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ON CONVEXITY AND WEAK CLOSENESS FOR THE SET OF Φ-SUPERHARMONIC FUNCTIONS

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Abstract

Convexity and weak closeness of the set of Φ-superharmonic functions in a bounded Lipschitz domain in $\mathbb{R}^n$ is considered. By using the fact of that Φ-superharmonic functions are just the solutions to an obstacle problem and establishing some special properties of the obstacle problem, it is shown that if Φ satisfies Δ_2-condition, then the set is not convex unless $\Phi(r) = Cr^2$ or $n = 1$. Nevertheless, it is found that the set is still weakly closed in the corresponding Orlicz-Sobolev space.

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1. Introduction

Let $\Phi(\cdot)$ be a Young's function satisfying:

(S1) $\Phi(\cdot) \in C^1[0, +\infty) \cap C^2(0, +\infty)$;
(S2) $\Phi(0) = \Phi'(0) = 0$;
(S3) $\Phi'(\cdot)$ increases strictly in $[0, +\infty)$, and $\lim_{r \to +\infty} \Phi'(r) = +\infty$.

Denote $\varphi = \Phi'$. Let $\psi = \varphi^{-1}$ be the inverse function of ψ and

$$\Psi(r) \triangleq \int_0^r \psi(s)\, ds, \quad \forall\, r \in \mathbb{R}^+ \equiv [0, +\infty). \tag{1.1}$$

One can easily verify that Ψ satisfies (S1)–(S3) too. We call (Φ, Ψ) a complementary pair in Young's sense. The well-known Young's inequality shows that

$$rs \le \Phi(r) + \Psi(s), \quad \forall\, r, s \in \mathbb{R}^+, \tag{1.2}$$

and the equality holds if and only if $s = \varphi(r)$.

In this paper, we will consider the set of Φ-superharmonic functions $\mathscr{H}_+^\Phi(\Omega)$ (see (3.2) in Section 3 for the definition). We will prove that, if both Φ and Ψ satisfy the Δ_2-condition (see Lemma 2.1 (iv) in Section 2 for details), then $\mathscr{H}_+^\Phi(a, b)$ is convex. Consequently, it is weakly closed in $W_0^{1,\Phi}(a, b)$ since the strong closeness holds naturally. When $n \geq 2$, if Φ satisfies Δ_2-condition and Ω is a bounded Lipschitz domain in Euclidean space $\mathbb{R}^n$, then $\mathscr{H}_+^\Phi(\Omega)$ is convex if and only if $\Phi(r) = Cr^2$ for some positive constant C. Nevertheless, $\mathscr{H}_+^\Phi(\Omega)$ is still weakly closed in $W_0^{1,\Phi}(\Omega)$. The main idea to get the results is as follows: we use the fact that $\mathscr{H}_+^\Phi(\Omega)$ is the set of solutions for obstacle problems. From this we get many important properties of a Φ-*superharmonic function.*

2. Preliminary properties of Orlicz spaces

In this section, we present some basic properties of Orlicz spaces. For further information about Orlicz spaces, see [1, 4, 8].

First, let us recall the definition of Orlicz spaces $L^\Phi(\Omega)$, $\mathscr{M}^\Phi(\Omega)$. Let Ω be a bounded domain in $\mathbb{R}^n$. We denote

$$L^\Phi(\Omega) \triangleq \left\{ v : \Omega \to \mathbb{R} \text{ measurable } \mid \exists\, t > 0, \text{ such that} \int_\Omega \Phi(t|v(x)|)\, dx < +\infty \right\},$$

equipped with the norm

$$\|v\|_{L^\Phi(\Omega)} \equiv \inf \left\{ t > 0 \,\middle|\, \int_\Omega \Phi\left(\frac{|v(x)|}{t}\right) dx \leq 1 \right\},$$

and

$$\mathscr{M}^\Phi(\Omega) \triangleq \left\{ v : \Omega \to \mathbb{R} \text{ measurable } \middle| \int_\Omega \Phi(t|v(x)|)\, dx < +\infty,\ \forall\, t > 0 \right\}.$$

The following results can be found in [4].

LEMMA 2.1. *Let Φ be Young's function satisfying* (S1)–(S3), *and Ψ be defined by* (1.1). *Then*

(i) *$L^\Phi(\Omega)$ is a Banach space and $L^\infty(\Omega) \subset \mathscr{M}^\Phi(\Omega) \subseteq L^\Phi(\Omega) \subset L^1(\Omega)$.*

(ii) *$\mathscr{M}^\Phi(\Omega)$ is a Banach subspace of $L^\Phi(\Omega)$ and $\mathscr{M}^\Phi(\Omega)$ is separable.*

(iii) *$(\mathscr{M}^\Psi(\Omega))^* = L^\Phi(\Omega)$, where X^* denotes the dual space of a normed linear space X.*

(iv) *$\mathscr{M}^\Phi(\Omega) = L^\Phi(\Omega)$ if and only if Φ satisfies Δ_2-condition (that is, there exist $\rho, \lambda > 0$ such that $\Phi(2r) \leq \lambda\Phi(r)$, for all $r \geq \rho$).*

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(v) *$L^\Phi(\Omega)$ is reflexive if and only if both Φ and Ψ satisfy Δ_2-condition; if and only if Φ satisfies Δ_2-condition, and Φ satisfies ∇_2-condition (that is, there exist ρ, $l > 1$ such that $\Phi(r) \le \Phi(lr)/(2l)$, for all $r \ge \rho$).*

Now, we recall the definition of Orlicz-Sobolev spaces. Denote

$$W^{1,\Phi}(\Omega) \triangleq \{v : \Omega \to \mathbb{R} \mid v, D_\alpha v \in L^\Phi(\Omega),\ \alpha = 1, 2, \dots, n\},$$

equipped with the norm

$$\|v\|_{W^{1,\Phi}(\Omega)} \equiv \sum_{\alpha=1}^{n} \|D_\alpha v\|_{L^\Phi(\Omega)} + \|v\|_{L^\Phi(\Omega)}.$$

Let $W_0^{1,\Phi}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $W^{1,\Phi}(\Omega)$. Denote

$$\|v\|_{W_0^{1,\Phi}(\Omega)} \equiv \||\nabla v|\|_{L^\Phi(\Omega)}, \quad \forall\, v \in W_0^{1,\Phi}(\Omega).$$

Then, $\|\cdot\|_{W_0^{1,\Phi}(\Omega)}$ is an equivalent norm to $\|\cdot\|_{W^{1,\Phi}(\Omega)}$ in $W_0^{1,\Phi}(\Omega)$. Moreover, we can get the following properties by straightforward generalization of the proof of the same properties for ordinary Sobolev spaces.

LEMMA 2.2. *Let Ω be a bounded domain, Φ be Young's function satisfying* (S1)–(S3).

(i) *Suppose that $u \in W_0^{1,\Phi}(\Omega)$. Then $|u| \in W_0^{1,\Phi}(\Omega)$. Consequently, $u^+ \equiv \max(u, 0) \in W_0^{1,\Phi}(\Omega)$, $u^- \equiv \max(-u, 0) \in W_0^{1,\Phi}(\Omega)$. Furthermore, if $u, v \in W_0^{1,\Phi}(\Omega)$, then $u \wedge v \equiv \min(u, v) \in W_0^{1,\Phi}(\Omega)$.*

(ii) *Suppose that Ω is a Lipschitz domain. Then $W_0^{1,\Phi}(\Omega) = W_0^{1,1}(\Omega) \cap W^{1,\Phi}(\Omega)$.*

If $L^\Phi(\Omega)$ is reflexive, then both $W^{1,\Phi}(\Omega)$ and $W_0^{1,\Phi}(\Omega)$ are reflexive, since they can be looked as closed subspaces of $(L^\Phi(\Omega))^{n+1}$. In general, $W^{1,\Phi}(\Omega)$ (or $W_0^{1,\Phi}(\Omega)$) needs not necessary to be reflexive. Moreover, we do not know if $W^{1,\Phi}(\Omega)$ (or $W_0^{1,\Phi}(\Omega)$) is the dual space of a normed space, though $L^\Phi(\Omega) = (\mathscr{M}^\Psi(\Omega))^*$ by Lemma 2.1 (iii). Thus, in general, a bounded series in $W^{1,\Phi}(\Omega)$ needs not necessary to have a subsequence converging weakly in $W^{1,\Phi}(\Omega)$. In some cases, we do not know if we can say weak* convergence. Nevertheless, we have:

LEMMA 2.3. *Let Ω be a bounded Lipschitz domain, Φ be Young's function satisfying* (S1)–(S3), *u_k be bounded in $W^{1,\Phi}(\Omega)$. Then there exists a subsequence u_{k_j}, and a function $u \in W^{1,\Phi}(\Omega)$, such that $u_{k_j} \to u$, weakly in $W^{1,1}(\Omega)$, and*

$$\int_\Omega \Phi(|\nabla u|)\, dx \le \liminf_{j\to+\infty} \int_\Omega \Phi(|\nabla u_{k_j}|)\, dx. \tag{2.1}$$

Moreover, if $u_k \in W_0^{1,\Phi}(\Omega)$ for $k = 1, 2, \dots$, then $u \in W_0^{1,\Phi}(\Omega)$.

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PROOF. Since u_k is bounded in $W^{1,\Phi}(\Omega)$, then u_k and $|\nabla u_k|$ are bounded in $L^\Phi(\Omega)$ (see Lemma 2.1 (i)). By Lemma 2.1 (ii) and (iii), $L^\Phi(\Omega)$ is the dual space of a separable Banach space $\mathscr{M}^\Psi(\Omega)$. Thus, we can choose a subsequence u_{k_j} such that

$$\begin{aligned} u_{k_j} &\to u, \quad \text{weakly}^* \text{ in } L^\Phi(\Omega), \\ \nabla u_{k_j} &\to \vec{h}, \quad \text{weakly}^* \text{ in } L^\Phi(\Omega, \mathbb{R}^n) \equiv (L^\Phi(\Omega))^n. \end{aligned}$$

Since $(\mathscr{M}^\Psi(\Omega))^* = L^\Phi(\Omega)$ and $(L^1(\Omega))^* = L^\infty(\Omega) \subset \mathscr{M}^\Psi(\Omega)$, we have

$$\begin{aligned} u_{k_j} &\to u, \quad \text{weakly in } L^1(\Omega), \\ \nabla u_{k_j} &\to \vec{h}, \quad \text{weakly in } L^1(\Omega, \mathbb{R}^n). \end{aligned}$$

Then it follows that $\vec{h} = \nabla u$ and

$$u_{k_j} \to u, \quad \text{weakly in } W^{1,1}(\Omega). \tag{2.2}$$

Combining with $u, \vec{h} \in L^\Phi(\Omega)$, we get $u \in W^{1,\Phi}(\Omega)$.

To prove (2.1), without loss of generality, we can suppose that

$$\lim_{j\to+\infty} \int_\Omega \Phi(|\nabla u_{k_j}(x)|)\, dx = M < +\infty. \tag{2.3}$$

On the other hand, by (2.2) and Mazur's theorem (see [7, page 120]), there exist $\alpha_{m,l} \geq 0$, $\sum_{l=1}^{k_m} \alpha_{m,l} = 1$, such that

$$\sum_{l=1}^{k_m} \alpha_{m,l} u_{k_{l+m}} \to u, \quad \text{strongly in } W^{1,1}(\Omega). \tag{2.4}$$

Thus, we can also suppose that

$$\sum_{l=1}^{k_m} \alpha_{m,l} \nabla u_{k_{l+m}} \to \nabla u, \quad \text{a.e. } \Omega. \tag{2.5}$$

By the convexity of Φ, we have

$$\int_\Omega \Phi\left(\left|\sum_{l=1}^{k_m} \alpha_{m,l} \nabla u_{k_{l+m}}(x)\right|\right) dx \leq \sum_{l=1}^{k_m} \alpha_{m,l} \int_\Omega \Phi(|\nabla u_{k_{l+m}}(x)|)\, dx. \tag{2.6}$$

Noting that $\Phi \geq 0$ and $\Phi \in C[0, +\infty)$, by (2.3), (2.5)–(2.6) and Fatou's lemma, we get

$$\int_\Omega \Phi(|\nabla u(x)|)\, dx \leq M,$$

and therefore (2.1) follows.

Finally, if $u_k \in W_0^{1,\Phi}(\Omega)$ for $k = 1, 2, \ldots$, then $u \in W_0^{1,1}(\Omega)$ by (2.2). Since we have obtained that $u \in W^{1,\Phi}(\Omega)$, we get $u \in W_0^{1,\Phi}(\Omega)$ by Lemma 2.2 (ii). □

3. Φ-superharmonic functions and obstacle problem

In the following sections we suppose that Φ satisfies Δ_2-condition. Thus, $L^\Phi(\Omega) = \mathscr{M}^\Phi(\Omega)$ by Lemma 2.1 (iv). Moreover,

$$\int_0^r \varphi(s)\,ds = \Phi(r) \geq \frac{1}{\lambda}\Phi(2r) = \frac{1}{\lambda}\int_0^{2r} \varphi(s)\,ds, \quad \forall\, r \in (\rho, +\infty),$$

where $\rho, \lambda > 0$ are given in Lemma 2.1 (iv). Therefore

$$r\varphi(r) \geq \Phi(r) \geq r\varphi(r)/\lambda, \quad r \in (\rho, +\infty), \tag{3.1}$$

since $\varphi(\cdot)$ is increasing in $[0, +\infty)$. Then, for any $v, w \in W_0^{1,\Phi}(\Omega)$, by (3.1) and Young's inequality (1.2),

$$\begin{aligned}\left|\varphi\left(|\nabla v|\right)\frac{\nabla v}{|\nabla v|}\cdot\nabla w\right| &\leq \varphi(|\nabla v|)|\nabla w| \leq \Psi(\varphi(|\nabla v|)) + \Phi(|\nabla w|)\\ &= |\nabla v|\varphi(|\nabla v|) - \Phi(|\nabla v|) + \Phi(|\nabla w|)\\ &\leq \max(\rho\varphi(\rho), (\lambda-1)\Phi(|\nabla v|)) + \Phi(|\nabla w|).\end{aligned}$$

Consequently, $\varphi(|\nabla v|)\nabla v \cdot \nabla w/|\nabla v|$ is integrable in Ω.

Now, we denote by

$$\mathscr{H}_+^\Phi(\Omega) \triangleq \{v \in W_0^{1,\Phi}(\Omega) \mid -\Delta_\Phi v \geq 0, \text{ in } \Omega\} \tag{3.2}$$

the set of all Φ-superharmonic functions, where

$$\Delta_\Phi v \triangleq \operatorname{div}\left(\varphi(|\nabla v|)\frac{\nabla v}{|\nabla v|}\right),$$

and we say that $-\Delta_\Phi v \geq 0$ (in Ω), if

$$\int_\Omega \varphi(|\nabla v|)\frac{\nabla v}{|\nabla v|}\cdot\nabla w\,dx \geq 0, \quad \forall\, w \in W_0^{1,\Phi}(\Omega), w \geq 0, \text{ a.e. } \Omega. \tag{3.3}$$

An element of $\mathscr{H}_+^\Phi(\Omega)$ is called a Φ-superharmonic function.

To study the set $\mathscr{H}_+^\Phi(\Omega)$, we consider the following obstacle problem:

PROBLEM (O). Let y be a measurable function in Ω, $\mathbb{K}(y) \triangleq \{v \in W_0^{1,\Phi}(\Omega) | v \geq y, \text{ a.e. } \Omega\}$. Find a $u \equiv T_\Phi(y) \in \mathbb{K}(y)$, such that

$$\int_\Omega \Phi(|\nabla u|)\,dx = \inf_{v\in\mathbb{K}(y)} \int_\Omega \Phi(|\nabla v|)\,dx. \tag{3.4}$$

One can easily check that $\mathbb{K}(y)$ is convex and closed in $W_0^{1,\Phi}(\Omega)$. It may be empty. When $\mathbb{K}(y) \neq \emptyset$, the following lemma characterizes the solution of Problem (O).

LEMMA 3.1. *Let Ω be a bounded Lipschitz domain, y be a measurable function, $\mathbb{K}(y) \neq \emptyset$, Φ satisfy* (S1)–(S3) *and Δ_2-condition. Then Problem* (O) *has a unique solution $u \equiv T_\Phi(y)$. Moreover, u is characterized by the following variational inequality*:

$$\int_\Omega \varphi(|\nabla u|)\frac{\nabla u}{|\nabla u|}\cdot\nabla(v-u)\,dx \geq 0, \quad \forall\, v \in \mathbb{K}(y). \tag{3.5}$$

PROOF. By Lemma 2.1 (iv), $L^\Phi(\Omega) = \mathscr{M}^\Phi(\Omega)$. Thus,

$$0 \leq \int_\Omega \Phi(|\nabla v|)\,dx < +\infty, \quad \forall\, v \in \mathbb{K}(y).$$

Let $u_k \in \mathbb{K}(y)$ satisfy

$$\lim_{k\to+\infty}\int_\Omega \Phi(|\nabla u_k|)\,dx = \inf_{v\in\mathbb{K}(y)}\int_\Omega \Phi(|\nabla v|)\,dx. \tag{3.6}$$

Then $\int_\Omega \Phi(|\nabla u_k|)\,dx \leq C$, for all $k = 1, 2, \ldots$, for some constant $C > 0$. Since $\Phi(0) = 0$ and Φ is convex,

$$\int_\Omega \Phi\Big(\frac{|\nabla u_k|}{C+1}\Big)\,dx \leq \int_\Omega \frac{1}{C+1}\Phi(|\nabla u_k|)\,dx < 1.$$

Therefore, $\|u_k\|_{W_0^{1,\Phi}(\Omega)} \leq C+1$, for all $k = 1, 2, \ldots$. Thus, by Lemma 2.3, we can suppose that $u_k \to u$, weakly in $W^{1,1}(\Omega)$, for some $u \in W_0^{1,\Phi}(\Omega)$, and

$$\int_\Omega \Phi(|\nabla u|)\,dx \leq \liminf_{k\to+\infty}\int_\Omega \Phi(|\nabla u_k|)\,dx. \tag{3.7}$$

On the other hand, it is easy to get $u \geq y$ from $u_k \geq y$. Consequently, $u \in \mathbb{K}(y)$, and it follows from (3.6) and (3.7) that

$$\int_\Omega \Phi(|\nabla u|)\,dx = \inf_{v\in\mathbb{K}(y)}\int_\Omega \Phi(|\nabla v|)\,dx,$$

that is, we get the existence of a solution. Since $\mathbb{K}(y)$ is convex and $G(\cdot)$ is strictly convex, such a solution must be unique.

Finally, to prove that $u \equiv T_\Phi(y)$ is characterized by (3.5), we modify the proof of Theorem 1.2 in [3, Chapter 1].

Let $u \equiv T_\Phi(y)$ be a solution of Problem (O) corresponding to y. Then, for all $v \in \mathbb{K}(y)$, $\alpha \in (0, 1)$, we have $u + \alpha(v - u) \in \mathbb{K}(y)$. Thus,

$$0 \leq \frac{1}{\alpha}\left\{\int_\Omega \Phi(|\nabla u + \alpha(\nabla v - \nabla u)|)\,dx - \int_\Omega \Phi(|\nabla u|)\,dx\right\}.$$

Let $\alpha \to 0^+$, we get (3.5).

On the other hand, suppose that $u \in \mathbb{K}(y)$ satisfies (3.5). Then, since $G(\cdot)$ is convex, we have

$$\begin{aligned}
&\frac{1}{\alpha}\int_\Omega [\Phi(|\alpha\nabla v + (1-\alpha)\nabla u)|) - \Phi(|\nabla u|)]\,dx \\
&\quad \leq \frac{1}{\alpha}\int_\Omega [\Phi(\alpha|\nabla v| + (1-\alpha)|\nabla u|) - \Phi(|\nabla u|)]\,dx \\
&\quad \leq \frac{1}{\alpha}\int_\Omega [\alpha\Phi(|\nabla v|) + (1-\alpha)\Phi(|\nabla u|) - \Phi(|\nabla u|)]\,dx \\
&\quad = \int_\Omega [\Phi(|\nabla v|) - \Phi(|\nabla u|)]\,dx, \quad \forall\, v \in \mathbb{K}(y).
\end{aligned}$$

Passing to the limit, we get

$$\begin{aligned}
\int_\Omega \Phi(|\nabla v|)\,dx - \int_\Omega \Phi(|\nabla u|)\,dx &\geq \int_\Omega \varphi(|\nabla u|)\frac{\nabla u}{|\nabla u|}\cdot\nabla(v-u)\,dx \\
&\geq 0, \quad \forall\, v \in \mathbb{K}(y).
\end{aligned}$$

Therefore, u is a solution of Problem (O), and we get the proof. □

Now, let us state a simple lemma before we establish the basic properties of T_Φ.

LEMMA 3.2. *Suppose $\phi(\cdot) \in C[0, +\infty)$, $\phi(0) = 0$, and ϕ increases strictly in $[0, +\infty)$. Then*

$$\left[\phi(|a|)\frac{a}{|a|} - \phi(|b|)\frac{b}{|b|}\right]\cdot(a-b) \geq 0, \quad \forall\, a, b \in \mathbb{R}^m, \tag{3.8}$$

and the equality holds if and only if $a = b$.

PROOF. Without loss of generality, we suppose that $a \neq 0$, $b \neq 0$. We have

$$\begin{aligned}
&\left[\phi(|a|)\frac{a}{|a|} - \phi(|b|)\frac{b}{|b|}\right]\cdot(a-b) \\
&\quad = \phi(|a|)|a| + \phi(|b|)|b| - \phi(|a|)\frac{a\cdot b}{|a|} - \phi(|b|)\frac{a\cdot b}{|b|} \\
&\quad \geq \phi(|a|)|a| + \phi(|b|)|b| - \phi(|a|)|b| - \phi(|b|)|a| \\
&\quad = [\phi(|a|) - \phi(|b|)](|a| - |b|) \geq 0.
\end{aligned}$$

Moreover, in the second inequality, the equality holds if and only if $a \cdot b = |a|\,|b|$, while in the last inequality, the equality holds if and only if $|a| = |b|$. Thus, (3.8) holds, and the equality holds if and only if $a = b$. $\square$

Now, we give some basic properties of T_Φ.

LEMMA 3.3. *Let Ω be a bounded Lipschitz domain, y be a measurable function, $\mathbb{K}(y) \neq \emptyset$, Φ satisfy* (S1)–(S3) *and Δ_2-condition. Then*

(i) $T_\Phi(y) \in \mathscr{H}_+^\Phi(\Omega)$.
(ii) $T_\Phi^2 = T_\Phi$, *that is,* $T_\Phi(T_\Phi(y)) = T_\Phi(y)$.
(iii) $T_\Phi(y) = y$ *if and only if* $y \in \mathscr{H}_+^\Phi(\Omega)$.
(iv) *Denote*

$$\mathbb{K}_+(y) \equiv \{v \in W^{1,\Phi}(\Omega) \mid v \geq y, \ a.e. \ \Omega\}, \quad and$$
$$\mathscr{S}_+^\Phi(\Omega) \equiv \{v \in W^{1,\Phi}(\Omega) \mid -\Delta v \geq 0, \ v \geq 0, \ a.e. \ \Omega\}.$$

Then $T_\Phi(y)$ is the smallest element in $\mathbb{K}_+(y) \cap \mathscr{S}_+^\Phi(\Omega)$, that is, $T_\Phi(y) \in \mathbb{K}_+(y) \cap \mathscr{S}_+^\Phi(\Omega)$ and $T_\Phi(y) \leq v$, a.e. Ω, for all $v \in \mathbb{K}_+(y) \cap \mathscr{S}_+^\Phi(\Omega)$. In particular, $T_\Phi(y)$ is the smallest element in $\mathbb{K}_+(y) \cap \mathscr{H}_+^\Phi(\Omega)$ since $\mathscr{H}_+^\Phi(\Omega) \subset \mathscr{S}_+^\Phi(\Omega)$.

(v) *Suppose $u_1 \in \mathscr{H}_+^\Phi(\Omega)$, $u_2, u_2, \ldots \in \mathscr{S}_+^\Phi(\Omega)$. Then $\underline{u} \equiv \inf_k u_k \in \mathscr{H}_+^\Phi(\Omega)$, and*

$$\int_\Omega \Phi(|\nabla \underline{u}|)\, dx \leq \int_\Omega \Phi(|\nabla u_1|)\, dx. \tag{3.9}$$

PROOF. By Lemma 3.1, $u \equiv T_\Phi(y)$ exists and is unique.

(i) For any $v \in W_0^{1,\Phi}(\Omega)$, $v \geq 0$, a.e. Ω, we have $u + v \in \mathbb{K}(y)$. Replacing v by $u + v$ in (3.5), we get

$$\int_\Omega \varphi(|\nabla u|) \frac{\nabla u}{|\nabla u|} \cdot \nabla v\, dx \geq 0, \quad \forall\, v \in W_0^{1,\Phi}(\Omega), \ v \geq 0.$$

Therefore, $u \in \mathscr{H}_+^\Phi(\Omega)$.

(ii) Obviously, $u \in \mathbb{K}(u)$. Therefore, $\mathbb{K}(u) \neq \emptyset$. On the other hand, for $v \in \mathbb{K}(u)$, we have $v \in \mathbb{K}(y)$. Thus, by the definition of u (see (3.4)),

$$\int_\Omega \Phi(|\nabla u|)\, dx \leq \int_\Omega \Phi(|\nabla v|)\, dx, \quad \forall\, v \in \mathbb{K}(u).$$

Consequently, $u = T_\Phi(u)$. That is, $T_\Phi^2 = T_\Phi$.

(iii) Let $T_\Phi(y) = y$. Then $y \in \mathscr{H}_+^\Phi(\Omega)$ by (i).

Now, suppose that $y \in \mathscr{H}_+^\Phi(\Omega)$. Then $y \in \mathbb{K}(y)$ and $\mathbb{K}(y) \neq \emptyset$. Since $-\Delta_\Phi y \geq 0$,

$$\int_\Omega \varphi(|\nabla y|) \frac{\nabla y}{|\nabla y|} \cdot \nabla v\, dx \geq 0, \quad \forall\, v \in W_0^{1,\Phi}(\Omega), \ v \geq 0.$$

For any $v \in \mathbb{K}(y)$, we have $v - y \geq 0$, a.e. Ω, and $v - y \in W_0^{1,\Phi}(\Omega)$. Thus,

$$\int_\Omega \varphi(|\nabla y|)\frac{\nabla y}{|\nabla y|} \cdot \nabla(v - y)\, dx \geq 0, \quad \forall\, v \in \mathbb{K}(y).$$

Therefore, $y = T_\Phi(y)$ by Lemma 3.1.

(iv) Suppose $v \in \mathbb{K}_+(y) \cap \mathscr{S}_+^\Phi(\Omega)$. Then $v \wedge u \in W_0^{1,1}(\Omega)$ since $W^{1,\Phi}(\Omega) \subset W^{1,1}(\Omega)$. Moreover,

$$\nabla(v \wedge u)(x) = \begin{cases} \nabla v(x), & \text{if } v(x) < u(x); \\ \nabla u(x), & \text{if } v(x) \geq u(x), \end{cases} \quad \text{a.e. } \Omega. \tag{3.10}$$

It follows that $v \wedge u \in W^{1,\Phi}(\Omega)$ since $v, u \in W^{1,\Phi}(\Omega)$. Consequently, $v \wedge u \in W_0^{1,\Phi}(\Omega)$ by Lemma 2.2 (ii). Thus, $v \wedge u \in \mathbb{K}(y)$. By Lemma 3.1, we get

$$\int_\Omega \varphi(|\nabla u|)\frac{\nabla u}{|\nabla u|} \cdot (\nabla(v \wedge u) - \nabla u)\, dx \geq 0. \tag{3.11}$$

On the other hand, since $-\Delta_\Phi v \geq 0$, and $u - (v \wedge u) \geq 0$, $u - (v \wedge u) \in W_0^{1,\Phi}(\Omega)$, we get (see (3.3))

$$\int_\Omega \varphi(|\nabla v|)\frac{\nabla v}{|\nabla v|} \cdot (\nabla u - \nabla(v \wedge u))\, dx \geq 0. \tag{3.12}$$

Combining with (3.11), we have

$$\int_\Omega \left[\varphi(|\nabla u|)\frac{\nabla u}{|\nabla u|} - \varphi(|\nabla v|)\frac{\nabla v}{|\nabla v|}\right] \cdot (\nabla u - \nabla(v \wedge u))\, dx \leq 0.$$

Then, by (3.10),

$$\int_{\{u>v\}} \left[\varphi(|\nabla u|)\frac{\nabla u}{|\nabla u|} - \varphi(|\nabla v|)\frac{\nabla v}{|\nabla v|}\right] \cdot (\nabla u - \nabla v)\, dx \leq 0.$$

Therefore, by Lemma 3.2, $\nabla u = \nabla v$, a.e. $\{u > v\}$, that is (see (3.10)), $\nabla u = \nabla(v \wedge u)$, a.e. Ω. Consequently, there exists a constant C, such that $u = v \wedge u + C$, a.e. Ω. Since $u, v \wedge u \in W_0^{1,\Phi}(\Omega)$, we have $C = 0$. Thus, $u = v \wedge u$, a.e. Ω, that is, $u \leq v$, a.e. Ω. On the other hand, it is easy to prove that any $v \in \mathscr{H}_+^\Phi(\Omega)$ satisfies $v \geq 0$, a.e. Ω. Thus $\mathscr{H}_+^\Phi(\Omega) \subset \mathscr{S}_+^\Phi(\Omega)$. Therefore, $T_\Phi(y)$ is also the smallest member in $\mathbb{K}_+(y) \cap \mathscr{H}_+^\Phi(\Omega)$ since $T_\Phi(y) \in \mathbb{K}_+(y) \cap \mathscr{H}_+^\Phi(\Omega)$.

(v) Since $0 \leq \underline{u} \leq u_1$, $\mathbb{K}(\underline{u}) \neq \emptyset$. Thus, $T_\Phi(\underline{u})$ uniquely exists. By (iv) and noting that $u_k \in \mathbb{K}_+(\underline{u}) \cap \mathscr{S}_+^\Phi(\Omega)$, we have $T_\Phi(\underline{u}) \leq u_k$, a.e. Ω. Thus,

$$T_\Phi(\underline{u}) \leq \underline{u} \equiv \inf_k u_k, \quad \text{a.e. } \Omega.$$

On the other hand, $T_\Phi(\underline{u}) \geq \underline{u}$ by the definition of T_Φ. Therefore, $\underline{u} = T_\Phi(\underline{u}) \in \mathscr{H}_+^\Phi(\Omega)$. Finally, since $u_1 \in \mathbb{K}(\underline{u})$, we get (3.9) from (3.4). $\square$

4. Convexity of $\mathscr{H}_+^{\Phi}(\Omega)$

First, we establish the following lemma.

LEMMA 4.1. *Let Ω be a bounded domain in $\mathbb{R}^n$, Φ satisfy* (S1)–(S3), *Δ_2-condition and ∇_2-condition. Let $W^{-1,\Psi}(\Omega)$ be the dual space of $W_0^{1,\Phi}(\Omega)$. Let $\nu \in W^{-1,\Psi}(\Omega)$ and $\nu \geq 0$ in Ω in the distribution sense. Then, there exists a sequence $v_k \in C^\infty(\bar{\Omega})$, such that $v_k \geq 0$ and*

$$v_k \to \nu, \quad \textit{strongly in } W^{-1,\Psi}(\Omega). \tag{4.1}$$

PROOF. By the assumptions and Lemma 2.1 (v), $W_0^{1,\Phi}(\Omega)$ is reflexive. Consequently, as the dual space of $W_0^{1,\Phi}(\Omega)$, $W^{-1,\Psi}(\Omega)$ is reflexive.

Let $\nu \in W^{-1,\Psi}(\Omega)$ and $\nu \geq 0$ in Ω in the distribution sense. According to [5, Chapter 1, Theorem V], ν is a nonnegative measure in Ω. For $k = 1, 2, \ldots$, denote $\Omega_k \equiv \{x \in \Omega | d(x, \partial\Omega) \geq 1/k\}$, where $d(x, \partial\Omega) \equiv \inf_{y\in\partial\Omega} |x - y|$. Let $\nu_k \equiv \nu\lfloor\Omega_k$ be the restriction of ν in Ω_k, that is, $\nu_k(A) \equiv \nu(\Omega_k \bigcap A)$ for any $A \subseteq \Omega$. Then, ν_k is a nonnegative measure in Ω (see [6, Chapter 1], for example). Moreover, for any $w \in C_c^\infty(\Omega)$, we have

$$|\langle \nu_k, w\rangle| \leq \langle \nu_k, |w|\rangle \leq \langle \nu, |w|\rangle \leq \|\nu\|_{W^{-1,\Psi}(\Omega)} \|w\|_{W_0^{1,\Phi}(\Omega)}. \tag{4.2}$$

Consequently, for any $w \in W_0^{1,\Phi}(\Omega)$, we can define $\langle \nu_k, w\rangle$ by choosing $w_j \in C_c^\infty(\Omega)$ converging strongly in $W_0^{1,\Phi}(\Omega)$ and defining

$$\langle \nu_k, w\rangle \triangleq \lim_{j\to+\infty} \langle \nu_k, w_j\rangle.$$

By (4.2),

$$\langle \nu_k, w\rangle \leq \|\nu\|_{W^{-1,\Psi}(\Omega)} \|w\|_{W_0^{1,\Phi}(\Omega)}, \quad \forall\, w \in W_0^{1,\Phi}(\Omega),$$

that is, $\nu_k \in W^{-1,\Psi}(\Omega)$ and

$$\|\nu_k\|_{W^{-1,\Psi}(\Omega)} \leq \|\nu\|_{W^{-1,\Psi}(\Omega)}. \tag{4.3}$$

On the other hand, we have $\langle \nu_k, w\rangle \to \langle \nu, w\rangle$, for all $w \in C_c^\infty(\Omega)$. Combining the above with (4.3), we can easily get $\langle \nu_k, w\rangle \to \langle \nu, w\rangle$, for all $w \in W_0^{1,\Phi}(\Omega)$, that is, $\nu_k \to \nu$, weakly in $W^{-1,\Psi}(\Omega)$.

By Mazur's Theorem, there exist $\alpha_{k,j} \geq 0$, $\sum_{j=1}^{N_k} \alpha_{k,j} = 1$, such that $\tilde{\nu}_k \equiv \sum_{j=1}^{N_k} \alpha_{k,j} \nu_j \to \nu$, strongly in $W^{-1,\Psi}(\Omega)$. For any $l \geq N_k$, we have

$$\begin{aligned} |\langle \nu_l - \nu, w\rangle| &\leq \langle \nu - \nu_l, |w|\rangle \leq \langle \nu - \tilde{\nu}_k, |w|\rangle \\ &\leq \|\nu - \tilde{\nu}_k\|_{W^{-1,\Psi}(\Omega)} \|w\|_{W_0^{1,\Phi}(\Omega)}, \quad \forall\, w \in W_0^{1,\Phi}(\Omega). \end{aligned}$$

Thus, $\|\nu_l - \nu\|_{W^{-1,\Psi}(\Omega)} \le \|\tilde{\nu}_k - \nu\|_{W^{-1,\Psi}(\Omega)}$, if $l \ge N_k$. Consequently,

$$\nu_k \to \nu, \quad \text{strongly in } W^{-1,\Psi}(\Omega). \tag{4.4}$$

Let

$$\eta(x) \equiv \begin{cases} \tau \exp[-1/(1-|x|^2)], & \text{if } |x| < 1; \\ 0, & \text{if } |x| \ge 1, \end{cases}$$

where $\tau > 0$ is chosen such that $\int_{\mathbb{R}^n} \eta(x)\,dx = 1$. Let $\zeta_k \in C_c^\infty(\mathbb{R}^n)$ satisfy $\zeta_k(x) = 1$, in Ω_k, and $\zeta_k(x) = 0$, in $\mathbb{R}^n \setminus \Omega_{2k}$. Then $\zeta_k \nu_k$ can be looked as a distribution and a nonnegative measure in $\mathbb{R}^n$. In fact, $\zeta_k \nu_k = \nu_k$, in Ω. For $j = 1, 2, \dots$, set $\eta_j(x) = j\,\eta(jx)$, and $\hat{\nu}_{kj} \equiv (\zeta_k \nu_k) * \eta_j$ (for the definition of convolution of generalized function, see [2]). Obviously, $\hat{\nu}_{kj}(x) \ge 0$. When $j \ge 12k$, we have $\hat{\nu}_{kj}(x) = 0$, for all $x \notin \Omega_{3k}$. Then, it is easy to prove that $\hat{\nu}_{kj} \in C_c^\infty(\Omega)$ (if $j \ge 12k$), and (as $j \to +\infty$)

$$\hat{\nu}_{kj} \to \nu_k, \quad \text{weakly in } W^{-1,\Psi}(\Omega).$$

Consequently, by Mazur's Theorem, we have $\tilde{\nu}_{kj} \in C^\infty(\bar{\Omega})$, such that $\tilde{\nu}_{kj} \ge 0$, and (as $j \to +\infty$) $\tilde{\nu}_{kj} \to \nu_k$, strongly in $W^{-1,\Psi}(\Omega)$. Thus, combining with (4.4), we have $j_k \ge 1$, such that $v_k \equiv \tilde{\nu}_{kj_k} \to \nu$, strongly in $W^{-1,\Psi}(\Omega)$. Thus, we get the proof. □

Now, we give a result in case $n = 1$.

THEOREM 4.2. *Let $a < b$, Φ satisfy* (S1)–(S3), *Δ_2-condition and ∇_2-condition. Then $u \in \mathscr{H}_+^\Phi(a,b)$ if and only if $u \in W_0^{1,\Phi}(a,b)$, and $-u'' \ge 0$, in (a,b). Consequently, $\mathscr{H}_+^\Phi(a,b)$ is convex.*

PROOF. We give a sketch of the proof.

Let $u \in \mathscr{H}_+^\Phi(a,b)$. Then $\nu \equiv -\Delta_\Phi u \in W^{-1,\Psi}(a,b)$, and $\nu \ge 0$ in the distribution sense. By Lemma 4.1, there exists $\nu_j \in C^\infty(\bar{\Omega})$, such that $\nu_j \ge 0$, and

$$\nu_j \to \nu, \quad \text{strongly in } W^{-1,\Psi}(a,b).$$

Let $u_j \in W_0^{1,\Phi}(a,b)$ be the unique solution of the following equation:

$$\begin{cases} -\left[\varphi\left(\sqrt{1/j^2 + |u_j'|^2}\right) u_j' / \sqrt{1/j^2 + |u_j'|^2}\right]' = \nu_j, & \text{in } (a,b); \\ u_j(a) = u_j(b) = 0. \end{cases} \tag{4.5}$$

Then $u_j \in C^\infty(a,b)$, and u_j is bounded in $W_0^{1,\Phi}(a,b)$. Since $W_0^{1,\Phi}(a,b)$ is reflexive, we can suppose that $u_j \to \tilde{u}$, weakly in $W_0^{1,\Phi}(a,b)$. Similarly as in Lemma 3.1, it follows from (4.5) that for all $v \in W_0^{1,\Phi}(a,b)$,

$$\int_a^b \Phi\left(\sqrt{1/j^2 + |u_j'|^2}\right) dx - \langle \nu_j, u_j \rangle \le \int_a^b \Phi\left(\sqrt{1/j^2 + |v'|^2}\right) dx - \langle \nu_j, v \rangle.$$

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Passing to the limit (see the proof of (2.1)), we get

$$\int_a^b \Phi(|\tilde{u}'|)\,dx - \langle \nu, \tilde{u}\rangle \le \int_a^b \Phi(|v'|)\,dx - \langle \nu, v\rangle, \quad \forall\, v \in W_0^{1,\Phi}(a,b),$$

that is,

$$\begin{cases} -\Delta_\Phi \tilde{u} = \nu, & \text{in } (a,b); \\ \tilde{u}(a) = \tilde{u}(b) = 0. \end{cases}$$

Therefore, $\tilde{u} = u$. Consequently, $u_j \to u$, weakly in $W_0^{1,\Phi}(a,b)$.

Since $u_j \in C^\infty(a,b)$, by (4.5),

$$-\left\{ \frac{\varphi'\left(\sqrt{1/j^2 + |u_j'|^2}\right)|u_j'|^2}{\sqrt{1/j^2 + |u_j'|^2}} + \frac{\varphi\left(\sqrt{1/j^2 + |u_j'|^2}\right)}{j^2(1/j^2 + |u_j'|^2)^{3/2}} \right\} u_j'' = \nu_j \ge 0, \quad \text{in } (a,b).$$

Thus, $-u_j'' \ge 0$, in (a,b). Passing to the limit, we get $-u'' \ge 0$, in (a,b). Similarly, if $u \in W_0^{1,\Phi}(a,b)$, and $-u'' \ge 0$, in (a,b), then we can prove that $u \in \mathscr{H}_+^\Phi(a,b)$. $\square$

When $n \ge 2$, we have:

THEOREM 4.3. *Let Ω be a bounded Lipschitz domain. Suppose that $n \ge 2$, Φ satisfies* (S1)–(S3) *and Δ_2-condition. Then $\mathscr{H}_+^\Phi(\Omega)$ is convex if and only if $\Phi(r) \equiv Cr^2$ for some positive constant C.*

PROOF. If $\Phi(r) \equiv Cr^2$, then $\Delta_\Phi = 2C\Delta$. Consequently, $\mathscr{H}_+^\Phi(\Omega)$ is convex.

On the other hand, suppose $\mathscr{H}_+^\Phi(\Omega)$ is convex. We want to prove $\Phi(r) \equiv Cr^2$, or equivalently, $h(r) \equiv r\varphi'(r) - \varphi(r) \equiv 0$. Without loss of generality, we suppose that $0 \in \Omega$, and therefore there exists an $a > 0$, such that the ball $B_a = B_a(0) \subset \Omega$. We will prove that $h(r) \equiv 0$ in two steps.

Step I. First, we claim that h does not change its sigh in $[0, +\infty)$.

To prove this, suppose that $r_0 > 0$ satisfies $h(r_0) > 0$. Then, by the continuity of h, there exists an $\varepsilon \in (0, r_0)$, such that

$$h(r) > 0, \quad \forall\, r \in (r_0 - \varepsilon, r_0 + \varepsilon). \tag{4.6}$$

Let

$$u_1(x) = -\frac{x_1^2 - x_2^2}{L} + r_0 x_1 + C_1, \quad x = (x_1, x_2, \dots, x_n) \in \Omega, \tag{4.7}$$

where L and C_1 are two large positive numbers, such that

$$|\nabla u_1(x)| \in (r_0 - \varepsilon, r_0 + \varepsilon), \quad \forall\, x \in \Omega, \tag{4.8}$$

and $u_1(x) \ge 1$, for all $x \in \Omega$.

Since $u_1 \in C^2(\Omega)$, $|\nabla u_1| \neq 0$ (see (4.8)), we have

$$-\Delta_\Phi u_1 = -h(|\nabla u_1|)\frac{\langle D^2 u_1 Du_1, Du_1\rangle}{|\nabla u_1|^3} - \varphi(|\nabla u_1|)\frac{\Delta u_1}{|\nabla u_1|}. \tag{4.9}$$

It is easy to verify that

$$\Delta u_1 = 0, \quad \text{in } \Omega, \tag{4.10}$$

$$\langle D^2 u_1 Du_1, Du_1\rangle < 0, \quad \text{in } \Omega. \tag{4.11}$$

Thus, $u_1 \in \mathscr{S}_+^\Phi(\Omega)$.

Let $v_M \in \mathscr{H}_+^\Phi(\Omega)$ be the solution of the following equation

$$\begin{cases} -\Delta_\Phi v_M = M, & \text{in } \Omega; \\ v_M|_{\partial\Omega} = 0. \end{cases} \tag{4.12}$$

We can prove that, if we choose M sufficiently large, then $v_M \geq \sup_{x\in B_a} u_1(x)$, in B_a.

By Lemma 3.3 (v), $u_1 \wedge v_M \in \mathscr{H}_+^\Phi(\Omega)$. Since $0 \in \mathscr{H}_+^\Phi(\Omega)$ and $\mathscr{H}_+^\Phi(\Omega)$ is convex, we have $t(u_1 \wedge v_M) \in \mathscr{H}_+^\Phi(\Omega)$, for $t \in (0, 1)$. Noting that

$$t(u_1 \wedge v_M) = tu_1, \quad \text{in } B_a, \ \forall\, t \in (0, 1),$$

we have

$$0 \leq -\Delta_\Phi(tu_1), \quad \text{in } B_a, \ \forall\, t \in (0, 1), \tag{4.13}$$

that is, (see (4.9) and (4.10)),

$$0 \leq -h(t|\nabla u_1|)\frac{\langle D^2 u_1 Du_1, Du_1\rangle}{|\nabla u_1|^3}, \quad \text{in } B_a, \ \forall\, t \in (0, 1).$$

Therefore, from (4.8) and (4.11), we get $h(r) \geq 0$, for all $r \in (0, r_0)$.

Similarly, if $h(r_0) < 0$ for some $r_0 > 0$, then $h(r) \leq 0$, for all $r \in (0, r_0)$. Therefore, we must have

$$h(r) \geq 0, \quad \forall\, r \geq 0, \tag{4.14}$$

or

$$h(r) \leq 0, \quad \forall\, r \geq 0. \tag{4.15}$$

Step II. By what we established in the first step, we can suppose that (4.14) holds without loss of generality. We claim that $h \equiv 0$. Otherwise, there exists an $r_0 > 0$, such that $h(r_0) > 0$.

Let u_1 be defined by (4.7). Since $h \geq 0$, we have (see (4.9)—(4.11))

$$-\Delta_\Phi(tLu_1) \geq 0, \quad \text{in } \Omega, \ \forall\, t > 0.$$

Let $u_2(x) = -r_0 x_1 + C_2$, such that $u_2 \geq 1$, in Ω. Then

$$-\Delta_\Phi(tLu_2) = 0, \quad \text{in } \Omega, \ \forall\, t > 0.$$

Let M be large enough, and v_M be defined by (4.12), such that

$$v_M \geq (Lr_0/a)\max(u_1, u_2), \quad \text{in } B_a.$$

Let $w_i \equiv (Lr_0 u_i/a) \wedge v_M$, $i = 1, 2$. Then $w_i = (Lr_0/a)u_i$, in B_a, $i = 1, 2$, and by Lemma 3.3 (v), $w_i \in \mathscr{H}_+^\Phi(\Omega)$, $i = 1, 2$. Therefore,

$$w \triangleq \frac{w_1 + w_2}{2} = \frac{r_0}{2a}(-x_1^2 + x_2^2), \quad \text{in } B_a,$$

and

$$|\nabla w| = \frac{r_0}{a}\sqrt{x_1^2 + x_2^2}, \quad \text{in } B_a.$$

Since h is continuous, there exists an $\varepsilon \in (0, r_0)$, such that

$$h(r) > 0, \quad \forall\, r \in (r_0 - \varepsilon, r_0 + \varepsilon).$$

Noting that

$$-\Delta_\Phi w = -h(|\nabla w|)\frac{-x_1^2 + x_2^2}{(x_1^2 + x_2^2)^{3/2}}, \quad \text{in } B_a,$$

and

$$h(|\nabla w|) = h\left(\frac{r_0}{a}\sqrt{x_1^2 + x_2^2}\right) > 0, \quad \text{in } \left\{x \in B_a \,\middle|\, \sqrt{x_1^2 + x_2^2} > \frac{r_0 - \varepsilon}{r_0}\, a\right\},$$

we see that $\{x \in B_a \mid -\Delta_\Phi w(x) < 0\}$ has positive measure. That is, $w \notin \mathscr{H}_+^\Phi(\Omega)$, contradicting the assumption. Therefore $h \equiv 0$ and consequently, $\Phi(r) \equiv Cr^2$. $\square$

5. Weak closeness of $\mathscr{H}_+^\Phi(\Omega)$

It is easy to prove that $\mathscr{H}_+^\Phi(\Omega)$ is strongly closed in $W_0^{1,\Phi}(\Omega)$. If $\mathscr{H}_+^\Phi(\Omega)$ is convex, then it is also weakly closed in $W_0^{1,\Phi}(\Omega)$ by Mazur's Theorem. Theorem 4.3 shows that when $n \geq 2$ and $\Phi(r) \not\equiv Cr^2$, $\mathscr{H}_+^\Phi(\Omega)$ is not convex. But we will prove that $\mathscr{H}_+^\Phi(\Omega)$ is still weakly closed in $W_0^{1,\Phi}(\Omega)$. More precisely, we have the following theorem.

THEOREM 5.1. *Let Ω be a bounded Lipschitz domain, Φ satisfy* (S1)–(S3) *and Δ_2-condition. Suppose that $u_k \in \mathscr{H}_+^\Phi(\Omega)$ is bounded in $W_0^{1,\Phi}(\Omega)$, and $u_k \to u$, weakly in $W_0^{1,1}(\Omega)$. Then $u \in \mathscr{H}_+^\Phi(\Omega)$.*

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PROOF. By the assumption of the theorem, we can suppose that $u_k \to u$, strongly in $L^1(\Omega)$. Thus, we can further suppose that

$$u_k \to u, \quad \text{a.e. } \Omega. \tag{5.1}$$

Therefore,

$$u(x) = \liminf_{k\to+\infty} u_k(x) = \lim_{k\to+\infty} \inf_{j\geq k} u_j(x), \quad \text{a.e. } x \in \Omega. \tag{5.2}$$

Denote $y_k(x) \equiv \inf_{j\geq k} u_j(x)$. Then

$$y_k \nearrow u, \quad a.e.\ \Omega. \tag{5.3}$$

Moreover, by Lemma 3.3 (v), $y_k \in \mathscr{H}_+^{\Phi}(\Omega)$ and

$$\int_\Omega \Phi(|\nabla y_k|)\,dx \leq \int_\Omega \Phi(|\nabla u_k|)\,dx \leq C.$$

Thus, by Lemma 2.3 and (5.3), we must have $y_k \to u$, weakly in $W_0^{1,1}(\Omega)$. Using Lemma 2.3 again, we get

$$\int_\Omega \Phi(|\nabla u|)\,dx \leq \liminf_{k\to+\infty} \int_\Omega \Phi(|\nabla y_k|)\,dx. \tag{5.4}$$

Since $y_k \in \mathscr{H}_+^{\Phi}(\Omega)$, $T_\Phi(y_k) = y_k$ by Lemma 3.3 (iii). By (5.3), $\mathbb{K}(u) \subseteq \mathbb{K}(y_k)$. Thus, we get, from (3.4), that

$$\int_\Omega \Phi(|\nabla y_k|)\,dx \leq \int_\Omega \Phi(|\nabla v|)\,dx, \quad \forall\, v \in \mathbb{K}(u). \tag{5.5}$$

Combining (5.5) with (5.4), we have

$$\int_\Omega \Phi(|\nabla u|)\,dx \leq \int_\Omega \Phi(|\nabla v|)\,dx, \quad \forall\, v \in \mathbb{K}(u),$$

that is, $u = T_\Phi(u)$. Consequently, $u \in \mathscr{H}_+^{\Phi}(\Omega)$. □

When Ω is only a bounded domain, if $W_0^{1,\Phi}(\Omega)$ is reflexive, then Theorem 5.1 still holds. In general, we have:

THEOREM 5.2. *Let Ω be a bounded domain, Φ satisfy* (S1)–(S3) *and* Δ_2*-condition. Suppose that $u_k \in \mathscr{S}_+^{\Phi}(\Omega) \cap W_0^{1,1}(\Omega)$ is bounded in $W^{1,\Phi}(\Omega)$, and $u_k \to u$, weakly in $W_0^{1,1}(\Omega)$. Then $u \in \mathscr{S}_+^{\Phi}(\Omega) \cap W_0^{1,1}(\Omega)$.*

Note that $\mathscr{S}_{+}^{\Phi}(\Omega) \cap W_0^{1,1}(\Omega) = \mathscr{H}_{+}^{\Phi}(\Omega)$ if Ω is a Lipschitz domain, Theorem 5.1 is a special case of Theorem 5.2.

Theorem 5.1 implies that $\mathscr{H}_{+}^{\Phi}(\Omega)$ is weakly closed in $W_0^{1,\Phi}(\Omega)$. In fact, when $W_0^{1,\Phi}(\Omega)$ is reflexive, a bounded sequence converging weakly in $W_0^{1,1}(\Omega)$ must converge weakly in $W^{1,\Phi}(\Omega)$. Thus, at this time, Theorem 5.1 is equivalent to say that $\mathscr{H}_{+}^{\Phi}(\Omega)$ is weakly closed in $W_0^{1,\Phi}(\Omega)$. However, when $W_0^{1,\Phi}(\Omega)$ is not reflexive, a bounded sequence in it need not have a subsequence converging weakly. In addition, we do not know, in $W_0^{1,\Phi}(\Omega)$, if we can always say weak* convergence, since we do not know if $W_0^{1,\Phi}(\Omega)$ is the dual space of some normed linear space. Nevertheless, by Lemma 2.3, a bounded sequence in $W_0^{1,\Phi}(\Omega)$ has a subsequence converging weakly in $W_0^{1,1}(\Omega)$. Thus, in application, for example, when we treat some variational problems, we will find that Theorem 5.1 is more useful than the result of $\mathscr{H}_{+}^{\Phi}(\Omega)$ being weakly closed in $W_0^{1,\Phi}(\Omega)$.

A typical case we are interested in is when $\Phi(r) \equiv r^p/p$ for some $p \in (1, +\infty)$. At this time, Δ_Φ is the so-called p-Laplacian and $\mathscr{H}_{+}^{\Phi}(\Omega)$ is denoted by $\mathscr{H}_{+}^{p}(\Omega)$. It is easy to verify that Φ satisfies Δ_2-condition and ∇_2-condition. Thus, $\mathscr{H}_{+}^{p}(\Omega)$ is convex if and only if $p = 2$ or $n = 1$. Moreover, it is weakly closed in $W_0^{1,p}(\Omega)$.

Another interesting case is when $\Phi(r) \equiv r\ln(1 + r)$. Then, Δ_Φ is called $L\ln L$-Laplacian and $\mathscr{H}_{+}^{\Phi}(\Omega)$ is denoted by $\mathscr{H}_{+}^{L\ln L}(\Omega)$. In this case, Φ satisfies Δ_2-condition. By Theorem 4.3, $\mathscr{H}_{+}^{L\ln L}(\Omega)$ is not convex when $n \geq 2$. By Theorem 5.1, $\mathscr{H}_{+}^{L\ln L}(\Omega)$ is weakly closed in $W_0^{1,L\ln L}(\Omega)$.

6. Generalization

In this section, we generalize the results obtained in Section 5. Let $G : \Omega \times \mathbb{R}^n \to \mathbb{R}$ be a measurable function satisfying

$$G(x, \cdot) \in C^1(\mathbb{R}^n) \cap C^2(\mathbb{R}^n \setminus \{0\}), \quad \forall\, x \in \Omega, \tag{6.1}$$

$$C_1\Phi(|\eta|) \leq G(x, \eta) \leq C_2\Phi(|\eta|), \quad \forall\, x \in \Omega,\ \eta \in \mathbb{R}^n, \tag{6.2}$$

and

$$G_{\eta_i\eta_j}(x, \eta)\xi_i\xi_j > 0, \quad \forall\, x \in \Omega,\ \eta, \xi \in (\mathbb{R}^n \setminus \{0\}), \tag{6.3}$$

where $C_2 > C_1 > 0$ are two constants.

Let $f \in W^{1,\Phi}(\Omega)$. Suppose that $b : \Omega \times \mathbb{R} \to \mathbb{R}$ is measurable in $x \in \Omega$ and continuous in $u \in \mathbb{R}$. Consider the following inequality

$$\begin{cases} -\operatorname{div}(G_\eta(x, \nabla u)) \geq b(x, u(x)), & \text{in } \Omega; \\ u|_{\partial\Omega} = f. \end{cases} \tag{6.4}$$

We have

THEOREM 6.1. *Let Ω be a bounded Lipschitz domain, Φ satisfies* (S1)–(S3) *and Δ_2-condition. Suppose that G satisfies* (6.1)–(6.3), *$f \in W^{1,\Phi}(\Omega)$, $b : \Omega \times \mathbb{R} \to \mathbb{R}$ is measurable in $x \in \Omega$ and continuous in $u \in \mathbb{R}$. Let $u_k(\cdot) \in W^{1,\Phi}(\Omega)$ satisfy* (6.4) *in the weak sense,*

$$u_k \to u, \quad \textit{weakly in } W^{1,1}(\Omega), \tag{6.5}$$

and $u_k(\cdot)$ be bounded in $W^{1,\Phi}(\Omega)$. Moreover, suppose that $b(\cdot, u_k(\cdot)) \in L^{\Psi}(\Omega)$, and there exists a $b_0(\cdot) \in L^{\Psi}(\Omega)$, such that $b(x, u_k(x)) \geq b_0(x)$, a.e. Ω, for all $k = 1, 2, \ldots$. Then $u \in W^{1,\Phi}(\Omega)$, and it satisfies (6.4) *in the weak sense too.*

PROOF. We give a sketch of the proof.

If $b(x, u)$ is independent of u, then the result can be obtained by a modification of the proof of Theorem 5.1.

In general, by (6.5), we can suppose that $u_k(x)$ converges to $u(x)$ for almost all $x \in \Omega$. Thus, by Egorov's Theorem (see [7, Chapter 0], for example), for any $\varepsilon > 0$, there exists a subset $E_\varepsilon \subset \Omega$, such that $|E_\varepsilon| < \varepsilon$ and $u_k(\cdot)$ converges to $u(\cdot)$ uniformly in $\Omega \setminus E_\varepsilon$, where $|E_\varepsilon|$ is the Lebesgue measure of E_ε. Let

$$\tilde{b}_\varepsilon(x) = \begin{cases} b(x, u(x)), & \text{if } x \in \Omega \setminus E_\varepsilon; \\ b_0(x), & \text{if } x \in E_\varepsilon. \end{cases}$$

Then for any $\delta > 0$, there exists a $K > 0$ such that $b(x, u_k(x)) \geq \tilde{b}_\varepsilon(x) - \delta$, a.e. $x \in \Omega$, for all $k \geq K$. Thus, for any $k > K$,

$$\begin{cases} -\operatorname{div}(G_\eta(x, \nabla u_k)) \geq \tilde{b}_\varepsilon(x) - \delta, & \text{in } \Omega; \\ u_k|_{\partial\Omega} = f. \end{cases}$$

Consequently,

$$\begin{cases} -\operatorname{div}(G_\eta(x, \nabla u)) \geq \tilde{b}_\varepsilon(x) - \delta, & \text{in } \Omega; \\ u|_{\partial\Omega} = f. \end{cases}$$

Let $\delta \to 0^+$ and $\varepsilon \to 0^+$, we get

$$\begin{cases} -\operatorname{div}(G_\eta(x, \nabla u)) \geq b(x, u(x)), & \text{in } \Omega; \\ u|_{\partial\Omega} = f. \end{cases}$$

□

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