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A NOTE ON A LOWER BOUND FOR THE MULTIPLICATIVE ODDS THEOREM OF OPTIMAL STOPPING

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Abstract

In this note we present a bound of the optimal maximum probability for the multiplicative odds theorem of optimal stopping theory. We deal with an optimal stopping problem that maximizes the probability of stopping on any of the last m successes of a sequence of independent Bernoulli trials of length N, where m and N are predetermined integers satisfying $1 \leq m < N$. This problem is an extension of Bruss' (2000) odds problem. In a previous work, Tamaki (2010) derived an optimal stopping rule. We present a lower bound of the optimal probability. Interestingly, our lower bound is attained using a variation of the well-known secretary problem, which is a special case of the odds problem.

Keywords: Optimal stopping; odd problem; lower bound; secretary problem; Maclaurin's inequality

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1. Introduction

Let $X_1, X_2, \dots, X_N$ denote a sequence of independent Bernoulli random variables. The outcome of each random variable is either a *success* or a *failure*. We let $X_j = 1$ if X_j is a success, and $X_j = 0$ otherwise. These random variables can be regarded as *indices* for the observation of an underlying discrete stochastic process. For example, we can assume they constitute the record process. A decision maker sequentially observes $X_1, X_2, \dots, X_N$ with the objective of correctly predicting, with the maximum probability, the occurrence of any of the last m successes at its respective occurrence time. We call the above problem a *multiplicative odds problem* of order m. We discuss the asymptotic lower bounds of the probability of '*win*' (i.e. obtaining any of the last m successes).

When $m = 1$, the multiplicative odds problem is equivalent to the well-known Bruss odds problem [1], which has an elegant and simple optimal stopping strategy known as the *odds theorem* or *sum-the-odds theorem*. A typical lower bound for an asymptotic optimal value (the probability of win), when N approaches ∞, has been shown to be e^{-1} by Bruss [2], which is equal to that for the classical secretary problem. One of the reason why the odds problem is popular in optimal stopping theory is that it includes the secretary problem as a special case.

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For a general case ($m \geq 2$), Tamaki [4] demonstrated the *sum-the-multiplicative-odds theorem*, which gives an optimal stopping rule obtained using a threshold strategy. Tamaki [4] also discussed the secretary problem and derived an asymptotic optimal value.

In this note we derive an asymptotic lower bound of the probability of win for the multiplicative odds problem. Our lower bound is equivalent to the asymptotic optimal value for the secretary problem obtained by Tamaki [4], which implies the tightness of our bound. A special feature of our proof is the application of Maclaurin's inequality [3] to obtain our bound.

2. Preliminaries

For any pair of positive integers k and N satisfying $1 \leq k \leq N$, and a vector $\boldsymbol{r} \in \mathbb{R}^N$, $e_k(\boldsymbol{r})$ denotes the kth *elementary symmetric function* of $\boldsymbol{r} = (r_1, r_2, \dots, r_N)$ defined by

$$e_k(\boldsymbol{r}) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq N} r_{i_1} r_{i_2} \cdots r_{i_k} = \sum_{\substack{B \subseteq \{1,2,\dots,N\} \\ \text{and } |B|=k}} \prod_{i \in B} r_i,$$

which is the sum of $\binom{N}{k}$ terms. We also define $e_0(\boldsymbol{r}) = 1$. The *kth elementary symmetric mean* of $\boldsymbol{r}$ is defined by

$$S_k(\boldsymbol{r}) = \frac{e_k(\boldsymbol{r})}{\binom{N}{k}}.$$

We now describe Maclaurin's inequalities, which play an important role in the next section.

Lemma 1. (Maclaurin's [3] inequalities.) *Every nonnegative vector* $\boldsymbol{r} \in \mathbb{R}_+^N$ *satisfies the chain of inequalities*

$$S_1(\boldsymbol{r}) \geq \sqrt{S_2(\boldsymbol{r})} \geq \sqrt[3]{S_3(\boldsymbol{r})} \geq \cdots \geq \sqrt[N]{S_N(\boldsymbol{r})}.$$

3. Lower bound

We deal with a sequence of independent 0/1 random variables $X_1, X_2, \dots, X_N$, where N is a given positive integer, with distribution $\mathbb{P}[X_k = 1] = p_k$, $\mathbb{P}[X_k = 0] = 1 - p_k = q_k$, $0 \leq p_k < 1$, for each k. We define $r_k = p_k/q_k$ for each k. The r_k are called *odds*. A multiplicative odds problem of order m provides a strategy to correctly predict, with the maximum probability, the occurrence of any of the last m successes at its respective occurrence time.

We begin by briefly reviewing the sum-the-multiplicative-odds theorem shown by Tamaki [4]. An optimal stopping rule for the multiplicative odds problem is obtained using a threshold strategy, i.e. it stops at the first success for which the sum of the m-fold multiplicative odds of success for future trials is less than or equal to 1. In particular, the optimal rule stops on the first success $X_i = 1$ with

$$i \geq i_* := \min\{k \geq 1 \mid e_m(r_{k+1}, r_{k+2}, \dots, r_N) \leq 1\}.$$

The corresponding probability of win is equal to

$$\begin{aligned} &q_{i_*} q_{i_*+1} \cdots q_N (e_m(\widetilde{\boldsymbol{r}}) + e_{m-1}(\widetilde{\boldsymbol{r}}) + \cdots + e_1(\widetilde{\boldsymbol{r}})) \\ &\quad = \frac{e_m(\widetilde{\boldsymbol{r}}) + e_{m-1}(\widetilde{\boldsymbol{r}}) + \cdots + e_1(\widetilde{\boldsymbol{r}})}{(1 + r_{i_*})(1 + r_{i_*+1}) \cdots (1 + r_N)}, \end{aligned}$$

where $\widetilde{\boldsymbol{r}} = (r_{i_*}, r_{i_*+1}, \dots, r_N)$.

In the rest of this section, we discuss the probability of win for a multiplicative odds problem under the above optimal stopping rule.

Theorem 1. *Let us consider the multiplicative odds problem of order m defined on $X_1, X_2, \ldots, X_N$, satisfying $m \leq N$ and $e_m(r_1, r_2, \ldots, r_N) \geq 1$. Under the optimal stopping rule, the probability of win is greater than or equal to*

$$\exp(-(m!)^{1/m}) \sum_{k=1}^{m} \frac{(m!)^{k/m}}{k!}.$$

Proof. It is obvious that the truncation of the subsequence $X_1, X_2, \ldots, X_{i_*-1}$ does not affect the probability of win. Thus, we only need to consider the case where

$$e_m(r_2, r_3, \ldots, r_N) \leq 1 \leq e_m(r_1, r_2, r_3 \ldots, r_N). \tag{1}$$

Under assumption (1), the optimal stopping rule satisfies $i_* = 1$ or 2. When we stop at the first success, the corresponding probability of win, denoted by $V_{m,N}$, gives a lower bound of the probability of win under the optimal stopping rule. It is clear that $V_{m,N}$ is equal to

$$V_{m,N} = \frac{e_m(\boldsymbol{r}) + e_{m-1}(\boldsymbol{r}) + \cdots + e_1(\boldsymbol{r})}{(1+r_1)(1+r_2)\cdots(1+r_N)}.$$

Thus, the greatest lower bound of the probability of win under the optimal stopping rule is greater than or equal to the optimal value of the optimization problem

(P1) minimise

$$V_{m,N} = \frac{e_m(\boldsymbol{r}) + e_{m-1}(\boldsymbol{r}) + \cdots + e_1(\boldsymbol{r})}{(1+r_1)(1+r_2)\cdots(1+r_N)}$$

such that $0 \leq r_k$ for all $k \in \{1, 2, \ldots, N\}$, $e_m(r_1, r_2, r_3 \ldots, r_N) \geq 1$, $e_m(r_2, r_3 \ldots, r_N) \leq 1$.

In the rest of this paper, we denote $(r_2, r_3, \ldots, r_N)$ by $\boldsymbol{r}_{-1}$ for simplicity. The objective function of (P1) becomes

$$\begin{aligned}
V_{m,N} &= \frac{\sum_{j=1}^{m} e_j(\boldsymbol{r})}{(1+r_1)(1+r_2)\cdots(1+r_N)} \\
&= \frac{\sum_{j=1}^{m}(e_j(\boldsymbol{r}_{-1}) + r_1 e_{j-1}(\boldsymbol{r}_{-1}))}{(1+r_1)(1+r_2)\cdots(1+r_N)} \\
&= \frac{\sum_{j=1}^{m}(e_j(\boldsymbol{r}_{-1}) - e_{j-1}(\boldsymbol{r}_{-1})) + (1+r_1)\sum_{j=1}^{m} e_{j-1}(\boldsymbol{r}_{-1})}{(1+r_1)(1+r_2)\cdots(1+r_N)} \\
&= \frac{(e_m(\boldsymbol{r}_{-1}) - 1)/(1+r_1) + \sum_{j=0}^{m-1} e_j(\boldsymbol{r}_{-1})}{(1+r_2)\cdots(1+r_N)}.
\end{aligned}$$

If we fix variables $\{r_2, r_3, \ldots, r_N\}$ to the values of a feasible solution of (P1), the minimum of $V_{m,N}$ is attained by setting the remaining variable r_1 to the value defined by $\min\{r_1 \geq 0 \mid e_m(\boldsymbol{r}) \geq 1\}$, since every feasible solution $\boldsymbol{r}$ satisfies $e_m(\boldsymbol{r}_{-1}) - 1 \leq 0$. Thus, problem (P1) has an optimal solution $\boldsymbol{r^*}$ satisfying $e_m(\boldsymbol{r^*}) = 1$, or, equivalently, $\sqrt[m]{S(\boldsymbol{r^*})} = c_*$, where $c_* = \binom{N}{m}^{-1/m}$. Applying this to Maclaurin's inequalities in Lemma 1 with $\boldsymbol{r}$ replaced by $\boldsymbol{r^*}$ yields

$$\begin{aligned}
e_k(\boldsymbol{r^*}) &\geq a_k \quad \text{for all } k,\ 1 \leq k \leq m, \\
e_k(\boldsymbol{r^*}) &\leq a_k \quad \text{for all } k,\ m+1 \leq k \leq N,
\end{aligned}$$

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where

$$a_k = \binom{N}{k} c_*^k \quad \text{for all } k,\ 1 \le k \le N.$$

Let $V_{m,N}^*$ denote the optimal value of (P1). From the above equality, we obtain an upper bound of $1/V_{m,N}^*$ as follows:

$$\begin{aligned}
\frac{1}{V_{m,N}^*} &= \frac{(1+r_1^*)(1+r_2^*)\cdots(1+r_N^*)}{\sum_{k=1}^m e_k(\boldsymbol{r^*})} \\
&= \frac{\sum_{k=0}^N e_k(\boldsymbol{r^*})}{\sum_{k=1}^m e_k(\boldsymbol{r^*})} \\
&= 1 + \frac{1 + \sum_{k=m+1}^N e_k(\boldsymbol{r^*})}{\sum_{k=1}^m e_k(\boldsymbol{r^*})} \\
&\le 1 + \frac{1 + \sum_{k=m+1}^N a_k}{\sum_{k=1}^m a_k} \\
&= \frac{1 + \sum_{k=1}^N a_k}{\sum_{k=1}^m a_k} \\
&= \frac{1 + \sum_{k=1}^N \binom{N}{k} c_*^k}{\sum_{k=1}^m a_k} \\
&= \frac{(1+c_*)^N}{\sum_{k=1}^m a_k}.
\end{aligned}$$

This implies that $V_{m,N}^* \ge \sum_{k=1}^m a_k/(1+c_*)^N$. It is easy to see that problem (P1) has an optimal solution $r_1 = r_2 = \cdots = r_N = c_*$ whose corresponding objective value attains the abovementioned lower bound, and, thus,

$$V_{m,N}^* = \frac{\sum_{k=1}^m a_k}{(1+c_*)^N}.$$

Finally, we consider a lower bound that is independent of N. Obviously, we have

$$V_{m,N}^* = \frac{\sum_{k=1}^m a_k}{(1+c_*)^N} \ge \mathrm{e}^{-Nc_*} \sum_{k=1}^m a_k = \exp\biggl(-\binom{N}{1} c_*\biggr) \sum_{k=1}^m a_k = \mathrm{e}^{-a_1} \sum_{k=1}^m a_k.$$

The greatest lower bound of the probability of win (under the optimal stopping rule) is nonincreasing with respect to N. Thus, $\lim_{N\to\infty} V_{m,N}^*$ gives a general lower bound. Since

$$\begin{aligned}
a_k &= \binom{N}{k} c_*^k \\
&= \binom{N}{k} \binom{N}{m}^{-k/m} \\
&= \frac{N!}{k!\,(N-k)!} \left(\frac{(N-m)!\,m!}{N!}\right)^{k/m} \\
&= \frac{(m!)^{k/m}}{k!} \frac{N!}{(N-k)!\,N^k} \left(\frac{(N-m)!\,N^m}{N!}\right)^{k/m}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(m!)^{k/m}}{k!}\left(1-\frac{0}{N}\right)\left(1-\frac{1}{N}\right)\cdots\left(1-\frac{k-1}{N}\right)\left(\frac{1}{(1-0/N)\cdots(1-(m-1)/N)}\right)^{k/m} \\
&\to \frac{(m!)^{k/m}}{k!} \quad \text{as } N \to \infty,
\end{aligned}
$$

we obtain

$$
\lim_{N\to\infty} V^*_{m,N} \geq \lim_{N\to\infty}\left(\mathrm{e}^{-a_1}\sum_{k=1}^{m} a_k\right) = \exp(-(m!)^{1/m})\sum_{k=1}^{m}\frac{(m!)^{k/m}}{k!}.
$$

This completes the proof.

The above theorem gives the very interesting result that our lower bound of the probability of win for the multiplicative odds problem is attained using the lower bound for the corresponding secretary problem (shown by Tamaki [4]), which is a special case of the multiplicative odds problem.

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