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ON COHERENCE OF ENDOMORPHISM RINGS

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Abstract

Let R be a ring and U a left R-module with $S = \mathrm{End}({}_RU)$. The aim of this paper is to characterize when S is coherent. We first show that a left R-module F is T_U-flat if and only if $\mathrm{Hom}_R(U, F)$ is a flat left S-module. This removes the unnecessary hypothesis that U is Σ-quasiprojective from Proposition 2.7 of Gomez Pardo and Hernandez ['Coherence of endomorphism rings', *Arch. Math. (Basel)* **48**(1) (1987), 40–52]. Then it is shown that S is a right coherent ring if and only if all direct products of T_U-flat left R-modules are T_U-flat if and only if all direct products of copies of ${}_RU$ are T_U-flat. Finally, we prove that every left R-module is T_U-flat if and only if S is right coherent with $\mathrm{wD}(S) \le 2$ and U_S is *FP*-injective.

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1. Introduction

Throughout this paper, all rings are associative with identity and all modules are unitary modules. For a ring R, ${}_RM$ (M_R) denotes a left (right) R-module. In what follows, U is a left R-module and $S = \mathrm{End}({}_RU)$. We denote by add ${}_RU$ the category consisting of all left R-modules isomorphic to direct summands of finite direct sums of copies of ${}_RU$ and by $\mathrm{pres}(U)$ the category of all finitely U-presented left R-modules, that is, of all left R-modules M admitting an exact sequence $U^n \to U^m \to M \to 0$ with m, n positive integers. Here H denotes $\mathrm{Hom}_R(U, -)$ and T means $U \otimes_S -$. Given a left R-module M and a left S-module A, define $\nu_M : TH(M) \to M$ and $\eta_A : A \to HT(A)$ via $\nu_M(u \otimes f) = f(u)$ and $\eta_A(a)(u) = u \otimes a$ for any $u \in U$, $f \in H(M)$ and $a \in A$. For a module M, M^I ($M^{(I)}$) is the direct product (sum) of copies of M indexed by a set I, $\mathrm{pd}(M)$ denotes the projective dimension of M, and the character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is denoted by M^+. As usual, we use $\mathrm{wD}(S)$ to denote the weak global dimension of a ring S. General background material can be found in [1, 7, 13, 16].

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Gomez Pardo and Hernandez [11] have given conditions under which S is a coherent ring assuming that ${}_RU$ is (Σ-)quasiprojective. Our aim is to characterize when S is coherent for a general left R-module ${}_RU$. We start by proving that a left R-module F is T_U-flat if and only if $H(F)$ is a flat left S-module. This removes the unnecessary hypothesis that U is Σ-quasiprojective from [11, Proposition 2.7]. Then it is shown that S is a right coherent ring if and only if all direct products of T_U-flat left R-modules are T_U-flat if and only if all direct products of copies of ${}_RU$ are T_U-flat. Moreover, if both ${}_RU$ and U_S are finitely presented, then we obtain that S is a right coherent ring if and only if F^{++} is T_U-flat for every T_U-flat left R-module F. Finally, we prove that every left R-module is T_U-flat if and only if S is right coherent with $\mathrm{wD}(S) \leq 2$ and U_S is FP-injective.

Next we recall some known notions and facts required in the paper.

A left R-module M is *quasiprojective* [1] if, for every quotient module L of M, the canonical homomorphism $\mathrm{Hom}_R(M, M) \to \mathrm{Hom}_R(M, L)$ is epic. On the other hand, M is called Σ-*quasiprojective* when every direct sum $M^{(I)}$ is quasiprojective. A left R-module F is called T_U-*flat* (see [11]) if for every homomorphism $f : K \to F$ with $K \in \mathrm{pres}(U)$, there exist homomorphisms $g : K \to U^n$ and $h : U^n \to F$ for some integer n such that $f = hg$. Note that if U is a finitely generated projective generator of the category of all left R-modules, the M is T_U-flat if and only if M is flat.

Let $\mathcal{C}$ be a class of left R-modules and M a left R-module. A homomorphism $\phi : M \to F$ with $F \in \mathcal{C}$ is called a $\mathcal{C}$-*preenvelope* of M [8] if for any homomorphism $f \colon M \to F'$ with $F' \in \mathcal{C}$, there is a homomorphism $g : F \to F'$ such that $g\phi = f$.

A left R-module M is *small* [7, p. 6] if the covariant functor $\mathrm{Hom}(M, -)$ commutes with arbitrary direct sums. It is well known that finitely generated modules are always small.

A right S-module N is called FP-injective [14] if $\mathrm{Ext}^1_S(F, N) = 0$ for every finitely presented right S-module F. When S_S is FP-injective, S is said to be right FP-injective.

A ring R is *right coherent* [4] when every finitely generated left ideal of R is finitely presented and *left IF* [6] when every injective left R-module is flat.

2. Coherence of endomorphism rings

Let U be a Σ-quasiprojective left R-module and F a left R-module, then $H(F)$ is a flat left S-module if and only if F is a T_U-flat module (see [11, Propsition 2.7]). In fact, this result is true for any left R-module U as shown by the following proposition.

PROPOSITION 2.1. *Let ${}_RU$ be a module with $S = \mathrm{End}({}_RU)$ and F be a left R-module. Then $H(F)$ is a flat left S-module if and only if F is a T_U-flat module.*

PROOF. Assume that $H(F)$ is a flat left S-module. Let $M \in \mathrm{pres}(U)$ and $\alpha : M \to F$ be an R-homomorphism. Then there is an exact sequence $0 \to K \to U^k \to U^l \to M \to 0$ with $k,\ l$ some positive integers. Let $Y = \mathrm{Coker}(H(U^k) \to H(U^l))$, then

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Y is a finitely presented left S-module. The exactness of $0 \to H(K) \to H(U^k) \to H(U^l) \to Y \to 0$ induces the following commutative diagram with exact rows.

$$\begin{array}{ccccccc} TH(U^k) & \longrightarrow & TH(U^l) & \longrightarrow & T(Y) & \longrightarrow & 0 \\ \downarrow{\scriptstyle \nu_{U^k}} & & \downarrow{\scriptstyle \nu_{U^l}} & & \downarrow{\scriptstyle \sigma} & & \\ U^k & \longrightarrow & U^l & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

Note that ν_{U^k} and ν_{U^l} are isomorphisms, and so the induced homomorphism σ is an isomorphism. Since $H(F)$ is a flat left S-module, there exist homomorphisms $f : Y \to S^n$ and $g : S^n \to H(F)$ for some integer n such that $H(\alpha)H(\sigma)\eta_Y = gf$. Note that $\nu_{T(Y)}T(\eta_Y) = 1_{T(Y)}$ by [7, Equality 2.1, p. 13] and the diagram

$$\begin{array}{ccccc} THT(Y) & \xrightarrow{TH(\sigma)} & TH(M) & \xrightarrow{TH(\alpha)} & TH(F) \\ \downarrow{\scriptstyle \nu_{T(Y)}} & & \downarrow{\scriptstyle \nu_M} & & \downarrow{\scriptstyle \nu_F} \\ T(Y) & \xrightarrow{\sigma} & M & \xrightarrow{\alpha} & F \end{array}$$

is commutative. Thus,

$$\begin{aligned} \nu_F T(g)T(f)\sigma^{-1} &= \nu_F T(gf)\sigma^{-1} \\ &= \nu_F T(H(\alpha)H(\sigma)\eta_Y)\sigma^{-1} \\ &= \nu_F TH(\alpha)TH(\sigma)T(\eta_Y)\sigma^{-1} \\ &= \alpha\nu_M TH(\sigma)T(\eta_Y)\sigma^{-1} \\ &= \alpha\sigma\nu_{T(Y)}T(\eta_Y)\sigma^{-1} \\ &= \alpha\sigma\sigma^{-1} = \alpha. \end{aligned}$$

Clearly, $T(f)\sigma^{-1} : M \to T(S^n)$ and $\nu_F T(g) : T(S^n) \to F$ are homomorphisms, and $T(S^n) \cong U^n$. So F is T_U-flat.

Conversely, suppose that F is T_U-flat and $f : X \to H(F)$ is an S-homomorphism with X a finitely presented left S-module. Note that $T(X) \in \mathrm{pres}(U)$, then there are R-homomorphisms $g : T(X) \to U^n$ and $h : U^n \to F$ satisfying $\nu_F T(f) = hg$. Since $H(\nu_F)\eta_{H(F)} = 1_{H(F)}$ by [7, Equality 2.1, p. 13], it follows that

$$H(h)(H(g)\eta_X) = H(hg)\eta_X = H(\nu_F)HT(f)\eta_X = H(\nu_F)\eta_{H(F)}f = f,$$

and hence f factors through $H(U^n) \cong S^n$. So $H(F)$ is a flat left S-module. □

The following corollary is an immediate consequence of Proposition 2.1.

COROLLARY 2.2. *Let U be a left R-module.*

(1) *$\bigoplus_{i=1}^n F_i$ is T_U-flat if and only if each F_i is T_U-flat for any positive integer n.*
(2) *If ${}_RU$ is small, then $\bigoplus_{i\in I} F_i$ is T_U-flat if and only if each F_i is T_U-flat for any index set I.*

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PROPOSITION 2.3. *Let ${}_RU$ be a module with $S = \mathrm{End}({}_RU)$. The following are equivalent.*

(1) *Every injective left R-module is T_U-flat.*
(2) *For any $M \in \mathrm{pres}(U)$, the injective envelope of M is T_U-flat.*
(3) *Any $M \in \mathrm{pres}(U)$ is finitely cogenerated by U.*

Moreover, if S is right coherent, then the above conditions are equivalent to:

(4) *U_S is FP-injective.*

PROOF. That condition (1) implies (2) is clear.

(2) $\Rightarrow$ (3). Let $M \in \mathrm{pres}(U)$ and $i : M \hookrightarrow E(M)$ be an injective envelope of M. By condition (2), there exist homomorphisms $\alpha : M \to U^n$ and $\beta : U^n \to E(M)$ for some positive integer n such that $\beta\alpha = i$. Note that α is monic, and so condition (3) holds.

(3) $\Rightarrow$ (1). For any homomorphism $\varphi : M \to E$ with $M \in \mathrm{pres}(U)$ and E injective, by condition (3) there is a monomorphism $M \to U^n$ for some integer n, and hence φ factors through U^n. So condition (1) follows.

Moreover, if S is right coherent, then by [13, Theorem 9.51] and the remark following it, we have U_S is FP-injective if and only if $H(E)$ is flat for any injective left R-module E. So the equivalence of (1) and (4) follows from Proposition 2.1. □

Specializing Proposition 2.3 to the case ${}_RU = {}_RR$ gives the following corollaries.

COROLLARY 2.4 (Part of [6, Theorem 1]). *The following are equivalent for a ring R.*

(1) *R is left IF.*
(2) *The injective envelope of every finitely presented left R-module is flat.*
(3) *Every finitely presented left R-module is a submodule of a free module.*

COROLLARY 2.5 [12, Theorem 3.10]. *If R is a right coherent ring, then R is left IF if and only if R is right FP-injective.*

Let M and N be left R-modules. There is a natural homomorphism

$$\sigma = \sigma_{M,N} : \mathrm{Hom}_R(M, U) \bigotimes_S \mathrm{Hom}_R(U, N) \to \mathrm{Hom}_R(M, N)$$

defined via $\sigma(f \otimes g)(m) = g(f(m))$ for all $f \in \mathrm{Hom}_R(M, U)$ and $g \in \mathrm{Hom}_R(U, N)$, $m \in M$.

It is easy to check that $\sigma_{M,N}$ is an isomorphism if $M \in \mathrm{add}_R\, U$ or $N \in \mathrm{add}_R\, U$.

LEMMA 2.6. *The following are equivalent.*

(1) *A left R-module F is T_U-flat.*
(2) *For any left R-module $M \in \mathrm{pres}(U)$, $\sigma_{M,F}$ is an epimorphism (isomorphism).*

PROOF. (1) $\Rightarrow$ (2). Let $M \in \mathrm{pres}(U)$ and F be T_U-flat. Then there is an exact sequence $U^n \to U^m \to M \to 0$ with m, n some positive integers, and

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so $0 \to \mathrm{Hom}_R(M, U) \to S^m \to S^n$ and $0 \to \mathrm{Hom}_R(M, F) \to \mathrm{Hom}_R(U^m, F) \to \mathrm{Hom}_R(U^n, F)$ are exact. Note that $\mathrm{Hom}_R(U, F) = H(F)$ is a flat left S-module by Proposition 2.1, and hence we obtain the following commutative diagram with exact rows.

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{Hom}_R(M, U) \bigotimes_S \mathrm{Hom}_R(U, F) & \longrightarrow & \mathrm{Hom}_R(U, F)^m & \longrightarrow & \mathrm{Hom}_R(U, F)^n \\ & \downarrow{\scriptstyle \sigma_{M,F}} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\ 0 \longrightarrow & \mathrm{Hom}_R(M, F) & \longrightarrow & \mathrm{Hom}_R(U^m, F) & \longrightarrow & \mathrm{Hom}_R(U^n, F) \end{array}$$

Thus condition (2) follows.

(2) $\Rightarrow$ (1). Let $M \in \mathrm{pres}(U)$ and $\alpha \in \mathrm{Hom}_R(M, F)$. By condition (2), there are $f_i \in \mathrm{Hom}_R(M, U)$ and $g_i \in \mathrm{Hom}_R(U, F)$ for all $i = 1, 2, \ldots, n$, such that $\alpha = \sigma_{M,F}(\sum_{i=1}^n f_i \otimes g_i)$. Define $f : M \to U^n$ via $f(m) = (f_i(m))$ for any $m \in M$ and $g : U^n \to F$ via $g((a_i)) = \sum_{i=1}^n g_i(a_i)$ for all $a_i \in U$. It is easy to check that $\alpha = gf$, as required. □

LEMMA 2.7. *Let U be a finitely presented left R-module. Then the class of T_U-flat left R-modules is closed under pure submodules and direct limits.*

PROOF. Let F be a T_U-flat left R-module and K a pure module of F, then there is an exact sequence $0 \to K \xrightarrow{i} F \xrightarrow{\pi} F/K \to 0$, where i is the canonical injection and π is the canonical projection. For any left R-module $M \in \mathrm{pres}(U)$ and any homomorphism $f : M \to K$, there are homomorphisms $g : M \to U^n$ and $h : U^n \to F$ for some integer n such that $if = hg$. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} & & M & \xrightarrow{g} & U^n & \xrightarrow{p} & \mathrm{Coker}(g) & \longrightarrow & 0 \\ & & {\scriptstyle f}\downarrow & \swarrow{\scriptstyle \gamma} & \downarrow{\scriptstyle h} & \swarrow{\scriptstyle \beta} & \downarrow{\scriptstyle \alpha} & & \\ 0 & \longrightarrow & K & \xrightarrow{i} & F & \xrightarrow{\pi} & F/K & \longrightarrow & 0 \end{array}$$

where α is the induced homomorphism. Note that $\mathrm{Coker}(g)$ is a finitely presented left R-module, then there exists a homomorphism $\beta : \mathrm{Coker}(g) \to F$ satisfying $\pi\beta = \alpha$. It follows that there is a homomorphism $\gamma : U^n \to K$ such that $\gamma g = f$, and so K is T_U-flat.

Suppose that $\{F_i\}_{i\in I}$ is a direct system of T_U-flat left R-modules over a directed index set I. Let $M \in \mathrm{pres}(U)$ and $f : M \to \lim_{\to} F_i$ be a homomorphism. Since U is finitely presented, so is M. By [10, Corollary 1.2.7], the epimorphism $\pi : \bigoplus_{i\in I} F_i \to \lim_{\to} F_i$ is pure. Thus, there is $g : M \to \bigoplus_{i\in I} F_i$ with $f = \pi g$. It follows that $\lim_{\to} F_i$ is T_U-flat since $\bigoplus_{i\in I} F_i$ is T_U-flat by Corollary 2.2(2). □

THEOREM 2.8. *Let ${}_RU$ be a module with $S = \mathrm{End}({}_RU)$. The following are equivalent.*

(1) *S is a right coherent ring.*

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(2) *All direct products of T_U-flat left R-modules are T_U-flat.*
(3) *All direct products of copies of ${}_RU$ are T_U-flat.*

Moreover, if ${}_RU$ and U_S are finitely presented, then the above conditions are also equivalent to the following.

(4) *Every left R-module has a T_U-flat preenvelope.*
(5) *F^{++} is T_U-flat for every T_U-flat left R-module F.*

PROOF. (1) $\Rightarrow$ (2). Let $\{F_i\}_{i\in I}$ be a family of T_U-flat left R-modules. Then

$$H\Big(\prod_{i\in I} F_i\Big) = \mathrm{Hom}_R\Big(U, \prod_{i\in I} F_i\Big) \cong \prod_{i\in I} \mathrm{Hom}_R(U, F_i) = \prod_{i\in I} H(F_i)$$

is a flat left S-module by Proposition 2.1 and condition (1). Thus, $\prod_{i\in I} F_i$ is T_U-flat by Proposition 2.1 again.

The implication (2) implies (3) is clear.

(3) $\Rightarrow$ (1). Note that, for any index set I, $S^I \cong \mathrm{Hom}_R(U, U^I)$ is a flat left S-module by Proposition 2.1 and condition (3). So condition (1) follows.

(2) $\Rightarrow$ (4). Let N be any left R-module. By [9, Lemma 5.3.12], for any homomorphism $f : N \to M$ where M is T_U-flat, there is a cardinal number $\aleph_\alpha$ and a pure submodule L of M such that $\mathrm{Card}(L) \le \aleph_\alpha$ and $f(N) \subseteq L$. Note that L is T_U-flat by Lemma 2.7, and so N has a T_U-flat preenvelope by condition (2) and [9, Proposition 6.2.1].

(4) $\Rightarrow$ (1). Let $M \in \mathrm{pres}({}_RU)$. Then M has a T_U-flat preenvelope $f : M \to F$ by condition (4). It follows that there are homomorphisms $\alpha : M \to \bar{U}$ and $\beta : \bar{U} \to F$ such that $f = \beta\alpha$ with $\bar{U} \in \mathrm{add}_R\, U$. It is easy to check that $\alpha : M \to \bar{U}$ is just an $\mathrm{add}_R\, U$-preenvelope of M. Thus condition (1) holds by [2, Proposition 5].

(1) $\Rightarrow$ (5). Let F be a T_U-flat left R-module. Then $\mathrm{Hom}_R(U, F) = H(F)$ is a flat left S-module by Proposition 2.1. Since S is right coherent by condition (1), $\mathrm{Hom}_R(U, F)^{++}$ is also a flat left S-module by [5, Theorem 1]. Note that $\mathrm{Hom}_R(U, F^{++}) \cong (F^+ \otimes_R U)^+ \cong \mathrm{Hom}_R(U, F)^{++}$, and hence F^{++} is T_U-flat by Proposition 2.1 again.

(5) $\Rightarrow$ (3). Note that $U^{(I)}$ is T_U-flat by Corollary 2.2, then $(U^{(I)})^{++}$ is T_U-flat by condition (5). Since $(U^+)^{(I)}$ is a pure submodule of $(U^+)^I$, $((U^+)^{(I)})^+$ is a direct summand of $((U^+)^I)^+ \cong (U^{(I)})^{++}$. It follows that $(U^{++})^I \cong ((U^+)^{(I)})^+$ is T_U-flat by Corollary 2.2 again. Note that U^I is a pure submodule of $(U^{++})^I$ by [5, Lemma 1(2)], so U^I is T_U-flat by Lemma 2.7. □

REMARK 2.9. Recall that a module ${}_RU$ is called a *generalized tilting module* [15] (now it is also called a *Wakamatsu tilting module* [3]) if it has the following properties:

(T1) there exists an exact sequence

$$\cdots \to P_i \to \cdots \to P_1 \to P_0 \to U \to 0$$

with each P_i finitely generated and projective for $i \ge 0$;

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(T2) ${}_RU$ is self-orthogonal, that is, $\mathrm{Ext}^i_R(U, U) = 0$ for $i \geq 1$;
(T3) there exists a $\mathrm{Hom}_R(-, U)$ exact sequence

$$0 \to {}_RR \to U_0 \to U_1 \to \cdots \to U_i \to \cdots$$

where each $U_i \in \mathrm{add}\,{}_RU$ for $i \geq 0$.

Wakamatsu [15] proved that ${}_RU$ is a Wakamatsu tilting module with $S = \mathrm{End}({}_RU)$ if and only if U_S is a Wakamatsu tilting module with $R = \mathrm{End}(U_S)$. So, for a Wakamatsu tilting module ${}_RU$, both ${}_RU$ and U_S are finitely presented.

REMARK 2.10. Let ${}_RU = {}_RR$ in Theorem 2.8, one obtains some known equivalent conditions for a ring to be right coherent.

We conclude this paper with the following theorem.

THEOREM 2.11. *Let ${}_RU$ be a module with $S = \mathrm{End}({}_RU)$. The following are equivalent.*

(1) *Every left R-module is T_U-flat.*
(2) *Every finitely U-presented left R-module belongs to add ${}_RU$.*
(3) *If ${}_SA$ is finitely presented, then $HT(A)$ is a finitely generated projective left S-module.*
(4) *S is right coherent with* $\mathrm{wD}(S) \leq 2$ *and U_S is FP-injective.*

PROOF. The equivalence of (1) and (2) holds by definition.

(2) ⇒ (3). Let ${}_SA$ be finitely presented. Then $T(A)$ is finitely U-presented, and so $T(A) \in \mathrm{add}_R\, U$ by condition (2). Thus, $HT(A)$ is a finitely generated projective left S-module.

(3) ⇒ (2). Let M be a finitely U-presented left R-module, then there is an exact sequence $0 \to K \to U^n \to U^m \to M \to 0$ with n, m positive integers. Note that $H(U^n) \cong S^n$ and $H(U^m) \cong S^m$, then we obtain an exact sequence $0 \to H(K) \to S^n \to S^m$ of left S-modules. Thus, $D = \mathrm{Coker}(S^n \to S^m)$ is a finitely presented left S-module, and so $HT(D)$ is a finitely generated projective left S-module by condition (3). It follows that $THT(D) \in \mathrm{add}_R\, U$. Since there is the commutative diagram with exact rows:

$$\begin{array}{ccccccc} U^n & \longrightarrow & U^m & \longrightarrow & M & \longrightarrow & 0 \\ \downarrow{\cong} & & \downarrow{\cong} & & \vdots & & \\ U^n & \longrightarrow & U^m & \longrightarrow & T(D) & \longrightarrow & 0 \end{array}$$

we have $M \cong T(D)$. Note that $T(D)$ is a direct summand of $THT(D)$ by [7, Equality 2.1, p. 13], so $M \in \mathrm{add}_R\, U$.

(2) ⇒ (4). Since condition (2) is equivalent to condition (1) by the foregoing proof, every left R-module is T_U-flat. So S is right coherent by Theorem 2.8. Thus, U_S is FP-injective by Proposition 2.3. Let A_S be finitely presented, then there is an exact sequence $S^k \to S^l \to A \to 0$ of right S-modules with k, l positive integers. Now we

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obtain an exact sequence $0 \to \mathrm{Hom}_S(A, U) \to U^l \to U^k$ of left R-modules which induces a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 & \longrightarrow & K & \longrightarrow & S^k & \longrightarrow & S^l \\
 & & \vdots h & & \downarrow \cong & & \downarrow \cong \\
0 & \longrightarrow & \mathrm{Hom}_R(D, U) & \longrightarrow & \mathrm{Hom}_R(U^k, U) & \longrightarrow & \mathrm{Hom}_R(U^l, U)
\end{array}$$

where $K = \mathrm{Ker}(S^k \to S^l)$, $D = \mathrm{Coker}(U^l \to U^k)$ and h is the induced homomorphism. Thus, $K \cong \mathrm{Hom}_R(D, U)$. Note that D is a finitely U-presented left R-module, then $D \in \mathrm{add}_R\, U$ by condition (2). It follows that K is a finitely generated projective right S-module, and hence $\mathrm{pd}(A_S) \leq 2$. Therefore, $\mathrm{wD}(S) = \sup\{\mathrm{pd}(A_S) \mid A_S \text{ is finitely presented}\} \leq 2$ by [14, Theorem 3.3].

(4) $\Rightarrow$ (1). Let M be any left R-module and E the injective envelope of M, then there is an exact sequence $0 \to M \to E \to C \to 0$ which induces the following exact commutative diagram.

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & H(M) & \longrightarrow & H(E) & \longrightarrow & H(C) & \longrightarrow & D & \longrightarrow & 0 \\
 & & & & & \pi\searrow & & \nearrow i & & & \\
 & & & & & & K & & & &
\end{array}$$

Since $\mathrm{wD}(S) \leq 2$, there are exact sequences

$$\begin{aligned}
0 \to \mathrm{Tor}_2^S(A, H(M)) &\to \mathrm{Tor}_2^S(A, H(E)) \to \\
&\mathrm{Tor}_2^S(A, K) \to \mathrm{Tor}_1^S(A, H(M)) \to \mathrm{Tor}_1^S(A, H(E))
\end{aligned} \qquad (*)$$

$$0 \to \mathrm{Tor}_2^S(A, K) \to \mathrm{Tor}_2^S(A, H(C)) \qquad (**)$$

for any right S-module A. Since S is right coherent and U_S is FP-injective, E is T_U-flat by Proposition 2.3. Hence, $H(E)$ is flat by Proposition 2.1. Thus, $\mathrm{Tor}_2^S(A, H(M)) = 0$ and $\mathrm{Tor}_2^S(A, K) \cong \mathrm{Tor}_1^S(A, H(M))$ by the exactness of the sequence ($*$). Similarly, we have $\mathrm{Tor}_2^S(A, H(C)) = 0$. Thus, $\mathrm{Tor}_2^S(A, K) = 0$ by the exactness of the sequence ($**$), and hence $\mathrm{Tor}_1^S(A, H(M)) = 0$. It follows that $H(M)$ is flat, and so M is T_U-flat by Proposition 2.1. □

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