11

The renormalization group

Renormalization invariance states that physical observables must be independent of the renormalization scheme chosen in their theoretical evaluation. The differential approach to renormalization invariance was pioneered by Stueckelberg–Peterman [75] and by Gell-Mann–Low [76], where it has been pointed out that the QED coupling constant is momentum dependent due to the definition of the renormalized charge. Such a consideration led to write a differential equation for the photon propagator. Later on, the study of the scaling behaviour in field theory (experimental observation of the Bjorken scaling [36] in deep inelastic scattering) gave rise to the Callan–Symanzik equation (CSE) [132], which is a very powerful technique for expressing the renormalization invariance constraints on the short-distance behaviour of the Green functions. The CSE takes into account the fact that scaling cannot be strictly implemented because of the necessity of a mass scale in the theory. In the ϵ-regularization, such a mass scale renders the coupling constants dimensionless (see Table 9.1). A generalization of the uses of the CSE to arbitrary Green functions has been proposed [123,171]. The central idea was to treat $g,\ m_i,\ \alpha_G$ as coupling constants of various interaction terms in the Lagrangian.

The meaning of the *renormalization group* can be seen from a simple example. Let us consider a field ϕ. One can renormalize it in two different renormalization schemes which we call R_1 and R_2. Then, the renormalized field in terms of the bare one is:

$$\phi_{R_1} = Z(R_1)\phi_B\ , \qquad \phi_{R_2} = Z(R_2)\phi_B\ , \tag{11.1}$$

where: $Z(R_i)$ is the renormalization constant for each scheme R_i, and ϕ_B is the *bare* field. As the bare field is by definition independent of the scheme, we can then, deduce:

$$\phi_{R_1} = Z(R_1, R_2)\phi_{R_2}\ , \tag{11.2}$$

with:

$$Z(R_1, R_2) \equiv Z(R_1)/Z(R_2)\ , \tag{11.3}$$

which should be finite as do the renormalized fields, despite the fact that the renormalization constants $Z(R_i)$ are divergent. Analogous reasoning can be applied for other parameters of the Lagrangian. The operation which relates quantities of two different renormalization schemes can be interpreted as a transformation from R_1 to R_2. The set of all these

https://doi.org/10.1017/9781009290296.018 Published online by Cambridge University Press

transformations is called the *renormalization group*. One can use the invariance of physical quantities under this group in order to study the asymptotic behaviour of the Green's functions. This can be done as shown below using the renormalization group equation.

11.1 The renormalization group equation

The ϵ-regularized Green function reads:

$$\Gamma^R(\nu, p_1, \ldots, p_N; g,\ \alpha_G,\ m_i) = Z_\Gamma \Gamma^B(\nu, p_1, \ldots, p_N; g,\ \alpha_G,\ m_i)\,. \tag{11.4}$$

The ν-independence of Γ_B implies the zero of the total derivative:

$$\nu \frac{d\Gamma_B}{d\nu} = 0\,, \tag{11.5}$$

which is equivalent to:

$$\left\{\nu\frac{\partial}{\partial\nu} + \nu\frac{d\alpha_s}{d\nu}\frac{\partial}{\partial\alpha_s} + \sum_j \frac{\nu}{m_j}\frac{dm_j}{d\nu}m_j\frac{\partial}{\partial m_j} + \nu\frac{d\alpha_G}{d\nu}\frac{\partial}{\partial\alpha_G} - \frac{1}{Z_\Gamma}\nu\frac{dZ_\Gamma}{d\nu}\right\}\Gamma^R = 0\,. \tag{11.6}$$

By introducing the universal β function and anomalous dimensions γ_i:

$$\alpha_s\beta(\alpha_s) = \nu\frac{d\alpha_s}{d\nu}\bigg|_{g_B,m_B\ \text{fixed}}\,,$$
$$\gamma_m = -\frac{\nu}{m_i^R}\frac{dm_i^R}{d\nu}\bigg|_{g_B,m_B\ \text{fixed}}\,,$$
$$\gamma_i = \frac{\nu}{Z_i}\frac{dZ_i}{d\nu}\bigg|_{g_B,m_B\ \text{fixed}}\,, \tag{11.7}$$

one can transform Eq. (11.6) into the renormalization group equation (RGE):

$$\left\{\nu\frac{\partial}{\partial\nu} + \beta(\alpha_s)\alpha_s\frac{\partial}{\partial\alpha_s} - \sum_j \gamma_m(\alpha_s)m_j\frac{\partial}{\partial m_j} + \beta_G\frac{\partial}{\partial\alpha_G} - \gamma_\Gamma\right\}\Gamma^R = 0\,. \tag{11.8}$$

For $N_G,\ N_{NP}$ and N_F external gluon, ghost and fermion lines:

$$\gamma_\Gamma = -\frac{1}{2}[N_G\gamma_{3YM} + N_F\gamma_{2F} + N_{FP}\tilde{\gamma}_3]\,. \tag{11.9}$$

The expressions of the previous universal parameters can be easily deduced from their definitions as we shall show below.

11.2 The β function and the mass anomalous dimension

Noticing that, in the $\overline{MS}$ scheme, $\beta(\alpha_s)$ is mass-independent, one can, therefore, write [110,111]:

$$\alpha_s\beta(\alpha_s,\epsilon) = \nu\frac{d\alpha_s^R}{d\nu} = \nu\frac{d}{d\nu}\left(\alpha_s^B\nu^{-\epsilon}Z_\alpha^{-1}\right) = -\epsilon\alpha_s^R - \alpha_s^R\frac{1}{Z_\alpha}\nu\frac{dZ_\alpha}{d\nu}\,. \tag{11.10}$$

https://doi.org/10.1017/9781009290296.018 Published online by Cambridge University Press

The fact that Z_α is ν-independent allows us to also write:

$$\left\{\alpha_s^R\beta(\alpha_s,\epsilon)+\epsilon\alpha_s^R+\left(\alpha_s^R\right)^2\beta(\alpha_s,\epsilon)\frac{\partial}{\partial\alpha_s^R}\right\}Z_\alpha=0\,. \tag{11.11}$$

Using the expression of the Z_α in terms of the $1/\epsilon$ poles into the previous differential equation, one gets from the finite terms:

$$\alpha_s^R\beta(\alpha_s,\epsilon)=-\epsilon\alpha_s^R+\left(\text{finite term}\equiv\alpha_s^R\beta(\alpha_s)\right)\,. \tag{11.12}$$

Using this relation into the $1/\epsilon$ term, one can deduce:

$$\beta(\alpha_s)=\alpha_s^R\frac{\partial Z_\alpha}{\partial\alpha_s^R}\,, \tag{11.13}$$

i.e., $\beta(\alpha_s)$ is nothing else than the coefficient of the $1/\epsilon$-term of Z_α. The different coefficients of β are given in Table 11.1, showing that β is negative for $n\leq 11$ where n is the number of flavours. We shall see in the discussion of the running coupling that this negativity is important for an asymptotically free theory. We apply the same reasoning for obtaining the quark mass anomalous dimension defined as:

$$\gamma_m(\alpha_s)=-\frac{\nu}{m^R}\frac{dm^R}{d\nu}\bigg|_{g^B,m^B\text{ fixed}}\equiv\frac{\nu}{Z_m}\frac{dZ_m}{d\nu}\,. \tag{11.14}$$

where B and R refer to renormalized and bare quantities. Using the fact that in the $\overline{MS}$ scheme, Z_m is only function of ν and α_s, one gets:

$$\nu\frac{dZ_m}{d\nu}\equiv\left\{\nu\frac{\partial}{\partial\nu}+\beta(\alpha_s,\epsilon)\alpha_s\frac{\partial}{\partial\alpha_s}\right\}Z_m\,. \tag{11.15}$$

To lowest order of α_s, noting that the only dependence on Z_m is from α_s, and using the previous expression of the β function in Eq. (11.10), the previous differential equation can be written as:

$$\nu\frac{dZ_m}{d\nu}=\left\{-\epsilon\alpha_s\frac{\partial}{\partial\alpha_s}+\alpha_s\beta(\alpha_s)\frac{\partial}{\partial\alpha_s}\right\}Z_m\,. \tag{11.16}$$

Using the expression of Z_m, which is generically given by:

$$Z_m=1+\sum_n\frac{1}{\hat\epsilon^n}Z_m^{(n)}\,, \tag{11.17}$$

one can obtain that the mass anomalous dimension is given by the opposite of the $1/\epsilon$ pole coefficient in our sign convention ($d=4-\epsilon$). Analogous reasoning applies to the other anomalous dimensions, i.e., they are the opposite of the $1/\hat\epsilon$-coefficient. Their expressions are given in Table 11.1. The coefficients of the quark mass anomalous dimension and β functions have been calculated in the $\overline{MS}$ scheme by: [133] (γ_2), [134] (β_2), [135] (γ_3 and β_3) and [136] (γ_4 and β_4).

11.3 Gauge invariance of $\beta(\alpha_s)$ and γ_m in the $\overline{MS}$ scheme

One can also prove the gauge invariance of β and γ_m. This property leads to a great simplicity in their evaluation, as one can perform the calculation in a given gauge like the Feynman gauge $\alpha_G = 1$. For completing the proof, we start from a dimensionless Green's function Γ associated to a gauge-invariant amplitude. Using the fact that the bare Green's function is independent of the renormalization scale ν and of the gauge α_G, one has the RGE:

$$\left\{\nu\frac{\partial}{\partial\nu} + \beta(\alpha_s)\alpha_s\frac{\partial}{\partial\alpha_s} - \gamma_m(\alpha_s)m\frac{\partial}{\partial m} + \beta_G\frac{\partial}{\partial\alpha_G}\right\}\Gamma^R = 0\,. \tag{11.18}$$

The fact that it is gauge invariant gives:

$$\left(\frac{\partial}{\partial\alpha_G} + \alpha_s\rho\frac{\partial}{\partial\alpha_s} + \sigma m\frac{\partial}{\partial m}\right)\Gamma^R = 0\,, \tag{11.19}$$

with:

$$\alpha_s\rho \equiv \left.\frac{d\alpha_s}{d\alpha_G}\right|_{g^B,\ \epsilon\ \text{fixed}} \quad \text{and} \quad \sigma \equiv \frac{1}{m}\left.\frac{dm}{d\alpha_G}\right|_{g^B,\ \epsilon\ \text{fixed}}\,. \tag{11.20}$$

We apply the commutators of the operators in Eqs. (11.18) and (11.19) into Γ^R:

$$[\{\ldots\}\,,(\ldots)]\Gamma^R = 0\,. \tag{11.21}$$

Eliminating $\partial\Gamma^R/\partial\alpha_G$ with the help of Eq. (11.19), one obtains a third independent RGE:

$$\left\{\left[D\bar\beta - \bar\beta\frac{\partial(\alpha_s\rho)}{\partial\alpha_s}\right]\alpha_s\frac{\partial}{\partial\alpha_s} + \left[D\bar\gamma_m - \bar\beta\alpha_s\frac{\partial\sigma}{\partial\alpha_s}\right]m\frac{\partial}{\partial m}\right\}\Gamma^R(\alpha_s,\ \alpha_G,\ m) = 0\,, \tag{11.22}$$

where:

$$D \equiv \frac{\partial}{\partial\alpha_G} + \alpha_s\rho\frac{\partial}{\partial\alpha_s}\,, \qquad \bar\beta \equiv \beta - \rho\beta_G\,, \qquad \bar\gamma_m \equiv \gamma_m - \sigma\beta_G\,. \tag{11.23}$$

However, Γ^R depends only on the two conditions in Eqs. (11.18) and (11.19). Therefore the third equation should be trivially satisfied:

$$\begin{aligned} D\bar\beta - \bar\beta\frac{\partial(\alpha_s\rho)}{\partial\alpha_s} &= 0 \\ D\bar\gamma_m - \bar\beta\alpha_s\frac{\partial\sigma}{\partial\alpha_s} &= 0\,. \end{aligned} \tag{11.24}$$

Therefore, the RGE becomes:

$$\left\{\nu\frac{\partial}{\partial\nu} + \bar\beta(\alpha_s)\alpha_s\frac{\partial}{\partial\alpha_s} - \bar\gamma_m(\alpha_s)m\frac{\partial}{\partial m}\right\}\Gamma^R = 0\,, \tag{11.25}$$

which shows that the physical consequences of the RGE are gauge invariant. Recalling that in the $\overline{MS}$ scheme:

$$g_B = \nu^{\epsilon/2}g_R\left(1 + \sum_n\frac{a_n}{\hat\epsilon^n}\right) \equiv \nu^{\epsilon/2}g_R Z_\alpha^{1/2}\,, \tag{11.26}$$

https://doi.org/10.1017/9781009290296.018 Published online by Cambridge University Press

and using the previous definition of ρ, one gets:

$$\rho = -\frac{1}{Z_\alpha}\frac{dZ_\alpha}{d\alpha_G}\bigg|_{g^B,\ \epsilon\ \text{fixed}} = -\frac{1}{Z_\alpha}\left\{\frac{\partial a_1}{\partial \alpha_G}\frac{1}{\epsilon} + \frac{\partial a_2}{\partial \alpha_G}\frac{1}{\epsilon^2} + \cdots\right\} . \tag{11.27}$$

Then:

$$\rho\left(1+\frac{a_1}{\epsilon}\right) = -\frac{\partial a_1}{\partial \alpha_G}\frac{1}{\epsilon} + \mathcal{O}\left(\frac{1}{\epsilon^2}\right) , \tag{11.28}$$

which is only satisfied *if and only if* $\rho = 0$ because ρ is independent of ϵ (see its definition and its relation with $\bar{\beta}$ and β). One should notice that it is also due to the fact that in the $\overline{MS}$ scheme, Z_α has no constant term other than 1 (the $\ln 4\pi - \gamma$ term being already absorbed into $1/\hat{\epsilon}$). Inserting $\rho = 0$ into Eq. (11.24), one gets the desired result:

$$\frac{\partial \beta}{\partial \alpha_G} = 0 , \tag{11.29}$$

showing that β is gauge independent. With similar proofs, one also obtains $\sigma = 0$, leading to the gauge independence of γ_m.

11.4 Solutions of the RGE

One can now solve the RGE. If D is the dimension of Γ in units of mass and if one scales the momenta $p_1, \ldots, p_N$ by a dimensionless factor λ, the Euler theorem on homogeneous function gives:

$$\left\{\lambda\frac{\partial}{\partial\lambda} + \sum_j m_j\frac{\partial}{\partial m_j} + \nu\frac{\partial}{\partial\nu} - D\right\}\Gamma^R(\lambda p_1, \ldots, \lambda p_N; \alpha_s, \alpha_G, m_j, \nu) = 0 . \tag{11.30}$$

Introducing for convenience the dimensionless variables:

$$t \equiv \ln\lambda \qquad x_j \equiv m_j/\nu , \tag{11.31}$$

one arrives at the desired form of the RGE:

$$\left\{-\frac{\partial}{\partial t} + \beta(\alpha_s)\alpha_s\frac{\partial}{\partial\alpha_s} - \sum_j(1+\gamma_m(\alpha_s))x_j\frac{\partial}{\partial x_j} + \beta_G\frac{\partial}{\partial\alpha_G} + D - \gamma_\Gamma\right\}$$
$$\times\,\Gamma^R(e^t p_1, \ldots, e^t p_N;\ \alpha_s,\ \alpha_G,\ x_j,\ \nu) = 0 , \tag{11.32}$$

with the solution:

$$\Gamma^R(e^t p_1, \ldots, e^t p_N;\ \alpha_s,\ \alpha_G,\ x_j,\ \nu)$$
$$= \lambda^D\Gamma^R(p_1, \ldots, p_N;\ \bar{\alpha}_s, \overline{\alpha_G}, \bar{x}_j,\ t=0)\exp\left\{-\int_0^t dt'\gamma_\Gamma[\bar{\alpha}_s(t', \alpha_s)]\right\} . \tag{11.33}$$

https://doi.org/10.1017/9781009290296.018 Published online by Cambridge University Press

Table 11.1. *Anomalous dimension* $\gamma_i = \frac{\nu}{Z_i}\frac{dZ_i}{d\nu}$ $\equiv$ *coefficient of* $-1/\hat\epsilon$ *and coefficients of the* β *function in the* $\overline{MS}$ *scheme for* $SU(N)_c \times SU(n)_F$

Fermion field	$\gamma_{2F} = \left(\frac{\alpha_a}{\pi}\right)\frac{N^2-1}{2N}\frac{\alpha_G}{2} + \mathcal{O}\left(\frac{\alpha_s}{\pi}\right)^2$
Gluon field	$\gamma_{3YM} = -\left(\frac{\alpha_s}{\pi}\right)\left\{\frac{N}{4}\left(\frac{13}{3}-\alpha_G\right) - \frac{2}{3}\left(\frac{1}{2}\right)n\right\}$
Ghost field	$\tilde\gamma_3 = -\left(\frac{\alpha_s}{\pi}\right)\frac{N}{8}(3-\alpha_G)$
Mass	$\gamma_m = [\gamma_1 \equiv 2]\left(\frac{\alpha_s}{\pi}\right) + \left[\gamma_2 \equiv \frac{1}{6}\left(\frac{101}{2} - \frac{5n}{3}\right)\right]\left(\frac{\alpha_s}{\pi}\right)^2$
	$+\left[\gamma_3 \equiv \frac{1}{96}\left[3747 - \left(160\zeta_3 - \frac{2216}{9}\right)n - \frac{140}{27}n^2\right]\right]\left(\frac{\alpha_s}{\pi}\right)^3$
	$+\left[\gamma_4 \equiv \frac{1}{128}\left[\frac{4603055}{162} + \frac{135680}{27}\zeta_5 - 8800\zeta_5\right.\right.$
	$+\left(-\frac{91723}{27} - \frac{34192}{9}\zeta_3 + 880\zeta_4 + \frac{18400}{9}\zeta_5\right)n$
	$\left.\left.+\left(\frac{5242}{243} + \frac{800}{9}\zeta_3 - \frac{160}{3}\zeta_4\right)n^2 + \left(-\frac{332}{243} + \frac{64}{27}\zeta_3\right)n^3\right]\right]\left(\frac{\alpha_s}{\pi}\right)^4$
	for $N = 3$; $\zeta_3 = 1.2020569\ldots, \zeta_4$
	$= 1.0823232\ldots, \zeta_5 = 1.0369277\ldots$
Coupling constant	$\beta(\alpha_s) \equiv \frac{\nu}{\alpha_s}\frac{d\alpha_s}{d\nu} = -\frac{\nu}{Z_\alpha}\frac{dZ_\alpha}{d\nu}$
	$= \left[\beta_1 = -\frac{1}{2}\left(11 - \frac{2}{3}n\right)\right]\left(\frac{\alpha_s}{\pi}\right) + \left[\beta_2 = -\frac{1}{4}\left(51 - \frac{19}{3}n\right)\right]\left(\frac{\alpha_s}{\pi}\right)^2$
	$+\left[\beta_3 = -\frac{1}{64}\left(2857 - \frac{5033}{9}n + \frac{325}{27}n^2\right)\right]\left(\frac{\alpha_s}{\pi}\right)^3$
	$+\left[\beta_4 = -\frac{1}{128}\left[\left(\frac{149753}{6} + 3564\zeta_3\right) - \left(\frac{1078361}{162} + \frac{6508}{27}\zeta_3\right)n\right.\right.$
	$\left.\left.+\left(\frac{50065}{162} + \frac{6472}{81}\zeta_3\right)n^2 + \frac{1093}{729}n^3\right]\right]\left(\frac{\alpha_s}{\pi}\right)^4$ for $N = 3$
Gauge	$\beta_G = \nu\frac{d\alpha_G}{d\nu} = -\alpha_G\gamma_{3YM}$
Three-gluon	$\gamma_{1YM} = -\left[\left(\frac{17}{6} - \frac{3}{2}\alpha_G\right)\frac{N}{4} - \frac{2}{3}\frac{1}{2}n\right]\left(\frac{\alpha_s}{\pi}\right)$
Ghost-gluon-ghost	$\tilde\gamma_1 = \alpha_G\frac{N}{4}\left(\frac{\alpha_s}{\pi}\right)$
Fermion-gluon-fermion	$\gamma_{1F} = \frac{1}{2}\left[(3+\alpha_G)\frac{N}{4} - \alpha_G\frac{N^2-1}{2N}\right]$

where $\bar\alpha_s$, $\bar\alpha_G$ and $\bar x_j$ are respectively the running QCD coupling, gauge and mass, solutions of the differential equations:

$$\frac{d\bar\alpha_s}{dt} = \bar\alpha_s\beta(\bar\alpha_s) \quad : \quad \bar\alpha_s(0,\alpha_s) = \alpha_s^R(\nu)\,,$$
$$\frac{d\bar\alpha_G}{dt} = \beta_G(\bar\alpha_s) \quad : \quad \bar\alpha_G(0,\alpha_s) = \alpha_G(\nu)\,, \tag{11.34}$$

and:

$$\frac{d\bar x_i}{dt} = -[1+\gamma_m(\bar\alpha_s)]\bar x_i(t) \quad : \quad \bar x_i(0,\alpha_s) = x_i^R(\nu)\,. \tag{11.35}$$

https://doi.org/10.1017/9781009290296.018 Published online by Cambridge University Press

Their explicit expressions will be given later on. One should notice that the Green function has acquired an extra dimension induced by the exponential factor, which explains the name *anomalous dimension*.

11.5 Weinberg's theorem

In connection with the power counting theorem, one can derive a theorem on the asymptotic behaviour of the Green's function at large external momenta. This theorem is known as *Weinberg's theorem* [137].

It states that if non-exceptional momenta[1] are parametrized as:

$$p_{i_l} = \lambda k_{i_l} \quad : \quad l = 1, m \ , \tag{11.36}$$

the renormalized Feynman amplitude of a Feynman diagram G behaves as:

$$\Gamma^R(p_1, \ldots, p_n) \sim \lambda^\alpha \ln \lambda^\beta \ , \tag{11.37}$$

when $\lambda \to \infty$ and k_i kept fixed. Here β is undetermined, while:

$$\alpha = \max d(H) \tag{11.38}$$

where $d(H)$ is the superficial degree of divergence of the subdiagram H consisting of continuous path of lines connected to the external lines with momenta $p_{i_1}, \ldots, p_{i_m}$. For a renormalizable theory like QCD, the constant $d(H)$ can be obtained from Eq. (9.15) by taking $r = 0$. In other word, the Weinberg theorem tells us that the asymptotic limit in the deep Euclidean region $\lambda \to \infty$ is given by the naïve power counting times a logarithmic factor.

11.6 The RGE for the two-point function in the $\overline{MS}$ scheme

In order to illustrate this discussion, let us consider the generic two-point correlator:

$$\Pi(q^2) \equiv i \int d^4x \, e^{iqx} \, \langle 0|\mathcal{T} J(x)_H \, (J_H(0))^\dagger \, |0\rangle \ , \tag{11.39}$$

where $J_H(x)$ is the hadronic current of quark and/or gluon fields. In $n = 4 - \epsilon$ dimension, $\Pi(q^2)$ acquires an extra $\nu^{-\epsilon}$ dimension. The renormalized two-point correlator is [28,110,111]:

$$\Pi_R(q^2, \ \alpha_s, \ m_i, \ \nu) \equiv \Pi_B\left(q^2, \ \alpha_s^B, \ m_i^B, \ \epsilon\right) - \nu^{-\epsilon} C\left(q^2, \ \alpha_s^B, \ m_i^B, \ \epsilon\right) \ , \tag{11.40}$$

[1] A momentum configuration $(p_1, \ldots, p_n)$ of momenta are non-exceptional if *no* non-trivial partial sum $p_{i_1} + p_{i_2} + \cdots p_{i_m}$ where, (i_j take any of the label $1, \ldots n$) vanishes. On the contrary, an example of vanishing trivial sum is $p_1 + p_2 + \cdots + p_n = 0$, which is due to the energy-momentum conservation.

https://doi.org/10.1017/9781009290296.018 Published online by Cambridge University Press

where in the $\overline{MS}$ scheme, C is the ϵ-pole terms:

$$C\left(q^2,\ \alpha_s^B,\ m_i^B,\ \epsilon\right) = \sum_k \frac{1}{\epsilon^k} C_k(q^2,\ \alpha_s,\ m_j)\ , \tag{11.41}$$

where, as usual, C_k are constants or polynomials in m_j^2/q^2. Using the fact that Π_B is independent of ν, implies the differential equation:

$$\left\{\nu\frac{\partial}{\partial\nu} + \beta(\alpha_s)\alpha_s\frac{\partial}{\partial\alpha_s} - \sum_j \gamma_m(\alpha_s) m_j \frac{\partial}{\partial m_j}\right\}\Pi^R(q^2, \alpha_s,\ m_i,\ \nu)$$
$$= -\nu\frac{d}{d\nu}\left(\nu^{-\epsilon}\sum_k \frac{1}{\epsilon^k}C_k\right). \tag{11.42}$$

Rewriting:

$$\nu\frac{d}{d\nu}\left(\nu^{-\epsilon}\sum_k\frac{1}{\epsilon^k}C_k\right) = \left\{\nu\frac{\partial}{\partial\nu} + \nu\frac{d\alpha_s}{d\nu}\frac{\partial}{\partial\alpha_s} - \sum_j\gamma_m(\alpha_s)m_j\frac{\partial}{\partial m_j}\right\}\nu^{-\epsilon}\sum_k\frac{1}{\epsilon^k}C_k\ , \tag{11.43}$$

using:

$$\nu\frac{d\alpha_s}{d\nu} = -\epsilon\alpha_s + \alpha_s\beta(\alpha_s)\ , \tag{11.44}$$

and the fact that the equation is finite for $\epsilon \to 0$, one gets:

$$\lim_{\epsilon\to 0} :\ \nu\frac{d}{d\nu}\left(\nu^{-\epsilon}\sum_k\frac{1}{\epsilon^k}C_k\right) = -\frac{\partial}{\partial\alpha_s}(\alpha_s C)\ , \tag{11.45}$$

and the set of recursive equations for $k \geq 1$:

$$\left\{\alpha_s\beta(\alpha_s) - \sum_i \gamma_m m_i\frac{\partial}{\partial m_i}\right\}C_k = \frac{\partial}{\partial\alpha_s}(\alpha_s C_{k+1})\ . \tag{11.46}$$

The dimensionless condition of Π reads:

$$\left\{\nu\frac{\partial}{\partial\nu} + \lambda\frac{\partial}{\partial\lambda} + \sum_j m_j\frac{\partial}{\partial m_j}\right\}\Pi(\lambda^2,\ \nu^2,\ \alpha_s, m_i,\ \nu) = 0\ , \tag{11.47}$$

where $t \equiv \ln\lambda$. Therefore, one arrives at the RGE for the two-point function:

$$\left\{-\frac{\partial}{\partial t} + \beta(\alpha_s)\alpha_s\frac{\partial}{\partial\alpha_s} - \sum_j(1+\gamma_m(\alpha_s))x_j\frac{\partial}{\partial x_j}\right\}\Pi(t, \alpha_s, x_i) = \frac{\partial}{\partial\alpha_s}(\alpha_s C) \equiv D\ , \tag{11.48}$$

https://doi.org/10.1017/9781009290296.018 Published online by Cambridge University Press

with the solution:

$$\Pi(t, \alpha_s, x_i) = \Pi\,(t = 0, \bar{\alpha}_s(t), \bar{x}_i(t)) - \int_0^t dt' D[t - t',\ \bar{\alpha}_s(t'),\ \bar{x}_i(t')]\ , \quad (11.49)$$

where $\bar{\alpha}_s$ and $\bar{x}_i$ are running parameters solutions of the differential equations given in Eq. (11.34), and which will be given explicitly in the following.

11.7 Running coupling

11.7.1 Lowest order expression and the definition of the QCD scale Λ

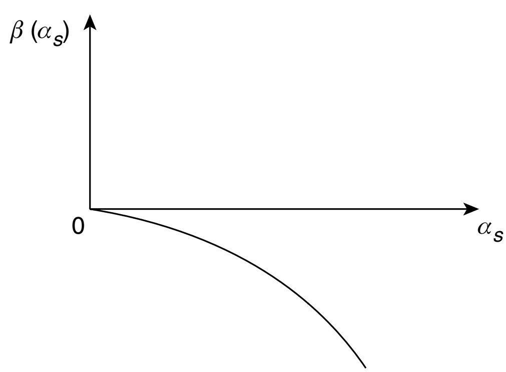
Solving the differential equation in Eq. (11.34), the expression of the running coupling, to one-loop accuracy is:

$$a_s^{(0)}(t, \alpha_s) = \frac{a_s(\nu)}{1 - \beta_1 a_s(\nu) t}\ , \quad (11.50)$$

where:

$$a_s \equiv \frac{\alpha_s}{\pi}\ ,$$
$$t \equiv \frac{1}{2} \ln \frac{-q^2}{\nu^2}\ , \quad (11.51)$$

and β_1 is the first coefficient of the β function given in Table 11.1. It shows that for $t \to +\infty$, $a_s^{(0)} \to 0$ for $\beta_1 < 0$, which is satisfied for the number of quark flavours $n_f \leq 11$. In this case, the theory is *asymptotically free* and the use of perturbation theory is legitimate. The point $\alpha_s = 0$ is an UV fixed point as shown in Fig. 11.1 because the β-function has a negative slope at the origin.



We can also re-write the solution as:

$$t = \int \frac{dz}{z} \frac{1}{\beta(z)} \equiv \varphi(z) + \text{constant} \quad (11.52)$$

https://doi.org/10.1017/9781009290296.018 Published online by Cambridge University Press

where the constant term is a renormalization group invariant (RGI) quantity, which one identifies as:

$$t - \varphi(z) \equiv \frac{1}{2} \ln \nu^2 + \frac{1}{\beta_1 a_s(\nu)} = \text{constant} \equiv \frac{1}{2} \ln \Lambda^2 , \tag{11.53}$$

where Λ is a RGI *but* renormalization scheme-dependent quantity. Therefore, the *running coupling*, in terms of Λ to one-loop accuracy, reads:

$$a_s^{(0)}(q^2) = \frac{1}{-\beta_1} \frac{1}{\frac{1}{2} \ln \frac{-q^2}{\Lambda^2}} . \tag{11.54}$$

11.7.2 Renormalization group invariance of the first two coefficients of β

Before discussing the high-order expression of the coupling, let us discuss the renormalization group invariance of the first two coefficients of the β function. Let β^a and β^b the β functions related to two different values of the subtraction ν_a and ν_b of the $\overline{MS}$ scheme. Using Eq. (11.52), we have:

$$t_b \equiv \frac{1}{2} \ln \frac{-q^2}{\nu_b^2} = \int_{\alpha_s(\nu_a)}^{\bar{\alpha}_s(t_b, \alpha_s(\nu_b^2))} \frac{dz}{z} \frac{1}{\beta^a(z)} \equiv \varphi(z) . \tag{11.55}$$

Applying the operator $\nu_b \partial/\partial \nu_b$ to both sides of Eq. (11.55), and using the fact that $\bar{\alpha}_s(t_b, \alpha_s(\nu_b^2))$ obeys the differential equation:

$$\left\{ \nu_b \frac{\partial}{\partial \nu_b} + \beta^b \alpha_s(\nu_b) \frac{\partial}{\partial \alpha_s(\nu_b)} \right\} \bar{\alpha}_s(t_b,\ \alpha_s(\nu_b)) = 0 , \tag{11.56}$$

one obtains:

$$-1 = -\left(\frac{1}{\alpha_s(\nu_a)\beta^a} \right) \beta^b \alpha_s(\nu_b) \frac{\partial \alpha_s(\nu_a)}{\partial \alpha_s(\nu_b)} \Longrightarrow \beta^a = \beta^b \left(\frac{\alpha_s(\nu_b)}{\alpha_s(\nu_a)} \right) \left(\frac{\partial \alpha_s(\nu_b)}{\partial \alpha_s(\nu_a)} \right) . \tag{11.57}$$

Using the α_s expansion:

$$\beta^a = \beta_1^a \left(\frac{\alpha_s}{\pi} \right) (\nu_a) + \beta_2^a \left(\frac{\alpha_s}{\pi} \right)^2 (\nu_a) + \cdots ,$$
$$\beta^b = \beta_1^b \left(\frac{\alpha_s}{\pi} \right) (\nu_b) + \beta_2^b \left(\frac{\alpha_s}{\pi} \right)^2 (\nu_b) + \cdots , \tag{11.58}$$

and the relation:

$$\alpha_s(\nu_a) = \alpha_s(\nu_b) + c\alpha_s^2(\nu_b) , \tag{11.59}$$

where c is an arbitrary constant depending on the subtraction scale, one can easily deduce:

$$\beta_1^a = \beta_1^b \quad \text{and} \quad \beta_2^a = \beta_2^b , \tag{11.60}$$

which achieves the proof of the RGI invariance of β_1 and β_2. The higher-order terms of the β function will be affected by the coefficient c and hence on the subtraction scale.

11.7.3 Higher order expression

The previous result can be extended to higher orders. To order α_s^2, one can write the solution of Eq. (11.52) as:

$$
\begin{aligned}
t &= \int \frac{dz}{z^2} \frac{\pi}{\beta_1 \left(1 + (\beta_2/\beta_1)(z/\pi)\right)} + \text{constant} \\
&= \frac{\pi}{\beta_1} \left\{ -\frac{1}{z} + \frac{\beta_2}{\beta_1} \frac{1}{\pi} \ln \left(\frac{1 + (\beta_2/\beta_1)(z/\pi)}{z} \right) \right\} + \text{constant} \,, \qquad (11.61)
\end{aligned}
$$

where the constant is a RGI quantity which has been fixed to be $\ln \Lambda$ to lowest order. At the two-loop level, it is convenient to fix it as in [138]:

$$
\text{constant} \equiv \ln \Lambda(1 \text{ loop}) - \frac{\beta_2}{\beta_1^2} \ln \left(-\frac{\beta_1}{2\pi} \right) . \qquad (11.62)
$$

Therefore, we get the RGI quantity to two loops:

$$
\ln \nu + \frac{1}{\beta_1 a_s} - \frac{\beta_2}{\beta_1^2} \ln \left(\frac{1 + (\beta_2/\beta_1)(a_s)}{a_s \pi} \right) = \ln \Lambda(\text{two loops}) - \frac{\beta_2}{\beta_1^2} \ln \left(-\frac{\beta_1}{2\pi} \right) . \qquad (11.63)
$$

Expanding Eq. (11.61), and inserting the expression of the running α_s to one loop, we deduce:

$$
a_s(q^2)^{(2)} = a_s^{(0)} \left\{ 1 - a_s^{(0)} \frac{\beta_2}{\beta_1} \ln \ln \frac{\nu^2}{\Lambda^2} \right\} . \qquad (11.64)
$$

It is not difficult to show that, to order α_s^2, one can relate the one- and two-loop values of Λ as:

$$
\Lambda(\text{two loops}) = \left(-\frac{\beta_1}{2\pi} \right)^{\beta_2/\beta_1^2} \Lambda(1 \text{ loop}) \qquad (11.65)
$$

To three-loop accuracy the running coupling can be parametrized as:

$$
\begin{aligned}
a_s(\nu) = a_s^{(0)} \Bigg\{ & 1 - a_s^{(0)} \frac{\beta_2}{\beta_1} \ln \ln \frac{\nu^2}{\Lambda^2} \\
& + \left(a_s^{(0)}\right)^2 \left[\frac{\beta_2^2}{\beta_1^2} \ln^2 \ \ln \frac{\nu^2}{\Lambda^2} - \frac{\beta_2^2}{\beta_1^2} \ln \ln \frac{\nu^2}{\Lambda^2} - \frac{\beta_2^2}{\beta_1^2} + \frac{\beta_3}{\beta_1} \right] + \mathcal{O}\left(a_s^3\right) \Bigg\} , \qquad (11.66)
\end{aligned}
$$

with β_i are the $\mathcal{O}(a_s^i)$ coefficients of the β function in the $\overline{MS}$ scheme for n_f flavours (see Table 2.2), which, for three flavours, read:

$$
\beta_1 = -9/2 \,, \qquad \beta_2 = -8 \,, \qquad \beta_3 = -20.1198 \,. \qquad (11.67)
$$

https://doi.org/10.1017/9781009290296.018 Published online by Cambridge University Press

Λ is a renormalization group invariant scale but is renormalization scheme dependent. The running coupling α_s has been measured from LEP, τ decays[2] and deep-inelastic scattering data. We shall discuss these determinations in the next chapter. The present *world average* is [16,139]:

$$\alpha_s(M_Z) = 0.1181 \pm 0.0027 \,. \tag{11.68}$$

11.8 Decoupling theorem

The decoupling theorem of Appelquist and Carazzone [140] states that the effect of heavy particles (fermion, boson) of mass $M_H^2 \gg -q^2$ can be ignored below their thresholds. However, in the MS and $\overline{MS}$ schemes, these heavy particles could contribute to the universal β and γ functions as they are mass independent, and therefore the MS and $\overline{MS}$ schemes do not a priori satisfy this theorem. In order to satisfy this theorem, one should modify the scheme. References [141–143] have proposed to absorb into the renormalization constant, not only the $1/\hat{\epsilon}$ pole but also terms of the type $\ln^n M_H/\nu$ coming from heavy fermion or boson loops (ν being the scale of the $\overline{MS}$ scheme). In such an effective theory, one can relate the QCD scale of n light quarks to the one with n light plus one heavy flavour. To one loop, this relation is:

$$\Lambda_{n+1} = \Lambda_n \left(\frac{M_H^2}{p^2}\right)^{\frac{1}{3\beta_1}} \,. \tag{11.69}$$

At the heavy quark threshold $p^2 = 4M_H^2$, one can see that the heavy quark effect tends to decrease slightly the value of Λ. One can see more explictly such effects in Table 11.2.

11.9 Input values of α_s and matching conditions

We shall discuss below, how this decoupling is used in the practical evaluation of the running coupling. In so doing, we run the value of $\alpha_s(M_Z)$ in the range given in Table 11.2, to lower scales by taking appropriately the threshold effects due to heavy quark productions. We run this value until $M_b = 4.6$–4.7 GeV, using the two-loop relation:

$$\frac{\alpha_s}{\pi} = a_s^{(0)} \left(1 - a_s^{(0)} \frac{\beta_2}{\beta_1} \ln\ln(-q^2/\Lambda^2)\right) \tag{11.70}$$

and for n_f flavours, we note that:

$$\beta_1 = -\frac{11}{2} + \frac{n_f}{3} \quad \text{and} \quad \beta_2 = -\frac{51}{4} + \frac{19}{12} n_f \,. \tag{11.71}$$

[2] This process gives so far the most precise measurement of α_s at M_Z as a modest accuracy at the τ-mass becomes a precise value at the Z-mass because the errors decrease faster than the running of α_s. Also, here, compared with some other determinations, we have relatively the best theoretical control including the perturbative corrections to order α_s^4, the non-perturbative condensates and the resummation of the asymptotic series.

Table 11.2. *Value of α_s and Λ to two-loops at different scales and flavours*

$\alpha_s(M_Z)$	Λ_5[MeV]	$\alpha_s(M_b)$	Λ_4[MeV]	$\alpha_s(M_c)$	Λ_3[MeV]	$\alpha_s(M_\tau)$
0.112	160	0.198	240	0.312	290	0.277
0.118	225	0.218	325	0.372	375	0.319
0.124	310	0.241	432	0.463	480	0.378
0.127	360	0.254	495	0.528	540	0.417

Following references [144,145], we do the matching condition $\alpha_s^{(5)} = \alpha_s^{(4)}$ at this b-mass, in order to extract α_s for four flavours. We continue iteratively this procedure for completing Table 11.2, which is one of the basic inputs of numerous phenomenological analyses discussed in this book. We use here the value of the perturbative pole mass to two-loops: $M_b = 4.62$ GeV and $M_c = 1.42$ GeV which we shall discuss later on. Notice that doing a similar procedure at the three-loop level, we reproduce the value of α_s given in [139]. In this case, one can use the three-loop relation at the subtraction scale M_H [146]:

$$\alpha_s^{(n_f-1)} = \alpha_s^{(n_f)}\left[1 - 0.291667a_s^2 - [5.32389 - (n_f - 1)0.26247]a_s^3\right] , \quad (11.72)$$

where: $a_s \equiv \alpha_s^{(n_f)}/\pi$.

11.10 Running gauge

The running gauge $\bar{\alpha}_G$ is the solution of the differential equation in Eq. (11.35). To leading order in α_s, it reads [110]:

$$\bar{\alpha}_G(-q^2) = \frac{\hat{\alpha}_G}{\left[\frac{1}{2}\ln(-q^2/\Lambda)\right]^{\delta/-\beta_1}} \left\{1 + \frac{N}{4\delta}\frac{\hat{\alpha}_G}{\left[\frac{1}{2}\ln(-q^2/\Lambda)\right]^{\delta/-\beta_1}}\right\}^{-1} , \quad (11.73)$$

where for $SU(N)_c \times SU(n)_F$:

$$\delta = \frac{13}{12}N - \frac{n}{3} . \quad (11.74)$$

$\hat{\alpha}_G$ is a renormalization group invariant parameter defined to one loop as:

$$\hat{\alpha}_G = \frac{\alpha_G(\nu)}{1 - \frac{N}{4\delta}\alpha_G(\nu)}\left(\frac{1}{-\beta_1 a_s(\nu)}\right)^{\delta/-\beta_1} . \quad (11.75)$$

It is interesting to notice that for $n \leq 9$, the running gauge tends to the Landau gauge ($\alpha_G = 0$) for $-q^2 \to \infty$. One also obtains:

$$\bar{\alpha}_G(q^2) = \alpha_G(\nu) , \quad (11.76)$$

for $\alpha_G = 0$ (Landau gauge) to all orders and for $\alpha_G = 4\delta$ (peculiar gauge) to lowest order in α_s.

11.11 Running masses

The running masses are solutions of the differential equation in Eq. (11.35). Analogously to Λ, one can also introduce an invariant mass $\hat{m}_i$ [28]. The expression of the running quark mass in terms of the invariant mass $\hat{m}_i$ is [28]:

$$\bar{m}_i(\nu) = \hat{m}_i\,(-\beta_1 a_s(\nu))^{-\gamma_1/\beta_1}\left\{1 + \frac{\beta_2}{\beta_1}\left(\frac{\gamma_1}{\beta_1} - \frac{\gamma_2}{\beta_2}\right)a_s(\nu)\right.$$
$$\left. + \frac{1}{2}\left[\frac{\beta_2^2}{\beta_1^2}\left(\frac{\gamma_1}{\beta_1} - \frac{\gamma_2}{\beta_2}\right)^2 - \frac{\beta_2^2}{\beta_1^2}\left(\frac{\gamma_1}{\beta_1} - \frac{\gamma_2}{\beta_2}\right) + \frac{\beta_3}{\beta_1}\left(\frac{\gamma_1}{\beta_1} - \frac{\gamma_3}{\beta_3}\right)\right]a_s^2(\nu) + \mathcal{O}\big(a_s^3\big)\right\}, \quad (11.77)$$

where γ_i are the $\mathcal{O}(a_s^i)$ coefficients of the quark-mass anomalous dimension (see Table 11.1). For three flavours, we have:

$$\gamma_1 = 2\,, \qquad \gamma_2 = 91/12\,, \qquad \gamma_3 = 24.8404\,. \quad (11.78)$$

As we shall see later on, QSSR is, at present, the most appropriate theoretical method for extracting the absolute values of the light quark masses. A long list of these determinations is given in the recent review [54] (see also [57] and the chapter on quark masses in this book), where the QSSR results are compared with the ones from chiral perturbation theory and lattice calculations. We only quote below the results:

$$\bar{m}_d(2\ \text{GeV}) = (6.5 \pm 1.2)\ \text{MeV}, \qquad \bar{m}_u(2\ \text{GeV}) = (3.6 \pm 0.6)\ \text{MeV}\,, \quad (11.79)$$

and:

$$\bar{m}_s(2\ \text{GeV}) = (117.4 \pm 23.4)\ \text{MeV}\,, \quad (11.80)$$

and the bounds from the positivity of the spectral functions:

$$90\ \text{MeV} \le \bar{m}_s(2\ \text{GeV}) \le 168\ \text{MeV}\,. \quad (11.81)$$

The running masses of the c and b quarks have been also extracted directly from the J/ψ and Υ sum rules. To two-loop (order α_s) accuracy, one obtains [149]:

$$\bar{m}_c(M_c) = \big(1.23^{+0.02}_{-0.04} \pm 0.03\big)\ \text{GeV} \qquad \bar{m}_b(M_b) = \big(4.23^{+0.03}_{-0.04} \pm 0.02\big)\ \text{MeV}\,. \quad (11.82)$$

From the D and B meson systems, one obtains to order α_s^2 [150]:

$$\bar{m}_c(M_c) = \big(1.10 \pm 0.04\big)\ \text{GeV} \qquad \bar{m}_b(M_b) = \big(4.05 \pm 0.06\big)\ \text{MeV}\,, \quad (11.83)$$

https://doi.org/10.1017/9781009290296.018 Published online by Cambridge University Press

which agree with the former within the errors though the central values are slightly lower. These results can be compared with different results based on non-relativistic and some other approaches [16].

11.12 The perturbative pole mass

The notion of perturbative pole mass can be useful in the phenomenology of the heavy quark systems. However, unlike in QED, where the pole mass is well-defined, due to the observation of the lepton, this definition is ambiguous in QCD due to confinement. Attempts to define the pole mass within perturbation theory have been done in the literature [141,133,148]. By analogy with QED, one can define the pole mass as the pole of the quark propagator. For definiteness, on can start with the bare quark propagator:

$$S_F(p) = \frac{1}{\hat{p} - M_B - i\epsilon} , \tag{11.84}$$

After interaction, one has:

$$S_F(p) = \left(\frac{1}{1-\Sigma_2}\right) \frac{1}{\hat{p} - M_B\left[1 + \frac{\Sigma_1}{1-\Sigma_2}\right]} \tag{11.85}$$

which shows explicitly the wave function and the mass renormalization constants in Eq. (9.20). An explicit evaluation of $\Sigma_{1,2}$ in the $\overline{MS}$ scheme gives:

$$\begin{aligned}
\Sigma_1^B = & (g_B \nu^{-\epsilon/2})^2 \frac{C_F}{(16\pi^2)^{1-\epsilon/4}} \int_0^1 dx \\
& \times \left[\Gamma(\epsilon/2) \left(\frac{\mathbf{R}^2}{\nu^2}\right)^{-\epsilon/2} [2(2-x) - \epsilon(1-x) + (1-\alpha_G)(1-2x)] \right. \\
& \left. + (1-\alpha_G) 2x(1-x) \frac{p^2}{M_B^2 - p^2 x} \right]
\end{aligned} \tag{11.86}$$

$$\begin{aligned}
\Sigma_2^B = & (g_B \nu^{-\epsilon/2})^2 \frac{C_F}{(16\pi^2)^{1-\epsilon/4}} \int_0^1 dx \\
& \times \left[\Gamma(\epsilon/2) \left(\frac{\mathbf{R}^2}{\nu^2}\right)^{-\epsilon/2} [-2x + \epsilon(1-x) + (1-\alpha_G)2(1-x)] \right. \\
& \left. + (1-\alpha_G) 2x(1-x) \frac{p^2}{M_B^2 - p^2 x} \right],
\end{aligned} \tag{11.87}$$

where:

$$\mathbf{R}^2 = (1-x)\left(M_B^2 - p^2 x\right) - i\epsilon'. \tag{11.88}$$

α_G is the covariant gauge parameter and $C_F = (N^2-1)/(2N)$ for $SU(N)_c$. These parametric integrals lead to:

$$\Sigma_1^B = \left(\frac{\alpha_s}{\pi}\right) C_F \frac{1}{2}\left\{\frac{3}{\hat{\epsilon}} + \frac{5}{2} - \frac{3}{2}\ln\frac{M_B^2 - p^2}{\nu^2} + \left(\frac{1}{2}\right)\frac{M_B^2}{-p^2}\left[1 - \left(4 + \frac{M_B^2}{-p^2}\right)\ln\left(1 - \frac{p^2}{M_B^2}\right)\right] + (1-\alpha_G)\left[-\frac{1}{2} - \frac{1}{2}\frac{M_B^2}{-p^2} + \frac{1}{2}\frac{M_B^2}{-p^2}\left(1 + \frac{M_B^2}{-p^2}\right)\ln\left(1 - \frac{p^2}{M_B^2}\right)\right]\right\}, \quad (11.89)$$

$$\Sigma_2^B = \left(\frac{\alpha_s}{\pi}\right) C_F \frac{1}{4}[-1 + (1-\alpha_G)]\left\{\frac{2}{\hat{\epsilon}} + 1 - \ln\frac{M_B^2 - p^2}{\nu^2} + \left(\frac{M_B^2}{-p^2}\right)^2 \ln\left(1 - \frac{p^2}{M_B^2}\right) - \frac{M_B^2}{-p^2}\right\}, \quad (11.90)$$

with:

$$1/\hat{\epsilon} \equiv 1/\epsilon + \frac{1}{2}(\ln 4\pi - \gamma), \quad (11.91)$$

which shows that Σ_2^B vanishes to order α_s in the Landau gauge $\alpha_G = 0$. Their asymptotic expressions are:

$$\Sigma_1^B\big|_{p^2 \gg M^2} = \left(\frac{\alpha_s}{\pi}\right) C_F \frac{1}{2}\left\{\frac{3}{\hat{\epsilon}} + \frac{5}{2} - \frac{3}{2}\ln\frac{-p^2}{\nu^2} + \frac{1}{2}(1-\alpha_G) + \mathcal{O}\left(\frac{M^2}{-p^2}\ln\frac{-p^2}{M^2}\right)\right\},$$
$$\Sigma_2^B\big|_{p^2 \gg M^2} = \left(\frac{\alpha_s}{\pi}\right) C_F \frac{1}{4}[-1 + (1-\alpha_G)]\left\{\frac{2}{\hat{\epsilon}} + 1 - \ln\frac{-p^2}{\nu^2} + \mathcal{O}\left(\frac{M^2}{-p^2}\ln\frac{-p^2}{M^2}\right)\right\}, \quad (11.92)$$

and:

$$\Sigma_1^B\big|_{p^2 \ll M^2} = \left(\frac{\alpha_s}{\pi}\right) C_F \frac{1}{2}\left\{\frac{3}{\hat{\epsilon}} - \frac{3}{2}\ln\frac{M_B^2}{\nu^2} + \frac{3}{4} + \frac{5}{6}\left(\frac{-p^2}{M_B^2}\right) + (1-\alpha_G)\left[-\frac{1}{4} - \frac{1}{12}\left(\frac{-p^2}{M_B^2}\right)\right]\right\},$$
$$\Sigma_2^B\big|_{p^2 \ll M^2} = \left(\frac{\alpha_s}{\pi}\right) C_F \frac{1}{4}[-1 + (1-\alpha_G)]\left\{\frac{2}{\hat{\epsilon}} - \ln\frac{M^2}{\nu^2} + \frac{1}{2} - \frac{2}{3}\left(\frac{-p^2}{M_B^2}\right)\right\}, \quad (11.93)$$

At $p^2 = M^2 = \nu^2$, one gets:

$$\Sigma_1^B\big|_{p^2 = M^2 = \nu^2} = \left(\frac{\alpha_s}{\pi}\right) C_F \frac{1}{2}\left[\frac{3}{\hat{\epsilon}} + 2\right], \quad (11.94)$$

which is gauge independent. It is related to the pole mass, which is defined at the pole $p^2 = M^2$ of the full quark propagator through Eq. (11.85). In terms of the running mass, the pole mass reads:

$$M_{\text{pole}} = \bar{m}(p^2)\left\{1 + \frac{\Sigma_1(p^2 = M^2)}{1 - \Sigma_2(p^2 = M^2)}\right\} , \tag{11.95}$$

Therefore, the previous expressions gives [148]:

$$M_{\text{pole}} = \bar{m}(p^2)\left\{1 + \left(\frac{4}{3} + \ln\frac{p^2}{M^2}\right)\left(\frac{\alpha_s}{\pi}\right)\right\} , \tag{11.96}$$

which is gauge and renormalization scheme independent. The IR finiteness of the result to order α_s^2 has been explicitly shown in [133]. The independence of M_{pole} on the choice of the regularization scheme has been demonstrated in [148]. The extension of the previous result to order α_s^2 is [151]:

$$\begin{aligned} M_{\text{pole}} = \bar{m}(p^2)\Bigg[& 1 + \left(\frac{4}{3} + \ln\frac{p^2}{M^2}\right)\left(\frac{\alpha_s}{\pi}\right) \\ & + \left[K_Q + \left(\frac{221}{24} - \frac{13}{36}n\right)\ln\frac{p^2}{M^2} + \left(\frac{15}{8} - \frac{n}{12}\right)\ln^2\frac{p^2}{M^2}\right]\left(\frac{\alpha_s}{\pi}\right)^2\Bigg] , \end{aligned} \tag{11.97}$$

where, in the RHS, M is the pole mass and:

$$K_Q = 17.1514 - 1.04137n + \frac{4}{3}\sum_{i \neq Q}\Delta\left(r \equiv \frac{m_i}{M_Q}\right) . \tag{11.98}$$

For $0 \leq r \leq 1$, $\Delta(r)$ can be approximated, within an accuracy of 1% by:

$$\Delta(r) \simeq \frac{\pi^2}{8}r - 0.597r^2 + 0.230r^3 , \tag{11.99}$$

while, its values in the following limiting cases are:

$$\begin{aligned} \Delta(r \to 0) &\simeq \frac{3}{4}\zeta(2)r + \mathcal{O}(r^2) , \\ \Delta(r \to \infty) &\simeq \frac{1}{4}\ln^2 r + \frac{13}{24}\ln r + \frac{1}{4}\zeta(2) + \frac{151}{288} + \mathcal{O}(r^{-2}\ln r) , \\ \Delta(r = 1) &\simeq \frac{3}{4}\zeta(2) - \frac{3}{8} . \end{aligned} \tag{11.100}$$

As, one can notice, the behaviour of $\Delta(r \to \infty)$ is quite bad, such that in the effective field theory where the heavy quark mass tends to infinity, one should write a well-defined relation in this limit. This can be achieved by introducing the coupling and light quark masses in the effective field theory in terms of the corresponding quantities in the full theory [144]:

$$\alpha_s^{\text{eff}}(\nu) = \alpha_s(\nu) C(\alpha_s(\nu), x)$$
$$m^{\text{eff}}(\nu) = m(\nu) H(\alpha_s(\nu), x)\,, \tag{11.101}$$

where $x \equiv \ln(\bar{m}_h^2/\nu^2)$ and:

$$C(\alpha_s, x) = 1 + \sum_{k\geq 1} C_k \left(\frac{\alpha_s}{\pi}\right)^k, \qquad C_k(x) = \sum_{0\leq i\leq k} C_{ik} x^i\,,$$
$$H(\alpha_s, x) = 1 + \sum_{k\geq 1} H_k \left(\frac{\alpha_s}{\pi}\right)^k, \qquad H_k(x) = \sum_{0\leq i\leq k} H_{ik} x^i\,, \tag{11.102}$$

with:

$$C_1 = \frac{x}{6}\,, \qquad C_2 = \frac{11}{72} + \frac{11}{24}x + \frac{x^2}{36}\,,$$
$$H_1 = 0\,, \qquad H_2 = \frac{89}{32} + \frac{5}{36}x + \frac{x^2}{12}\,, \tag{11.103}$$

and by expressing α_s^{eff} in terms of the pole mass:

$$\alpha_s^{\text{eff}} = \alpha_s \left\{1 + \frac{X}{6}\left(\frac{\alpha_s}{\pi}\right) + \left(-\frac{7}{24} + \frac{19X}{24} + \frac{X^2}{36}\right)\left(\frac{\alpha_s}{\pi}\right)^2\right\}\,, \tag{11.104}$$

where $X \equiv \ln(M_h^2/\nu^2)$. In this way, the previous expression becomes:

$$M_{\text{pole}} = \bar{m}(p^2)\left[1 + \left(\frac{4}{3} + \ln\frac{p^2}{\bar{m}^2}\right)\left(\frac{\alpha_s}{\pi}\right) + \left[K_Q(\bar{m}_f/\bar{m}) + \left(\frac{173}{24} - \frac{13}{36}n\right)\ln\frac{p^2}{\bar{m}^2} + \left(\frac{15}{8} - \frac{n}{12}\right)\ln^2\frac{p^2}{\bar{m}^2}\right]\left(\frac{\alpha_s}{\pi}\right)^2\right]\,, \tag{11.105}$$

where $\bar{m}$ is the running mass of the finite mass heavy quark, n is the number of finite mass quark flavours and the summation in K_Q through $\Delta(\bar{m}_f/\bar{m})$ runs over the $n-1$ lightest quarks. For instance, in the case of the bottom quark mass, one uses $n = 5$, and deduce:

$$M_b = \bar{m}_b(p^2)\left[1 + \left(\frac{4}{3} + \ln\frac{p^2}{\bar{m}_b^2}\right)\left(\frac{\alpha_s^{\text{eff}}}{\pi}\right) + \left[K_Q(\bar{m}_f/\bar{m}_b) + \frac{389}{72}\ln\frac{p^2}{\bar{m}_b^2} + \frac{35}{24}\ln^2\frac{p^2}{\bar{m}^2}\right]\left(\frac{\alpha_s}{\pi}\right)^2\right]\,, \tag{11.106}$$

where, by neglecting the u and d quark masses:

$$K_Q(\bar{m}_f/\bar{m}_b) = 9.278 + \frac{4}{3}\sum_{f\equiv s,c}\Delta(\bar{m}_f/\bar{m}_b)\,. \tag{11.107}$$

https://doi.org/10.1017/9781009290296.018 Published online by Cambridge University Press

Finally, a recent order α_s^3 evaluation leads to [152]:

$$\bar{m}(M_{\text{pole}}) = M_{\text{pole}} \left[1 - \frac{4}{3}\left(\frac{\alpha_s}{\pi}\right) + [-14.3323 - 1.0414n]\left(\frac{\alpha_s}{\pi}\right)^2 \right.$$
$$\left. + [-198.7068 + 26.9239n - 0.65269n^2]\left(\frac{\alpha_s}{\pi}\right)^3 \right] . \quad (11.108)$$

However, one should be careful when using the previous mass in the OPE, as, in order to be consistent, one should use the same truncations in the mass definition and in the hadronic correlator to be analysed. For this reason, the re-summed result obtained to leading order in $(\beta_1\alpha_s)$ term within the large n_f-limit [154], should also be used with care. Using the previous relation with the pole and running mass as well as a direct estimate of the two-loop order α_s running mass from the ψ and Υ-sum rules, one obtains the value of the pole mass to two-loop accuracy [149]:[3]

$$M_c^{PT2} = (1.42 \pm 0.03) \text{ GeV} , \qquad M_b^{PT2} = (4.62 \pm 0.02) \text{ GeV} . \quad (11.109)$$

It is informative to compare these values with those of the pole masses from non-relativistic sum rules to two loops [149]:

$$M_c^{NR} = \left(1.45^{+0.04}_{-0.03} \pm 0.03\right) \text{ GeV} , \qquad M_b^{NR} = \left(4.69^{+0.02}_{-0.01} \pm 0.02\right) \text{ GeV} , \quad (11.110)$$

and, recently, to three loops of order α_s^2 [4] including a resummation of the Coulombic corrections [156]:

$$M_b^{NR} = (4.60 \pm 0.02) \text{ GeV} , \quad (11.111)$$

in good agreement with the former results.

If one uses the value of the running mass obtained to three-loop accuracy [156], and the three-loop relation between the pole and the running mass, one obtains:[5]

$$M_b^{PT3} \simeq (4.7 \pm 0.07 \pm 0.02) \text{ GeV} , \quad (11.112)$$

which, although slightly higher, is in agreement within the errors with the two-loop result.

Recent extension of the sum rules analysis [157,159] have led to more accurate values of the pole mass. The one using the relation between the pole and the 1S meson mass gives [159]:

$$M_b^{PT3} \simeq (4.71 \pm 0.03) \text{ GeV} , \quad (11.113)$$

in agreement with the two-loop α_s result given in Eq. (11.109).

One can also compare the previous values with the *dressed mass*:

$$M_b^{nr} = (4.94 \pm 0.10 \pm 0.03) \text{ GeV} , \quad (11.114)$$

[3] We shall discuss these different points in more details in the chapter on quark masses.
[4] This result can be considered to be an improvment of the Voloshin value of 4.8 GeV [155].
[5] This value is slightly lower than the one given in [149], as the value of the running mass used there is higher. However, the results agree within the errors.

https://doi.org/10.1017/9781009290296.018 Published online by Cambridge University Press

obtained from a non-relativistic Balmer formula based on a $\bar{b}b$ Coulomb potential and including higher order α_s^4-corrections [94], or the mass obtained from the fit of the spectra within potential models [12]:

$$M_b^{\rm pot} \simeq (4.8 \sim 4.9)\ \text{GeV}. \tag{11.115}$$

This non-relativistic mass is slightly higher than the one from the sum rules. One can remark that the mass difference is :

$$M_b^{nr} - M_b^{PT} \approx (100 \sim 200)\ \text{MeV}\ . \tag{11.116}$$

The interpretation of this mass difference is not very well understood. If one has in mind that the non-relativistic pole mass contains a non-perturbative part, which can be of the same origin as the one induced by the truncation of the perturbative series at large order, then one might eventually consider this value as a phenomenological estimate of the renormalon contribution, which is comparable in strength with the estimate of about 100–133 MeV from the summation of higher-order corrections of large-order perturbation theory [154].

An extension of the previous analysis of the J/ψ and Υ-systems to the case of the D, B and D^*, B^* mesons leads to the value to order α_s [149]:

$$M_b^{PT2} = (4.63 \pm 0.08)\ \text{GeV}\ , \tag{11.117}$$

in good agreement with the previous results, but less accurate. This result has been confirmed by recent estimates to order α_s^2 [150]:

$$M_c^{PT3} = (1.47 \pm 0.06)\ \text{GeV}\ , \qquad M_b^{PT3} = (4.69 \pm 0.06)\ \text{GeV}\ , \tag{11.118}$$

11.12.1 The b and c pole mass difference

One can also use the previous results, in order to deduce the mass difference between the b and c (non)-relativistic pole masses:

$$M_b(M_b) - M_c(M_c) = (3.22 \pm 0.03)\ \text{GeV}\ , \tag{11.119}$$

in good agreement (within the errors) with potential model expectations [12,16], and with the heavy quark symmetry (HQET) result from the B and D mass difference [164] (see also Chapter 44):

$$M_b(M_b) - M_c(M_c) \simeq (\bar{M}_B - \bar{M}_D)\left\{1 - \frac{\lambda_1}{2\bar{M}_B\bar{M}_D} + \mathcal{O}\left(\frac{1}{M_Q^3}\right)\right\} \simeq (3.4 \pm 0.04)\ , \tag{11.120}$$

where one has used the QSSR estimate of the heavy quark kinetic term inside the meson [165,166]:

$$\lambda_1 \simeq -(0.5 \pm 0.2)\ \text{GeV}^2. \tag{11.121}$$

A direct comparison of this mass difference with the one from the analysis of the inclusive B-decays needs however a better understanding of the mass definition and of the value of the scale entering into these decay processes. If one chooses to evaluate these pole masses at the scale $\nu = M_b$, which might be a natural scale for this process, one obtains to two-loop accuracy:

$$M_c(\nu = M_b) = (1.08 \pm 0.04) \text{ GeV}, \tag{11.122}$$

which leads to the mass difference:

$$M_b - M_c|_{\nu=M_b} = (3.54 \pm 0.05) \text{ GeV} . \tag{11.123}$$

11.13 Alternative definitions to the pole mass

It has been argued that the pole masses can be affected by non-perturbative terms induced by the resummation of the QCD perturbative series [154] (see chapter on power corrections) and alternative definitions free from such ambiguities have been proposed (residual mass [158] (see also [160]) and 1S mass [159]). Assuming that the QCD potential has no linear power corrections, the residual or potential-subtracted (PS) mass is related to the pole mass as:

$$M_{\rm PS} = M_{\rm pole} + \frac{1}{2}\int_{|\vec{q}|<\mu} \frac{d^3\vec{q}}{(2\pi)^3} V(\vec{q}) . \tag{11.124}$$

The 1S mass is defined as half of the perturbative component to the ${}^3S_1$ $\bar{Q}Q$ ground state, which is half of its static energy $\langle 2M_{\rm pole} + V\rangle$.[6] The running and short distance pole mass defined at a given order of PT series will be used in the following discussions in this book.

11.14 $\overline{MS}$ scheme and RGE for the pseudoscalar two-point correlator

In order to illustrate the discussions in the previous sections, let us consider the two-point correlator:

$$\Psi_5(q^2) \equiv i \int d^4x \; e^{iqx} \; \langle 0|\mathcal{T} J_P(x) \,(J_P(0))^\dagger \,|0\rangle , \tag{11.125}$$

where:

$$J_P = (m_i + m_j)\bar{\psi}_i(i\gamma_5)\psi_j , \tag{11.126}$$

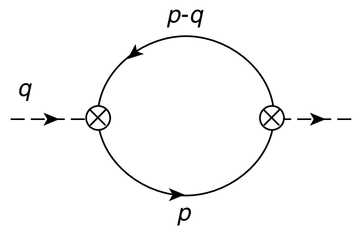
is the light quark pseudoscalar current.

[6] These definitions might still be affected by a dimension-two term advocated in [162,161,438], which might limit their accuracy [163].

https://doi.org/10.1017/9781009290296.018 Published online by Cambridge University Press

11.14.1 Lowest order perturbative calculation

We shall be concerned with Fig. 8.1 discussed in Section 8.2.5 for massless quarks (Fig. 11.2):



Using Feynman rules, it reads:

$$i\nu^\epsilon \Psi_5(q^2) = (m_i + m_j)^2(-1)N \int \frac{d^n p}{(2\pi)^n} \times \mathbf{Tr}\left\{(i\gamma_5)\frac{i}{\hat{p} - m_i + i\epsilon'}(i\gamma_5)\frac{i}{\hat{p} - \hat{q} - m_j + i\epsilon'}\right\} . \tag{11.127}$$

Parametrizing the quark propagators à la Feynman (Appendix E) and using the properties of the Dirac matrices (Appendix D) and momentum integrals (Appendix F) in n-dimensions, one obtains for the bare correlator:

$$\nu^\epsilon \Psi_5^B(q^2) = (m_i + m_j)^2 \frac{N}{4\pi^2} \int_0^1 dx \left(\frac{2}{\epsilon} + \ln 4\pi - \gamma\right)\left(\frac{\mathbf{R}^2 - i\epsilon'}{\nu^2}\right)^{-\epsilon/2} \times \left\{\left(3 + \frac{\epsilon}{2}\right) q^2 x(1-x) - 2\left(1 + \frac{\epsilon}{4}\right)\left(m_i^2 x + m_j^2(1-x) + m_i m_j\right)\right\} , \tag{11.128}$$

where:

$$\mathbf{R}^2 \equiv -q^2 x(1-x) + m_i^2 x + m_j^2(1-x) , \tag{11.129}$$

and $\gamma = 0.5772\ldots$ is the Euler constant. Two limiting cases are particularly interesting:

$$\nu^\epsilon \Psi_5^B\left(q^2 \gg m_{i,j}^2\right) = (m_i + m_j)^2 q^2 \frac{N}{8\pi^2}\left[\left(\frac{2}{\epsilon} + \ln 4\pi - \gamma - \ln\left(\frac{-q^2}{\nu^2}\right)\right) \times \left[1 + 2\frac{(m_i^2 + m_j^2 - m_i m_j)}{-q^2}\right] + 2 + \frac{\epsilon}{4}\ln^2\left(\frac{-q^2}{\nu^2}\right) - \frac{\epsilon}{2}(\ln 4\pi - \gamma + 2)\ln\left(\frac{-q^2}{\nu^2}\right)\right], \tag{11.130}$$

https://doi.org/10.1017/9781009290296.018 Published online by Cambridge University Press

and:

$$\nu^\epsilon \Psi_5^B(q^2=0) = (m_i+m_j)\frac{N}{4\pi^2}\left[\left(m_i^3 \ln\frac{m_i^2}{\nu^2} + m_j^3 \ln\frac{m_j^2}{\nu^2}\right) - \left(\frac{2}{\epsilon} + \ln 4\pi - \gamma - 1\right)\left(m_i^3 + m_j^3\right)\right]. \tag{11.131}$$

The case $q = 0$ is useful for the Ward identity discussed in Eq. (2.17) and for the definition of the scale-invariant condensate which will be discussed in Part VII.

One can explicitly check the Ward identity perturbatively by evaluating the longitudinal part of the axial-vector current correlator defined in Eq. (2.18). One obtains:

$$q_\mu q_\nu \Pi_5^{\mu\nu} = \frac{N}{8\pi^2} q^2 \int_0^1 dx\, \left[m_i^2 x + m_j^2(1-x) + m_i m_j\right] \left(\frac{\mathbf{R}^2 - i\epsilon'}{\nu^2}\right)^{-\epsilon/2} \Gamma(\epsilon/2)\,, \tag{11.132}$$

which by comparison gives:

$$q_\mu q_\nu \Pi_5^{\mu\nu} = \Psi_5(q^2) - (m_i+m_j)\frac{N}{4\pi^2}\left(m_i^3 \ln\frac{m_i^2}{\nu^2} + m_j^3 \ln\frac{m_j^2}{\nu^2}\right)\,. \tag{11.133}$$

Finally, one can extract the spectral function by using:

$$\ln \mathbf{R}^2 = \ln |\mathbf{R}^2| - i\pi\theta(-\mathbf{R}^2)\,. \tag{11.134}$$

Therefore, one can deduce:

$$\begin{aligned}\mathrm{Im}\Psi_5(t) &= \mathrm{Im}\left(q_\mu q_\nu \Pi_5^{\mu\nu}\right)\\ &= \frac{N}{8\pi^2}(m_i+m_j)^2 t \left(1 - \frac{(m_i-m_j)^2}{t}\right)\\ &\quad\times \lambda^{1/2}\left(1,\ \frac{m_i^2}{t},\ \frac{m_j^2}{t}\right)\theta[t-(m_i+m_j)^2]\,.\end{aligned} \tag{11.135}$$

11.14.2 Two-loop perturbative calculation in the $\overline{MS}$ scheme

For a pedagogical illustration, we consider a massless quark inside the quark loop. The corresponding two-loop perturbative contribution comes from Fig. 11.3.

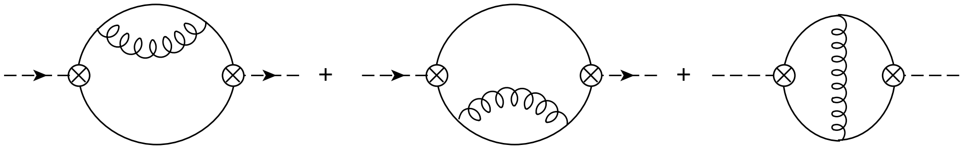


Fig. 11.3. Two-loop perturbative contribution to the pseudoscalar two-point function.

https://doi.org/10.1017/9781009290296.018 Published online by Cambridge University Press

A routine application of the previous rules leads to [167]:

$$\Psi_5^B(q^2) = \nu^\epsilon \frac{3}{8\pi^2}\left(m_i^B + m_j^B\right)^2 q^2 \left[\frac{2}{\epsilon} + \ln 4\pi - \gamma + 2 - \ln\left(\frac{-q^2}{\nu^2}\right)\right.$$
$$- \frac{\epsilon}{2}(\ln 4\pi - \gamma + 2)\ln\left(\frac{-q^2}{\nu^2}\right) + \frac{\epsilon}{4}\ln^2\left(\frac{-q^2}{\nu^2}\right)$$
$$\left. + \left(\frac{g^B \nu^{-\epsilon/2}}{4\pi^2}\right)^2 \left[\frac{4}{\epsilon^2} + \frac{4}{\epsilon}(\ln 4\pi - \gamma) + \frac{29}{3\epsilon} + \mathcal{O}(1)\right]\left(\frac{-q^2}{\nu^2}\right)^{-\epsilon}\right]. \quad (11.136)$$

Introducing the renormalized parameter (we shall omit the index R):

$$g^B \nu^{-\epsilon/2} = g\left[1 + \mathcal{O}\left(\frac{\alpha_s}{\pi}\right)\right],$$
$$m_i^B = m_i\left[1 - \frac{2}{\epsilon}\left(\frac{\alpha_s}{\pi}\right)\right], \quad (11.137)$$

one can deduce [167]:

$$\Psi_5(q^2) = \frac{3}{8\pi^2}(m_i + m_j)^2 q^2 \left[\frac{2}{\epsilon} + \ln 4\pi - \gamma + 2 - \ln\left(\frac{-q^2}{\nu^2}\right)\right.$$
$$+ \left(\frac{\alpha_s}{\pi}\right)\left[-\frac{4}{\epsilon^2} + \frac{5}{3\epsilon} + \ln^2\left(\frac{-q^2}{\nu^2}\right)\right.$$
$$\left.\left. - \left(\frac{17}{3} + 2(\ln 4\pi - \gamma)\right)\ln\left(\frac{-q^2}{\nu^2}\right)\right]\right]. \quad (11.138)$$

This expression tells us that the lowest order term proportional to ϵ induce via the mass renormalization a non-zero finite term. It also shows how the non-local:

$$\frac{1}{\epsilon}\ln\left(\frac{-q^2}{\nu^2}\right) \quad (11.139)$$

pole has disappeared after renormalization. The disappearance of this term is a double check of the calculation as well. Finally, one can also use the RGE for checking the ln-coefficient. This can be done by working with the RGE of the two-point correlator given in Section 11.6. In so doing, we consider the coefficient of the $1/\epsilon$-terms:

$$D = D_0 + \left(\frac{\alpha_s}{\pi}\right) D_1, \quad (11.140)$$

with:

$$D_0 = -\frac{3}{8\pi^2}(x_i + x_j)^2 2e^{2t},$$
$$D_1 = -\frac{3}{8\pi^2}(x_i + x_j)^2 \frac{10}{3} e^{2t}. \quad (11.141)$$

https://doi.org/10.1017/9781009290296.018 Published online by Cambridge University Press

where $x_i \equiv m_i/\nu$ is a *dimensionless* mass and $t \equiv -1/2 \ln(-q^2/\nu^2)$. Expressing Ψ_5 in terms of x_i, one has:

$$\Psi_5(t, \alpha_s, x_{i,j}) = -\frac{3}{8\pi^2}(x_i + x_j)^2 e^{2t} q^4 \times \left[-2t + \ln 4\pi - \gamma + 2 + \left(\frac{\alpha_s}{\pi}\right)\left(4at^2 + 2bt + c\right)\right], \quad (11.142)$$

where a, b, c have to be determined. Using the RGE, one obtains the constraint:

$$D_0 = -\frac{3}{8\pi^2}(x_i + x_j)^2 2e^{2t} ,$$
$$D_1 = -\frac{3}{8\pi^2}(x_i + x_j)^2 e_{2t} \times [-8at - 2b - 2\gamma_1(\ln 4\pi - \gamma + 2) + 2\gamma_1 2t] , \quad (11.143)$$

where $\gamma_1 = 2$ is the mass anomalous dimension. The fact that D_1 cannot depend on t implies:

$$-4a + 2\gamma_1 = 0 \implies a = 1 . \quad (11.144)$$

The relation between C_1 and D given in Eq. (11.48) implies:

$$C_1^{(0)} = D_0 . \quad (11.145)$$

$C_1^{(1)}$ is not fixed by the RGE but we know it from the previous calculation:

$$C_1^{(1)} = \frac{3}{8\pi^2}(x_i + x_j)^2 e^{2t} \frac{5}{3} , \quad (11.146)$$

while we deduce from Eq. (11.48):

$$2C_1^{(1)} = D_1 . \quad (11.147)$$

The recursive relation implies:

$$C_2^{(1)} = \frac{3}{8\pi^2}(x_i + x_j)^2 e^{2t} 2\gamma_1 . \quad (11.148)$$

Inserting the previous expressions into the one of D_1, one can deduce:

$$-2b - 2\gamma_1(\ln 4\pi - \gamma + 2) = \frac{10}{3} . \quad (11.149)$$

One can see that the RGE and an explicit evaluation of the $1/\epsilon$-coefficient to order α_s allows one to fix the coefficients of the $1/\epsilon^2$, $\ln^2$ and $\ln$ at that order. This impressive result allows to have a double check of the direct calculation.