Bull. Aust. Math. Soc. (First published online 2024), page 1 of 8*
doi:10.1017/S0004972723001338
*Provisional—final page numbers to be inserted when paper edition is published

BOUNDING ZETA ON THE 1-LINE UNDER THE PARTIAL RIEMANN HYPOTHESIS

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(Received 14 October 2023; accepted 10 November 2023)

Abstract

We provide explicit bounds for the Riemann zeta-function on the line $\mathrm{Re}\, s = 1$, assuming that the Riemann hypothesis holds up to height T. In particular, we improve some bounds in finite regions for the logarithmic derivative and the reciprocal of the Riemann zeta-function.

2020 *Mathematics subject classification*: primary 11M06; secondary 11M26, 11Y35.

Keywords and phrases: Riemann zeta-function, Riemann hypothesis, partial Riemann hypothesis.

1. Introduction

A classical problem in analytic number theory is to find explicit bounds for the Riemann zeta-function. In particular, bounds on the line $\mathrm{Re}\, s = 1$ are of great interest, owing to their usefulness in estimations for the Möbius function and the von Mangoldt function. The main purpose of this paper is to obtain new bounds for the Riemann zeta-function in ranges where currently it is challenging to get computational verification.

1.1. Background. Let $\zeta(s)$ be the Riemann zeta-function. Unconditionally, it is known that, as $t \to \infty$,

$$\frac{1}{\zeta(1+it)} = O(\log t) \quad \text{and} \quad \frac{\zeta'(1+it)}{\zeta(1+it)} = O(\log t). \tag{1.1}$$

Currently, the best explicit bounds for $1/\zeta(1+it)$ and $\zeta'(1+it)/\zeta(1+it)$ are given by

$$\left|\frac{1}{\zeta(1+it)}\right| \leq 42.9 \log t \quad \text{and} \quad \left|\frac{\zeta'(1+it)}{\zeta(1+it)}\right| \leq 40.14 \log t, \tag{1.2}$$

for $t \geq 133$. The first bound in (1.2) was established by Carneiro *et al.* in [2, Proposition A.2], and the second bound was established by Trudgian in [15]. There are improvements in the orders of magnitude of the mentioned estimates (see, for example, [14, page 135]), but it appears that those bounds are better when t is astronomically large and then they will not be useful for computational purposes.

On the other hand, assuming the Riemann hypothesis (RH), Littlewood proved in [7] that, as $t \to \infty$,

$$\left|\frac{1}{\zeta(1+it)}\right| \leq \left(\frac{12e^{\gamma}}{\pi^2} + o(1)\right)\log\log t,$$

where γ is the Euler–Mascheroni constant. An explicit version of this result has been given by Lamzouri *et al.* [6, page 2394], establishing that

$$\left|\frac{1}{\zeta(1+it)}\right| \leq \frac{12e^{\gamma}}{\pi^2}\left(\log\log t - \log 2 + \frac{1}{2} + \frac{1}{\log\log t} + \frac{14\log\log t}{\log t}\right),$$

for $t \geq 10^{10}$. Moreover, recently, Chirre *et al.* [3, Theorem 5] proved under RH that

$$\left|\frac{\zeta'(1+it)}{\zeta(1+it)}\right| \leq 2\log\log t - 0.4989 + 5.35\,\frac{(\log\log t)^2}{\log t},$$

for $t \geq 10^{30}$. Some generalisations of these estimates for families of L-functions can be found in [8, 10].

1.2. Bounds for zeta under partial RH. We are interested in obtaining bounds for the Riemann zeta-function, but assuming only a partial verification of RH. For $T > 0$, we say that RH is true up to height T if all nontrivial zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ such that $|\gamma| \leq T$ satisfy $\beta = 1/2$. The best current result of this type is given by Platt and Trudgian [12] who verified numerically, in a rigorous way using interval arithmetic, that RH is true up to height $T = 3 \cdot 10^{12}$.

THEOREM 1.1. *For a fixed δ with $0 < \delta < 1$, define*

$$E_{\delta}(T) = \left(\frac{1}{\delta^2} + 1\right)\frac{\log T}{2\pi T}. \tag{1.3}$$

Assume RH up to height $T \geq 10^9$. Then, for $10^6 \leq t \leq (1-\delta)T$,

$$\left|\frac{\zeta'(1+it)}{\zeta(1+it)}\right| \leq 2\log\log t + 1.219 + \frac{16.108}{(\log\log t)^2} + 1.057E_{\delta}(T) \tag{1.4}$$

and

$$\left|\frac{1}{\zeta(1+it)}\right| \leq 2e^{\gamma}\left(\log\log t + 3.404 + \frac{9.378}{\log\log t}\right)\cdot\exp(0.793E_{\delta}(T)). \tag{1.5}$$

From Theorem 1.1, we can also derive explicit versions of (1.1) in a finite but large range where computational verification is difficult. In fact, by Platt and Trudgian's result, we can take $T = 3 \cdot 10^{12}$ and, letting $\delta = 10^{-5}$, it follows unconditionally that

$$\left|\frac{\zeta'(1+it)}{\zeta(1+it)}\right| \leq 0.639 \cdot \log t \quad \text{and} \quad \frac{1}{|\zeta(1+it)|} \leq 2.506 \cdot \log t,$$

for $10^6 \leq t \leq 2.99997 \cdot 10^{12}$. This improves (1.2) in the range $10^6 \leq t \leq 2.99997 \cdot 10^{12}$.

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We mention that (1.5) is derived from an upper bound for $|\log\zeta(1+it)|$ (see (3.4)), which allows us to deduce that

$$|\zeta(1+it)| \le 2e^{\gamma}\Big(\log\log t + 3.404 + \frac{9.378}{\log\log t}\Big)\cdot\exp(0.793E_{\delta}(T)). \tag{1.6}$$

Currently, the best unconditional explicit bound for $\zeta(1+it)$ is given by Patel, who proved in [11, Theorem 1.1] that, for $t \ge 3$,

$$|\zeta(1+it)| \le \min\Big\{\log t, \frac{\log t}{2} + 1.93, \frac{\log t}{5} + 44.02\Big\}. \tag{1.7}$$

So, (1.6) improves (1.7) if RH is verified up to height T for T sufficiently large.

The proof of Theorem 1.1 is carried out in Section 3 and partly follows the conditional proofs of [9, Section 13.2]. Here, an explicit formula is used that relates the zeros of $\zeta(s)$ and the prime numbers. This formula is unconditional and contains a certain sum involving the nontrivial zeros. Assuming RH, this sum is bounded without much effort. In our case, the novelty is in bounding the contribution of the nontrivial zeros, since we only assume RH up to height T. We split this sum into two parts, the zeros with ordinates $|\gamma| \le T$ and $|\gamma| > T$, and analyse them separately. These sums are studied in Section 2. The proof of Theorem 1.1 is short, and the constants involved can be improved slightly.

Throughout the paper, we use the notation $\alpha = O^*(\beta)$, which means that $|\alpha| \le \beta$.

2. The sum over the nontrivial zeros

To bound the sum related to the nontrivial zeros of $\zeta(s)$ with ordinates $|\gamma| \le T$, we use the following lemma.

LEMMA 2.1. *Assume RH up to height $T > 0$. Then, for $t \ge 10^6$ and $1 \le \alpha \le 3/2$,*

$$\sum_{|\gamma|\le T} \frac{\alpha-\frac{1}{2}}{(\alpha-\frac{1}{2})^2+(t-\gamma)^2} \le \operatorname{Re}\frac{\zeta'(\alpha+it)}{\zeta(\alpha+it)} + \frac{\log t}{2}.$$

PROOF. Letting $s = \alpha + it$ and using the fractional decomposition of $\zeta(s)$ (see [9, Corollary 10.14]), we get

$$\sum_{\gamma} \frac{\alpha-\operatorname{Re}\rho}{(\alpha-\operatorname{Re}\rho)^2+(t-\gamma)^2} = \operatorname{Re}\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{2}\operatorname{Re}\frac{\Gamma'}{\Gamma}\Big(\frac{s}{2}+1\Big) - \frac{\log\pi}{2} + \frac{\alpha-1}{(\alpha-1)^2+t^2}.$$

From the bound

$$\frac{\Gamma'}{\Gamma}(z) = \log z - \frac{1}{2z} + O^*\Big(\frac{1}{4|z|^2}\Big) \quad \text{for } \operatorname{Re} z \ge 0$$

(see [4, Lemma 3.11, page 67]), it follows that

$$\sum_{\gamma} \frac{\alpha-\operatorname{Re}\rho}{(\alpha-\operatorname{Re}\rho)^2+(t-\gamma)^2} \le \operatorname{Re}\frac{\zeta'(s)}{\zeta(s)} + \frac{\log t}{2}. \tag{2.1}$$

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On the other hand, splitting the sum over the zeros and using the fact that $\operatorname{Re}\rho < 1$, we get

$$\begin{aligned}\sum_{\gamma} \frac{\alpha - \operatorname{Re}\rho}{(\alpha - \operatorname{Re}\rho)^2 + (t-\gamma)^2} &= \sum_{|\gamma|\le T} \frac{\alpha - \frac{1}{2}}{(\alpha - \frac{1}{2})^2 + (t-\gamma)^2} + \sum_{|\gamma|>T} \frac{\alpha - \operatorname{Re}\rho}{(\alpha - \operatorname{Re}\rho)^2 + (t-\gamma)^2} \\ &\ge \sum_{|\gamma|\le T} \frac{\alpha - \frac{1}{2}}{(\alpha - \frac{1}{2})^2 + (t-\gamma)^2}. \end{aligned} \tag{2.2}$$

By combining (2.1) and (2.2), we arrive at the desired result. □

To bound the sum related to the nontrivial zeros of $\zeta(s)$ with ordinates $|\gamma| > T$, we use the auxiliary function

$$E(t,T) := \sum_{|\gamma|>T} \frac{1}{(\gamma - t)^2},$$

where t does not coincide with an ordinate of a zero of $\zeta(s)$. This function measures, in a certain sense, the difference between the bounds under RH up to height T and the bounds under RH. In fact, for a fixed $t \ge 0$, we see that

$$\lim_{T\to\infty} E(t,T) = 0.$$

To estimate $E(t,T)$, the parameter t must not be close to the ordinates of the zeros, and we need to take T sufficiently large to reduce the contribution. Here, we bound this term using a sum studied by Brent, Platt and Trudgian in [1].

LEMMA 2.2. *Fix $0 < \delta < 1$ and $T \ge 10^9$. Then, for $0 \le t \le (1-\delta)T$,*

$$0 < E(t,T) \le E_\delta(T),$$

where $E_\delta(T)$ was defined in (1.3).

PROOF. Since $t \le (1-\delta)T$, we find that

$$E(t,T) = \sum_{|\gamma|>T} \frac{1}{(\gamma - t)^2} = \sum_{\gamma>T} \frac{1}{(\gamma - t)^2} + \sum_{\gamma>T} \frac{1}{(\gamma + t)^2} \le \left(\frac{1}{\delta^2} + 1\right) \sum_{\gamma>T} \frac{1}{\gamma^2}.$$

By [1, Theorem 1 and Example 1],

$$\left| {\sum_{\gamma\ge T}}' \frac{1}{\gamma^2} - \frac{1}{2\pi}\int_T^\infty \frac{\log(t/2\pi)}{t^2}\,dt \right| \le \frac{0.14 + 0.56\log T}{T^2},$$

where the prime symbol $'$ indicates that if $\gamma = T$, then it is counted with weight $1/2$. Thus,

$$\sum_{\gamma>T} \frac{1}{\gamma^2} \le \frac{1}{2\pi T}\log\left(\frac{T}{2\pi}\right) + \frac{1}{2\pi T} + \frac{0.14 + 0.56\log T}{T^2}.$$

Hence, using $T \ge 10^9$ concludes the proof. □

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3. Proof of Theorem 1.1

3.1. Bounding $\zeta'(s)/\zeta(s)$. Assume that RH is true up to height $T \ge 10^9$. Let $t \ge 10^6$ and $1 \le \alpha \le 3/2$. Given $x, y \ge 2$ and $s = \alpha + it$, the unconditional formula [9, Equation 13.35] states that $\zeta'(s)/\zeta(s)$ is equal to

$$-\sum_{\rho} \frac{(xy)^{\rho-s} - x^{\rho-s}}{(\rho-s)^2 \log y} - \sum_{k=1}^{\infty} \frac{(xy)^{-2k-s} - x^{-2k-s}}{(2k+s)^2 \log y} + \frac{(xy)^{1-s} - x^{1-s}}{(1-s)^2 \log y} - \sum_{n \le xy} \frac{\Lambda(n)}{n^s} w(n), \quad (3.1)$$

where $w(n)$ is the function defined in [9, page 433] satisfying $|w(n)| \le 1$. We bound each term of (3.1). Since $|(xy)^{\rho-s} - x^{\rho-s}| \le x^{\operatorname{Re}\rho-\alpha}(y^{\operatorname{Re}\rho-\alpha} + 1)$ and $\operatorname{Re}\rho < 1$,

$$\begin{aligned}
\left|\sum_{\rho} \frac{(xy)^{\rho-s} - x^{\rho-s}}{(\rho-s)^2 \log y}\right| &= \left|\sum_{|\gamma| \le T} \frac{(xy)^{\frac{1}{2}+i\gamma-s} - x^{\frac{1}{2}+i\gamma-s}}{(\frac{1}{2}+i\gamma-s)^2 \log y} + \sum_{|\gamma|>T} \frac{(xy)^{\rho-s} - x^{\rho-s}}{(\rho-s)^2 \log y}\right| \\
&\le \frac{x^{\frac{1}{2}-\alpha}(y^{\frac{1}{2}-\alpha}+1)}{\log y} \sum_{|\gamma| \le T} \frac{1}{(\alpha-\frac{1}{2})^2 + (t-\gamma)^2} + \frac{x^{1-\alpha}(y^{1-\alpha}+1)}{\log y} E(t,T). \\
&\le \frac{x^{\frac{1}{2}-\alpha}(y^{\frac{1}{2}-\alpha}+1)}{(\alpha-\frac{1}{2})\log y} \cdot \operatorname{Re}\frac{\zeta'(s)}{\zeta(s)} + \frac{x^{\frac{1}{2}-\alpha}(y^{\frac{1}{2}-\alpha}+1)\log t}{2(\alpha-\frac{1}{2})\log y} + \frac{x^{1-\alpha}(y^{1-\alpha}+1)}{\log y} E_\delta(T),
\end{aligned}$$

from the assumption that $10^6 \le t \le (1-\delta)T$ and Lemmas 2.1 and 2.2. We estimate the next terms in (3.1) trivially as

$$\left|\sum_{k=1}^{\infty} \frac{(xy)^{-2k-s} - x^{-2k-s}}{(2k+s)^2 \log y}\right| \le \frac{0.3}{t^2}, \qquad \left|\frac{(xy)^{1-s} - x^{1-s}}{(1-s)^2 \log y}\right| \le \frac{2.9}{t^2}$$

and

$$\left|\sum_{n \le xy} \frac{\Lambda(n)}{n^s} w(n)\right| \le \sum_{n \le xy} \frac{\Lambda(n)}{n^\alpha}.$$

Inserting these bounds in (3.1), we arrive at

$$\begin{aligned}
\left|\frac{\zeta'(s)}{\zeta(s)}\right| &\le \frac{x^{\frac{1}{2}-\alpha}(y^{\frac{1}{2}-\alpha}+1)}{(\alpha-\frac{1}{2})\log y}\left|\frac{\zeta'(s)}{\zeta(s)}\right| \\
&\quad + \frac{x^{\frac{1}{2}-\alpha}(y^{\frac{1}{2}-\alpha}+1)\log t}{2(\alpha-\frac{1}{2})\log y} + \sum_{n \le xy} \frac{\Lambda(n)}{n^\alpha} + \frac{x^{1-\alpha}(y^{1-\alpha}+1)}{\log y} E_\delta(T) + \frac{3.2}{t^2}.
\end{aligned}$$

Now, let $\lambda_0 = 1.2784\ldots$ be the point where the function $\lambda \mapsto (1+e^\lambda)/\lambda$ reaches its minimum value $\mathcal{A}_0 = 3.5911\ldots$ in $(0,\infty)$. Take

$$y = \exp\left(\frac{\lambda_0}{\alpha-\frac{1}{2}}\right) \ge 2 \quad \text{and} \quad x = \frac{\log^2 t}{y} \ge 2.$$

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Note that

$$\frac{x^{\frac{1}{2}-\alpha}(y^{\frac{1}{2}-\alpha}+1)}{(\alpha-\frac{1}{2})\log y}=\mathcal{A}_0(\log t)^{1-2\alpha}<1 \quad \text{and} \quad \frac{x^{1-\alpha}(y^{1-\alpha}+1)}{\log y}\le\frac{2}{\log y}=\frac{2\alpha-1}{\lambda_0}.$$

Therefore, we get

$$\left|\frac{\zeta'(\alpha+it)}{\zeta(\alpha+it)}\right|\le(1+\epsilon(\alpha,t))\left[\frac{\mathcal{A}_0}{2}(\log t)^{2-2\alpha}+\sum_{n\le\log^2 t}\frac{\Lambda(n)}{n^\alpha}+\frac{(2\alpha-1)}{\lambda_0}E_\delta(T)+\frac{3.2}{t^2}\right], \tag{3.2}$$

where $\epsilon(\alpha,t)$ is defined as

$$\epsilon(\alpha,t):=\frac{1}{\mathcal{A}_0^{-1}(\log t)^{2\alpha-1}-1}.$$

3.2. Bounding $\zeta'(1+it)/\zeta(1+it)$. Letting $\alpha=1$ in (3.2), it follows that

$$\left|\frac{\zeta'(1+it)}{\zeta(1+it)}\right|\le(1+\epsilon(1,t))\left[\frac{\mathcal{A}_0}{2}+\sum_{n\le\log^2 t}\frac{\Lambda(n)}{n}+\frac{E_\delta(T)}{\lambda_0}+\frac{3.2}{t^2}\right].$$

To bound the sum over the primes in the above expression, we use the estimate (see [5, Lemma 10])

$$\sum_{n\le X}\frac{\Lambda(n)}{n}\le\log X-\gamma+\frac{1.3}{\log^2 X} \quad \text{for all } X>1.$$

Finally, using $t\ge 10^6$, we arrive at (1.4).

3.3. Bounding $\log\zeta(1+it)$. By the fundamental calculus theorem,

$$\log\zeta(1+it)=\log\zeta\left(\frac{3}{2}+it\right)-\int_1^{3/2}\frac{\zeta'(\alpha+it)}{\zeta(\alpha+it)}\,d\alpha.$$

Since $|\log\zeta(\frac{3}{2}+it)|\le\log\zeta(\frac{3}{2})$, we obtain

$$|\log\zeta(1+it)|\le\int_1^{3/2}\left|\frac{\zeta'(\alpha+it)}{\zeta(\alpha+it)}\right|d\alpha+\log\zeta\left(\frac{3}{2}\right).$$

To bound the right-hand side of this inequality, we use $\epsilon(\alpha,t)\le\epsilon(1,t)$ in $1\le\alpha\le 3/2$ and integrate (3.2) from 1 to 3/2 to obtain

$$|\log\zeta(1+it)|\le(1+\epsilon(1,t))\left[\frac{\mathcal{A}_0}{4\log\log t}+\sum_{n\le\log^2 t}\frac{\Lambda(n)}{n\log n}-\sum_{n\le\log^2 t}\frac{\Lambda(n)}{n^{\frac{3}{2}}\log n}+\frac{3E_\delta(T)}{4\lambda_0}\right]$$
$$+\log\zeta\left(\frac{3}{2}\right),$$

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where we have used $-\mathcal{A}_0/(4\log t\log\log t) + 1.6/t^2 < 0$. Furthermore, since

$$\log\zeta\left(\frac{3}{2}\right) = \sum_{n\geq 1}\frac{\Lambda(n)}{n^{\frac{3}{2}}\log n} = 0.960\ldots,$$

we have

$$(1+\epsilon(1,t))\sum_{n>\log^2 t}\frac{\Lambda(n)}{n^{\frac{3}{2}}\log n} \leq \epsilon(1,t)\sum_{n\geq 1}\frac{\Lambda(n)}{n^{\frac{3}{2}}\log n},$$

for $t \geq 10^6$. This implies that

$$|\log\zeta(1+it)| \leq (1+\epsilon(1,t))\left[\frac{\mathcal{A}_0}{4\log\log t} + \sum_{n\leq\log^2 t}\frac{\Lambda(n)}{n\log n} + \frac{3E_\delta(T)}{4\lambda_0}\right]. \tag{3.3}$$

To bound the sum over the primes, we use [13, Equation (3.30)] to see that, for $x > 1$,

$$\sum_{n\leq x}\frac{\Lambda(n)}{n\log n} \leq \sum_{p\leq x}\sum_{k=1}^{\infty}\frac{1}{kp^k} = \log\prod_{p\leq x}\left(1-\frac{1}{p}\right)^{-1} \leq \log\log x + \gamma + \frac{1}{\log^2 x}.$$

Inserting this in (3.3), we arrive at

$$\begin{aligned}&|\log\zeta(1+it)|\\ &\quad\leq (1+\epsilon(1,t))\left[\log\log\log t + \log(2e^\gamma) + \frac{\mathcal{A}_0}{4\log\log t} + \frac{1}{4(\log\log t)^2} + \frac{3E_\delta(T)}{4\lambda_0}\right].\end{aligned}$$

Thus,

$$|\log\zeta(1+it)| \leq \log\log\log t + \log(2e^\gamma) + \frac{3.404}{\log\log t} + 0.793E_\delta(T). \tag{3.4}$$

Taking exponentials in this inequality and using the estimate $e^x \leq 1 + x + 0.8093x^2$ for $0 \leq x \leq 1.297$, we obtain

$$\exp(|\log\zeta(1+it)|) \leq 2e^\gamma\left(\log\log t + 3.404 + \frac{9.378}{\log\log t}\right)\cdot\exp(0.793E_\delta(T)).$$

Since $\log|\zeta(1+it)|^{-1} \leq |\log\zeta(1+it)|$ and $\log|\zeta(1+it)| \leq |\log\zeta(1+it)|$, we deduce (1.5) and (1.6), respectively.

Acknowledgement

I am grateful to Harald Helfgott for encouraging me in this project and for helpful discussions related to the material in this paper.

References

[1] R. Brent, D. Platt and T. Trudgian, ‘Accurate estimation of sums over zeros of the Riemann zeta-function’, *Math. Comp.* **90**(332) (2021), 2923–2935.

[2] E. Carneiro, A. Chirre, H. A. Helfgott and J. Mejía-Cordero, ‘Optimality for the two-parameter quadratic sieve’, *Acta Arith.* **203**(3) (2022), 195–226.

https://doi.org/10.1017/S0004972723001338 Published online by Cambridge University Press

[3] A. Chirre, A. Simonič and M. Valås Hagen, 'Conditional bounds for the logarithmic derivative of Dirichlet L-functions', *Indag. Math.*, to appear, https://doi.org/10.1016/j.indag.2023.07.005.

[4] H. A. Helfgott, 'The ternary Goldbach problem', *Ann. of Math. Stud.*, to appear. Second preliminary version available at https://webusers.imj-prg.fr/harald.helfgott/anglais/book.html.

[5] D. R. Johnston, O. Ramaré and T. Trudgian, 'An explicit upper bound for $L(1,\chi)$ when χ is quadratic', *Res. Number Theory* **9** (2023), Article No. 72.

[6] Y. Lamzouri, X. Li and K. Soundararajan, 'Conditional bounds for the least quadratic non-residue and related problems', *Math. Comp.* **84**(295) (2015), 2391–2412.

[7] J. E. Littlewood, 'On the function $1/\zeta(1+it)$', *Proc. Lond. Math. Soc. (3)* **27**(5) (1928), 349–357.

[8] A. Lumley, 'Explicit bounds for L-functions on the edge of the critical strip', *J. Number Theory* **188** (2018), 186–209.

[9] H. L. Montgomery and R. C. Vaughan, *Multiplicative Number Theory: I. Classical Theory*, Cambridge Studies in Advanced Mathematics, 97 (Cambridge University Press, Cambridge, 2006).

[10] N. Palojärvi and A. Simonič, 'Conditional estimates for L-functions in the Selberg class', Preprint, 2023, arXiv:2211.01121.

[11] D. Patel, 'An explicit upper bound for $|\zeta(1+it)|$', *Indag. Math.* **33**(5) (2022), 1012–1032.

[12] D. Platt and T. Trudgian, 'The Riemann hypothesis is true up to $3\cdot 10^{12}$', *Bull. Lond. Math. Soc.* **53**(3) (2021), 792–797.

[13] J. B. Rosser and L. Schoenfeld, 'Approximate formulas for some functions of prime numbers', *Illinois J. Math.* **6** (1962), 64–94.

[14] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd edn, Oxford Science Publications (Oxford University Press, Oxford, 1986).

[15] T. Trudgian, 'Explicit bounds on the logarithmic derivative and the reciprocal of the Riemann zeta-function', *Funct. Approx. Comment. Math.* **52**(2) (2015), 253–261.

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