Can. J. Math. Vol. **48** (4), 1996 pp. 710–736

NECESSARY AND SUFFICIENT CONDITIONS FOR MEAN CONVERGENCE OF LAGRANGE INTERPOLATION FOR ERDŐS WEIGHTS

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ABSTRACT. We investigate mean convergence of Lagrange interpolation at the zeros of orthogonal polynomials $p_n(W^2, x)$ for Erdős weights $W^2 = e^{-2Q}$. The archetypal example is $W_{k,\alpha} = \exp(-Q_{k,\alpha})$, where

$$Q_{k,\alpha}(x) := \exp_k(|x|^\alpha),$$

$\alpha > 1$, $k \geq 1$, and $\exp_k = \exp\Big(\exp\big(\exp(\ldots)\big)\Big)$ is the k-th iterated exponential. Following is our main result: Let $1 < p < \infty$, $\Delta \in \mathbb{R}$, $\kappa > 0$. Let $L_n[f]$ denote the Lagrange interpolation polynomial to f at the zeros of $p_n(W^2,x) = p_n(e^{-2Q},x)$. Then for

$$\lim_{n\to\infty} \|(f - L_n[f])W(1+Q)^{-\Delta}\|_{L_p(\mathbb{R})} = 0$$

to hold for every continuous function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ satisfying

$$\lim_{|x|\to\infty} (fW)(x)(\log|x|)^{1+\kappa} = 0,$$

it is necessary and sufficient that

$$\Delta > \max\Big\{0, \frac{2}{3}\Big(\frac{1}{4} - \frac{1}{p}\Big)\Big\}.$$

1. **Introduction and results.** In the past twenty years, there has begun to develop a general theory of orthogonal polynomials, and associated approximation theory, for weights on $\mathbb{R}$ [8], [18]. In several aspects of the investigations, it has been helpful to distinguish between *Erdős weights* and *Freud weights*.

Freud weights have the form $W^2 = e^{-2Q}$, where $Q\colon \mathbb{R} \longrightarrow \mathbb{R}$ is even and of polynomial growth at infinity. The archetypal example is

$$W_\beta(x) := \exp\big(-Q_\beta(x)\big), \quad Q_\beta(x) := \frac{1}{2}|x|^\beta, \quad \beta > 0. \tag{1.1}$$

Erdős weights have the form $W^2 = e^{-2Q}$, where $Q\colon \mathbb{R} \longrightarrow \mathbb{R}$ is even and of faster than polynomial growth at infinity. The archetypal example is

$$W_{k,\alpha}(x) := \exp\big(-Q_{k,\alpha}(x)\big), \tag{1.2}$$

Received by the editors June 25, 1995; revised August 10, 1994, June 22, 1995.
AMS subject classification: Primary: 42C15, 42C05; secondary: 65D05.
Key words and phrases: Erdős weights, Lagrange interpolation, mean convergence, L_p norms.

where

$$Q_{k,\alpha}(x) := \exp_k(|x|^\alpha), \quad k \geq 1, \quad \alpha > 0. \tag{1.3}$$

Here $\exp_k = \exp\Big(\exp\big(\exp(\ldots)\big)\Big)$ denotes the k-th iterated exponential.

Given a weight $W: \mathbb{R} \longrightarrow \mathbb{R}$ such as those above, we can define orthonormal polynomials

$$p_n(x) = p_n(W^2, x) = \gamma_n x^n + \cdots, \quad \gamma_n = \gamma_n(W^2) > 0,$$

satisfying

$$\int_{-\infty}^{\infty} p_n(W^2, x) p_m(W^2, x) W^2(x)\, dx = \delta_{mn}.$$

To those unfamiliar with the theory of weights on $\mathbb{R}$, writing W^2, rather than say w, for a weight, might seem strange. However the square reflects the L_2 norm, and facilitates formulation of theorems. We denote the zeros of p_n by

$$-\infty < x_{nn} < x_{n-1,n} < x_{n-2,n} < \cdots < x_{2n} < x_{1n} < \infty.$$

The Lagrange interpolation polynomial to a function $f: \mathbb{R} \longrightarrow \mathbb{R}$ at $\{x_{jn}\}_{j=1}^n$ is denoted by $L_n[f]$. Thus if $\mathcal{P}_m$ denotes the class of polynomials of degree $\leq m$, and $\ell_{jn} \in \mathcal{P}_{n-1}$, $1 \leq j \leq n$, are the *fundamental polynomials* of Lagrange interpolation at $\{x_{jn}\}_{j=1}^n$, so

$$\ell_{jn}(x_{kn}) = \delta_{jk},$$

then

$$L_n[f](x) = \sum_{j=1}^{n} f(x_{jn}) \ell_{jn}(x). \tag{1.4}$$

For a large class of Freud weights, mean convergence of Lagrange interpolation was investigated by several authors [1], [4], [11], [17]. The possibility of obtaining identical necessary and sufficient conditions for mean convergence of L_n arises from bounds obtained for $p_n(W^2, \cdot)$ by A. L. Levin and the second author [6]. For notational simplicity, we recall the result of Matjila and the second author [11] only for W_β^2, $\beta > 1$.

THEOREM 1.1. *Let $W_\beta(x) := \exp(-\frac{1}{2}|x|^\beta)$, $\beta > 1$. Given $f: \mathbb{R} \longrightarrow \mathbb{R}$, let $L_n[f]$ denote the Lagrange interpolation polynomial to f at the zeros of $p_n(W^2, x)$. Let $1 < p < \infty$, $\Delta \in \mathbb{R}$, $\alpha > 0$, and*

$$\tau := \frac{1}{p} - \min\{1, \alpha\} + \max\left\{0, \frac{\beta}{6}\left(1 - \frac{4}{p}\right)\right\}.$$

Then for

$$\lim_{n \to \infty} \|(f - L_n[f])(x) W(x)(1 + |x|)^{-\Delta}\|_{L_p(\mathbb{R})} = 0,$$

to hold for every continuous function $f: \mathbb{R} \longrightarrow \mathbb{R}$ satisfying

$$\lim_{|x| \to \infty} (fW)(x)(1 + |x|)^\alpha = 0,$$

it is necessary and sufficient that

$$\Delta > \tau \text{ if } 1 < p \leq 4;$$
$$\Delta > \tau \text{ if } p > 4 \text{ and } \alpha = 1;$$
$$\Delta \geq \tau \text{ if } p > 4 \text{ and } \alpha \neq 1.$$

In describing analogous results for Erdős weights, we need a class of weights W^2 for which suitable bounds are available for $p_n(W^2, \cdot)$. These were found in [7] and L_p analogues were found in [10]. For our purposes, the following subclass of the weights from [7] is suitable:

DEFINITION 1.2. Let $W := e^{-Q}$, where $Q: \mathbb{R} \longrightarrow \mathbb{R}$ is even, continuous, Q'' exists in $(0, \infty)$, $Q^{(j)} \geq 0$ in $(0, \infty)$, $j = 0, 1, 2$, and the function

$$T(x) := 1 + xQ''(x)/Q'(x) \tag{1.5}$$

is increasing in $(0, \infty)$, with

$$\lim_{x \to \infty} T(x) = \infty; \quad T(0+) := \lim_{x \to 0+} T(x) > 1. \tag{1.6}$$

Moreover, we assume that for some $C_1, C_2, C_3 > 0$,

$$C_1 \leq T(x) \Big/ \left(\frac{xQ'(x)}{Q(x)}\right) \leq C_2, \quad x \geq C_3, \tag{1.7}$$

and for every $\varepsilon > 0$,

$$T(x) = O\big(Q(x)^{\varepsilon}\big), \quad x \longrightarrow \infty. \tag{1.8}$$

Then we write $W \in \mathcal{E}_1$.

The new restrictions over those in [7] are (1.8) and $Q \geq 0$. The latter is easily achieved by replacing Q by $Q + |Q(0)|$. The former is needed in simplifying the formulation of our theorems. The principal example of $W = e^{-Q} \in \mathcal{E}_1$ is $W_{k,\alpha} = \exp(-Q_{k,\alpha})$ given by (1.3) with $\alpha > 1$. For this W,

$$T(x) = T_{k,\alpha}(x) = \alpha\left[1 + x^{\alpha} \sum_{\ell=1}^{k} \prod_{j=1}^{\ell-1} \exp_j(x^{\alpha})\right], \quad x \geq 0. \tag{1.9}$$

Here (1.7) holds in the stronger form

$$\lim_{x \to \infty} T(x) \Big/ [xQ'(x)/Q(x)] = 1, \tag{1.10}$$

and (1.8) holds in the stronger form

$$\lim_{x \to \infty} T(x) \Big/ \left[\prod_{j=1}^{k} \log_j Q(x)\right] = \alpha. \tag{1.11}$$

Here, and in the sequel, $\log_k = \log\big(\log(\ldots)\big)$ denotes the k-th iterated logarithm. For $\alpha \leq 1$, the second part of (1.6) fails, but this can be circumvented by considering $W_{k,\alpha/2}(A+x^2)$, with A large enough to guarantee $T(0+) > 1$.

Another (more slowly decaying) example of $W = e^{-Q} \in \mathcal{E}_1$ is given by

$$Q(x) := \exp\Big[\big(\log(A+x^2)\big)^{\beta}\Big], \quad \beta > 1, \quad A \text{ large enough,} \tag{1.12}$$

for which

$$T(x) = \frac{2x^2}{A+x^2}\left[\frac{\beta-1}{\log(A+x^2)} + \beta\{\log(A+x^2)\}^{\beta-1}\right] + \frac{2A}{A+x^2}. \tag{1.13}$$

Again (1.7) holds in the stronger form (1.10), while (1.8) holds in the stronger form

$$\lim_{x\to\infty} T(x)\log x/\log Q(x) = \beta. \tag{1.14}$$

The first results for mean convergence of Lagrange interpolation for a class of Erdős weights appeared in [13]. However in the sufficient conditions for convergence, the restrictions there both on W and on the growth of f, are more severe, because the correct bounds on $p_n(W^2,\cdot)$ were not available. Moreover, there could be no necessary conditions in [13].

Following is our main result:

THEOREM 1.3. *Let $W := e^{-Q} \in \mathcal{E}_1$. Let $L_n[\cdot]$ denote the Lagrange interpolation polynomial to f at the zeros of $p_n(W^2,\cdot)$. Let $1 < p < \infty$, $\Delta \in \mathbb{R}$, $\kappa > 0$. Then for*

$$\lim_{n\to\infty} \|(f - L_n[f])W(1+Q)^{-\Delta}\|_{L_p(\mathbb{R})} = 0, \tag{1.15}$$

to hold for every continuous function $f\colon \mathbb{R} \to \mathbb{R}$ satisfying

$$\lim_{|x|\to\infty} |fW|(x)(\log|x|)^{1+\kappa} = 0, \tag{1.16}$$

it is necessary and sufficient that

$$\Delta > \max\left\{0, \frac{2}{3}\left(\frac{1}{4} - \frac{1}{p}\right)\right\}. \tag{1.17}$$

At first, the choice of the extra weighting factor $(1+Q)$ in (1.15) may seem rather severe. After all, Q grows faster than any polynomial. However, even if f vanishes outside a fixed finite interval, we need such a factor if $p > 4$:

THEOREM 1.4. *Let W, L_n be as above and $p > 4$. Suppose that measurable $U\colon \mathbb{R} \to \mathbb{R}$ satisfies*

$$\liminf_{x\to\infty} U(x)x^{-(\frac{3}{2}-\frac{1}{p})}Q(x)^{\frac{2}{3}(\frac{1}{4}-\frac{1}{p})} > 0. \tag{1.18}$$

Then there exists continuous $f\colon \mathbb{R} \longrightarrow \mathbb{R}$, *vanishing outside* $[-2,2]$, *such that*

$$\limsup_{n\to\infty} \|L_n[f]WU\|_{L_p(\mathbb{R})} = \infty. \tag{1.19}$$

So for $p > 4$, no growth restriction on f, however severe, allows us a weighting factor weaker than a power of $1 + Q$. One can formulate versions of Theorem 1.3 for $p > 4$ that involve $\Delta = \frac{2}{3}(\frac{1}{4} - \frac{1}{p})$, and then one has to introduce extra factors in (1.15), such as negative powers of $1 + |x|$ and negative powers of T or $\log(2 + Q)$. Unfortunately one then needs extra hypotheses on T to avoid very complicated formulations. One of the complicating features here is that T may grow faster than any power of $|x|$ (as in (1.9) for $k \geq 2$), like a power of x (as in (1.9) for $k = 1$), or slower than any power of $|x|$ (as in (1.13)). Moreover, one has to compare T to $\log Q$. We spare the reader the details.

For $p \leq 4$, the weighting factor $1 + Q$ is unnecessarily strong. Let us recall the Erdős-Turan theorem, as extended by Shohat (see [3, Ch. 2, p. 97]). If $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ is Riemann integrable in each finite interval, and there exists an even entire function G with all non-negative Maclaurin series coefficients such that

$$\lim_{|x|\to\infty} f^2(x)/G(x) = 0,$$

and

$$\int_{-\infty}^{\infty} G(x)W^2(x)\,dx < \infty,$$

then

$$\lim_{n\to\infty} \|(f - L_n[f])W\|_{L_2(\mathbb{R})} = 0. \tag{1.20}$$

For the nice weights here, a result of Clunie and Kövari [2, Thm. 4, p. 19] allows us to choose G with

$$G(x) \sim W^{-2}(x)(1 + |x|)^{-1}\big(\log(2 + |x|)\big)^{-1-\kappa}, \quad x \in \mathbb{R}, \quad \kappa > 0.$$

Here and in the sequel, the notation involving $\sim$ means that the ratio of the two sides is bounded above and below by positive constants independent of x. (Later on, the dependence will be on n and possibly other parameters). Thus we can ensure that (1.20) holds provided

$$\lim_{|x|\to\infty} (fW)(x)(1 + |x|)^{1/2}\big(\log(2 + |x|)\big)^{1/2+\kappa/2} = 0.$$

Thus our result does not extend the classical result for $p = 2$.

Actually, the extension from continuous functions to Riemann integrable ones can be completed in the context of the present paper, but would substantially lengthen the proofs, so is omitted. Our main emphasis in any event, is the weighting factors required on L_n or f.

Using different methods, we can prove results of the form

$$\lim_{n\to\infty} \|(f - L_n[f])W\|_{L_p(\mathbb{R})} = 0, \tag{1.21}$$

with $p < 4$, extending the classical Erdős-Turan result. We shall present these in a subsequent paper.

This paper is organized as follows: In Section 2, we gather technical estimates from other papers. In Section 3, we present some quadrature sum estimates. In Section 4, we prove the sufficiency part of Theorem 1.3, and in Section 5, we prove the necessity part of Theorem 1.3, and also prove Theorem 1.4.

We close this section by introducing more notation. Given Q as above, the *Mhaskar-Rahmanov-Saff* number a_u is the positive root of the equation

$$u = \frac{2}{\pi}\int_0^1 a_u t Q'(a_u t)\,dt \Big/ \sqrt{1-t^2}, \quad u > 0. \tag{1.22}$$

For example, for W_β, $a_u = C(\beta)u^{1/\beta}$, $u > 0$. It is instructive to see how a_u, $T(a_u)$, $Q(a_u)$ grow for the example $Q = Q_{k,\alpha}$ of (1.3). Here

$$a_u \sim (\log_k u)^{1/\alpha}; \tag{1.23}$$

$$T(a_u) \sim \prod_{j=1}^{k} \log_j u; \tag{1.24}$$

$$Q(a_u) \sim u\left\{\prod_{j=1}^{k} \log_j u\right\}^{-1/2}. \tag{1.25}$$

To the unfamiliar, one of the uses of a_u is in the identity [14]

$$\|PW\|_{L_\infty(\mathbb{R})} = \|PW\|_{L_\infty[-a_n,a_n]}, \quad P \in \mathcal{P}_n. \tag{1.26}$$

Here and in the sequel, $\mathcal{P}_n$ denotes the polynomials of degree $\leq n$. There are also several L_p analogues [15], [6], [7], for example, there exists $C > 0$ such that for $n \geq 1$ and $P \in \mathcal{P}_n$, [6], [7]

$$\|PW\|_{L_p(\mathbb{R})} \leq C\|PW\|_{L_p[-a_n,a_n]}. \tag{1.27}$$

In the sequel, $C, C_1, C_2, \ldots$ denote constants independent of n, x and $P \in \mathcal{P}_n$. The same symbol does not necessarily denote the same constant in different occurrences.

The n-th *Christoffel function* for a weight W^2 is

$$\begin{aligned}\lambda_n(x) = \lambda_n(W^2, x) &= \inf_{P\in\mathcal{P}_{n-1}} \int_{-\infty}^{\infty} (PW)^2(t)\,dt \big/ P^2(x)\\ &= 1 \Big/ \sum_{j=0}^{n-1} p_j^2(x).\end{aligned} \tag{1.28}$$

The *Christoffel numbers* are

$$\lambda_{jn} := \lambda_n(W^2, x_{jn}), \quad 1 \leq j \leq n. \tag{1.29}$$

The fundamental polynomials ℓ_{jn} of (1.4) admit the representation

$$\ell_{jn}(x) = \lambda_{jn}\frac{\gamma_{n-1}}{\gamma_n}p_{n-1}(x_{jn})\frac{p_n(x)}{x - x_{jn}}. \tag{1.30}$$

The reproducing kernel for W^2 is

$$\begin{aligned} K_n(x,t) = K_n(W^2,x,t) &= \sum_{j=0}^{n-1} p_j(x)p_j(t) \\ &= \frac{\gamma_{n-1}}{\gamma_n}\frac{p_n(x)p_{n-1}(t) - p_n(t)p_{n-1}(x)}{x-t} \end{aligned} \tag{1.31}$$

(the Christoffel-Darboux formula).

Given measurable $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ with $f(x)x^jW^2 \in L_1(\mathbb{R})\ \forall j \geq 0$, the *n-th partial sum of its orthonormal expansion* with respect to W^2 is denoted by $S_n[f](x)$, and admits the representation

$$S_n[f](x) = \int_{-\infty}^{\infty} K_n(x,t)f(t)W^2(t)\,dt. \tag{1.32}$$

If we introduce the *Hilbert transform* of $g \in L_1(\mathbb{R})$ by

$$H[g](x) := \lim_{\varepsilon \to 0+} \int_{|x-t|\geq\varepsilon} \frac{g(t)}{x-t}\,dt, \tag{1.33}$$

(this exists a.e. [20]), then we may use the Christoffel-Darboux formula for $K_n(x,t)$ to rewrite (1.32) as

$$S_n[f] = \frac{\gamma_{n-1}}{\gamma_n}\{p_nH[fp_{n-1}W^2] - p_{n-1}H[fp_nW^2]\}. \tag{1.34}$$

Finally, we define some auxiliary quantities:

$$\delta_n := \big(nT(a_n)\big)^{-2/3}, \quad n \geq 1. \tag{1.35}$$

This quantity is useful in describing the behaviour of $p_n(e^{-2Q},\cdot)$ near x_{1n}. For example,

$$|x_{1n}/a_n(Q) - 1| \leq \frac{L}{2}\delta_n. \tag{1.36}$$

Here L is independent of n. We often use the fact that δ_n is much smaller than any power of $1/T(a_n)$, see Section 2. We also use the function

$$\Psi_n(x) := \max\left\{\sqrt{1 - \frac{|x|}{a_n} + L\delta_n}, \frac{1}{T(a_n)\sqrt{1 - \frac{|x|}{a_n} + L\delta_n}}\right\}, \quad |x| \leq a_n, \tag{1.37}$$

and set

$$\Psi_n(x) := \Psi_n(a_n), \quad |x| \geq a_n. \tag{1.38}$$

This function is used in describing spacing of zeros of p_n, behaviour of Christoffel functions, and so on.

2. **Technical estimates.** In this section, we gather technical estimates from various sources. We begin by recalling a number of estimates from [7]. Throughout, we assume that $W := e^{-Q} \in \mathcal{E}_1$.

LEMMA 2.1. *(a) Uniformly for $n \geq 1$ and $|x| \leq a_n$,*

$$\lambda_n(W^2, x) \sim \frac{a_n}{n} W^2(x)\Psi_n(x). \tag{2.1}$$

(b) For $n \geq 1$,

$$|x_{1n}/a_n - 1| \leq C\delta_n. \tag{2.2}$$

Uniformly for $n \geq 2$ and $1 \leq j \leq n-1$,

$$x_{jn} - x_{j+1,n} \sim \frac{a_n}{n}\Psi_n(x_{jn}). \tag{2.3}$$

(c) For $n \geq 1$,

$$\sup_{x \in \mathbf{R}} |p_n W|(x)\left|1 - \frac{|x|}{a_n}\right|^{1/4} \sim a_n^{-1/2}, \tag{2.4}$$

and

$$\sup_{x \in \mathbf{R}} |p_n W|(x) \sim a_n^{-1/2}\big(nT(a_n)\big)^{1/6}. \tag{2.5}$$

(d) Let $0 < p \leq \infty$ and $K > 0$. There exists $C > 0$ and $n_1 > 0$ such that for $n \geq n_1$ and $P \in \mathcal{P}_n$,

$$\|PW\|_{L_p(\mathbf{R})} \leq C\|PW\|_{L_p[-a_n(1-K\delta_n), a_n(1-K\delta_n)]}. \tag{2.6}$$

Moreover, given $r > 1$, there exists $C > 0$ such that for $n \geq 1$ and $P \in \mathcal{P}_n$,

$$\|PW\|_{L_p(|x| \geq a_{rn})} \leq e^{-Cn/T(a_n)^{1/2}}\|PW\|_{L_p[-a_n, a_n]}. \tag{2.7}$$

(e) For $n \geq 1$,

$$\frac{\gamma_{n-1}}{\gamma_n} \sim a_n. \tag{2.8}$$

(f) Uniformly for $n \geq 2$ and $1 \leq j \leq n-1$,

$$1 - |x_{jn}|/a_n + L\delta_n \sim 1 - |x_{j+1,n}|/a_n + L\delta_n, \tag{2.9}$$

and

$$\Psi_n(x_{jn}) \sim \Psi_n(x_{j+1,n}). \tag{2.10}$$

Here, L is chosen so large that (1.36) is true.

(g) Uniformly for $n \geq 2$ and $2 \leq j \leq n-1$,

$$\frac{a_n^{3/2}}{n}\Psi_n(x_{jn})(1-|x_{jn}|/a_n+L\delta_n)^{1/2}|p_n'W|(x_{jn}) \sim a_n^{1/2}|p_{n-1}W|(x_{jn}) \sim (1-|x_{jn}|/a_n+L\delta_n)^{1/4}. \tag{2.11}$$

PROOF. (a) This is part of Theorem 1.2 in [7, p. 204].

(b) (2.2) is part of Corollary 1.3 in [7, p. 205]. We note however that the proof there actually establishes

$$1-x_{1n}/a_n \leq C\delta_n$$

which is the more difficult part of (2.2). The (easier) converse inequality

$$1-x_{1n}/a_n \geq C\delta_n$$

is not discussed in [7], but requires only a little extra effort. Next (2.3) is Corollary 1.3 in [10]. (A weaker form of (2.3) appears in Corollary 1.3 in [7, p. 205]).

(c) This is Corollary 1.4 (a) in [7, p. 205].

(d) This is Theorem 1.5 in [7, p. 206]. We note that there is a (minor) oversight in the proof of Theorem 1.5 in [7], for $0 < p < \infty$. The proof in [7, pp. 231–236] correctly shows that

$$\|PW\|_{L_p[-a_{4n},a_{4n}]} \leq C\|PW\|_{L_p[-a_n(1-K\delta_n),a_n(1-K\delta_n)]},$$

with C independent of n and P. To estimate $\|PW\|_{L_p(\mathbb{R}\setminus[-a_{4n},a_{4n}])}$, an appeal is made to Lemma 2.5 in [7, p. 215], and unfortunately that lemma is incorrect. It should actually read as follows: For $r > 0$ and $s > 1$, $n \geq 1$ and $P \in \mathcal{P}_n$,

$$\|PW|Q'|^r\|_{L_p(|x|\geq a_{sn})} \leq e^{-Cn/T(a_n)^{1/2}}\|PW\|_{L_p[-a_{sn},a_{sn}]}.$$

This assertion is easily proved using the method of [7, pp. 231 ff.]. The case $r = 0$ gives (2.7).

(e) This is (10.33) in [7, p. 285].

(f) (2.9) is (9.9) in [7, p. 265] and (2.10) follows immediately from (2.9). ■

(g) This is Corollary 1.4 (b) in [7, p. 205]. ■

Next, we recall some results from [9], [10], involving mostly the fundamental polynomials of Lagrange interpolation:

LEMMA 2.2. *(a) Let $0 < p < \infty$. Then for $n \geq 2$,*

$$\|p_nW\|_{L_p(\mathbb{R})} \sim a_n^{\frac{1}{p}-\frac{1}{2}} \times \begin{cases} 1, & p<4, \\ (\log n)^{\frac{1}{4}}, & p=4, \\ \big(nT(a_n)\big)^{\frac{2}{3}(\frac{1}{4}-\frac{1}{p})}, & p>4. \end{cases} \tag{2.12}$$

(b) Uniformly for $n \geq 1$, $1 \leq j \leq n$, $x \in \mathbb{R}$,

$$|\ell_{jn}(x)| \sim \frac{a_n^{3/2}}{n}(\Psi_nW)(x_{jn})(1-|x_{jn}|/a_n+L\delta_n)^{1/4}\left|\frac{p_n(x)}{x-x_{jn}}\right|. \tag{2.13}$$

(c) Uniformly for $n \geq 1$, $1 \leq j \leq n$, $x \in \mathbb{R}$,

$$|\ell_{jn}(x)|W(x)W^{-1}(x_{jn}) \leq C. \tag{2.14}$$

(d) For $n \geq 2$, $1 \leq j \leq n-1$, $x \in [x_{jn}, x_{j+1,n}]$,

$$\ell_{jn}(x)W(x)W^{-1}(x_{jn}) + \ell_{j+1,n}(x)W(x)W^{-1}(x_{j+1,n}) \geq 1. \tag{2.15}$$

PROOF. (a) This is Theorem 1.1 in [10].
(b), (c) These are Theorem 1.2 in [10].
(d) is a special case of the main result of [9]. ■

Next, some technical estimates on growth of a_u, $Q(a_u)$, $T(a_u)$, *etc.*:

LEMMA 2.3. *(a) Given* $r > 0$, *there exists* x_0 *such that for* $x \geq x_0$ *and* $j = 0, 1, 2$, $Q^{(j)}(x)/x^r$ *is increasing in* $[x_0, \infty)$.

(b) Uniformly for $u \geq C$ *and* $j = 0, 1, 2$,

$$a_u^j Q^{(j)}(a_u) \sim uT(a_u)^{j-1/2}. \tag{2.16}$$

(c) Let $0 < \alpha < \beta$. *Then uniformly for* $u \geq C$, $j = 0, 1, 2$,

$$T(a_{\alpha u}) \sim T(a_{\beta u}); \quad Q^{(j)}(a_{\alpha u}) \sim Q^{(j)}(a_{\beta u}). \tag{2.17}$$

(d) Given fixed $r > 1$,

$$a_{ru}/a_u \geq 1 + \frac{\log r}{T(a_{ru})}, \quad u \in (0, \infty). \tag{2.18}$$

Moreover,

$$a_{ru} \sim a_u, \quad u \in (1, \infty). \tag{2.19}$$

(e) Uniformly for $t \in (C, \infty)$,

$$\frac{a_t'}{a_t} \sim \frac{1}{tT(a_t)}. \tag{2.20}$$

(f) Uniformly for $u \in (C, \infty)$, *and* $v \in [\frac{u}{2}, 2u]$, *we have*

$$\left|\frac{a_u}{a_v} - 1\right| \sim \left|\frac{u}{v} - 1\right| \frac{1}{T(a_u)}. \tag{2.21}$$

PROOF. (a) This is Lemma 2.1 (iii) in [7, p. 207].
(b)–(f) are part of Lemma 2.2 in [7, pp. 208–209]. ■

Our final lemma in this section concerns estimates that specifically follow from (1.8): Recall that δ_n was defined by (1.35).

LEMMA 2.4. *(a) Let* $\varepsilon > 0$. *Then*

$$a_n \le Cn^{\varepsilon}; \quad T(a_n) \le Cn^{\varepsilon}, \quad n \ge 1. \tag{2.22}$$

(b) Given $A > 0$, *we have*

$$\delta_n \le CT(a_n)^{-A}, \quad n \ge 1. \tag{2.23}$$

(c) Let $0 < \eta < 1$. *Uniformly for* $n \ge 1$, $0 < |x| \le a_{\eta n}$, $|x| = a_s$, *we have*

$$C_1 \le T(x)\Big(1 - \frac{|x|}{a_n}\Big) \le C_2 \log\frac{n}{s}. \tag{2.24}$$

PROOF. (a) From (2.16) for $j = 0$, we have

$$Q(a_n) \sim nT(a_n)^{-1/2} \le nT(a_1)^{-1/2}.$$

Since Q grows faster than any power of x (Lemma 2.3 (a)), we deduce

$$a_n \le n^{\varepsilon},$$

for n large enough. Also (1.8) then shows that

$$T(a_n) = O\big(Q(a_n)^{\varepsilon}\big) \le Cn^{\varepsilon}.$$

(b) This follows as

$$\delta_n \le n^{-2/3}T(a_1)^{-2/3},$$

that is δ_n decays faster than a power of n, while $T(a_n)$ grows slower than any power of n.

(c) Firstly if $|x|/a_n \le 1/2$, then

$$T(x)\Big(1 - \frac{|x|}{a_n}\Big) \ge T(0+)\frac{1}{2} > \frac{1}{2}.$$

If $|x|/a_n \ge 1/2$, write $|x| = a_s$, so that as $s \le \eta n$,

$$T(x)\Big(1 - \frac{|x|}{a_n}\Big) \ge T(a_s)\Big(1 - \frac{a_s}{a_{s/\eta}}\Big) \ge C_1,$$

by Lemma 2.3 (d). So we have the lower bound in (2.24). We proceed to the upper bound. We can assume that $x = a_s$, $s \ge 1$, and $n \ge n_0$. Then using the inequality

$$1 - u \le |\log u|, \quad u \in (0,1),$$

we obtain

$$\begin{aligned} 1 - \frac{|x|}{a_n} &\le \Big|\log\frac{a_s}{a_n}\Big| = \int_s^n \frac{a_t'}{a_t}\,dt \\ &\le C\int_s^n \frac{dt}{tT(a_t)} \le \frac{C}{T(a_s)}\log\frac{n}{s} = \frac{C}{T(x)}\log\frac{n}{s}. \end{aligned}$$

■

3. **Quadrature sum estimates.** We present two quadrature sum estimates, the first of which is really part of a Lebesgue function type estimate. The second involves quadrature sums for polynomials.

LEMMA 3.1. *Let* $\beta \in (0, \frac{1}{4})$ *and*

$$\Sigma_n(x) := \sum_{|x_{kn}| \geq a_{\beta n}} |\ell_{kn}(x)| W^{-1}(x_{kn}). \tag{3.1}$$

We have for $|x| \leq a_{\beta n/2}$ *and* $|x| \geq a_{2n}$,

$$(\Sigma_n W)(x) \leq C. \tag{3.2}$$

Moreover, for $a_{\beta n/2} \leq |x| \leq a_{2n}$

$$(\Sigma_n W)(x) \leq C\{\log n + a_n^{1/2} |p_n W|(x) T(a_n)^{-1/4}\}. \tag{3.3}$$

PROOF. Let $\Sigma_n^*(x)$ denote the sum $\Sigma_n(x)$ omitting those terms x_{kn} for which $x \in [x_{k+2,n}, x_{k-2,n}]$ (if there are any such k). Here and in the sequel, we set for $\ell \geq 1$,

$$x_{1-\ell,n} := x_{1n} + \ell\delta_n; \quad x_{n+\ell,n} := x_{nn} - \ell\delta_n. \tag{3.4}$$

Of course the sum $\Sigma_n - \Sigma_n^*$ consists of at most 4 terms. Each of these 4 terms admits the bound in Lemma 2.2 (c). So

$$|(\Sigma_n - \Sigma_n^*) W|(x) \leq C_1. \tag{3.5}$$

Next, by (2.13) and (2.3),

$$(\Sigma_n^* W)(x) \sim a_n^{1/2} |p_n W|(x) \sum_{|x_{kn}| \geq a_{\beta_n}}^{*} \frac{(x_{kn} - x_{k+1,n})}{|x - x_{kn}|} \left(1 - \frac{|x_{kn}|}{a_n} + L\delta_n\right)^{1/4}. \tag{3.6}$$

Here the $*$ indicates that the sum omits those k for which $x \in [x_{k+2,n}, x_{k-2,n}]$. Now (*cf.* (2.9)),

$$1 - \frac{|t|}{a_n} + L\delta_n \sim 1 - \frac{|x_{kn}|}{a_n} + L\delta_n, \quad t \in [x_{k+1,n}, x_{kn}], \tag{3.7}$$

uniformly in k and n. Next, if $x \notin [x_{k+2,n}, x_{k-2,n}]$, and $t \in [x_{k+1,n}, x_{kn}]$,

$$\left|\frac{x-t}{x-x_{kn}} - 1\right| = \left|\frac{t - x_{kn}}{x - x_{kn}}\right| \leq \frac{x_{kn} - x_{k+1,n}}{|x_{k\pm 2,n} - x_{kn}|} \leq C.$$

Similarly we may bound $(x - x_{kn})/(x - t)$. So

$$|x - t| \sim |x - x_{kn}|, \quad t \in [x_{k+1,n}, x_{kn}], \quad x \notin [x_{k+2,n}, x_{k-2,n}]. \tag{3.8}$$

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In view of the spacing of the zeros (Lemma 2.1 (b)), we deduce that

$$(3.9)\quad \begin{aligned}(\Sigma_n^* W)(x) &\sim a_n^{1/2}|p_nW|(x)\int_{\substack{a_{\beta n}\le|t|\le a_n\\ |t-x|\ge C\frac{a_n}{n}\Psi_n(x)}}\frac{\left(1-\frac{|t|}{a_n}+L\delta_n\right)^{1/4}}{|t-x|}\,dt\\ &= a_n^{1/2}|p_nW|(x)\int_{\substack{a_{\beta n}/a_n\le|s|\le 1\\ |s-\frac{x}{a_n}|\ge\frac{C}{n}\Psi_n(x)}}\frac{(1-|s|+L\delta_n)^{1/4}}{\left|s-\frac{x}{a_n}\right|}\,ds.\end{aligned}$$

Note that since δ_n is much smaller than $1/T(a_n)$,

$$1-s+L\delta_n\le C_2\left(1-\frac{a_{\beta n}}{a_n}\right)\le C_3/T(a_n).$$

(See Lemma 2.3 (f)). Then we obtain the bound

$$(\Sigma_n^* W)(x)\le Ca_n^{1/2}|p_nW|(x)T(a_n)^{-1/4}\int_{\dots}\frac{ds}{\left|s-\frac{x}{a_n}\right|}.$$

(The range of integration is the same as in (3.9)). Now if $0\le x\le a_{\beta n/2}$ or $x\ge a_{2n}$, then for $n\ge n_0$, we can bound the integral above by

$$\int_{a_{\beta n}/a_n}^{1}\frac{ds}{\left|s-\frac{x}{a_n}\right|}\le\left(1-\frac{a_{\beta n}}{a_n}\right)\max\left\{\left|1-\frac{a_{2n}}{a_n}\right|^{-1},\left|\frac{a_{\beta n}}{a_n}-\frac{a_{\beta n/2}}{a_n}\right|^{-1}\right\}\le C_4,$$

by Lemma 2.3 (f). In this case the bound (2.4) gives

$$(\Sigma_n^* W)(x)\le C_5\left\{1+\left|1-\frac{|x|}{a_n}\right|^{-1/4}T(a_n)^{-1/4}\right\}\le C_6.$$

So we have (3.2). Now let us turn to the more difficult case where $a_{\beta n/2}\le x\le a_{2n}$. We bound the integral in (3.9) as follows:

$$\begin{aligned}&\int_{\substack{a_{\beta n}/a_n\le|s|\le 1\\ |s-\frac{x}{a_n}|\ge\frac{C}{n}\Psi_n(x)}}\frac{(1-|s|+L\delta_n)^{1/4}}{\left|s-\frac{x}{a_n}\right|}\,ds\\ &\quad\le C_7\left[\int_{\substack{a_{\beta n}/a_n\le|s|\le 1\\ |s-\frac{x}{a_n}|\ge\frac{C}{n}\Psi_n(x)}}\frac{(1-s)^{1/4}}{\left|s-\frac{x}{a_n}\right|}\,ds+\delta_n^{1/4}\int_{\substack{a_{\beta n}/a_n\le|s|\le 1\\ |s-\frac{x}{a_n}|\ge\frac{C}{n}\Psi_n(x)}}\frac{ds}{\left|s-\frac{x}{a_n}\right|}\right]\\ &\quad=: C_7[I_1+I_2].\end{aligned}$$

Now since $\frac{1}{n}\Psi_n(x)$ is bounded below by a power of n, we see that

$$I_2\le C_8\delta_n^{1/4}\log n.$$

If $x\ge a_n$, we estimate

$$I_1\le\int_{a_{\beta n}/a_n}^{1}\frac{(1-s)^{1/4}}{|s-1|}\,ds\le C_9T(a_n)^{-1/4}.$$

If $x < a_n$, we make the substitution $1 - s = (1 - \frac{x}{a_n})v$ to get

$$I_1 = \Big(1 - \frac{x}{a_n}\Big)^{1/4} \int_{\substack{v\in[0,(1-a_{\beta n}/a_n)/(1-x/a_n)]\\ |v-1|\geq C\Psi_n(x)/[n(1-x/a_n)]}} \frac{v^{1/4}}{|v-1|}\,dv$$

$$\leq C_{10}\Big(1 - \frac{x}{a_n}\Big)^{1/4}\bigg\{\int_{\substack{v\in[0,2]\\ |v-1|\geq C\Psi_n(x)/[n(1-x/a_n)]}} \frac{dv}{|v-1|} + \int_2^{(1-a_{\beta n}/a_n)/(1-x/a_n)} v^{-3/4}\,dv\bigg\}$$

$$\leq C_{11}\Big\{\Big(1 - \frac{x}{a_n}\Big)^{1/4}\log n + T(a_n)^{-1/4}\Big\}.$$

Combining our estimates for I_1, I_2 and using the bound

$$a_n^{1/2}|p_nW|(x)\delta_n^{1/4} \leq C,$$

which follows from (2.5), we deduce (3.3) from (3.9). ∎

In our second quadrature sum estimate, we need the kernel function for the Chebyshev weight

$$v(t) := (1 - t^2)^{-1/2}, \quad t \in (-1, 1). \tag{3.10}$$

If $p_j(v,x) = \sqrt{2/\pi}\,T_j(x)$ is the j-th orthonormal polynomial for v (at least for $j \geq 1$), then

$$K_n(v,x,t) := \sum_{j=0}^{n-1} p_j(v,x)p_j(v,t) \tag{3.11}$$

admits the following estimates [19, p. 36], [16, p. 108]:

$$K_n(v,x,x) \sim n, \qquad |x| \leq 1. \tag{3.12}$$

Also

$$|K_n(v,x,t)| \leq C\min\Big\{n, \frac{\sqrt{1-x^2}+\sqrt{1-t^2}}{|x-t|}\Big\}, \quad x,t \in [-1,1]. \tag{3.13}$$

LEMMA 3.2. *Let $0 < \eta < 1$. Let $\phi\colon \mathbb{R} \longrightarrow (0,\infty)$ be a continuous function with the following property: For $n \geq 1$, there exist polynomials R_n of degree $\leq n$ such that*

$$C_1 \leq \phi(t)/R_n(t) \leq C_2, \quad |t| \leq a_{4n}. \tag{3.14}$$

Then for $n \geq n_0$ and $P \in \mathcal{P}_n$,

$$\sum_{|x_{jn}|\leq a_{\eta n}} \lambda_{jn}|PW^{-1}|(x_{jn})\phi(x_{jn}) \leq C\int_{-a_{4n}}^{a_{4n}} |PW|\phi. \tag{3.15}$$

PROOF. Essentially the proof is the same as in [13], and the ideas appeared much earlier [16], [17], but we include the details.

STEP 1: AN L_1 CHRISTOFFEL FUNCTION TYPE ESTIMATE. We first note that for $P_1 \in \mathcal{P}_{4n-1}$,

$$(P_1W)^2(x) \le \lambda_{4n}^{-1}(W^2,x)W^2(x)\int_{-\infty}^{\infty}(P_1W)^2(t)\,dt$$
$$\le C_1\frac{n}{a_n}\Psi_{4n}(x)^{-1}\int_{-a_{4n}}^{a_{4n}}(P_1W)^2(t)\,dt,$$

by Lemma 2.1 (a), (d). We deduce that

$$\|P_1W\Psi_{4n}^{1/2}\|^2_{L_\infty[-a_{4n},a_{4n}]} \le C_1\frac{n}{a_n}\int_{-a_{4n}}^{a_{4n}}|P_1W\Psi_{4n}^{-1/2}|(t)\,dt\|P_1W\Psi_{4n}^{1/2}\|_{L_\infty[-a_{4n},a_{4n}]}$$

and hence that for $|x| \le a_{4n}$,

$$|P_1W\Psi_{4n}^{1/2}|(x) \le C_1\frac{n}{a_n}\int_{-a_{4n}}^{a_{4n}}|P_1W\Psi_{4n}^{-1/2}|(t)\,dt.$$

Now we apply this, for fixed $|x| \le a_{4n}$, to

$$P_1(t) := P_2(t)K_n^2\Big(v,\frac{x}{a_{4n}},\frac{t}{a_{4n}}\Big),$$

where $P_2 \in \mathcal{P}_{2n}$. We obtain, using (3.12) that

$$|P_2W\Psi_{4n}^{1/2}|(x) \le C_2\frac{1}{na_n}\int_{-a_{4n}}^{a_{4n}}|P_2W\Psi_{4n}^{-1/2}|(t)K_n^2\Big(v,\frac{x}{a_{4n}},\frac{t}{a_{4n}}\Big)\,dt.$$

In particular, applying this to $P_2 := PR_n$, where $P \in \mathcal{P}_n$, and using (3.14), we obtain

$$|PW\Psi_{4n}^{1/2}\phi|(x) \le \frac{C_3}{na_n}\int_{-a_{4n}}^{a_{4n}}|PW\phi\Psi_{4n}^{-1/2}|(t)K_n^2\Big(v,\frac{x}{a_{4n}},\frac{t}{a_{4n}}\Big)\,dt. \tag{3.16}$$

STEP 2: THE GENERAL QUADRATURE SUM BOUNDED IN TERMS OF A SPECIAL QUADRATURE SUM. We take (3.16) for $x = x_{jn}$, multiply by $\lambda_{jn}W^{-2}(x_{jn})\Psi_{4n}^{-1/2}(x_{jn})$, and sum over all $|x_{jn}| \le a_{\eta n}$. Using our estimate for the Christoffel function $\lambda_n(W^2,\cdot)$ in Lemma 2.1 (a), we obtain

$$\sum_{|x_{jn}|\le a_{\eta n}}\lambda_{jn}|PW^{-1}|(x_{jn})\phi(x_{jn}) \le C_4\int_{-a_{4n}}^{a_{4n}}|PW\phi|(t)\Sigma_n(t)\,dt, \tag{3.17}$$

where

$$\Sigma_n(t) := n^{-2}\sum_{|x_{jn}|\le a_{\eta n}}\Psi_n(x_{jn})\Psi_{4n}^{-1/2}(x_{jn})K_n^2\Big(v,\frac{x_{jn}}{a_{4n}},\frac{t}{a_{4n}}\Big)\Psi_{4n}^{-1/2}(t). \tag{3.18}$$

Clearly the result follows once we show that

$$\Sigma_n(t) \le C_5, \quad |t| \le a_{4n}. \tag{3.19}$$

STEP 3: ESTIMATION OF $\Sigma_n(t)$. First note that for $|x| \le a_{\eta n}$,

$$\Psi_n(x) \sim \Psi_{4n}(x) \sim \Big(1-\frac{|x|}{a_{4n}}\Big)^{1/2}.$$

This follows easily from the fact that $1 - |x|/a_{4n} \geq 1 - |x|/a_n \geq C_6/T(a_n)$ for this range. Moreover,

$$\Psi_{4n}(t) \geq \left(1 - \frac{|t|}{a_{4n}} + L\delta_n\right)^{1/2}$$

for $|t| \leq a_{4n}$. Let us set

$$y_{jn} := x_{jn}/a_{4n}; \quad T := t/a_{4n}.$$

Then we have, using also (3.13) and the spacing in Lemma 2.1 (b), that

(3.20)

$$\begin{aligned}
&\Sigma_n(t)\left(1 - \frac{|t|}{a_{4n}} + L\delta_n\right)^{1/4} \\
&\leq \frac{C_7}{na_n} \sum_{|x_{jn}| \leq a_{\eta n}} (x_{jn} - x_{j+1,n})\left(1 - \frac{|x_{jn}|}{a_{4n}}\right)^{-1/4} K_n^2\left(v, \frac{x_{jn}}{a_{4n}}, \frac{t}{a_{4n}}\right) \\
&\leq C_8 n^{-1} \sum_{|y_{jn}| \leq a_{\eta n}/a_{4n}} (y_{jn} - y_{j+1,n})(1 - |y_{jn}|)^{-1/4} \min\left\{n, \frac{\sqrt{1 - y_{jn}^2} + \sqrt{1 - T^2}}{|y_{jn} - T|}\right\}^2 \\
&\leq C_9 n^{-1} \int_{-1}^{1} (1 - |y|)^{-1/4} \min\left\{n, \frac{\sqrt{1 - y^2} + \sqrt{1 - T^2}}{|y - T|}\right\}^2 dy
\end{aligned}$$

In bounding the sum in terms of the integral, we have used (2.9). Let us assume that $1 - n^{-2} \geq T \geq 0$. Then we can continue the above as

$$\begin{aligned}
\Sigma_n(t)(1 - T)^{1/4} &\leq C_{10} n^{-1} \Big\{ n^2 \int_{y \in [0,1]: |y - T| \leq \frac{1}{n}(1-T)^{1/2}} (1 - y)^{-1/4}\, dy \\
&\quad + \int_{y \in [0,1]: |y - T| > \frac{1}{n}(1-T)^{1/2}} (1 - y)^{-1/4} \frac{1 - y + 1 - T}{|y - T|^2}\, dy \Big\} \\
&= C_{10} n^{-1} \Big\{ n^2 (1 - T)^{3/4} \int_{w: |1 - w| \leq \frac{1}{n}(1-T)^{-1/2}} w^{-1/4}\, dw \\
&\quad + (1 - T)^{-1/4} \int_{w: |1 - w| \geq \frac{1}{n}(1-T)^{-1/2}} w^{-1/4} \frac{|1 + w|}{|1 - w|^2}\, dw \Big\} \\
&\qquad\qquad \text{(substitution } 1 - y = (1 - T)w\text{)} \\
&\leq C_{11}(1 - T)^{1/4}.
\end{aligned}$$

Here we have used that fact that

$$\frac{1}{n}(1 - T)^{-1/2} \leq 1.$$

So in this case, we have (3.19). In the remaining case where $1 - n^{-2} \leq T < 1$, we continue (3.20) as

$$\begin{aligned}
\Sigma_n(t)(L\delta_n)^{1/4} &\leq C_{12} n^{-1} \Big\{ n^2 \int_{y \in [0,1]: |y - T| \leq 4n^{-2}} (1 - y)^{-1/4}\, dy \\
&\quad + \int_0^{1 - 2n^{-2}} (1 - y)^{-1/4} \frac{1 - y + n^{-2}}{|y - T|^2}\, dy \Big\} \\
&\leq C_{13} n^{-1/2}.
\end{aligned}$$

Since $\delta_n^{1/4}$ decays scarcely faster than $n^{-1/6}$, we again have (3.19). ∎

4. **Proof of the sufficiency conditions.** In proving the sufficiency conditions, we split our functions into pieces that vanish inside or outside $[-a_{n/9}, a_{n/9}]$. Throughout, we let χ_S denote the characteristic function of a set S. Also, we set for some fixed $\kappa > 0$,

$$\phi(x) := \left(\log(2+x^2)\right)^{-1-\kappa}. \tag{4.1}$$

Throughout, we assume that $W = e^{-Q} \in \mathcal{E}_1$, that $1 < p < \infty$, and

$$\Delta > \max\left\{0, \frac{2}{3}\left(\frac{1}{4} - \frac{1}{p}\right)\right\}. \tag{4.2}$$

LEMMA 4.1. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions from $\mathbb{R} \longrightarrow \mathbb{R}$ such that for $n \geq 1$,*

$$f_n(x) = 0, \quad |x| < a_{n/9}; \tag{4.3}$$

$$|f_n W|(x) \leq \phi(x), \quad x \in \mathbb{R}. \tag{4.4}$$

Then

$$\lim_{n \longrightarrow \infty} \|L_n[f_n] W(1+Q)^{-\Delta}\|_{L_p(\mathbb{R})} = 0. \tag{4.5}$$

PROOF. Firstly for $|x| \leq a_{n/18}$ or $|x| \geq a_{2n}$, Lemma 3.1 (with $\beta = 1/9$) and (4.4), (4.5) show that

$$\begin{aligned} |L_n[f_n]W|(x) &\leq \phi(a_{n/9}) \sum_{|x_{kn}| \geq a_{n/9}} |\ell_{kn}(x)| W^{-1}(x_{kn}) W(x) \\ &\leq C_1 \phi(a_{n/9}). \end{aligned}$$

So

$$\begin{aligned} \|L_n[f_n]W(1+Q)^{-\Delta}\|_{L_p(|x| \leq a_{n/18}) \cup (|x| \geq a_{2n})} &\leq C_1 \phi(a_{n/9}) \|(1+Q)^{-\Delta}\|_{L_p(\mathbb{R})} \\ &\leq C_2 \phi(a_{n/9}). \end{aligned}$$

Here we have used the fact that Q grows faster than any power of x (Lemma 2.3 (a)). Next, for $a_{n/18} \leq |x| \leq a_{2n}$, Lemma 3.1 gives

$$|L_n[f_n]W|(x) \leq C_3 \phi(a_{n/9}) \{\log n + a_n^{1/2} |p_n W|(x) T(a_n)^{-1/4}\}.$$

Also for this range of x,

$$Q(x) \sim Q(a_n) \sim n T(a_n)^{-1/2}.$$

https://doi.org/10.4153/CJM-1996-037-1 Published online by Cambridge University Press

So

$$
\begin{aligned}
&\|L_n[f_n]W(1+Q)^{-\Delta}\|_{L_p(a_{n/18}\le|x|\le a_{2n})}\\
&\quad\le C_4\phi(a_{n/9})\big(nT(a_n)^{-1/2}\big)^{-\Delta}\{\log n(a_{2n}-a_{n/18})^{1/p}\\
&\qquad + a_n^{1/2}T(a_n)^{-1/4}\|p_nW\|_{L_p(\mathbb{R})}\}\\
&\quad\le C_5\phi(a_{n/9})\big(nT(a_n)^{-1/2}\big)^{-\Delta}(\log n)\big(a_n/T(a_n)\big)^{1/p}\\
&\qquad + C_5\phi(a_{n/9})\big(nT(a_n)^{-1/2}\big)^{-\Delta}T(a_n)^{-1/4}a_n^{1/p}\begin{cases}1, & p<4\\ (\log n)^{1/4}, & p=4\\ \big(nT(a_n)\big)^{(2/3)(1/4-1/p)}, & p>4\end{cases}
\end{aligned}
$$

by Lemma 2.2 (a) and Lemma 2.3 (f). Since $T(a_n)$ and a_n grow slower than any positive power of n (Lemma 2.4 (a)), we see that the last right-hand side is $o\big(\phi(a_{n/9})\big)=o(1)$, because of (4.2). ■

Next, we deal with functions that vanish outside $[-a_{n/9},a_{n/9}]$. We separately estimate the weighted L_p norms of their Lagrange interpolants over $[-a_{n/8},a_{n/8}]$ and $\mathbb{R}\setminus[-a_{n/8},a_{n/8}]$.

LEMMA 4.2. *Let $\{g_n\}_{n=1}^{\infty}$ be a sequence of measurable functions from $\mathbb{R}\longrightarrow\mathbb{R}$ such that for $n\ge 1$,*

$$g_n(x)=0,\quad |x|\ge a_{n/9}; \tag{4.6}$$

$$|g_nW|(x)\le\phi(x),\quad x\in\mathbb{R}. \tag{4.7}$$

Then

$$\lim_{n\longrightarrow\infty}\|L_n[g_n]W(1+Q)^{-\Delta}\|_{L_p(|x|\ge a_{n/8})}=0. \tag{4.8}$$

PROOF. For $x\ge a_{n/8}$,

$$
\begin{aligned}
|L_n[g_n](x)| &\le \sum_{|x_{kn}|\le a_{n/9}}|\ell_{kn}(x)|W^{-1}(x_{kn})\phi(x_{kn})\\
&\le C_1a_n^{1/2}|p_n(x)|\sum_{|x_{kn}|\le a_{n/9}}(x_{kn}-x_{k+1,n})\frac{\Big(1-\frac{|x_{kn}|}{a_n}+L\delta_n\Big)^{1/4}}{|x-x_{kn}|}\phi(x_{kn})\\
&\qquad\qquad\text{(by Lemma 2.2 (b) and (2.3))}\\
&\le C_2a_n^{1/2}|p_n(x)|\int_{-a_{n/9}}^{a_{n/9}}\frac{\Big(1-\frac{|t|}{a_n}+L\delta_n\Big)^{1/4}}{|x-t|}\phi(t)\,dt.
\end{aligned}
$$

Here we have used the monotonicity of ϕ and (3.8). Next, for $t\in[0,a_{n/9}]$, $x\ge a_{n/8}$,

$$0\le\frac{a_n-t}{x-t}=1+\frac{a_n/x-1}{1-t/x}\le 1+\frac{a_n/a_{n/8}-1}{1-a_{n/9}/a_{n/8}}\le C_3,$$

by Lemma 2.3 (f). Moreover,

$$1 - |t|/a_n \geq C_4/T(a_n) >> \delta_n.$$

So

$$|L_n[g_n](x)| \leq C_5 a_n^{1/4} |p_n(x)| \int_0^{a_{n/9}} \frac{(a_n - t)^{1/4}}{x - t} \phi(t)\, dt$$
$$\leq C_6 a_n^{1/4} |p_n(x)| \int_0^{a_{n/9}} (x - t)^{-3/4} \phi(t)\, dt.$$

Here if $t = a_s$, $n/9 \geq s \geq 1$, we have for $x \geq a_{n/8}$,

$$x - t = x(1 - t/x) \geq a_{n/8}(1 - a_s/a_{9s/8}) \geq C_7 a_n / T(a_s).$$

So,

$$|L_n[g_n](x)| \leq C_8 a_n^{-1/2} |p_n(x)| \int_0^{a_{n/9}} T(t)^{3/4} \phi(t)\, dt.$$

Thus

$$\|L_n[g_n] W(1+Q)^{-\Delta}\|_{L_p(|x| \geq a_{n/8})}$$
$$\leq C_9 a_n^{-1/2} \left[\int_0^{a_{n/9}} T(t)^{3/4} \phi(t)\, dt \right] Q(a_{n/8})^{-\Delta} \|p_n W\|_{L_p(\mathbb{R})}.$$

It is easy to see that the integral involving ϕ in the last right-hand side grows slower than any power of n. Then using (4.2) and the estimate on $\|p_n W\|_{L_p(\mathbb{R})}$ provided by Lemma 2.2 (a), we obtain (4.8). ■

We now turn to the most difficult part of the sufficiency proof, namely the estimation of $\|L_n[g_n] W(1+Q)^{-\Delta}\|_{L_p(|x| \leq a_{n/8})}$. We present the most technical part of this as a separate lemma. Recall the notation (1.31)–(1.34) for partial sums $S_n[\cdot]$ of orthonormal expansions with respect to W^2.

LEMMA 4.3. *Let $\sigma \colon \mathbb{R} \longrightarrow \mathbb{R}$ be a bounded measurable function. Then*

$$\|S_n[\sigma \phi W^{-1}] W(1+Q)^{-\Delta}\|_{L_p(|x| \leq a_{n/8})} \leq C \|\sigma\|_{L_\infty(\mathbb{R})}, \tag{4.9}$$

for $n \geq 1$. Here C is independent of σ and n.

PROOF. We split this into several steps. Part of the difficulty lies in that we cannot simply estimate Hilbert transforms in L_p with the weight $(1+Q)^{-\Delta}$, as it does not satisfy Muckenhoupt's A_p condition [20]. We may assume that $\|\sigma\|_{L_\infty(\mathbb{R})} = 1$.

STEP 1: SPLIT $S_n[\cdot](x)$ INTO SEVERAL TERMS DEPENDING ON THE LOCATION OF x. First note that by (1.34) and by our estimates for $\frac{\gamma_{n-1}}{\gamma_n}$ and p_n (see Lemma 2.1 (c), (e)),

$$|S_n[\sigma \phi W^{-1}] W|(x) \leq C_1 a_n^{1/2} \left(1 - \frac{|x|}{a_n}\right)^{-1/4} \sum_{j=n-1}^{n} |H[\sigma \phi p_j W]|(x). \tag{4.10}$$

Now let us choose $\ell := \ell(n)$ such that

$$2^\ell \leq n/8 \leq 2^{\ell+1}. \tag{4.11}$$

Note that our choice of $\ell = \ell(n)$ guarantees that

$$2^{\ell+3} \le n. \tag{4.12}$$

Define

$$\mathcal{T}_k := [a_{2^k}, a_{2^{k+1}}], \quad k \ge 1. \tag{4.13}$$

The reason for this choice of intervals is that

$$Q(x) \sim Q(a_{2^k}) \sim 2^k T(a_{2^k})^{-1/2}, \quad x \in \mathcal{T}_k, \tag{4.14}$$

uniformly in k. For $j = n-1, n$ and $x \in \mathcal{T}_k$, we split

$$\begin{aligned} H[\sigma\phi p_j W](x) &= \left[\int_{-\infty}^{0} + \int_{0}^{a_{2^{k-1}}} + \text{P.V.}\int_{a_{2^{k-1}}}^{a_{2^{k+2}}} + \int_{a_{2^{k+2}}}^{\infty}\right] \frac{(\sigma\phi p_j W)(t)}{x-t}\,dt \\ &:= I_1(x) + I_2(x) + I_3(x) + I_4(x). \end{aligned} \tag{4.15}$$

Here P.V. stands for principal value.

STEP 2: ESTIMATION OF I_1 AND I_2 FOR $x \in \mathcal{T}_k$. We see that (recall $x \ge a_2$)

$$\begin{aligned} |I_1(x)| &\le \int_0^\infty \frac{|p_j W\phi|(-t)}{t+x}\,dt \\ &\le C_2 a_n^{-1/2} \int_0^{\frac{1}{2}a_n} \frac{\phi(t)}{t+a_2}\,dt + C_2 a_n^{-1} \int_{\frac{1}{2}a_n}^\infty |p_j W|(t)\,dt \\ &\le C_3 a_n^{-1/2}. \end{aligned}$$

Here we have used the bound (2.4), the bound for $\|p_n W\|_{L_1(\mathbb{R})}$ in Lemma 2.2 (a), and also the form of ϕ (recall (4.1)), which guarantees that

$$\int_0^\infty \frac{\phi(t)}{1+t}\,dt < \infty. \tag{4.16}$$

Next the bound (2.4) gives

$$\begin{aligned} |I_2(x)| &\le \int_0^{a_{2^{k-1}}} \frac{|p_j W\phi|(t)}{x-t}\,dt \\ &\le C_4 a_n^{-1/2}(1 - x/a_n)^{-1/4} \int_0^{a_{2^{k-1}}} \frac{dt}{x-t} \\ &= C_4 a_n^{-1/2}(1 - x/a_n)^{-1/4} \log(1 - a_{2^{k-1}}/x)^{-1}. \end{aligned}$$

Now

$$1 - a_{2^{k-1}}/x \ge 1 - a_{2^{k-1}}/a_{2^k} \ge C_5/T(a_{2^k}) \ge C_6/T(x).$$

Thus

$$|I_2(x)| \le C_7 a_n^{-1/2}(1 - x/a_n)^{-1/4} \log\big(C_8 T(x)\big).$$

STEP 3: ESTIMATION OF I_4 FOR $x \in \mathcal{T}_k$. Now using our bound (2.4) again and considering separately $2a_{2^{k+2}} \leq$ or $> \frac{1}{2}a_n$ gives

$$\begin{aligned}|I_4(x)| &\leq \int_{a_{2^{k+2}}}^{\infty} \frac{|p_j W\phi|(t)}{t-x}\,dt \\ &\leq C_9\Big[a_n^{-1/2}\int_{a_{2^{k+2}}}^{2a_{2^{k+2}}} |1-t/a_n|^{-1/4}\,\frac{dt}{t-x} \\ &\quad + a_n^{-1/2}\int_{2a_{2^{k+2}}}^{\max\{2a_{2^{k+2}},\frac{1}{2}a_n\}} \frac{\phi(t)}{t}\,dt + \int_{\frac{1}{2}a_n}^{\infty}\frac{|p_jW|(t)}{t}\,dt\Big] \\ &\leq C_{10}a_n^{-1/2}[1+J],\end{aligned}$$

where

$$J := \int_{a_{2^{k+2}}}^{2a_{2^{k+2}}} |1-t/a_n|^{-1/4}\,\frac{dt}{t-x}.$$

(We have used (4.16) and the bound on the L_1 norm of p_nW). Here if $|1-t/a_n| \leq \frac{1}{2}\big(1-\frac{x}{a_n}\big)$, then

$$|t-x| = a_n|(1-x/a_n)-(1-t/a_n)| \geq \frac{1}{2}a_n(1-x/a_n).$$

Thus

$$\begin{aligned}J &\leq C_{11}\Big[(1-x/a_n)^{-1/4}\int_{\substack{|1-t/a_n|\geq\frac{1}{2}(1-x/a_n)\\ t\in[a_{2^{k+2}},2a_{2^{k+2}}]}} \frac{dt}{t-x} \\ &\quad + a_n^{-1}(1-x/a_n)^{-1}\int_{\substack{|1-t/a_n|\leq\frac{1}{2}(1-x/a_n)\\ t\in[a_{2^{k+2}},2a_{2^{k+2}}]}} |1-t/a_n|^{-1/4}\,dt\Big] \\ &\leq C_{12}\Big[(1-x/a_n)^{-1/4}\log\Big(1+\frac{a_{2^{k+2}}}{a_{2^{k+2}}-x}\Big) \\ &\quad + (1-x/a_n)^{-1}\int_{|1-s|\leq\frac{1}{2}(1-x/a_n)} |1-s|^{-1/4}\,ds\Big] \\ &\leq C_{13}(1-x/a_n)^{-1/4}\log\big(C_{14}T(x)\big).\end{aligned}$$

STEP 4: ESTIMATION OF $\|S_n\cdots\|_{L_p(\mathcal{T}_k)}$. Combining our estimates for $I_j, j=1,2,4$ gives

$$|I_1+I_2+I_4|(x) \leq C_{14}a_n^{-1/2}(1-x/a_n)^{-1/4}\log\big(C_{15}T(x)\big).$$

Together with (4.10), (4.14) and (4.15), this gives

$$\begin{aligned}&\|S_n[\sigma\phi W^{-1}]W(1+Q)^{-\Delta}\|_{L_p(\mathcal{T}_k)} \\ &\quad\leq Q(a_{2^k})^{-\Delta}(1-a_{2^{k+1}}/a_n)^{-1/4} \\ &\qquad\times\Big\{(1-a_{2^{k+1}}/a_n)^{-1/4}\log\big(C_{15}T(a_{2^{k+1}})\big)(a_{2^{k+1}}-a_{2^k})^{1/p} \\ &\qquad + a_n^{1/2}\sum_{j=n-1}^{n}\|\mathrm{P.V.}\int_{a_{2^{k-1}}}^{a_{2^{k+2}}}\frac{(\sigma\phi p_jW)(t)}{x-t}\,dt\|_{L_p(\mathcal{T}_k)}\Big\}\end{aligned}$$

https://doi.org/10.4153/CJM-1996-037-1 Published online by Cambridge University Press

We use M. Riesz' theorem on boundedness of the Hilbert transform from $L_p(\mathbb{R})$ to $L_p(\mathbb{R})$ [20] to deduce that

$$\|\text{P.V.}\int_{a_{2^{k-1}}}^{a_{2^{k+2}}} \frac{(\sigma\phi p_j W)(t)}{x-t}\,dt\|^p_{L_p(\mathcal{T}_k)} \le C_{16}\int_{a_{2^{k-1}}}^{a_{2^{k+2}}} |\sigma\phi p_j W|^p(t)\,dt$$
$$\le C_{17} a_n^{-p/2}(1-a_{2^{k+2}}/a_n)^{-p/4}(a_{2^{k+2}}-a_{2^{k-1}}).$$

Next, note that, in view of (4.12), $n \ge 2^{k+3}$ for $k \le \ell$, so

$$1-a_{2^{k+1}}/a_n \ge 1-a_{2^{k+2}}/a_n \ge 1-a_{2^{k+2}}/a_{2^{k+3}} \ge C_{18}/T(a_{2^k}).$$

Moreover,

$$a_{2^{k+1}}-a_{2^k} \le a_{2^{k+2}}-a_{2^{k-1}} \le C_{19}a_{2^k}/T(a_{2^k}).$$

Hence

$$\text{(4.17)}\qquad \begin{aligned}&\|S_n[\sigma\phi W^{-1}]W(1+Q)^{-\Delta}\|_{L_p(\mathcal{T}_k)}\\ &\le C_{20}Q(a_{2^k})^{-\Delta}T(a_{2^k})^{1/2}\log\big(C_{15}T(a_{2^{k+1}})\big)\big(a_{2^k}/T(a_{2^k})\big)^{1/p}.\end{aligned}$$

STEP 5: COMPLETION OF THE PROOF. The estimation of $S_n[\cdot](x)$ for $x \in -\mathcal{T}_k = [-a_{2^{k+1}}, -a_{2^k}]$ is exactly the same as for $x \in \mathcal{T}_k$. Since we have (4.14), and since a_{2^k}, $T(a_{2^k})$ grow much slower than $Q(a_{2^k})$ (Lemma 2.4 (a)), we obtain

$$\|S_n[\sigma\phi W^{-1}]W(1+Q)^{-\Delta}\|^p_{L_p(a_2\le|x|\le a_{n/8})} \le \sum_{k=1}^{\ell}\|S_n[\sigma\phi W^{-1}]W(1+Q)^{-\Delta}\|^p_{L_p(\mathcal{T}_k)}$$
$$\le C_{21}\sum_{k=1}^{\ell}2^{-kp\Delta/2} \le C_{22}.$$

The estimation of $\|S_n[\sigma\phi W^{-1}]W(1+Q)^{-\Delta}\|^p_{L_p(|x|\le a_2)}$ is similar but easier. For $x \in [-a_2, a_2]$, we split

$$H[\sigma\phi p_j W](x) = \left[\int_{-\infty}^{-2a_2} + \text{P.V.}\int_{-2a_2}^{2a_2} + \text{P.V.}\int_{-2a_2}^{2a_2} + \int_{2a_2}^{\infty}\right]\frac{(\sigma\phi p_j W)(t)}{x-t}\,dt.$$

The first and third integrals may be estimated as we did I_1 before, and the second is estimated as we did I_3. ∎

Armed with this lemma, we can complete the estimation of $L_n[g_n]$ over $[-a_{\beta n}, a_{\beta n}]$:

LEMMA 4.4. *Let $\varepsilon \in (0,1)$. Let $\{g_n\}$ be as in Lemma 4.2, except that rather than (4.7), we assume that*

$$\text{(4.18)}\qquad |g_n W|(x) \le \varepsilon\phi(x), \quad x\in\mathbb{R}, \quad n \ge 1.$$

Then

$$\text{(4.19)}\qquad \limsup_{n\to\infty}\|L_n[g_n]W(1+Q)^{-\Delta}\|_{L_p(|x|\le a_{n/8})} \le C\varepsilon,$$

where C is independent of n, $\{g_n\}$ and ε.

PROOF. Let

$$\chi_n := \chi_{[-a_{n/8}, a_{n/8}]};$$
$$h_n := \operatorname{sign}(L_n[g_n])|L_n[g_n]|^{p-1}\chi_n W^{p-2}(1+Q)^{-\Delta p}$$

and

$$\sigma_n := \operatorname{sign} S_n[h_n].$$

We shall show that

(4.20) $$\|L_n[g_n]W(1+Q)^{-\Delta}\|_{L_p(|x|\le a_{n/8})} \le C\varepsilon\|S_n[\sigma_n\phi W^{-1}]W(1+Q)^{-\Delta}\|_{L_p(|x|\le a_{n/8})}.$$

Then Lemma 4.3 gives the result. Now using orthogonality of $f - S_n[f]$ to $\mathcal{P}_{n-1}$, and the Gauss quadrature formula, we see that

$$\begin{aligned}
\|L_n[g_n]W(1+Q)^{-\Delta}\|^p_{L_p(|x|\le a_{n/8})} &= \int_{\mathbb{R}} L_n[g_n]h_n W^2 \\
&= \int_{\mathbb{R}} L_n[g_n]S_n[h_n]W^2 \\
&= \sum_{j=1}^{n} \lambda_{jn} g_n(x_{jn}) S_n[h_n](x_{jn}) \\
&= \sum_{|x_{kn}|<a_{n/9}} \lambda_{jn} g_n(x_{jn}) S_n[h_n](x_{jn}) \\
&\le \varepsilon \sum_{|x_{kn}|<a_{n/9}} \lambda_{jn} |S_n[h_n](x_{jn})| W^{-1}(x_{jn})\phi(x_{jn}) \\
&\le C\varepsilon \int_{\mathbb{R}} |S_n[h_n]|\phi W,
\end{aligned}$$

by Lemma 3.2. Note that it is easy to verify the approximation property of Lemma 3.2 for ϕ (in fact Jackson's Theorem gives polynomials of degree $o(n)$ satisfying (3.14)). We can continue this as

$$\begin{aligned}
&= C\varepsilon \int_{\mathbb{R}} S_n[h_n]\sigma_n \phi W^{-1} W^2 \\
&= C\varepsilon \int_{\mathbb{R}} h_n S_n[\sigma_n \phi W^{-1}] W^2 \\
&= C\varepsilon \int_{-a_{n/8}}^{a_{n/8}} h_n S_n[\sigma_n \phi W^{-1}] W^2
\end{aligned}$$

for (see the definition of h_n), h_n has support inside $[-a_{n/8}, a_{n/8}]$. Using Hölder's inequality with $q = p/(p-1)$, we continue this as

$$\begin{aligned}
&\le C\varepsilon \left(\int_{-a_{n/8}}^{a_{n/8}} |h_n W(1+Q)^{\Delta}|^q\right)^{1/q} \left(\int_{-a_{n/8}}^{a_{n/8}} |S_n[\sigma_n\phi W^{-1}]W(1+Q)^{-\Delta}|^p\right)^{1/p} \\
&= C\varepsilon \|L_n[g_n]W(1+Q)^{-\Delta}\|^{p-1}_{L_p(|x|\le a_{n/8})} \|S_n[\sigma_n\phi W^{-1}]W(1+Q)^{-\Delta}\|_{L_p(|x|\le a_{n/8})}.
\end{aligned}$$

Cancelling the $p-1$-st power of $\|L_n \cdots\|$ gives (4.20). ■

We can now turn to the

PROOF OF THE SUFFICIENCY PART OF THEOREM 1.3. Let $f: \mathbb{R} \longrightarrow \mathbb{R}$ be continuous and satisfy (1.16). Let $\varepsilon \in (0, 1)$. We can choose a polynomial P such that

$$\|(f-P)W\phi^{-1}\|_{L_\infty(\mathbb{R})} \leq \varepsilon.$$

(Compare [5]). Then for n large enough

$$\begin{aligned} &\|(f-L_n[f])W(1+Q)^{-\Delta}\|_{L_p(\mathbb{R})} \\ &\leq \|(f-P)W(1+Q)^{-\Delta}\|_{L_p(\mathbb{R})} + \|L_n[P-f]W(1+Q)^{-\Delta}\|_{L_p(\mathbb{R})} \\ &\leq \varepsilon\|\phi(1+Q)^{-\Delta}\|_{L_p(\mathbb{R})} + \|L_n[P-f]W(1+Q)^{-\Delta}\|_{L_p(\mathbb{R})}. \end{aligned} \tag{4.21}$$

The first norm in (4.21) is of course finite as $\Delta > 0$, and Q grows faster than any power of x. Next, let

$$\chi_n := \chi_{[-a_{n/9}, a_{n/9}]},$$

and write

$$P - f = (P-f)\chi_n + (P-f)(1-\chi_n) =: g_n + f_n.$$

By Lemma 4.1,

$$\lim_{n\to\infty} \|L_n[f_n]W(1+Q)^{-\Delta}\|_{L_p(\mathbb{R})} = 0.$$

Also Lemmas 4.2 and 4.4 together give

$$\limsup_{n\to\infty} \|L_n[g_n]W(1+Q)^{-\Delta}\|_{L_p(\mathbb{R})} \leq C\varepsilon,$$

with C independent of ε. Substituting the estimates for $L_n[f_n]$, $L_n[g_n]$ into (4.21) and then using the arbitrariness of ε, gives (1.15). ∎

5. Proof of the necessary conditions.

We begin with the

PROOF OF THE NECESSITY PART OF THEOREM 1.3. Fix $1 < p < \infty$, $\Delta \in \mathbb{R}$, $\kappa > 0$, $\delta > 1+\kappa$, and assume the conclusion of Theorem 1.3 is true, that is (1.15) holds for every continuous f satisfying (1.16). Let X be the space of all continuous functions $f: \mathbb{R} \longrightarrow \mathbb{R}$ with

$$\|f\|_X := \sup_{x\in\mathbb{R}} |fW|(x)\big(\log(2+|x|)\big)^{\delta} < \infty.$$

Moreover, let Y be the space of all measurable functions $f: \mathbb{R} \longrightarrow \mathbb{R}$ with

$$\|f\|_Y := \|fW(1+Q)^{-\Delta}\|_{L_p(\mathbb{R})} < \infty.$$

Each $f \in X$ satisfies (1.16), so the conclusion of Theorem 1.3 ensures that

$$\lim_{n\to\infty} \|f - L_n[f]\|_Y = 0.$$

Since X is a Banach space, the uniform boundedness principle gives

$$\|f - L_n[f]\|_Y \leq C\|f\|_X, \tag{5.1}$$

with C independent of n and f. In particular as $L_1[f] = f(0)$ (recall that $p_1(x) = \gamma_1 x$), we deduce that for $f \in X$ with $f(0) = 0$,

$$\|f\|_Y \le C\|f\|_X.$$

So for such f,

$$\|L_n[f]\|_Y \le 2C\|f\|_X. \tag{5.2}$$

Choose g_n continuous in $\mathbb{R}$, with $g_n = 0$ in $[0,\infty) \cup (-\infty, -\frac{1}{2}a_n]$, with

$$\|g_n\|_X = \sup_{x\in\mathbb{R}} |g_n W|(x)\big(\log(2+|x|)\big)^{\delta} = 1,$$

and for $x_{jn} \in (-\frac{1}{2}a_n, 0)$,

$$(g_n W)(x_{jn})\big(\log(2+|x_{jn}|)\big)^{\delta} \operatorname{sign}\big(p_n'(x_{jn})\big) = 1.$$

For example, $(g_n W)(x)\big(\log(2+|x|)\big)^{\delta}$ can be chosen to be piecewise linear. Then for $x \in [1, a_n]$,

$$\begin{aligned}
|L_n[g_n](x)| &= \left| \sum_{x_{jn}\in(-\frac{1}{2}a_n,0)} g_n(x_{jn}) \frac{p_n(x)}{p_n'(x_{jn})(x - x_{jn})} \right| \\
&= |p_n(x)| \sum_{x_{jn}\in(-\frac{1}{2}a_n,0)} \frac{\big(\log(2+|x_{jn}|)\big)^{-\delta}}{|p_n' W|(x_{jn})(x + |x_{jn}|)} \\
&\ge C_1 a_n^{1/2} |p_n(x)| (\log a_n)^{-\delta} a_n^{-1} \sum_{x_{jn}\in(-\frac{1}{2}a_n,0)} (x_{jn} - x_{j+1,n}) \\
&\qquad \text{(by Lemma 2.1 (g) and (b))} \\
&\ge C_2 a_n^{1/2} |p_n(x)| (\log a_n)^{-\delta}.
\end{aligned}$$

Then by (5.2),

$$\begin{aligned}
2C = 2C\|g_n\|_X &\ge \|L_n[g_n]\|_Y \\
&\ge C_3 a_n^{1/2} (\log a_n)^{-\delta} \|p_n W (1+Q)^{-\Delta}\|_{L_p[1,a_n]} \\
&\ge C_4 a_n^{1/p} (\log a_n)^{-\delta} Q(a_n)^{-\max\{\Delta,0\}} \begin{cases} 1, & p < 4 \\ (\log n)^{1/4}, & p = 4 \\ \big(nT(a_n)\big)^{\frac{2}{3}(\frac{1}{4}-\frac{1}{p})}, & p > 4 \end{cases}
\end{aligned}$$

Here we used the monotonicity of Q, Lemma 2.2 (a) and Lemma 2.1 (d). Note that $[-1,1]$ does not give a big contribution to the L_p norm of $p_n W$. We obtain a contradiction if $\Delta \le 0$, for all p. Also, for $p > 4$, assuming $\Delta > 0$, we obtain from Lemma 2.3 (b),

$$2C \ge C_5 a_n^{1/p} (\log a_n)^{-\delta} T(a_n)^{\frac{\Delta}{2}+\frac{2}{3}(\frac{1}{4}-\frac{1}{p})} n^{-\Delta+\frac{2}{3}(\frac{1}{4}-\frac{1}{p})}.$$

Since the terms involving a_n and $T(a_n)$ grow to ∞ with n, we see that necessarily

$$\Delta > \frac{2}{3}\Big(\frac{1}{4} - \frac{1}{p}\Big). \qquad \blacksquare$$

PROOF OF THEOREM 1.4. This is similar to the previous proof. We let X be the Banach space of continuous functions $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ vanishing outside $[-2,2]$, with norm

$$\|f\|_X := \|fW\|_{L_\infty[-2,2]}.$$

We let Y be the space of all measurable $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ with

$$\|f\|_Y := \|fWU\|_{L_p(\mathbb{R})} < \infty.$$

Assume that we cannot find f satisfying (1.19). Then the uniform boundedness principle gives (5.1) for all $f \in X$. Again, when $f(0) = 0$, we obtain (5.2). We now choose $g_n \in X$, with $\|g_n\|_X = 1$; with

$$(g_n W)(x_{jn}) \operatorname{sign}\big(p_n'(x_{jn})\big) = 1, \quad x_{jn} \in \Big[-1, -\frac{1}{2}\Big];$$

$g_n = 0$ in $(-\infty, -2] \cup [0, \infty)$ and

$$(g_n W)(x_{jn}) \operatorname{sign}\big(p_n'(x_{jn})\big) \ge 0, \quad x_{jn} \in [-2,2].$$

Much as before, we deduce that for $x \ge 1$,

$$|L_n[g_n](x)| \ge C a_n^{1/2} |p_n(x)|/x.$$

Also by hypothesis, there exist C_1 and C_2 such that

$$U(x) \ge C_1 x^{\frac{3}{2}-\frac{1}{p}} Q(x)^{-\frac{2}{3}(\frac{1}{4}-\frac{1}{p})}, \quad x \ge C_2.$$

Hence by (5.2),

$$\begin{aligned} 2C = 2C\|g_n\|_X &\ge \|L_n[g_n]\|_Y \\ &\ge C_1 \|L_n[g_n](x) W(x) x^{\frac{3}{2}-\frac{1}{p}} Q(x)^{-\frac{2}{3}(\frac{1}{4}-\frac{1}{p})}\|_{L_p[C_2,a_n]} \\ &\ge C_2 a_n^{\frac{1}{2}-\frac{1}{p}} Q(a_n)^{-\frac{2}{3}(\frac{1}{4}-\frac{1}{p})} \|p_n W\|_{L_p[a_{n/2},a_n]} \\ &\ge C_3 T(a_n)^{\frac{1}{4}-\frac{1}{p}}, \end{aligned}$$

much as before, by Lemma 2.2 (a) and (2.4). Of course this is impossible for large n, and we have a contradiction. $\blacksquare$

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