# ON THE ABSOLUTE NÖRLUND SUMMABILITY OF A FOURIER SERIES

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(Received 28 January 1966)

## 1

Let $\sum_{n=0}^{\infty} a_n$ be a given infinite series and $\{s_n\}$ the sequence of its partial sums. Let $\{p_n\}$ be a sequence of constants, real or complex, and let us write

$$P_n = p_0 + p_1 + p_2 + \cdots + p_n.$$

If

$$\sigma_n = \frac{1}{P_n} \sum_{\nu=0}^{n} p_{n-\nu} s_\nu \to \sigma \qquad (P_n \neq 0)$$

as $n \to \infty$, then we say that the series is summable by the Nörlund method $(N, p_n)$ to $\sigma$. And the series $\sum a_n$ is said to be absolutely summable $(N, p_n)$ or summable $|N, p_n|$ if $\{\sigma_n\}$ is of bounded variation, i.e.,

$$\sum_{n=0}^{\infty} |\Delta \sigma_n| = \sum_{n=0}^{\infty} |\sigma_n - \sigma_{n+1}| < \infty.$$

## 2

Suppose that $\varphi(t)$ is an even and integrable function, periodic with period $2\pi$. Let

$$\varphi(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt.$$

In this note, we prove a theorem for the absolute Nörlund summability [1] of the series $a_2/2 + \sum_{n=1}^{\infty} a_n$.

THEOREM. *Let $\{p_n\}$ be a sequence of positive constants. If $\{\nabla p_n\} = \{(p_n - p_{n-1})\}$ is monotonic and bounded, and if*

(i) $$\sum_{n=2}^{\infty} \frac{n}{P_n (\log n)^{\Delta}} < \infty$$

*for some $\Delta > 0$, and*

[1] For further results concerning the absolute Nörlund summability of a Fourier series, cf. [2].

(ii) $$\left(\log \frac{1}{t}\right)^{A} |\varphi(t)| = O(1)$$

*as* $t \to 0+$, *then the series* $a_0/2+\sum_{n=1}^{\infty} a_n$ *is summable* $|N, p_n|$.

## 3

In the proof of the theorem, the following lemmas are required.

LEMMA 1. *If* $\{p_n\}$ *is defined as in the theorem, and if the series*

$$\sum \frac{|t_n|}{P_n} < \infty,$$

*where* $t_n = \sum_{\nu=0}^{n} s_\nu$, *then* $\sum a_n$ *is summable* $|N, p_n|$.

This lemma is due to Bhatt [1] with improvement.

PROOF. We have

$$\begin{aligned}
\sigma_n - \sigma_{n+1} &= \frac{1}{P_n}\sum_{\nu=0}^{n} p_{n-\nu}s_\nu - \frac{1}{P_{n+1}}\sum_{\nu=0}^{n+1} p_{n+1-\nu}s_\nu \\
&= \Delta\left(\frac{1}{P_n}\right)\sum_{\nu=0}^{n} p_{n-\nu}s_\nu + \frac{1}{P_{n+1}}\sum_{\nu=0}^{n}(p_{n-\nu}-p_{n+1-\nu})s_\nu - \frac{p_0 s_{n+1}}{P_{n+1}} \\
&= I_n + J_n + K_n,
\end{aligned}$$

say. By Abel's transformation,

$$\begin{aligned}
\sum_{n=1}^{m} |I_n| &\leqq \left|\sum_{n=1}^{m} \Delta\left(\frac{1}{P_n}\right)\sum_{\nu=0}^{n-1}(\nabla p_{n-\nu})t_\nu\right| + \sum_{n=1}^{m}\Delta\left(\frac{1}{P_n}\right)p_0|t_n| \\
&\leqq A\sum_{n=1}^{m}\Delta\left(\frac{1}{P_n}\right)\sum_{\nu=0}^{n-1}|t_\nu| + A\sum_{n=1}^{m}\frac{|t_n|}{P_n} \\
&\leqq A\sum_{\nu=0}^{m-1}|t_\nu|\sum_{n=\nu+1}^{m}\Delta\left(\frac{1}{P_n}\right) + A \\
&\leqq A\sum_{\nu=0}^{m-1}\frac{|t_\nu|}{P_\nu} + A \\
&\leqq A.
\end{aligned}$$

By Abel's transformation once more,

$$\begin{aligned}
\sum_{n=1}^{m}|J_n| &\leqq \sum_{n=1}^{m}\frac{1}{P_n}\sum_{\nu=0}^{n-1}|\nabla p_{n-\nu}-\nabla p_{n+1-\nu}|\,|t_\nu| + \sum_{n=1}^{m}\frac{1}{P_{n+1}}|p_0-p_1|\,|t_n| \\
&\leqq \sum_{\nu=0}^{m-1}|t_\nu|\sum_{n=\nu+1}^{m}\frac{|\nabla p_{n-\nu}-\nabla p_{n+1-\nu}|}{P_{n+1}} + A
\end{aligned}$$

https://doi.org/10.1017/S1446788700005607 Published online by Cambridge University Press

$$
\begin{aligned}
&\leqq \sum_{\nu=0}^{m-1} \frac{|t_\nu|}{P_\nu} \sum_{n=\nu+1}^{m} |\nabla p_{n-\nu} - \nabla p_{n+1-\nu}| + A \\
&\leqq A \sum_{\nu=0}^{m-1} \frac{|t_\nu|}{P_\nu} + A \\
&\leqq A,
\end{aligned}
$$

since $\{\nabla p_n\}$ is monotonic and bounded. Finally,

$$
\begin{aligned}
\sum_{n=1}^{m} |K_n| &\leqq \sum_{n=1}^{m} \frac{p_0}{P_{n+1}} |t_{n+1} - t_n| \\
&\leqq A \sum_{n=1}^{m} \frac{|t_n|}{P_n} \\
&\leqq A.
\end{aligned}
$$

Since the $A$'s are independent of $m$, the lemma follows.

LEMMA 2. *If (ii) is satisfied, then*

$$
t_n = O\{n(\log n)^{-A}\}
$$

*as* $n \to \infty$.

PROOF. Choose $0 < r < \frac{1}{2}$ and write

$$
\begin{aligned}
\pi t_n &= \int_0^\pi \varphi(t) \left\{\frac{\sin (n+1)(t/2)}{\sin (t/2)}\right\}^2 dt \\
&= \int_0^{n^{-r}} + \int_{n^{-r}}^{\pi} \\
&= I_1 + I_2,
\end{aligned}
$$

say. We have

$$
\begin{aligned}
|I_1| &\leqq \int_0^{n^{-r}} |\varphi(t)| \left\{\frac{\sin (n+1)(t/2)}{\sin (t/2)}\right\}^2 dt \\
&\leqq \sup_{0 \leqq t \leqq n^{-r}} |\varphi(t)| \int_0^\pi \left\{\frac{\sin (n+1)(t/2)}{\sin (t/2)}\right\}^2 dt \\
&= \pi(n+1) \sup_{0 \leqq t \leqq n^{-r}} |\varphi(t)| \\
&= O\left\{\frac{n}{(\log n)^A}\right\}
\end{aligned}
$$

as $n \to \infty$ by (ii), since

$$
\frac{1}{\pi(n+1)} \int_0^\pi \left\{\frac{\sin (n+1)(t/2)}{\sin (t/2)}\right\}^2 dt \equiv 1.
$$

$$I_2 = 2\int_{n^{-r}}^{\pi} \varphi(t) \left\{\frac{\sin (n+1)(t/2)}{t}\right\}^2 dt+o(1)$$
$$= 2I_3+o(1),$$

say. By integration by parts,

$$I_3 = \left\{\Phi(t)\, \frac{\sin^2 (n+1)(t/2)}{t^2}\right\}_{n^{-r}}^{\pi} - \frac{n+1}{2}\int_{n^{-r}}^{\pi} \frac{\Phi(t)}{t} \cdot \frac{\sin (n+1)t}{t}\, dt$$
$$+2\int_{n^{-r}}^{\pi} \frac{\Phi(t)}{t} \left\{\frac{\sin (n+1)(t/2)}{t}\right\}^2 dt$$
$$= O(1)- \frac{n+1}{2} I_4+2I_5,$$

say. In order to estimate $I_5$, let us construct a function:

$$\psi(t) = \frac{1}{t^{\eta}(\log 1/t)^{\Delta}},$$

where $0 < \eta < 1$. This function is monotonic decreasing in $(\delta, e^{-\Delta/\eta})$ for every $\delta > 0$. If we write

$$\Phi(t) = \frac{t}{(\log 1/t)^{\Delta}}\, h(t),$$

then $h(t) = O(1)$ as $t \to +0$. We write

$$I_5 = \int_{n^{-r}}^{e^{-\Delta/\eta}} + \int_{e^{-\Delta/\eta}}^{\pi}$$
$$= I_6+I_7,$$

say. We have

$$|I_6| = \left|\int_{n^{-r}}^{e^{-\Delta/\eta}} \frac{h(t)}{(\log 1/t)^{\Delta}} \left\{\frac{\sin (n+1)(t/2)}{t}\right\}^2 dt\right|$$
$$\leqq \int_{n^{-r}}^{e^{-\Delta/\eta}} \frac{|h(t)|}{t^2(\log 1/t)^{\Delta}}\, dt$$
$$= \int_{n^{-r}}^{e^{-\Delta/\eta}} \frac{\psi(t)|h(t)|}{t^{2-\eta}}\, dt$$
$$= \psi(n^{-r}) \int_{n^{-r}}^{\tau} \frac{|h(t)|}{t^{2-\eta}}\, dt \qquad (n^{-r} < \tau < e^{-\Delta/\eta})$$
$$= O\left\{\psi(n^{-r}) \int_{n^{-r}}^{\tau} t^{\eta-2} dt\right\}$$
$$= O\{\psi(n^{-r})n^{r(1-\eta)}\}$$
$$= O\left\{\frac{n}{(\log n)^{\Delta}}\right\}$$

as $n \to \infty$. $I_7 = o(1)$ by Riemann-Lebesgue's theorem, Finally,

$$I_4 = \int_{n^{-\tau}}^{\pi} \frac{\Phi(t)}{t^2} \sin(n+1)t\, dt$$
$$= n^{2\tau} \int_{n^{-\tau}}^{\tau} \Phi(t) \sin(n+1)t\, dt$$
$$= O(n^{-2\tau-1}),$$

since $\Phi(t)$ is an integral. Hence,

$$(n+1)I_4 = O(n^{2\tau}) = O\left\{\frac{n}{(\log n)^A}\right\}$$

as $n \to \infty$.

## 4

By Lemma 2, we have

$$\sum_{\nu=n}^{\infty} \frac{|t_\nu|}{P_\nu} = O\left\{\sum_{\nu=n}^{\infty} \frac{\nu}{P_\nu (\log \nu)^A}\right\} = o(1)$$

as $n \to \infty$ by (i).

The theorem follows from Lemma 1.

## References

[1] S. N. Bhatt, 'An aspect of local property of $|N, p_n|$ summability of a Fourier series', *Indian Journ. Math.*, 5 (1963), 87–91.
[2] L. McFadden, 'Absolute Nŏrlund summability', *Duke Math. Journ.*, 9 (1942), 168–207.

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