BULL. AUSTRAL. MATH. SOC. 16R50
VOL. 49 (1994) [85–90]

SKEW FIELDS WITH A NON-TRIVIAL GENERALISED POWER CENTRAL RATIONAL IDENTITY

KATSUO CHIBA

Let D be a skew field with uncountable centre K. The main result in the present paper is as follows: If D satisfies a non-trivial generalised power central rational identity, then D is finite dimensional over K. As a corollary we obtain the following result. Let a be an element of D such that $\left(a^{-1}x^{-1}ax\right)^{q(x)} \in K$ for all $x \in D \setminus \{0\}$ where $q(x)$ is a positive integer depending on x. Then $a \in K$.

Several authors [7, 8, 10] have studied skew fields with a certain power central rational identity. In this paper we shall study skew fields with uncountable centre which satisfy a general power central rational identity.

Let D be a skew field with centre K and $K\langle X\rangle$ the free K-algebra on a finite set $X = \{x_1, x_2, \ldots, x_n\}$. We denote by $D\langle X\rangle = D *_K K\langle X\rangle$ the free product of D and $K\langle X\rangle$ over K and by $D(X)$ the universal skew field of fractions of $D\langle X\rangle$. Let $d = (d_i)$ be an element of D^n and $\alpha_d\colon D\langle X\rangle \to D$ the D-ring homomorphism defined by $\alpha_d(x_i) = d_i$, $i = 1, 2, \ldots, n$. We denote by Σ_d the set of all matrices over $D\langle X\rangle$ which are mapped by α_d to invertible matrices over D. Let Σ_d^{-1} be the set of all entries of inverses A^{-1} over $D(X)$ for all $A \in \Sigma_d$. Then Σ_d^{-1} is a ring and it contains $D\langle X\rangle$ as a subring. Moreover, there is a D-ring homomorphism $\beta_d\colon \Sigma_d^{-1} \to D$ which extends α_d and satisfies that any element of Σ_d^{-1} not in the kernel of β_d has an inverse in Σ_d^{-1} (see [4, Chapter 7]). Let $f = f(x_i)$ be an element of $D(X)$. If f belongs to Σ_d^{-1}, we say f is defined at (d_i) and write $f(d_i)$ instead of $\beta_d(f)$. We say D satisfies a generalised power central rational identity (abbreviated GPCRI) if there is an element f in $D(X)$ satisfying the following condition: if f is defined at $(d_i) \in D^n$ then $f(d_i)^q \in K$ for some positive integer q which depends only on (d_i). Furthermore, if $f^p \notin K$ for any positive integer p, we say D satisfies a non-trivial GPCRI f.

The purpose of this paper is to prove the following theorem.

THEOREM 1. *Let D be a skew field with uncountable centre K. If D satisfies a non-trivial GPCRI, then D is finite dimensional over K.*

In [7] Herstein conjectured that any element a of D which satisfies $\left(a^{-1}x^{-1}ax\right)^{q(x)} \in K$ for all $x \in D \setminus \{0\}$ where $q(x)$ depends on x must be central and in [8, p.489] he

Received 16 February 1993

https://doi.org/10.1017/S0004972700016117 Published online by Cambridge University Press

settled the conjecture for the case in which K is an uncountable field of characteristic 0. As a corollary of Theorem 1, we settle this conjecture here for the case in which K is an uncountable field of arbitrary characteristic.

COROLLARY 2. *Let D be a skew field with uncountable centre K. Let a be an element of D such that $(a^{-1}x^{-1}ax)^{q(x)} \in K$ for all $x \in D \setminus \{0\}$ where $q(x)$ is a positive integer depending on x. Then $a \in K$.*

Furthermore, by Theorem 1, we obtain [**10**, Theorem].

COROLLARY 3. *Let D be a skew field with uncountable centre K. Suppose that there is a non-trivial word w in a free group such that every value of w over D is periodic over K. Then D is commutative.*

NOTATIONS AND TERMINOLOGY. Let E be a skew field which contains D and the centre of E contains K, the centre of D. Let $(e_i) \in E^n$ and $f = f(x_i) \in D(X)$. In the same fashion as we previously defined, we say f is defined at (e_i), we use an expression $f(e_i)$ and we say E satisfies a GPCRI f. Let k be a field, R a k-algebra and $R *_k k\langle X\rangle$ the free product of R and $k\langle X\rangle$ over k, where $k\langle X\rangle$ is a free algebra on a finite set $X = \{x_1, x_2, \ldots, x_n\}$. Let $f \in R *_k k\langle X\rangle$. We say that f is a generalised power central identity (abbreviated GPCI) of R if for each $(r_i) \in R^n$, there exists a positive integer q, which depends on (r_i), such that $f(r_i)^q$ is a central element of R. We shall denote by $D\langle X\rangle[t]$ the polynomial ring over $D\langle X\rangle$ in a central indeterminate t and by $D(X)(t)$ (respectively $D(t)$) the quotient skew field of the polynomial ring $D(X)[t]$ (respectively $D[t]$). The Laurent series skew field over $D(X)$ (respectively D) is denoted by $D(X)((t))$ (respectively $D((t))$). There is a natural embedding of $D(X)(t)$ (respectively $D(t)$) into $D(X)((t))$ (respectively $D((t))$), so we shall think of $D(X)(t)$ (respectively $D(t)$) as a subring of $D(X)((t))$ (respectively $D((t))$). If R is a semifir, we denote by $U(R)$ the universal skew field of fractions of R.

To prove Theorem 1, we need several lemmas. We begin with the following.

LEMMA 4. *Let D be a skew field with uncountable centre K and $D(t_1, t_2, \ldots, t_m)$ be a quotient skew field of the polynomial ring $D[t_1, t_2, \ldots, t_m]$, where t_i, $i = 1, 2, \ldots, m$, are central indeterminates. If D satisfies a GPCRI $f = f(x_i) \in D(X)$ then $D(t_1, t_2, \ldots, t_m)$ also satisfies the GPCRI f.*

PROOF: We first show that $D(t_1)$ satisfies the GPCRI f. Suppose $f = f(x_i)$ is defined at $(h_i(t_1)) \in D(t_1)^n$. Since K is uncountable, f is defined at $(h_i(u)) \in D^n$ for uncountably many elements $u \in K$. Then, by the Pigeon-Hole Principle we can find a positive integer q such that $f(h_i(u))^q \in K$ for infinitely many elements $u \in K$. By [**10**, Lemma 1], $f(h_i(t_1))^q$ is central in $D(t_1)$. Thus $D(t_1)$ satisfies the GPCRI f. Since $D(t_1, t_2, \ldots, t_{i+1}) = D(t_1, t_2, \ldots, t_i)(t_{i+1})$, by induction on m it follows that $D(t_1, t_2, \ldots, t_m)$ satisfies the GPCRI f. This completes the proof. ∎

Lemma 5. *Let D be a skew field with uncountable centre K and L a field containing K. Suppose $g = g(x_i) \in D *_K K\langle X\rangle = D\langle X\rangle$ is a GPCI of D. Then g is also a GPCI of $D \otimes_K L$.*

Proof: Let $a_i \in D \otimes_K L$, $i = 1, 2, \ldots, n$. Then there are a polynomial ring $D[t_1, t_2, \ldots, t_m]$, n elements $h_i \in D[t_1, t_2, \ldots, t_m]$, $i = 1, 2, \ldots, n$, and a D-ring homomorphism ϕ: $D[t_1, t_2, \ldots, t_m] \to D \otimes_K L$ such that $\phi(t_j) \in L$, $j = 1, 2, \ldots, m$, and $\phi(h_i) = a_i$, $i = 1, 2, \ldots, n$. It is clear that g is a GPCRI of D. Hence, by Lemma 4, $D(t_1, t_2, \ldots, t_m)$ satisfies the GPCRI g. Therefore we can find a positive integer q such that $g(h_i)^q \in K[t_1, t_2, \ldots, t_m]$, the centre of $D[t_1, t_2, \ldots, t_m]$, and hence $g(a_i)^q = \phi(g(h_i)^q) \in L$. Thus $D \otimes_K L$ satisfies the GPCI g. ∎

Lemma 6. *Let D be a skew field with uncountable centre K. If D satisfies a GPCI in $D\langle X\rangle \setminus D$, then D is finite dimentional over K.*

Proof: Let $g = g(x_i) \in D\langle X\rangle \setminus D$ be a GPCI of D, let the X-degree of g be m, and let $D\langle X\rangle[t]$ be the polynomial ring over $D\langle X\rangle$ in a central indeterminate t. Then we can express $g(x_it) \in D\langle X\rangle[t]$ in the form

$$g(x_it) = g_m(x_i)t^m + g_{m-1}(x_i)t^{m-1} + \ldots + g_0$$

where $g_j(x_i) \in D\langle X\rangle$ is homogeneous and of X-degree j for $j = 0, 1, 2, \ldots, m$. Let $(d_i) \in D^n$. Since K is uncountable, by the Pigeon-Hole Principle, we can find a positive integer q such that $g(d_iu)^q \in K$ for infinitely many elements $u \in K$. By a van der Monde determinant argument, we have that $g_m(d_i)^q \in K$. Thus we may assume that g is homogeneous. Let us write g in the following form:

$$g = \Sigma_i\, e_i x_{i1}\, d_{i1} x_{i2} d_{i2} \ldots x_{im} d_{im}$$

where $x_{ij} \in \{x_1, x_2, \ldots, x_n\}$, $\{e_i, d_{ij}\} \subset D$ and $e_i \neq 0$. Let us denote the elements d_{ij} by $d_1, d_2, \ldots, d_h$. We may assume that $d_1, d_2, \ldots, d_h$ are K-linearly independent. Now, let L be a maximal commutative subfield of D. Then $R = D \otimes_K L$ is a dense ring of linear transformations on D considered as a right vector space over L. Assume $[D : K] = \infty$. Then, by [1, Corollary 8*], $D \otimes_K L$ has no finite ranked transformation. By [1, Lemma 11] we obtain $m + 1$ elements $v_0, v_1, v_2, \ldots, v_m$ in D such that the elements of $V = \{d_iv_j : i = 1, 2, \ldots, h, \quad j = 0, 1, 2, \ldots, m\}$ are right L-linearly independent. Consider the finite set of the x's which appear in the monomial $e_1x_{11}d_{11}x_{12}d_{12} \ldots x_{1m}d_{1m}$. Without loss of generality we may assume that these are $x_1, x_2, \ldots, x_u$. Since $R = D \otimes_K L$ acts densely on D, we can find u elements c_k $k = 1, 2, \ldots, u$, in R which act on V such that

(1) $c_kd_{11}v_1 = e_1^{-1}v_m$ if $x_{11} = x_k$,

(2) $c_kd_{1j}v_j = v_{j-1}$ if $x_{1j} = x_k$ for $j = 2, \ldots, m$,

(3) $c_kd_\lambda v_\mu = 0$ otherwise.

Let $c = g(c_1, c_2, \ldots, c_u, 0, \ldots, 0)$. Then we have $cv_m = v_m$. By Lemma 5, g is a GPCI of R. Hence there is a positive integer s such that $c^s = 1$. On the other hand, by the definitions of c_k, $k = 1, 2, \ldots, u$, it follows that $cv_0 = 0$, a contradiction. Thus $[D : K] < \infty$. This proves the lemma. $\square$

LEMMA 7. *If $f = f(x_i) \in D(X) \setminus D$ is defined at $(d_i) \in D^n$, then f is defined at $(d_i + x_i t) \in D(X)((t))^n$ and $f(d_i + x_i t)$ has the representation:*

$$f(d_i + x_i t) = f_0 + f_1 t + f_2 t^2 + \ldots$$

where $f_0 = f(d_i)$, $f_i \in D\langle X\rangle$ is homogeneous with X-degree i for $i \geqq 1$. Moreover, there exists $i \geqq 1$ such that $f_i \neq 0$.

PROOF: It is easy to show that f is defined at $(d_i + x_i t) \in D(X)((t))^n$ and $f(d_i + x_i t)$ has the above representation. We show that $f_i \neq 0$ for some $i \geqq 1$. Let

$$R_1 = \left\{q(t)p(t)^{-1} \in D(X)(t);\ p(t),\ q(t) \in D(X)[t] \text{ and } p(1) \text{ invertible in } D(X)\right\}.$$

Then, by [2, Lemma 5], R_1 is a subring of $D(X)(t)$ and there is a ring homomorphism $\phi\colon R_1 \to D(X)$ such that $\phi\left(q(t)p(t)^{-1}\right) = q(1)p(1)^{-1}$. Clearly we have $f(d_i + x_i t) \in R_1$. Let $\psi\colon D(X) \to D(X)$ be the D-automorphism defined by $\psi(x_i) = x_i - d_1$, $i = 1, 2, \ldots, n$. Suppose $f(d_i + x_i t) = f_0 \in D$. Then we have $\psi\phi(f(d_i + x_i t)) = f(x_i) = f_0 \in D$, a contradiction. This proves the lemma. $\square$

We are now ready to prove Theorem 1.

PROOF OF THEOREM 1: Assume to the contrary that $[D : K] = \infty$. Let $f = f(x_i) \in D(X) \setminus D$ be a non-trivial GPCRI of D. Then by [5, Theorem 7.2.7] we can find an element $(d_i) \in D^n$ such that f is defined at (d_i) and $f(d_i) \neq 0$. Since f is a GPCRI of D, there is a positive integer p such that $f(d_i)^p \in K$. We show $f(x_i)^p \notin D$. Suppose $f(x_i)^p \in D$. Since $f(x_i)^p$ is defined at (d_i), it follows that $f(x_i)^p = f(d_i)^p \in K$, contradicting the fact that $f(x_i)$ is a non-trivial GPCRI of D. Hence, by Lemma 7 we have the representation in $D(X)((t))$:

$$f(d_i + x_i t)^p = f_0 + f_m t^m + f_{m+1} t^{m+1} + \ldots$$

where $f_0 = f(d_i)^p \neq 0$, $0 \neq f_m = f_m(x_i) \in D\langle X\rangle$ with f_m homogeneous of X-degree m. It is easy to see that $f(x_i)$ is defined at $(d_i + e_i t) \in D(t)^n$ for any $(e_i) \in D^n$. By Lemma 4 $D(t)$ satisfies the GPCRI f, so for each $(e_i) \in D^n$ we can find an integer r such that $f(d_i + e_i t)^{pr} \in K$. If the characteristic of D is zero, then the first two terms in $f(d_i + e_i t)^{pr} = \{f_0 + f_m(e_i)t^m + \ldots\}^r$ are $f_0^r + r f_0^{r-1} f_m(e_i) t^m$, so that $f_m(e_i)$ must be central. If the characteristic of D is $\kappa \neq 0$, then we write $r = kN$ where k is a power of κ and N is prime to κ. Then the first two terms in $f(d_i + e_i t)^{pr} = \{f_0 + f_m(e_i)t^m + \ldots\}^r$ are $f_0^r + N f_0^{r-k} f_m(e_i)^k t^{mk}$ so that $f_m(e_i)^k$ is central. Thus $f_m(x_i)$ is a GPCI of D. By Lemma 6, D is finite dimensional over K, a contradiction. This completes the proof. ∎

For the proof of Corollaries 2 and 3, we recall

LEMMA 8. *Let F be a free group on the set $X = \{x_1, x_2, \ldots, x_n\}$ and $K[F]$ the group algebra over K. Then there is a natural isomorphism $D(X) = U(D *_K K\langle X\rangle) \simeq U(D *_K K[F])$.*

PROOF: Let $K(X) = U(K\langle X\rangle)$ and $K(F) = U(D[F])$. Then, by [**9**, Theorem 2] we have a natural isomorphism $K(X) \simeq K(F)$. By [**5**, Lemma 5.4.1 (ii)], we have natural isomorphisms $U(D *_K K\langle X\rangle) \simeq U(D *_K K(X))$ and $U(D *_K K[F]) \simeq U(D *_K K(F))$. Thus we have a natural isomorphism $U(D *_K K\langle X\rangle) \simeq U(D *_K K[F])$. ∎

PROOF OF COROLLARY 2: Assume $a \notin K$. Then also $a^{-1} \notin K$. By Lemma 8 and [**3**, Corollary 8.1] $a^{-1}x_1^{-1}ax_1$ is a non-trivial GPCRI of D. Then $[D : K] < \infty$ by Theorem 1. Hence $[D(t) : K(t)] < \infty$, where $K(t)$ is the centre of $D(t)$, as is well known. By Lemma 4, $D(t)$ satisfies the GPCRI $a^{-1}x_1^{-1}ax_1$, and hence, by [**7**, Sublemma] for each $d \in D$ we can find a positive integer q, which depends on d, such that $\{a^{-1}(1+dt)^{-1}a(1+dt)\}^q = \{1 + (d - a^{-1}da)t + \ldots\}^q = 1$. By the same argument as in the proof of Theorem 1, we obtain a positive integer N such that $(d - a^{-1}da)^N = 0$. Hence $d - a^{-1}da = 0$. Therefore we have $a \in K$, a contradiction. This completes the proof. ∎

PROOF OF COROLLARY 3: Let w be a non-trivial word in a free group of rank n such that every value of w over D is periodic over K. By Lemma 8, w is a non-trivial GPCRI of D. Then, by Theorem 1 $[D : K] < \infty$. Suppose D is not commutative. Then, by [**6**, Theorem 2.1] $D \setminus \{0\}$ contains a free subgroup G of rank two. As is well-known, G contains a free subgroup of rank n, which is a contradiction. This completes the proof. ∎

REFERENCES

[1] S.A. Amisur, 'Generalized polynomial identities and pivotal monomials', *Trans. Amer. Math. Soc.* **114** (1965), 210–226.

[2] S.A. Amisur, 'Rational identities and applications to algebra and geometry', *J. Algebra* **3** (1966), 304–359.

[3] G.M. Bergman, 'The diamond lemma for ring theory', *Adv. in Math.* **29** (1978), 178–218.

[4] P.M. Cohn, *Free rings and their relations* (Academic Press, New York and London, 1971).

[5] P.M. Cohn, *Skew field constructions*, London Math. Soc. Lecture Notes No. 27 (Cambridge University Press, Cambridge, 1977).

[6] J.Z. Goncalves, 'Free groups in subnormal subgroups and the residual nilpotence of the groups of units of groups rings', *Canad. Math. Bull.* **27** (1984), 365–370.

[7] I.N. Herstein, 'Multiplicative commutators in division rings', *Israel J. Math.* **31** (1978), 180–188.

[8] I.N. Hetstein, 'Multicative commutators in division rings II', *Rend. Circ. Mat. Palermo II* **29** (1980), 485–489.

[9] J. Lewin, 'Fields of fractions for group algebras of free groups', *Trans. Amer. Math. Soc.* **192** (1974), 339–346.

[10] L. Makar-Limanov and P. Malcolmson, 'Words periodic over the center of a division ring', *Proc. Amer. Math. Soc.* **93** (1985), 590–592.

Niihama National College of Technology
7 - 1 Yagumo-Cho
Niihama 792
Japan