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NON-STANDARD 3-SPHERES LOCALLY FOLIATED BY ELASTIC HELICES

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Abstract. In this note we use the Hopf map to construct a family of metrics in the 3-sphere parametrized on the space of positive smooth functions in the 2-sphere. All these metrics make the Hopf map a Riemannian submersion. Also, the fibres are all geodesics if and only if the metric comes from a constant function and so, we have a Berger 3-sphere. Every geodesic in a 3-dimensional Riemannian manifold is a minimum for each elastic energy functional. Therefore, we characterize those functions on the 2-sphere that locally give metrics which have all the fibres being elastica, i.e., critical points of those functionals. Some applications are given including one to the Willmore-Chen variational problem.

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1. Introduction. The *Willmore-Chen functional* [**5**] is defined on the space of inmersions, $I(N, P)$, of an n-dimensional compact smooth manifold N into a semi-Riemannian manifold $(P, \bar{g})$ by

$$\mathcal{W}(\varphi) = \int_N (\bar{g}(H, H) - \tau_e)^{n/2} dv,$$

where H and τ_e denote the mean curvature vector field and the extrinsic scalar curvature function of φ, respectively, and dv is the volume element of $\varphi^*(\bar{g})$ on N.

Since the group of conformal transformations of $(P, \bar{g})$ preserves this functional [**4**], it is also called the *conformal total tension functional*, an it states a variational problem in $(P, [\bar{g}])$, where $[\bar{g}]$ is the conformal structure defined by $\bar{g}$. The critical points of $\mathcal{W}$ are known as *Willmore-Chen submanifolds*. Certainly, this is the natural extension to highest dimensions of the Willmore functional which corresponds with $n = 2$, and now its critical points are the Willmore surfaces [**10**].

The reduction of symmetry method gives a strong relationship between this variational problem and another one associated with a certain elastic energy functional. For example, let P be a principal fibre G-bundle (G being an r-dimensional compact Lie group) endowed with a principal flat connection over a semi-Riemannian manifold (M, g). If $\bar{g}$ is a metric on P obtained by the *Kaluza-Klein mechanism*, then the principle of symmetric criticality [**9**] can be used to produce symmetric solutions to the Willmore-Chen variational problem in $(P, [\bar{g}])$. These solutions are associated with the critical points of the elastic energy functional

$$\mathcal{F}^r(\gamma) = \int_\gamma \kappa^{r+1} ds,$$

defined on the space of closed curves γ in (M, g), where κ denotes the corresponding curvature function [2]. We call *r-elasticae* to the critical points of $\mathcal{F}^r$, and again observe that this notion naturally extends the classical one of free elastic curves, which is obtained for $r = 1$ [7]. Every closed geodesic in (M, g) is automatically an r-elastica.

On the other hand, if $(\mathbb{S}^2, g)$ is the standard round 2-sphere with radius 1/2, the usual Hopf map $\pi : \mathbb{S}^3 \longrightarrow \mathbb{S}^2$ is a principal fibre $\mathbb{S}^1$-bundle which admits a canonical principal connection ω with non-trivial holonomy. For every positive smooth function f on $\mathbb{S}^2$, we construct on $\mathbb{S}^3$ the metric $\bar{g}^f = \pi^*(g) + (f \cdot \pi)^2 \omega^*(dt^2)$. It is not difficult to see that all the fibres in $(\mathbb{S}^3, \bar{g}^f)$ are geodesics if and only if $f = a$ is a constant and so, $\bar{g}^a$ is a Berger metric, i.e. $(\mathbb{S}^3, \bar{g}^a)$ is up to a constant factor, isometric to a distance sphere in $\mathbb{C}P^2$ or its dual.

In this note, we study the following natural problem.

Given an open subset U in $\mathbb{S}^2$, characterize those funtions f such that all the fibres in $\pi^{-1}(U)$ are r-elastica in $(\mathbb{S}^3, \bar{g}^f)$.

We also obtain some applications including one that shows the existence of nontrivial conformal structures which are foliated by equivariant Willmore-Chen submanifolds.

2. Some preliminaires. Let $\pi : \mathbb{S}^3 \longrightarrow \mathbb{S}^2$ be the usual Hopf fibration. Here $\mathbb{S}^3$ is viewed as the unit 3-sphere in $\mathbb{C}^2$ so that $\bar{g}$ will denote its standard metric of constant curvature 1. We define a global vector field V on $\mathbb{S}^3$ by: $V(z) = iz$ for any $z \in \mathbb{S}^3$. We use V and $\bar{g}$ to define the canonical principal connection ω in this principal fibre $\mathbb{S}^1$-bundle. In particular, if we choose on the base $\mathbb{S}^2$ the metric g of constant Gaussian curvature 4, then $\pi : (\mathbb{S}^3, \bar{g}) \longrightarrow (\mathbb{S}^2, g)$ is a Riemannian submersion with geodesic fibres. The following O'Neill formulae are well known. (See [8].)

$$\bar{\nabla}_{\bar{X}} \bar{Y} = \overline{\nabla_X Y} - \bar{g}(i\bar{X}, \bar{Y})V, \tag{2.1}$$

$$\bar{\nabla}_{\bar{X}} V = \bar{\nabla}_V \bar{X} = i\bar{X}, \tag{2.2}$$

$$\bar{\nabla}_V V = 0, \tag{2.3}$$

where $\bar{\nabla}$ and ∇ stand for the Levi-Civita connection of $\bar{g}$ and g, respectively, and overbars means horizontal liftings.

For any positive smooth function f on $\mathbb{S}^2$ and $\varepsilon = \pm 1$, we define the semi-Riemannian generalized Kaluza-Klein metric $\bar{g}^f$ on $\mathbb{S}^3$ by

$$\bar{g}^f = \pi^*(g) + \varepsilon (f \cdot \pi)^2 \omega^*(dt^2), \tag{2.4}$$

where dt^2 is the standard metric on $\mathbb{S}^1$. Then $\pi : (\mathbb{S}^3, \bar{g}^f) \longrightarrow (\mathbb{S}^2, g)$ is still a semi-Riemannian submersion. Notice that $\bar{g}^f$ is Riemannian or Lorentzian according to ε is $+1$ or -1, respectively. Although in this note we will work in the Riemannian case, similar conclusions can be obtained in the Lorentzian one. For the sake of simplicity, we shall write f instead of $f \cdot \pi$. Let $T = \frac{1}{f} V$ be the $\bar{g}^f$-unit tangent vector field to the fibres. Then, a standard computation involving some well-known facts from the theory of semi-Riemannian submersions allows us to obtain the corresponding O'Neill formulae:

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$$\bar{\nabla}^f_{\bar{X}}\bar{Y} = \overline{\nabla_X Y} - \bar{g}^f(i\bar{X}, \bar{Y})V, \tag{2.5}$$

$$[T, \bar{X}] = \frac{\bar{X}(f)}{f}T, \tag{2.6}$$

$$\bar{\nabla}^f_T T = -grad(\log f), \tag{2.7}$$

where $\bar{\nabla}^f$ and *grad* stand for the Levi-Civita connection and the gradient of $\bar{g}^f$, respectively.

3. The fibres in a generalized Kaluza-Klein metric. Recall that a helix in a semi-Riemannian manifold is a curve which has constant all its curvature functions. Notice also that the fibres are geodesics, and so helices, in a generalized Kaluza-Klein metric on $\mathbb{S}^3$ if and only if f is a constant (see equation (2.7)). More generally, if $p \in \mathbb{S}^2$, then $\pi^{-1}(p)$ is a geodesic in $\bar{g}^f$ if and only if p is a critical point of f. Otherwise, let κ and N be the curvature and the unit principal normal of $\pi^{-1}(p)$ in $\bar{g}^f$, respectively. Then, we combine equation (2.7) with the first Frenet equation of the fibre to obtain

$$-grad(f) = f\kappa N. \tag{3.1}$$

In particular we observe that the fibres have constant curvature $\kappa = \dfrac{|grad(f)|}{f}$.

Let τ and B be the torsion and the unit binormal of a fibre in $\bar{g}^f$. Then we combine the formula (2.6) with the second Frenet equation to have

$$\bar{\nabla}^f_N T = -\tau B. \tag{3.2}$$

Let Σ be the set of critical points of f. It is not difficult to see that $\{T, N\}$ span an involutive distribution on $\mathbb{S}^3 - \pi^{-1}(\Sigma)$. Furthermore, every leaf of this foliation can be regarded as a Hopf tube shaped on a curve on $\mathbb{S}^2$, i.e., the leaves are as $\pi^{-1}(\gamma)$, where γ is an inmersed curve in $\mathbb{S}^2$. Notice that these tubes, S_γ, can be parametrized by $\Phi : I \times \mathbb{R} \longrightarrow \mathbb{S}^3$ as follows:

$$\Phi(s, t) = e^{it}\bar{\gamma}(s),$$

where I is the domain of γ and $\bar{\gamma}$ denotes a horizontal lift of γ. It should be observed that in this parametrization, the coordinate curves $t =$ constant generate the N-flow while those curves obtained for $s =$ constant are fibres. The unit normal vector field to S_γ in $(\mathbb{S}^3, \bar{g}^f)$ coincides with the unit binormal to the fibre. Now, one can compute [**1**] the shape operator, A^f, of S_γ in $(\mathbb{S}^3, \bar{g}^f)$. In the orthonormal basis $\{T = \frac{1}{f}\Phi_t$, $N = \Phi_s\}$, it is given by the matrix:

$$A^f = \begin{pmatrix} -B(\log(f)) & f \\ f & \rho \end{pmatrix},$$

where ρ stand for the curvature of γ in $(\mathbb{S}^2, g)$.

On the other hand, formula (3.2) shows that $\bar{\nabla}^f_N T$ is normal to S_γ and so

$$\bar{\nabla}^f_N T = \bar{g}^f(A^f(N), T)B = fB.$$

Now, we compare this formula with (3.2) to deduce the following result.

PROPOSITION 1. *For any positive smooth function f on $\mathbb{S}^2$, the fibres of $\pi : (\mathbb{S}^3, \bar{g}^f) \longrightarrow (\mathbb{S}^2, g)$ are helices in $(\mathbb{S}^3, \bar{g}^f)$ with curvature κ and torsion τ given by*

$$\kappa = \frac{|\,grad(f)\,|}{f} \quad \text{and} \quad \tau = -f$$

REMARK 1. Notice that the fibres of π in Proposition 1 are trivially helices because the $\mathbb{S}^1$-action on $\mathbb{S}^3$ is carried out throughout isometries of $(\mathbb{S}^3, \bar{g}^f)$. However, we shall need these particular values of κ and τ in the next section.

4. Elasticity of fibres. Let Ω be the manifold of regular closed curves in a semi-Riemannian manifold $(M, d\sigma^2)$. For any natural number r, define an elastic energy functional $\mathcal{F}^r : \Omega \longrightarrow \mathbb{R}$ by

$$\mathcal{F}^r(\gamma) = \int_\gamma (\kappa^2)^{\frac{r+1}{2}} ds,$$

where κ denotes the curvature function of $\gamma \in \Omega$, and we write the integrand in this form to point out that it is an even function of the curvature. The variational problems associated with these functionals were considered in [**2**], [**3**]. The critical points of $\mathcal{F}^r$ are called *r-elasticae* (or *r-elastic curves*), and the Euler-Lagrange equations characterizing these curves were computed there.

In particular, since the fibres of $\pi : (\mathbb{S}^3, \bar{g}^f) \longrightarrow (\mathbb{S}^2, g)$ are helices, we use those equations to deduce that a fibre is an r-elastica if and only if

$$\kappa^r((r+1)\bar{R}^f(N, T)T + (r\kappa^2 - (r+1)\tau^2)N) = 0, \tag{4.1}$$

where $\bar{R}^f$ denotes the curvature operator associated with $\bar{g}^f$.

As a consequence of this formula, we see that every geodesic fibre is automatically an r-elastica for any natural number r. In other words, for any $p \in \Sigma$, the fibre $\pi^{-1}(p)$ is an r-elastica in $(\mathbb{S}^3, \bar{g}^f)$ for arbitrary r.

Let U be an open subset of $\mathbb{S}^2 - \Sigma$. The problem is to characterize those positive smooth functions f on U in order for $\pi^{-1}(p)$ to be an r-elastica in $(\mathbb{S}^3, \bar{g}^f)$ for any $p \in U$. To solve this problem, we only need to compute the curvature term appearing in equation (4.1). A straightforward calculus involving some formulae obtained in the last section gives

$$\bar{R}^f(N, T)T = (N(\kappa) + \tau^2 + \frac{N(f)}{f}\,\kappa)N + \kappa\bar{\nabla}^f_N N,$$

and so it can be combined with equation (4.1) and Proposition 1 to deduce the following.

PROPOSITION 2. *Let U be an open subset of $\mathbb{S}^2 - \Sigma$. Then all the fibres in $\pi^{-1}(U)$ are r-elastica in $(\mathbb{S}^3, \bar{g}^f)$ if and only if*

(1) *the unitary field given by* $N = -\dfrac{grad(f)}{|\,grad(f)\,|}$ *defines a unit speed geodesic flow on* $\pi^{-1}(U)$,

(2) *along this N-flow, f evolves according to*

$$(r+1)fN(N(f)) - r(N(f))^2 = 0.$$

Corollary 1. *Let p a point of* $\mathbb{S}^2$ *and denote by* $-p$ *its antipode. We define* $U = \mathbb{S}^2 - \{p, -p\}$ *and* $f : U \longrightarrow \mathbb{R}$ *by* $f(x) = (d(x,p))^{r+1}$, *where* $d(x,p)$ *denotes the distance in* $\mathbb{S}^2$ *from x to p. Then,* $(\pi^{-1}(U), \bar{g}^f)$ *admits a foliation with leaves being r-elastica. Furthermore, this is a subfoliation of a foliation in* $(\pi^{-1}(U), \bar{g}^f)$ *with leaves being flat tori with constant mean curvature.*

In the next result, we choose (U, f) as in Corollary 1.

Corollary 2. *Let G be a compact Lie group of dimension r endowed with a bi-invariant metric* $d\sigma^2$. *Let H be a closed subgroup of the fundamental group* $\pi_1(\pi^{-1}(U))$ *and* $\phi : \pi_1(\pi^{-1}(U))/H \longrightarrow G$ *a monomorphism.*

(1) *There exists a principal fibre G-bundle,* $\eta : P \longrightarrow \pi^{-1}(U)$ *which admits a principal flat connection* θ.

(2) *The metric* $h = \eta^*(\bar{g}^f + \theta^*(d\sigma^2))$ *on P defines a conformal structure,* $[h]$, *on P which is foliated by* $(r+1)$*-dimensional G-invariant, Willmore-Chen submanifolds which have constant mean curvature in the metric h.*

Proof. The way to construct (P, θ) is well-known [**6**]. To show the second statement, we first notice that the space of $(r+1)$-dimensional compact G-invariant submanifolds of P can be identified with $Q = \{\eta^{-1}(\alpha) \mid \alpha$ is a closed immersed curve in $\pi^{-1}(U)\}$. The Willmore-Chen functional $\mathcal{W} : \bar{Q} \longrightarrow \mathbb{R}$ is defined on the space $\bar{Q}$ of $(r+1)$-dimensional compact submanifolds of P and it only depends on the conformal structure. Since the natural action of G on P is carried out throughout isometries of (P, h), it preserves $\mathcal{W}$ and hence, we can apply the principle of symmetric criticality (see [**9**]). Therefore, to obtain G-invariant Willmore-Chen submanifolds in $(P, [h])$ we only need to compute critical points of $\mathcal{W}$ but restricted to Q. However, this restriction can be computed to obtain that $\mathcal{W}(\eta^{-1}(\alpha))$ is a constant multiple of $\mathcal{F}^r(\alpha)$ (see [**2**]). Consequently, $\eta^{-1}(\alpha)$ is Willmore-Chen in $(P, [h])$ if and only if α is an r-elastica in $(\pi^{-1}(U), \bar{g}^f)$. Now the second statement follows from Corollary 1.

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