \mu(\Omega)^{1/p} = \pi_p(T').$$

We need to show that for all $x \in E$ with $\|x\| \leqq 1$, $j(|Tx|) \leqq f$. That is, for all y' with $0 \leqq y' \in L'$, $y'(|Tx|) \leqq f(y')$. If $0 \leqq \omega' \in L'$, then for $\|x\| \leqq 1$, we have by (g)

$$(12) \quad |\omega'(Tx)| \leqq ||T'\omega'|| \leqq \left(\int_\Omega |y''(\omega')|^p d\mu(y'')\right)^{1/p} \leqq f(\omega').$$

Now if $0 \leqq y' \in L'$ and $\|x\| \leqq 1$, (12) yields:

$$y'(|Tx|) = \sup\{\omega'(Tx) + z'(Tx)|\omega', z' \geqq 0, \omega' + z' = y'\}$$
$$\leqq \sup\{|\omega'(Tx)| + |z'(Tx)| \mid \omega', z' \geqq 0, \omega' + z' = y'\}$$
$$\leqq \sup\{f(\omega') + f(z')|\omega', z' \geqq 0, \omega' + z' = y'\} = f(y').$$

PROPOSITION 11. *Let F be a Banach space and L any Banach lattice. Let $A \in L(F, L)$ be a p-convex operator and $T \in L(E, F)$ with $T' \in \pi_p(F', E')$.*

Then there exists f, $0 \leqq f \in L''$ *with*

$$\|f\| \leqq K^{(p)}(A)\pi_p(T')$$

such that

$$AT(B(E)) \subset [-f, f].$$

Proof. By the proof of Proposition 5 we have the following factorization:

$$\begin{tikzcd} E \arrow[r, "T"] & F \arrow[r, "A"] \arrow[dr, "A_1"'] & L \arrow[r, "j"] & L'' \\ & & M \arrow[ur, "Q"'] & \end{tikzcd}$$

where $\|A_1\| \leqq 1$, Q is a lattice homomorphism, $\|Q\| \leqq K^{(p)}(A)$ and M is a p-convex Banach lattice. By Proposition 10, there exists g, $0 \leqq g \in M$, $\|g\| \leqq \pi_p((A_1T)') \leqq \pi_p(T')$ and

$$A_1(T(B(E))) \subset [-g, g].$$

Taking $f = Q(g)$ completes the proof.

COROLLARY 12. *If* E' *is of cotype* q' $(1 < q \leqq 2)$, *and* $T \in L(E, L)$ (L *any Banach lattice*), *then for every* $S \in L(G, E)$ *with* $S' \in \pi_r(G, E)$ $(r < q)$, TS *is order bounded in* L''.

THEOREM 13. *Let* $T \in M_{p,q}(E, F)$. *Then for any Banach space* G *and operator* $S \in \pi_{q'}(F, G)$ *we have*

$$T'S' \in I_{p'}(G', E') \textit{ and } i_{p'}(T'S') \leqq \mu_{p,q}(T)\pi_{q'}(S).$$

Proof. By Proposition 5, T has a factorization of the form (5) with

$$\|Q\| \leqq \mu_{p,q}(T)(1 + \epsilon).$$

Let $S \in \pi_{q'}(F, G)$; the diagram dual to (5) is

$$\begin{tikzcd} E' & F' \arrow[l, "T'"'] \arrow[d, "V^1"] & G' \arrow[l, "S'"'] \\ L' \arrow[u, "U'"] & M' \arrow[l, "Q'"'] & \end{tikzcd}$$

where $V^1 = V'|_{F'}$.

By Proposition 10 there is $0 \leqq f \in M'$, $\|f\| \leqq \pi_{q'}(S)$ such that $V^1S'(B(G')) \subset [-f, f]$.

Let $C(K)$ be the completion of $I(f)$ with respect to the norm for which $[-f, f]$ is the unit ball, and let $i: C(K) \to M'''$ be the extension of the inclusion.

$$\|i\| \leqq \|f\| \leqq \pi_{q'}(S).$$

Let $j: M' \to M'''$ and $k: M''' \to M'$ be the cannonical injection and projection.

We have the diagram

$$\begin{tikzcd}
E' & F' \arrow[l, "T'"'] & G' \arrow[l, "S'"'] \arrow[dl, "V^2"] \\
L' \arrow[u, "U'"] & C(K) \arrow[l, "Q^1"'] &
\end{tikzcd}$$

where V^2 is the operator that jV^1S' induces into $C(K)$ ($\|V^2\| \leqq 1$). Q^1 is $Q'ki$ and of course Q^1 is a homomorphism from $C(K)$ into L'. From p'-concavity of L' we have for $\phi_1, \ldots, \phi_m \in C(K)$:

$$\begin{aligned}(\textstyle\sum \|Q^1\phi_k\|^{p'})^{1/p'} &\leqq \|(\textstyle\sum |Q^1\phi_k|^{p'})^{1/p'}\|_{L'} \\ &\leqq \|Q^1\| \, \|(\textstyle\sum |\phi_k|^{p'})^{1/p'}\|_{C(K)} = \|Q^1\| \epsilon_{p'}((\phi_k)_{k=1}{}^n).\end{aligned}$$

Hence Q^1 is p'-summing and

$$\pi_{p'}(Q^1) = \|Q^1\| \leqq \mu_{p,q}(T)(1 + \epsilon)\|i\|.$$

Since Q^1 is defined on a $C(K)$ space we have

$$i_{p'}(Q^1) \leqq \mu_{p,q}(T)(1 + \epsilon)\|i\| \leqq (1 + \epsilon)\mu_{p,q}(T)\pi_{q'}(S),$$

hence

$$\begin{aligned} i_{p'}(T'S') = i_{p'}(U'Q^1V^2) &\leqq \|U'\| i_{p'}(Q^1)\|V^2\| \\ &\leqq (1 + \epsilon)\mu_{p,q}(T)\pi_{q'}(S).\end{aligned}$$

3. Relations with uniform convexity and uniform smoothness. For further information concerning vector lattices we refer to [**18**].

The following construction was introduced in [**1**] and further investigated in [**10, 11**].

Let X_0 and X_1 be Banach lattices which are embedded lattice-isomorphically as ideals of the same complete vector lattice W. It is easy to check that in this case $X_0 + X_1$ and $X_0 \cap X_1$ (definitions of [**1**]) are Banach lattices, $X_i (i = 0, 1)$ are ideals of $X_0 + X_1$, $X_0 \cap X_1$ is an ideal of the other three, and all four are ideals of W.

Let $0 < \theta < 1$. We define:

(1) $X_\theta = X_0{}^{1-\theta}X_1{}^\theta = \{x \in W | \, |x| \leqq \lambda |u|^{1-\theta}|v|^\theta$ for some $u \in X_0$, $v \in X_1$, $\lambda \geqq 0$; with $\|u\|_{X_0} \leqq 1$, $\|v\|_{X_1} \leqq 1\}$.

X_θ is a Banach lattice equipped with the norm

$$\|x\|_{X_\theta} = \inf \{\lambda | \lambda \text{ as in } (1)\}.$$

It is easy to check that the following set-theoretic inclusions hold:

$$X_0 \cap X_1 \subset \overset{i}{\hookrightarrow} X_\theta \subset \overset{j}{\hookrightarrow} X_0 + X_1$$

and that i and j are very strong lattice homomorphism of norms not greater than 1.

The following result is from [**11**].

THEOREM (Lozanovskii). *Suppose X_0, X_1 and W are as above. Then there exists an order complete vector lattice V and lattice-isomorphic embeddings of X_0' and X_1' as ideals of V such that $(X_\theta)'$ is isometric and lattice isomorphic to $(X_0')^{1-\theta}(X_1^1)^\theta$.*

THEOREM 1. *Under the assumptions of the preceding theorem,*

a) *Suppose that $X_i (i = 0, 1)$ has an upper-p_i estimate (u-p_i-est.) $M^{(p_i)}(X_i)$ and lower-q_i-estimate (l-q_i-est.) $M_{(q_i)}(X_i)$. Let p_θ, q_θ satisfy*

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}\,;\frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1} \quad (0 < \theta < 1).$$

Then X_θ has u-p_θ-est. and l-q_θ-est. and

$$M^{(p_\theta)}(X_\theta) \leqq [M^{(p_0)}(X_0)]^{1-\theta}[M^{(p_1)}(X_1)]^\theta,$$
$$M_{(q_\theta)}(X_\theta) \leqq [M_{(q_0)}(X_0)]^{1-\theta}[M_{(q_1)}(X_1)]^\theta.$$

b) *Suppose X_i is p_i-convex and q_i-concave. Then X_θ is p_θ-convex and q_θ-concave and*

$$K^{(p_\theta)}(X_\theta) \leqq [K^{(p_0)}(X_0)]^{1-\theta}[K^{(p_1)}(X_1)]^\theta,$$
$$K_{(q_\theta)}(X_\theta) \leqq [K_{(q_0)}(X_0)]^{1-\theta}[K_{(q_1)}(X_1)]^\theta.$$

Proof. We prove only a); the proof of b) is similar. By duality and using Lozanovskii's theorem it is enough to prove the first assertion and the first inequality. Let $x_1, \ldots, x_n \in X_\theta$. For each i let $0 \leqq u_i \in X_0$, $0 \leqq v_i \in X_1$ with $\|u_i\|_{X_0} \leqq 1$, $\|v_i\|_{X_1} \leqq 1$ and $\lambda_i \geqq 0$ be such that for all $1 \leqq i \leqq n$

$$|x_i| \leqq \lambda_i u_i^{1-\theta} v_i^\theta \quad \text{and} \quad \lambda_i \leqq \|x_i\|_{X_\theta}(1 + \epsilon).$$

Then

$$\begin{aligned}\bigvee_{i=1}^n |x_i| &\leqq \bigvee_{i=1}^n \lambda_i^{(p_\theta(1-\theta))/p_0} u_i^{1-\theta} \lambda_i^{(p_\theta\theta)/p_1} v_i^\theta \\ &\leqq \left(\bigvee_{i=1}^n \lambda_i^{(p_\theta(1-\theta))/p_0} u_i^{1-\theta}\right)\left(\bigvee_{i=1}^n \lambda_i^{(p_\theta\theta)/p_1} v_i^\theta\right) \\ &= \left(\bigvee_{i=1}^n \lambda_i^{(p_\theta)/p_0} u_i\right)^{1-\theta}\left(\bigvee_{i=1}^n \lambda_i^{(p_\theta)/p_1} v_i\right)^\theta.\end{aligned}$$

Hence

$$\Big\| \bigvee_{i=1}^{n} |x_i| \Big\|_{X_\theta} \leqq \Big\| \bigvee_{i=1}^{n} \lambda_i^{(p_\theta)/p_0} u_i \Big\|^{1-\theta} \Big\| \bigvee_{i=1}^{n} \lambda_i^{(p_\theta)/p_1} v_i \Big\|^{\theta}$$

$$\leqq [M^{(p_0)}(X_0)]^{1-\theta} \left(\sum_{i=1}^{n} \lambda_i^{p_\theta} \right)^{(1-\theta)/p_0} [M^{(p_1)}(X_1)]^{\theta} \cdot \left(\sum_{i=1}^{n} \lambda_i^{p_\theta} \right)^{(\theta)/p_1}$$

$$= [M^{(p_0)}(X_0)]^{1-\theta} [M^{(p_1)}(X_1)]^{\theta} \left(\sum \lambda_i^{p_\theta} \right)^{1/p_\theta}$$

$$\leqq (1 + \epsilon) [M^{(p_0)}(X_0)]^{1-\theta} [M^{(p_1)}(X_1)]^{\theta} \left(\sum \|x_i\|_{X_\theta}^{p_\theta} \right)^{1/p_\theta}.$$

Since ϵ is arbitrary, the last inequality proves the assertion.

PROPOSITION 2. *Let L, M be Banach lattices, L with a u-p-est., M with an l-q-est.* ($1 < p, q < \infty$) *and M-order complete. Let Q: $L \rightarrow M$ be a very strong homomorphism of Banach lattices. Then for every* $0 < \theta < 1$, *Q has a factorization*

$$\begin{tikzcd} L \arrow[rr, "Q"] \arrow[dr, "Q_1"'] & & M \\ & X \arrow[ur, "Q_2"'] & \end{tikzcd}$$

where Q_1, Q_2 are very strong homomorphisms, $\|Q_1\| \, \|Q_2\| = \|Q\|$, X has an u-p_θ-est and l-q_θ-est

$$\left(\frac{1}{p_\theta} = \frac{1-\theta}{p} + \theta, \frac{1}{q_0} = \frac{\theta}{q} \right) \quad \textit{and}$$

$$M^{(p_\theta)}(X) \leqq [M^{(p)}(L)]^{1-\theta}; \quad M_{(q_\theta)}(X) \leqq [M_{(q)}(M)]^{\theta}.$$

A similar assertion is true if we replace upper and lower estimates by convexity and concavity, in the assumptions and in the result.

Proof. We may suppose Q is one to one; this is because the quotient Banach lattice $L/\text{Ker } Q$ has

$$M^{(p)}(L/\text{Ker } Q) \leqq M^{(p)}(L).$$

We may therefore consider L as an ideal of M on which the L-norm is another norm. We may, then, apply Theorem 1 (with $X_0 = L$, $X_1 = M$, $W = M$, $p_0 = p$, $p_1 = 1$, $q_0 = \infty$, $q_1 = q$). We have then the factorization

$$\begin{tikzcd} L \arrow[rr, "Q"] \arrow[dr, "Q_1"'] & & M \\ & X_\theta \arrow[ur, "Q_2"'] & \end{tikzcd}$$

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where $X_\theta = L^{1-\theta}M^\theta$ and Q_1, Q_2 are the set-theoretic inclusion operators. It is easily checked that $\|Q_1\| \leqq \|Q\|^{1-\theta}$, $\|Q_2\| \leqq \|Q\|^\theta$; the estimations of the upper and lower estimates of X_θ are those of Theorem 1.

COROLLARY 3. *Under the assumptions of Proposition 2 with $1 < p \leqq 2 \leqq q < \infty$, for all $r < p$, Q factors through a uniformly smooth Banach lattice with modulus of smoothness of power type r. Also for all $s > q$, Q factors through a uniformly convex Banach lattice with modulus of convexity of power type s.*

Proof. It is a result of [**9**] that a Banach lattice L with an u-p-est. and l-q-est. $(1 < p \leqq 2 \leqq q < \infty)$ can be given two lattice norms, $\| \, \|_1$ and $\| \, \|_2$, both equivalent to the original norm, such that $(L, \| \, \|_1)$ has modulus of smoothness of power type p and $(L, \| \, \|_2)$ has modulus of convexity of power type q. This, together with Proposition 2, proves the assertion.

PROPOSITION 4. *Let E be a subspace of a Banach space F such that the following hold:*

i) *F has* G.L.-l.u.st *and cotype $q < \infty$.*

ii) *E' has cotype $p' < \infty$ $(p^{-1} + (p')^{-1} = 1)$.*

Then

iii) *For all $s > q$, E can be equivalently renormed to be uniformly convex with modulus of convexity of power type s.*

iv) *For all $r < p$, E can be equivalently renormed to be uniformly smooth with modulus of smoothness of power type r.*

v) $1/p(E) + 1/q(E') = 1$ *where* $p(E) = \sup \{p | E$ *is of type* $p\}$, $q(E') = \inf \{q | E'$ *is of cotype* $q\}$.

Proof. Under the assumptions i) and ii) we have the following factorization

$$\begin{tikzcd} E \arrow[r, "i"] \arrow[dr, "U"'] & F \arrow[r, "j_1"] & F'' \\ & L \arrow[ur, "V"'] & \end{tikzcd}$$

where i is the inclusion map, L is a Banach lattice and U, V are operators which are r-convex and s-concave respectively, for all $r < p$ and $s > q$. Therefore for all $r < p$ and $s > q$, we have $M_{r,s}(\mathrm{i}) < \infty$. iii) and iv) are therefore consequences of the preceding propositions and Proposition 2.5 (note that in the proof of Proposition 2.5, M, being bi-dual, is order complete). Note that E is isomorphic to a subspace of Banach lattices with the desired moduli of convexity and smoothness. v) is a direct consequence of iv).

Remark 5. Recently Pisier used in [**15**] the same method of interpolation used here. A further use of interpolation leads to a special case of one of his main results (Theorem 3.4 and Corollary 3.5):

PROPOSITION. *Let F be a Banach space with* G.L.-l.u.st *and cotype $q < \infty$. Let X be a subspace of F such that X' has cotype $p' < \infty$. Then for every $\alpha > \frac{1}{2} - 1/(p' + q)$ $(1/p + 1/p' = 1)$ holds:*

(P_α) There exists $C_\alpha > 0$ such that for every subspace Y of X and operator $u \in L(Y, Z)$ of rank $n < \infty$, there is an extension $\tilde{u} \in L(X, Z)$ of u with

$$\gamma_2(\tilde{u}) \leqq C_\alpha n^\alpha \|u\|.$$

Proof. By the preceding results, for every $\tilde{q}, \tilde{p}, \theta$ with $q < \tilde{q} < \infty$, $1 < \tilde{p} < p$, $0 < \theta < 1$ there is a Banach lattice L which is s-convex and r-concave with

$$\frac{1}{s} = \frac{1-\theta}{\tilde{p}} + \theta, \frac{1}{r} = \frac{\theta}{\tilde{q}}$$

such that X is isomorphic to a subspace of L. Now, the result of [**15**] shows that X satisfies (P_α) for

$$\alpha = \max\left(\frac{1}{s} - \frac{1}{2}, \frac{1}{2} - \frac{1}{r}\right) = \max\left(\frac{1 + \theta(\tilde{p} - 1)}{\tilde{p}} - \frac{1}{2}, \frac{1}{2} - \frac{\theta}{\tilde{q}}\right).$$

It is left only to choose $\theta = \tilde{q}/(\tilde{p}' + \tilde{q})$ to complete the proof.

One should compare the estimate of α obtained here with the estimate $1/p - 1/q$ obtained by König, Retherford and Tomczak-Jaegerman in [**6**] (see [**16**]) for general type-p, cotype-q Banach spaces. For p and q close to 2 the last estimate is better than α, however α is always smaller than $1/2$ if $p > 1$ and $q < \infty$.

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