Can. J. Math., Vol. XXIX, No. 6, 1977, pp. 1171–1197

CERTAIN VARIETIES AND QUASIVARIETIES OF COMPLETELY REGULAR SEMIGROUPS

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1. Introduction and summary. We adopt the following definition of a *completely regular semigroup* S: for every element a of S, there exists a unique element a^{-1} of S such that

(1) $\quad a = aa^{-1}a, \quad a^{-1} = a^{-1}aa^{-1}, \quad aa^{-1} = a^{-1}a.$

This property is equivalent to S being a union of its (maximal) subgroups, and for this reason, these semigroups are frequently called *unions of groups*. Another equivalent definition is that they are semilattices of completely simple semigroups, a result due to Clifford [**2**], which is of fundamental importance for studying their structure, and hence they are occasionally referred to as *Clifford semigroups*. Further characterizations of these semigroups can be found in the books [**3**] and [**14**].

The class $\mathscr{C}\mathscr{R}$ of completely regular semigroups does not form a variety, for it is not closed under taking subsemigroups. However, if we consider elements of $\mathscr{C}\mathscr{R}$ as algebras with two operations $\langle S, \cdot, ^{-1}\rangle$, where $\cdot$ is the given semigroup operation and $x \rightarrow x^{-1}$ is the unary operation on S satisfying conditions (1), then $\mathscr{C}\mathscr{R}$ constitutes a variety of universal algebras. Note that for $x \in S$ and S in $\mathscr{C}\mathscr{R}$, x^{-1} is the inverse of x in the maximal subgroup of S containing x. The purpose of this work is to study certains subvarieties and subquasivarieties of the variety $\mathscr{C}\mathscr{R}$ of universal algebras just introduced. We first present two diagrams of the objects under study.

The notation introduced in the two diagrams is fixed throughout the paper. Section 2 contains some special notation and terminology. We start with a study of subvarieties of orthodox bands of groups, normal bands of groups, and of orthodox normal bands of groups. The principal results in Sections 3–5 are the isomorphisms:

$$\mathscr{V}(\mathscr{O}\mathscr{B}\mathscr{G}) \cong \mathscr{V}(\mathscr{B}) \times \mathscr{V}(\mathscr{G}),$$
$$\mathscr{V}(\mathscr{N}\mathscr{B}\mathscr{G}) \cong \mathscr{V}(\mathscr{Y}) \times \mathscr{V}(\mathscr{C}\mathscr{S}),$$
$$\mathscr{V}(\mathscr{O}\mathscr{N}\mathscr{B}\mathscr{G}) \cong Y_2{}^3 \times \mathscr{V}(\mathscr{G}),$$

where $\mathscr{V}(\mathscr{S})$ is the lattice of all subvarieties of a variety $\mathscr{S}$ of completely regular semigroups, and Y_2 is a 2-element semilattice. These sections also contain various characterizations of the semigroups under study, as well as a description of an equational base for the join of a variety of bands and a

Received November 9, 1976 and in revised form, March 11, 1977.

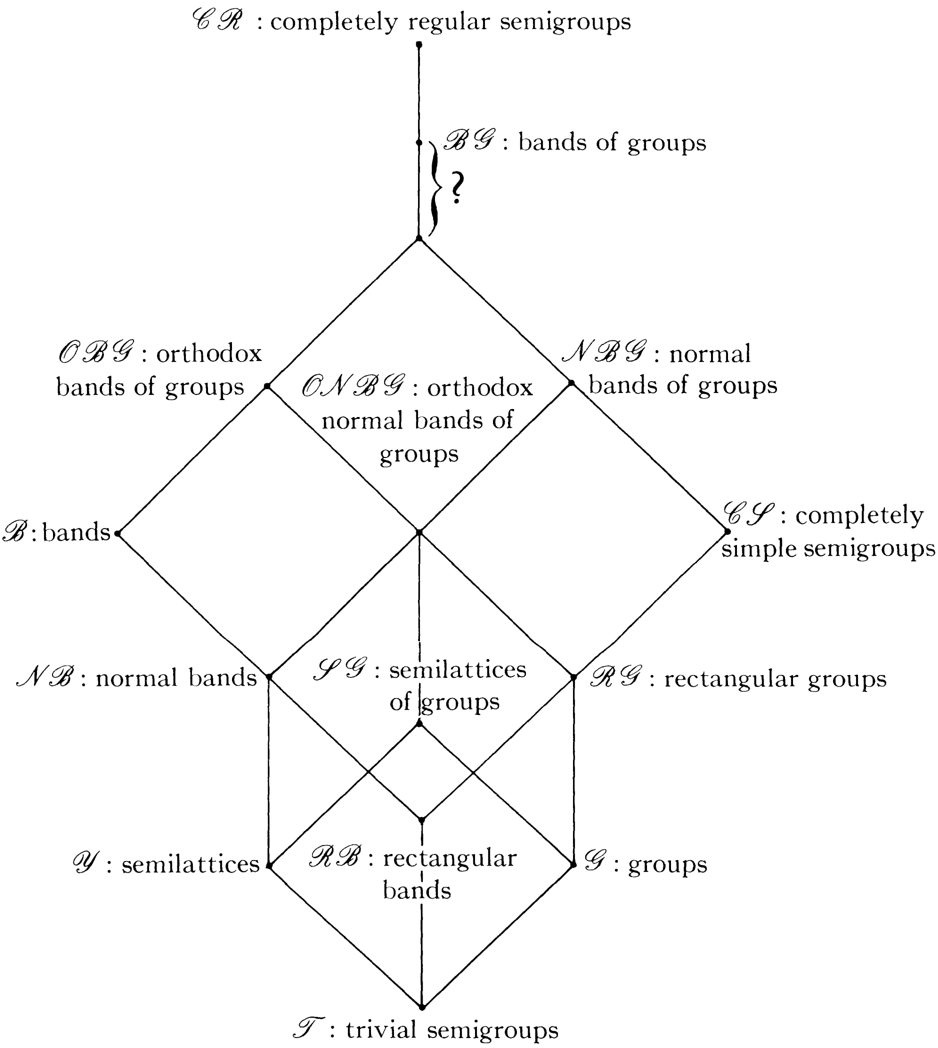
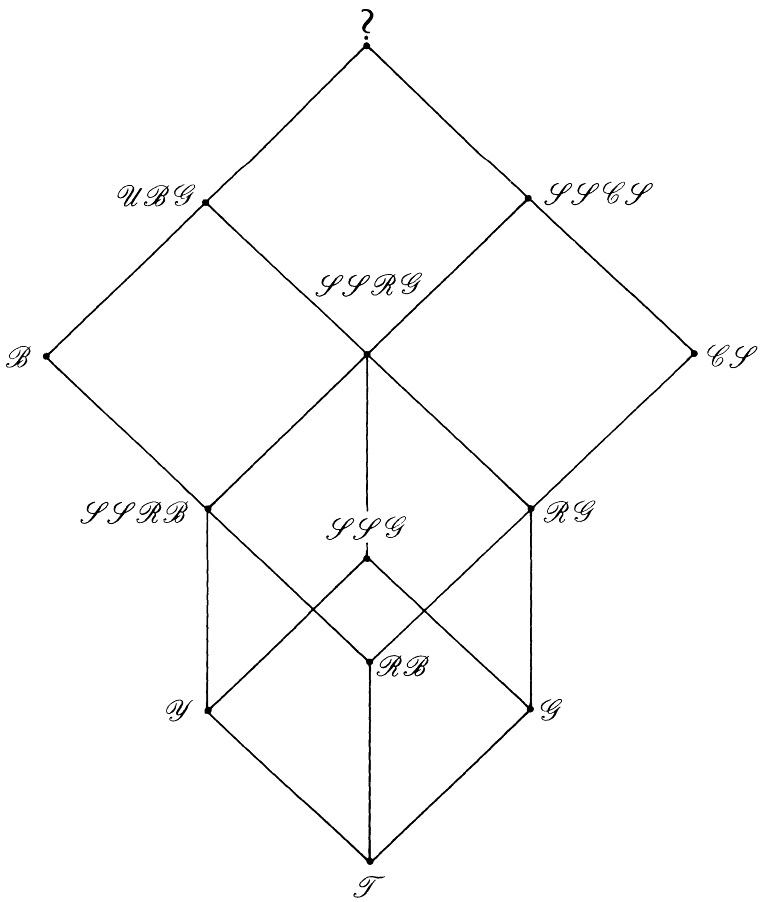


Diagram 1: Varieties

variety of groups, and the join of $\mathscr{Y}$ and a variety of completely simple semigroups. This study shows that Diagram 1 represents a sublattice of the lattice of all varieties of completely regular semigroups.

Sections 6–8 contain a similar analysis for quasivarieties which are joins of some of the pairs of varieties considered in Sections 3–5. We establish the following isomorphisms:

$$\mathscr{Q}(\mathscr{U}\mathscr{B}\mathscr{G}) \cong \mathscr{Q}(\mathscr{B}) \times \mathscr{Q}(\mathscr{G}),$$
$$\mathscr{Q}(\mathscr{S}\mathscr{S}\mathscr{C}\mathscr{S}) \cong \mathscr{Q}(\mathscr{Y}) \times \mathscr{Q}(\mathscr{C}\mathscr{S}),$$
$$\mathscr{Q}(\mathscr{S}\mathscr{S}\mathscr{R}\mathscr{G}) \cong Y_2^3 \times \mathscr{Q}(\mathscr{G}),$$



$\mathscr{UBG}$ = unitary bands of groups
$\mathscr{SSCS}$ = sturdy semilattices of completely simple semigroups
$\mathscr{SSRG}$ = sturdy semilattices of rectangular groups
$\mathscr{SSG}$ = sturdy semilattices of groups
$\mathscr{SSRB}$ = sturdy semilattices rectangular bands

Diagram 2: Quasivarieties

as well as characterize the joins of quasivarieties of bands and groups, etc., as above for varieties. This study yields that Diagram 2 represents a sublattice of the lattice of all quasi-varieties of completely simple semigroups.

We prove, in Sections 9-11, that each of the varieties in Diagram 1 is the homomorphic closure of the quasivarieties at the corresponding vertex in Diagram 2. We define here a certain congruence on a regular semigroup which is a subdirect product of a band and a completely simple semigroup. The principal results are the statements that all congruences on regular semigroups which are subdirect products of a band and a group or a semilattice and a completely simple semigroup are of the type constructed.

Certain properties of these congruences are studied in Section 12. The main result here is that a special type of this congruence still yields all semigroups in the three varieties mentioned above.

Section 13 contains a few problems which suggest themselves naturally in this study.

It seems remarkable that the form of Diagrams 1 and 2 occur repeatedly in [**16**] where further references to this type of diagram can be found.

2. Notation and terminology. For a variety $\mathscr{S}$ of completely regular semigroups, the lattice of all subvarieties of $\mathscr{S}$ will be denoted by $\mathscr{V}(\mathscr{S})$ and the join in it by $\vee$.

An *implication* is an ordered pair $(\{u_\alpha = v_\alpha\}_{\alpha \in A}, u = v)$ of a family of equations and a single equation, to be denoted by

$$\{u_\alpha = v_\alpha\}_{\alpha \in A} \Rightarrow u = v.$$

A semigroup $\mathscr{S}$ *satisfies* this implication if for any substitution of variables in all u_α, v_α, u and v, whenever $u_\alpha = v_\alpha$ is true for all α, then also $u = v$ is true. The class of all semigroups satisfying all implications in a family $\mathscr{I}$ of implications is a *quasivariety*, to be denoted by $[\mathscr{I}]$. The notation $[\mathscr{I}, \mathscr{J}, \ldots]$ stands for the quasivariety of all semigroups satisfying all implications in $\mathscr{I}$, all implications in $\mathscr{J}$, etc. For a quasi-variety $\mathscr{S}$ of semigroups, $\mathscr{Q}(\mathscr{S})$ will denote the lattice of all quasivarieties contained in $\mathscr{S}$ ordered by inclusion; the join will be denoted by $\bigvee$. If all implications in $\mathscr{I}$ are identities, then $[\mathscr{I}]$ is a variety.

The above definitions and notation will be used here only for the universal algebra $\mathscr{C}\mathscr{R}$ of completely regular semigroups.

In forming the join $\bigvee$ we will often use subdirect products. We denote by $S \lhd S_1 \times S_2 \times \ldots \times S_n$ that a semigroup S is a subdirect product of semigroups $S_1, S_2, \ldots, S_n$. The cross $\times$ will be used for direct products of semigroups as well as of lattices. In particular Y^n stands for $Y \times Y \times \ldots \times Y$, n times.

The class of regular semigroups will be denoted by $\mathscr{R}$. A semigroup S in $\mathscr{R}$ is *orthodox* if its idempotents E_S form a subsemigroup. If ρ is a congruence on a semigroup S, the quotient semigroup will be denoted by S/ρ and the natural homomorphism by $\rho^{\#}$. If S/ρ is a semilattice (band or group), then ρ is a *semilattice* (*band* or *group*) *congruence* on S. We denote by η the least semilattice congruence on any semigroup. If each ρ-class is a group, then S is a *band of groups*; if also S/ρ is a normal band, then S is a *normal band of groups*.

If S is a semilattice Y of semigroups S_α and the multiplication in S is determined by a transitive system of homomorphisms $\phi_{\alpha,\beta}$, then S is a *strong semi-*

lattice of semigroups S_α, to be denoted by $[Y; S_\alpha, \phi_{\alpha,\beta}]$ briefly by $[Y; S_\alpha]$. If also all $\phi_{\alpha,\beta}$ are one-to-one, then S is a *sturdy semilattice of semigroups* S_α.

For complete definitions concerning these concepts, as well as for undefined terms and notation, we refer the reader to books [**3**] and [**14**]. A comprehensive review of results on varieties of semigroups is given in Evans [**4**].

3. Varieties of orthodox bands of groups. We have proved in [**17**] that $\mathscr{V}(\mathscr{O}\mathscr{B}\mathscr{G}) \cong \mathscr{V}(\mathscr{B}) \times \mathscr{V}(\mathscr{G})$. Note that $\mathscr{O}\mathscr{B}\mathscr{G}$ is denoted by $\mathscr{C}$ in [**17**]. We will give below an alternate proof of this result, and will subsequently use its method of proof several times.

We have seen in [**17**] that an S in $\mathscr{C}\mathscr{R}$ is a band of groups (equivalently the Green relation $\mathscr{H}$ on S is a congruence) if and only if S satisfies the identity

(2) $\quad (a^2bc^2)\,(a^2bc^2)^{-1} = (abc)\,(abc)^{-1}.$

It follows from [**14**, IV.3.1] that S in $\mathscr{C}\mathscr{R}$ is orthodox if and only if it satisfies the identity

(3) $\quad ab = abb^{-1}a^{-1}ab.$

Now [**17**, Proposition 1] asserts that the conjunction of (2) and (3) is equivalent to the single identity

(4) $\quad (ab)\,(ab)^{-1} = aa^{-1}bb^{-1}.$

Some notation from [**17**] will be very handy.

3.1 *Notation.* Let $\mathscr{I}$ be a family of identities on $\mathscr{B}$. For each identity $u = v$ in $\mathscr{I}$, we formally substitute each variable x occurring in $u = v$ by xx^{-1}, and denote the new identity by $\bar{u} = \bar{v}$. Let

$$\bar{\mathscr{I}} = \{\bar{u} = \bar{v} \mid u = v \text{ in } \mathscr{I}\}$$

and consider $\bar{\mathscr{I}}$ as a family of identities on $\mathscr{C}\mathscr{R}$.

Next let $\mathscr{I}$ be a family of identities on $\mathscr{G}$. Let $u = v$ be an identity in $\mathscr{I}$. We may suppose that both u and v contain the same set $\{x_1, x_2, \ldots, x_n\}$ of variables. We formally set $e = (x_1x_2 \ldots x_n)\,(x_1x_2 \ldots x_n)^{-1}$ and substitute each occurrence of x_i in $u = v$ by ex_ie. We then obtain an identity to be denoted by $\hat{u} = \hat{v}$. Let

$$\hat{\mathscr{I}} = \{\hat{u} = \hat{v} \mid u = v \text{ in } \mathscr{I}\}$$

and consider $\hat{\mathscr{I}}$ as a family of identities on $\mathscr{C}\mathscr{R}$. Note that any choice of the order in which the variables x_i are written will do.

The next proposition is essentially [**17**, corollary to the theorem and Proposition 2]; for the sake of completeness, we furnish a (better) proof for it.

3.2 Proposition. *If* $\mathscr{V}' = [\mathscr{I}'] \in \mathscr{V}(\mathscr{B})$ *and* $\mathscr{V}'' = [\mathscr{I}''] \in \mathscr{V}(\mathscr{G})$, *then*

$$\mathscr{V}' \vee \mathscr{V}'' = [\overline{\mathscr{I}'}, \widehat{\mathscr{I}''}, (4)]$$
$$= \{S \in \mathscr{O}\mathscr{B}\mathscr{G} | E_S \in \mathscr{V}' \text{ and } G_e \in \mathscr{V}'' \text{ for all } e \in E_S\}.$$

Proof. Denote the three sets in the statement of the proposition by A_1, A_2, A_3 in the given order.

First let $S \in A_3$. Then E_S satisfies $\mathscr{I}'$ and hence S satisfies $\overline{\mathscr{I}'}$. Let $u = v$ be an identity in $\mathscr{I}''$, with the notation as in the second part of Notation 3.1 above. Letting $y_i = ex_ie$ for $i = 1, 2, \ldots, n$, we obtain from $\hat{u} = \hat{v}$ an identity of the form $u = v$ with all variables ranging over the group G_e. The hypothesis that $G_e \in \mathscr{V}''$ implies that $u = v$ is valid in G_e, and hence $\hat{u} = \hat{v}$ is valid in S. Consequently S satisfies $\widehat{\mathscr{I}''}$. Since S is in $\mathscr{O}\mathscr{B}\mathscr{G}$, it must statisfy identity (4). Hence $S \in A_2$, and thus $A_3 \subseteq A_2$.

Next let $S \in A_2$. Since S satisfies identity (4), we must have $S \in \mathscr{O}\mathscr{B}\mathscr{G}$. Further, S satisfies $\overline{\mathscr{I}'}$, which evidently implies that E_S satisfies $\mathscr{I}'$. Similarly, since S satisfies $\widehat{\mathscr{I}''}$, each maximal subgroup of S must satisfy $\mathscr{I}''$. Hence $S \in A_3$, and consequently $A_2 \subseteq A_3$.

It is clear that $\mathscr{V}', \mathscr{V}'' \subseteq A_3$. The equality $A_2 = A_3$ implies that A_3 is a variety, which then shows that $\mathscr{V}' \vee \mathscr{V}'' \subseteq A_3$ by the very definition of the join $\vee$. Hence $A_1 \subseteq A_3$.

Finally let $S \in A_3$. According to [**15**, Theorem 3.2], we can write S as a subdirect product of a band B and a semilattice of groups T, where $B \cong E_S$ and $B/\eta \cong T/\eta$. We represent T as a semilattice $Y \cong B/\eta$ of groups G_α, where G_α are isomorphic to maximal subgroups of S. The hypothesis implies that $G_\alpha \in \mathscr{V}''$. It follows from [**14**, III.7.2] that T is a subdirect product of semigroups T_α, where $T_\alpha = G_\alpha$ or $T_\alpha = G_\alpha^0$. This implies that Y is nontrivial. Since $Y \cong B/\eta$ and $B \in \mathscr{V}'$, it follows that $\mathscr{V}'$ contains all semilattices. Let $Y_2 = \{0, 1\}$ be a 2-element semilattice, and ρ be the Rees congruence on the direct product $G_\alpha \times Y_2$ relative to its kernel (i.e., ρ identifies all elements of the form $(g, 0)$). Then $G_\alpha^0 \cong (G_\alpha \times Y_2)/\rho$ which proves that $G_\alpha^0 \in \mathscr{V}' \vee \mathscr{V}''$. Consequently, in any case, $T_\alpha \in \mathscr{V}' \vee \mathscr{V}''$, and thus $T \in \mathscr{V}' \vee \mathscr{V}''$ since T is a subdirect product of various T_α. Finally, S is a subdirect product of $B \cong E_S$, where E_S is in $\mathscr{V}'$, and T which is in $\mathscr{V}' \vee \mathscr{V}''$; we deduce that S is in $\mathscr{V}' \vee \mathscr{V}''$. Therefore $A_3 \subseteq A_1$, and the proof is complete.

We now come to the main result of this section.

3.3. THEOREM. *The function*

$$\chi : \mathscr{V} \to (\mathscr{V} \cap \mathscr{B}, \mathscr{V} \cap \mathscr{G}) \quad (\mathscr{V} \in \mathscr{V}(\mathscr{O}\mathscr{B}\mathscr{G}))$$

is an isomorphism of $\mathscr{V}(\mathscr{O}\mathscr{B}\mathscr{G})$ *onto* $\mathscr{V}(\mathscr{B}) \times \mathscr{V}(\mathscr{G})$.

Proof. It is clear that χ maps $\mathscr{V}(\mathscr{O}\mathscr{B}\mathscr{G})$ into $\mathscr{V}(\mathscr{B}) \times \mathscr{V}(\mathscr{G})$ and is inclusion preserving. If $\mathscr{V}', \mathscr{V}'' \in \mathscr{V}(\mathscr{O}\mathscr{B}\mathscr{G})$ satisfy $\mathscr{V}'\chi = \mathscr{V}''\chi$, then by [**17**, Lemma 1], we have

$$\mathscr{V}' = (\mathscr{V}' \cap \mathscr{B}) \vee (\mathscr{V}' \cap \mathscr{G}) = (\mathscr{V}'' \cap \mathscr{B}) \vee (\mathscr{V}'' \cap \mathscr{G}) = \mathscr{V}''$$

and hence χ is one-to-one. Let $\mathscr{V}' \in \mathscr{V}(\mathscr{B})$, $\mathscr{V}'' \in \mathscr{V}(\mathscr{G})$ and $\mathscr{V} = \mathscr{V}' \vee \mathscr{V}''$. Then by Proposition 3.2, we have

$$\begin{aligned}\mathscr{V} \cap \mathscr{B} &= (\mathscr{V}' \vee \mathscr{V}'') \cap \mathscr{B} \\ &= \{S \in \mathscr{O}\mathscr{B}\mathscr{G} | E_S \in \mathscr{V}' \text{ and } G_e \in \mathscr{V}'' \text{ for all } e \in E_S\} \cap \mathscr{B} \\ &= \mathscr{V}'\end{aligned}$$

and analogously $\mathscr{V} \cap \mathscr{G} = \mathscr{V}''$. Hence $\mathscr{V} \in \mathscr{V}(\mathscr{O}\mathscr{B}\mathscr{G})$ and $\mathscr{V}\chi = (\mathscr{V}', \mathscr{V}'')$. Therefore χ is a (lattice) isomorphism of $\mathscr{V}(\mathscr{O}\mathscr{B}\mathscr{G})$ onto $\mathscr{V}(\mathscr{B}) \times \mathscr{V}(\mathscr{G})$.

The lattice $\mathscr{V}(\mathscr{B})$ of all varieties of bands has been constructed by Birjukov [**1**], Fennemore [**5**] and Gerhard [**6**]. As contrasted to this, the lattice $\mathscr{V}(\mathscr{G})$ is not known. The book by Neumann [**11**] contains an extensive study of varieties of groups. Note that the lattice of all varieties of abelian groups is known.

4. Varieties of normal bands of groups. These semigroups, symbolized by $\mathscr{N}\mathscr{B}\mathscr{G}$, are by definition semigroups S which have a congruence ρ such that each ρ-class is a group and S/ρ is a normal band, i.e. satisfies the identity $axya = ayxa$. It is easy to see that this is equivalent to the Green relation $\mathscr{H}$ being a congruence such that $S/\mathscr{H}$ is a normal band. Since in any completely regular semigroup $a \mathscr{H} b$ if and only if $aa^{-1} = bb^{-1}$, the semigroups in $\mathscr{N}\mathscr{B}\mathscr{G}$ are characterized by identity (2) and

(5) $\quad (axya)(axya)^{-1} = (ayxa)(ayxa)^{-1}.$

We will characterize these semigroups in various manners below. We start with some auxiliary statements.

4.1 LEMMA. *Let S be a regular semigroup satisfying $axya \mathscr{H} ayxa$ for all $a, x, y \in S$. Then S is completely regular and $\mathscr{H}$ is a congruence.*

Proof. Let $a \in S$; there is $x \in S$ such that $a = axa$. Then $a = axa = ax(ax)a \mathscr{H} a^2x^2a$ so that $a \in a^2xaS \subseteq a^2S$. Similarly, we have $a \in Sa^2$, which shows that S is completely regular.

Next let $x \mathscr{H} y$ and $a \in S$. Then $x = yu$ and thus $xa = (yua)t(yua)$ for some $t \in S$. The hypothesis yields $xa = yu(at)yua \mathscr{H} y(at)uyua$ and thus $xa \in yaS$, which implies, by symmetry, that $xaS = yaS$.

Trivially $Sx = Sy$ implies $Sxa = Sya$. Consequently $xa \mathscr{H} ya$. A dual proof shows that also $ax \mathscr{H} ay$. Therefore $\mathscr{H}$ is a congruence.

4.2 LEMMA. *Let S be a completely regular semigroup for which $xy \mathscr{H} yx$ implies $xay \mathscr{H} yax$ for all $a, x, y \in S$. Then $\mathscr{H}$ is a congruence.*

Proof. Let $x \mathscr{H} y$ and $a \in S$. Then $xS = yS$ so that

$$xyS = x^2S = xS = yS = y^2S = yxS$$

and analogously $Sxy = Syx$. Hence $xy \mathscr{H} yx$ which by hypothesis implies

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$xay\mathscr{H} yax$. In particular, we have $xayS = yaxS$. There exists $u \in S$ such that $xa = (xa)^2u$. Hence

$$xa = (xax)au \in xa(xS) = xayS = yaxS \subseteq yaS$$

which implies $xaS \subseteq yaS$ and, by symmetry, $xaS = yaS$. Trivially $Sxa = Sya$, and thus $xa\mathscr{H} ya$. By symmetry, we conclude that $\mathscr{H}$ is a congruence.

4.3 LEMMA. *The following implications on a band are equivalent.*

(i) $axya = ayxa$.

(ii) $axyb = ayxb$.

(ii) $xy = yx \Rightarrow xay = yax$.

Proof. A proof of this can be found in [**18**].

From the preceding lemmas, we deduce

4.4 PROPOSITION. *Semigroups in $\mathscr{N}\mathscr{B}\mathscr{G}$ can be characterized within $\mathscr{C}\mathscr{R}$ by satisfaction of any of the following implications*:

(i) $(axya)(axya)^{-1} = (ayxa)(ayxa)^{-1}$,

(ii) $(axyb)(axyb)^{-1} = (ayxb)(ayxb)^{-1}$,

(iii) $(xy)(xy)^{-1} = (yx)(yx)^{-1} \Rightarrow (xay)(xay)^{-1} = (yax)(yax)^{-1}$.

Further characterizations of normal bands of groups can be found in [**14**, IV.4.3]. The following notation will be presently useful. Recall that $\mathscr{C}\mathscr{S}$ is the variety of completely simple semigroups.

4.5 *Notation.* If $\mathscr{I}$ is a family of identities on $\mathscr{C}\mathscr{S}$, let

$$\mathscr{I}^* = \{uv = vu,\ uvu = vuv | u = v \text{ in } \mathscr{I}\}$$

and consider $\mathscr{I}^*$ as a family of identities on $\mathscr{C}\mathscr{R}$.

4.6 PROPOSITION. *If $\mathscr{V} = [\mathscr{I}] \in \mathscr{V}(\mathscr{C}\mathscr{S})$, then*

$$\mathscr{Y} \vee \mathscr{V} = [\mathscr{I}^*, (5)] = \{[Y; S_\alpha] \mid Y \in \mathscr{Y}, S_\alpha \in \mathscr{V} \textit{ for all } \alpha \in Y\}.$$

Proof. Denote by A_1, A_2, A_3 the sets in the statement of the proposition in the given order. Any semilattice trivially satisfies $\mathscr{I}^*$ and (5). If $S \in \mathscr{V}$, then S satisfies $\mathscr{I}$ and thus $\mathscr{I}^*$, and is a completely simple semigroup so that it also satisfies (5). Consequently $\mathscr{Y}, \mathscr{V} \subseteq A_2$ which implies that $A_1 \subseteq A_2$.

Next let $S \in A_2$. Then S is a normal band of groups and is hence, by [**14**, IV.4.3], a strong semilattice of completely simple semigroups, say $S = [Y; S_\alpha]$. The hypothesis implies that each S_α satisfies $\mathscr{I}^*$. Let $u = v$ be an identity in $\mathscr{I}$. Then $uv = vu$ and $uvu = vuv$ are identities in $\mathscr{I}^*$ so each S_α satisfies them. Singling out any S_α, and considering u and v as elements of S_α, we see that $uv = vu$ implies that $u\mathscr{H} v$ and thus $uvu = vuv$ implies that $u = v$ since every $\mathscr{H}$-class of S_α is a group. It follows that S_α satisfies $u = v$. Consequently S_α satisfies $\mathscr{I}$ and thus $S_\alpha \in \mathscr{V}$ for all $\alpha \in Y$. Thus $A_2 \subseteq A_3$.

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Finally let $S \in A_3$. Then $S = [Y; S_\alpha]$ with $S_\alpha \in \mathscr{V}$ for all $\alpha \in Y$. It follows from [**14**, III.7.2] that S is a subdirect product of semigroups T_α where $T_\alpha = S_\alpha$ or $T_\alpha = S_\alpha^0$. As in the proof of Proposition 3.2, one sees that $S_\alpha^0 \in \mathscr{Y} \vee \mathscr{V}$, which shows that $T_\alpha \in \mathscr{Y} \vee \mathscr{V}$ for all $\alpha \in Y$. But then $S \in \mathscr{Y} \vee \mathscr{V}$ which proves that $S \in A_1$. Therefore $A_3 \subseteq A_1$ and the proof is complete.

We are now ready for the main result of this section.

4.7 THEOREM. *The function*

$$\chi : \mathscr{V} \rightarrow (\mathscr{V} \cap \mathscr{Y}, \mathscr{V} \cap \mathscr{C}\mathscr{S}) \quad (\mathscr{V} \in \mathscr{V}(\mathscr{N}\mathscr{B}\mathscr{G}))$$

is an isomorphism of $\mathscr{V}(\mathscr{N}\mathscr{B}\mathscr{G})$ *onto* $\mathscr{V}(\mathscr{Y}) \times \mathscr{V}(\mathscr{C}\mathscr{S})$.

Proof. It is clear that χ maps $\mathscr{V}(\mathscr{N}\mathscr{B}\mathscr{G})$ into $\mathscr{V}(\mathscr{Y}) \times \mathscr{V}(\mathscr{C}\mathscr{S})$ and that it is order preserving. For $\mathscr{V} \in \mathscr{V}(\mathscr{N}\mathscr{B}\mathscr{G})$, we assert

$$(6) \qquad \mathscr{V} = (\mathscr{V} \cap \mathscr{Y}) \vee (\mathscr{V} \cap \mathscr{C}\mathscr{S}).$$

If $\mathscr{Y} \not\subseteq \mathscr{V}$, this is trivial since then $\mathscr{V} \subseteq \mathscr{C}\mathscr{S}$ and $\mathscr{V} \cap \mathscr{Y} = \mathscr{T}$. Assume that $\mathscr{Y} \subseteq \mathscr{V}$. The right hand side of (6) is obviously contained in the left hand side. Let $S \in \mathscr{V}$. Then $S = [Y; S_\alpha]$ with $S_\alpha \in \mathscr{V}$ for all $\alpha \in Y$ in light of [**14**, IV.4.3]. Similarly as in the proof of Proposition 4.6, we have that $S \in \mathscr{Y} \vee (\mathscr{V} \cap \mathscr{C}\mathscr{S})$. This establishes (6) which easily implies that χ is one-to-one.

Finally, let $\mathscr{V}' \in \mathscr{V}(\mathscr{Y})$ and $\mathscr{V}'' \in \mathscr{V}(\mathscr{C}\mathscr{S})$. If $\mathscr{V}' = \mathscr{T}$, than $\mathscr{V}''\chi = (\mathscr{T}, \mathscr{V}'')$. If $\mathscr{V}' = \mathscr{Y}$, then by Proposition 4.6, we obtain

$$(\mathscr{Y} \vee \mathscr{V}'')\chi = ((\mathscr{Y} \vee \mathscr{V}'') \cap \mathscr{Y}'', (\mathscr{Y} \vee \mathscr{V}'') \cap \mathscr{C}\mathscr{S}) = (\mathscr{Y}, \mathscr{V}'')$$

where $\mathscr{Y} \vee \mathscr{V}'' \in \mathscr{N}\mathscr{B}\mathscr{G}$. It follows that χ is an isomorphism of $\mathscr{V}(\mathscr{N}\mathscr{B}\mathscr{G})$ onto $\mathscr{V}(\mathscr{Y}) \times \mathscr{V}(\mathscr{C}\mathscr{S})$.

Letting Y_2 be a 2-element semilattice, and noting that $\mathscr{Y}$ has no non-trivial proper subvarieties, we have $\mathscr{V}(\mathscr{Y}) \cong Y_2$, and by the theorem, $\mathscr{V}(\mathscr{N}\mathscr{B}\mathscr{G}) \cong Y_2 \times \mathscr{V}(\mathscr{C}\mathscr{S})$. Nothing seems to be known about the lattice $\mathscr{V}(\mathscr{C}\mathscr{S})$.

5. Orthodox normal bands of groups. By definition, these are bands of groups which are also orthodox semigroups, and thus the idempotents must form a normal band. According to [**14**, IV.4.6], a completely regular semigroup whose idempotents form a normal band is necessarily a band of groups, and thus an orthodox normal band of groups. These semigroups are thus characterized, within $\mathscr{C}\mathscr{R}$, by the identity

$$(7) \qquad axx^{-1}yy^{-1}a = ayy^{-1}xx^{-1}a.$$

As for varieties of these semigroups, we have the following results. Recall the symbolism $\mathscr{O}\mathscr{N}\mathscr{B}\mathscr{G}$ and $\mathscr{N}\mathscr{B}$, and let $\mathscr{O}\mathscr{G}$ stand for orthodox completely regular semigroups.

5.1 PROPOSITION. *If* $\mathscr{V}' = [\mathscr{I}'] \in \mathscr{V}(\mathscr{N}\mathscr{B})$ *and* $\mathscr{V}'' = [\mathscr{I}''] \in \mathscr{V}(\mathscr{G})$, *then* $\mathscr{V}' \vee \mathscr{V}'' = [\bar{\mathscr{I}}', \widehat{\mathscr{I}'', (1)}] = \{S \in \mathscr{O}\mathscr{G} | E_S \in \mathscr{V}'$ *and* $G_e \in \mathscr{V}''$ *for all* $e \in E_S\}$

Proof. This is a special case of Proposition 3.2 except for the fact that in the square bracket we have (7) instead of (4) and in the third set $S \in \mathscr{O}\mathscr{G}$ (i.e., S is only orthodox) instead of $S \in \mathscr{O}\mathscr{B}\mathscr{G}$. These modifications are justified by the remarks made above.

5.2 COROLLARY. $\mathscr{O}\mathscr{N}\mathscr{B}\mathscr{G} = \mathscr{N}\mathscr{B} \vee \mathscr{G} = \mathscr{Y} \vee \mathscr{R}\mathscr{G} = \mathscr{O}\mathscr{B}\mathscr{G} \cap \mathscr{N}\mathscr{B}\mathscr{G}$

$= \{S \in \mathscr{C}\mathscr{R} | E_S$ *is a normal band*$\}$.

Proof. This follows easily from Proposition 5.1.

It is well known that $\mathscr{N}\mathscr{B} = \mathscr{R}\mathscr{B} \vee \mathscr{Y}$ and that $\mathscr{R}\mathscr{B} = \mathscr{L}\mathscr{Z} \vee \mathscr{R}\mathscr{Z}$, the join of left zero semigroups and right zero semigroups, and that $\mathscr{V}(\mathscr{Y}) \cong \mathscr{V}(\mathscr{L}\mathscr{Z}) \cong \mathscr{V}(\mathscr{R}\mathscr{Z}) \cong Y_2$, a 2-element semilattice. The next corollary now follows from the above.

5.3 COROLLARY.

$$
\begin{aligned}
\mathscr{V}(\mathscr{O}\mathscr{N}\mathscr{B}\mathscr{G}) &\cong Y_2^3 \times \mathscr{V}(\mathscr{G}),\\
\mathscr{V}(\mathscr{N}\mathscr{B}) &\cong Y_2^3,\\
\mathscr{V}(\mathscr{R}\mathscr{G}) &\cong Y_2^2 \times \mathscr{V}(\mathscr{G}),\\
\mathscr{V}(\mathscr{S}\mathscr{G}) &\cong Y_2 \times \mathscr{V}(\mathscr{G}),\\
\mathscr{V}(\mathscr{R}\mathscr{B}) &\cong Y_2^2,\\
\mathscr{V}(\mathscr{Y}) &\cong Y_2.
\end{aligned}
$$

This is as much as we can say about Diagram 1. As we have already mentioned, the lattice $\mathscr{V}(\mathscr{C}\mathscr{S})$ remains unknown; in fact the unknown part is the portion above $\mathscr{R}\mathscr{G}$. We have been also unable to compute the join of $\mathscr{O}\mathscr{B}\mathscr{G}$ and $\mathscr{N}\mathscr{B}\mathscr{G}$. With this join, Diagram 1 represents a sublattice of $\mathscr{V}(\mathscr{C}\mathscr{R})$. It is likely that $\mathscr{O}\mathscr{B}\mathscr{G} \vee \mathscr{N}\mathscr{B}\mathscr{G}$ is strictly less than $\mathscr{B}\mathscr{G}$, but this question remains open.

So far, Sections 3, 4 and 5 were concerned with the lattices of varieties $\mathscr{O}\mathscr{B}\mathscr{G}, \mathscr{N}\mathscr{B}\mathscr{G}$ and $\mathscr{O}\mathscr{N}\mathscr{B}\mathscr{G}$, respectively, with

$$\mathscr{O}\mathscr{B}\mathscr{G} = \mathscr{B} \vee \mathscr{G}, \quad \mathscr{N}\mathscr{B}\mathscr{G} = \mathscr{Y} \vee \mathscr{C}\mathscr{S}, \quad \mathscr{O}\mathscr{N}\mathscr{B}\mathscr{G} = \mathscr{Y} \vee \mathscr{R}\mathscr{G}.$$

The next three sections concern the joins of $\mathscr{B}$ and $\mathscr{G}$, $\mathscr{Y}$ and $\mathscr{C}\mathscr{S}$, and $\mathscr{Y}$ and $\mathscr{R}\mathscr{G}$, but in the lattice $\mathscr{Q}(\mathscr{C}\mathscr{R})$, of quasivarieties of completely regular semigroups. For these quasivarieties we will establish results analogous to those we have seen for varieties. We will prove later that the variety in any vertex of Diagram 1 is a homomorphic closure of the quasivariety in the corresponding vertex in Diagram 2. In the latter diagram, the join is built by subdirect products.

6. Quasivarieties of unitary bands of groups. It is proved in [**9**] that the set of idempotents of a regular semigroup is left unitary if and only if it is right unitary. For the sake of brevity we call a completely regular semigroup S *unitary* if its set E_S of idempotents is left (equivalently, right) unitary. We write this in terms of an implication

(8) $\quad x^2 = x,\ (xa)^2 = xa \Rightarrow a^2 = a,$

and denote the quasivariety of unitary bands of groups by $\mathscr{U}\mathscr{B}\mathscr{G}$. It is also proved in [9] that for a regular semigroup S, if E_S is unitary, it forms a subsemigroup. We can thus say that "unitary implies orthodox" for (completely) regular semigroups. According to [**15**, Theorem 4.1], semigroups in $\mathscr{U}\mathscr{B}\mathscr{G}$ coincide with regular semigroups which are subdirect products of a band and a group. Compare the following with Notation 3.1.

6.1 *Notation.* If $\mathscr{I}$ is a family of implications on bands, we substitute every variable x occurring in $\mathscr{I}$ by xx^{-1}, denote the new family of implications by $\overline{\mathscr{I}}$ and consider it over $\mathscr{C}\mathscr{R}$. If $\{u_i = v_i\}_{i=1}^n \Rightarrow u = v$ is an implication from a family $\mathscr{I}$ of implications on groups, we may assume that all the words u_i, v_i, u, v contain the same variables $x_1, x_2, \ldots, x_m$. We then let $e = (x_1x_2 \ldots x_m)\cdot(x_1x_2 \ldots x_m)^{-1}$ and substitute each occurrence of x_i by ex_ie. We do this for each implication in $\mathscr{I}$, denote the new family of implications by $\widehat{\mathscr{I}}$ and consider it over $\mathscr{C}\mathscr{R}$.

6.2 PROPOSITION. *If* $\mathscr{Q}' = [\mathscr{I}'] \in \mathscr{Q}(\mathscr{B})$ *and* $\mathscr{Q}'' = [\mathscr{I}''] \in \mathscr{Q}(\mathscr{G})$, *then*

$$\begin{aligned}\mathscr{Q}' \vee \mathscr{Q}'' &= [\overline{\mathscr{I}'}, \widehat{\mathscr{I}''}, (4), (8)]\\ &= \{S \in \mathscr{R} | S \lhd B \times G, B \in \mathscr{Q}', G \in \mathscr{Q}''\}.\end{aligned}$$

Proof. Denote by A_1, A_2, A_3 the three sets in the statement of the proposition in the given order.

Let $S \in A_3$ and $S \lhd B \times G$ with $B \in \mathscr{Q}'$, $G \in \mathscr{Q}''$. Then B satisfies the family of implications $\mathscr{I}'$ and hence also $\overline{\mathscr{I}'}$. Similarly G satisfies $\mathscr{I}''$ and hence also $\widehat{\mathscr{I}''}$. But then S satisfies both $\overline{\mathscr{I}'}$ and $\widehat{\mathscr{I}''}$ being a subdirect product of B and G. Both B and G satisfy identity (4) and implication (8) trivially, and hence S satisfies both (4) and (8). This shows that $S \in A_2$ so that $A_3 \subseteq A_2$.

Now let $S \in A_2$. In view of [**15**, Theorem 4.1], we may represent S as a subdirect product of a band B and a group G because S satisfies (4) and (8). Since S satisfies $\overline{\mathscr{I}'}$, it is clear that E_S satisfies $\mathscr{I}'$ and thus $E_S \in \mathscr{Q}'$. Let $\{u_i = v_i\}_{i=1}^n \Rightarrow u = v$ be an implication in $\mathscr{I}''$ and let $g_1, g_2, \ldots, g_m$ be a set of elements of G satisfying the equations $\{u_i = v_i\}_{i=1}^n$. For $j = 1, 2, \ldots, m$, there exist $e_j \in B$ such that $(e_j, g_j) \in S$, where we assume that $S \subseteq B \times G$. Since S is regular, we must have $(e_j, 1) \in S$, where 1 is the identity element of G. Let $e = e_1e_2 \ldots e_m$. Then $(e, 1) \in S$ and thus

$$(e, g_j) = (e, 1)(e_j, g_j)(e, 1) \in S$$

for $j = 1, 2, \ldots, m$. Consequently the elements $(e, g_1), (e, g_2), \ldots, (e, g_m)$

satisfy the equations $\{u_1 = v_i\}_{i=n}^n$ and thus also the equations $\{\hat{u}_i = \hat{v}_i\}_{i=1}^n$. By hypothesis, these elements then satisfy $\hat{u} = \hat{v}$, and hence also $u = v$, since all these elements are in the same maximal subgroup of S. But then the elements $g_1, g_2, \ldots, g_m$ satisfy the equation $u = v$. Consequently G satisfies $\mathcal{I}''$ and thus $G \in \mathcal{Q}''$. Therefore $S \in A_3$ which proves that $A_2 \subseteq A_3$.

We have proved so far that $A_2 = A_3$. Now $A_3 \subseteq A_1$ since the latter is closed under subdirect products and in A_3 all semigroups are necessarily completely regular. It is clear that both $\mathcal{Q}'$ and $\mathcal{Q}''$ are contained in A_3. Since $A_2 = A_3$, we have that A_3 is a quasivariety, so that $\mathcal{Q}', \mathcal{Q}'' \subseteq A_3$ implies that $\mathcal{Q}' \vee \mathcal{Q}'' \subseteq A_3$. Consequently $A_1 \subseteq A_3$ which completes the proof.

We are now able to prove the principal result of this section, viz.

6.3 Theorem. *The function*

$$\chi : \mathcal{Q} \to (\mathcal{Q} \cap \mathcal{B}, \mathcal{Q} \cap \mathcal{G}) \quad (\mathcal{Q} \in \mathcal{Q}(\mathcal{U}\mathcal{B}\mathcal{G}))$$

is an isomorphism of $\mathcal{Q}(\mathcal{U}\mathcal{B}\mathcal{G})$ *onto* $\mathcal{Q}(\mathcal{B}) \times \mathcal{Q}(\mathcal{G})$.

Proof. The argument goes along the same lines as in the proof of Theorem 3.3 now using Proposition 6.2 and the assertion that

$$\mathcal{Q} = (\mathcal{Q} \cap \mathcal{B}) \vee (\mathcal{Q} \cap \mathcal{G}) \quad (\mathcal{Q} \in \mathcal{Q}(\mathcal{U}\mathcal{B}\mathcal{G})).$$

Indeed, let $S \in \mathcal{Q}$. Using the arguments as in the proof of Proposition 6.2, we show that S represented as a subdirect product of a band B and a group G, has the properties $B \cong E_S \in \mathcal{Q} \cap B$ and $G \in \mathcal{Q} \cap \mathcal{G}$. This proves one inclusion, the other is trivial.

The lattices $\mathcal{Q}(\mathcal{B})$ and $\mathcal{Q}(\mathcal{G})$ are not known. However, the lattice $\mathcal{Q}(\mathcal{N}\mathcal{B})$ of quasivarieties of normal bands has been determined by Gerhard and Shafaat [7] (see also Shafaat [**19**]). Yamada [**20**] classified all implications on bands in two variables; for a proof, see [**18**].

7. Quasivarieties of sturdy semilattices of completely simple semigroups. By definition, these are the semigroups which are strong semilattices of completely simple semigroups, i.e., the multiplication among the completely simple components is determined by a transitive system of homomorphisms, and in addition, all these homomorphisms are one-to-one. We have reserved the symbol $[Y; S_\alpha, \phi_{\alpha,\beta}]$ for a strong semilattice Y of semigroups S_α determined by homomorphisms $\phi_{\alpha,\beta}$. The following statement provides several alternative characterizations of semigroups at hand.

7.1 Proposition. *The following conditions on a semigroup S are equivalent.*

(i) $S \in \mathcal{S}\mathcal{S}\mathcal{C}\mathcal{S}$.

(ii) $S \in \mathcal{N}\mathcal{B}\mathcal{G}$ *and satisfies the implication*

$$\left.\begin{array}{l} a = xby,\ e = e^2 = eaa^{-1} = aa^{-1}e = ebb^{-1} = bb^{-1}e \\ b = waz,\ ae = be \end{array}\right\} \Rightarrow a = b.$$

https://doi.org/10.4153/CJM-1977-118-7 Published online by Cambridge University Press

(iii) *S is regular and a subdirect product of a semilattice and a completely simple semigroup.*

(iv) $S \in \mathcal{N}\mathcal{B}\mathcal{G}$ *and satisfies for all* $a, x, y \in S$,

$$xay \mathcal{H} yax \Rightarrow xy \mathcal{H} yx,$$
$$xy = x^2y \Rightarrow x = x^2.$$

Proof. (i) *implies* (ii). First let $S = [Y; S_\alpha, \phi_{\alpha,\beta}]$ be a strong semilattice of completely simple semigroups. Let $a, b \in S_\alpha$, $\alpha \geqq \beta$, and $a\phi_{\alpha,\beta} = b\phi_{\alpha,\beta}$. In view of the proof of [**14**, III.4.7], $a\phi_{\alpha,\beta} = ae$ where $e \in E_{S_\beta}$ has the property that $e \leqq aa^{-1}$; analogously $b\phi_{\alpha,\beta} = fb$ where $f \in E_{S_\beta}$ and $f \leqq bb^{-1}$. Taking into account that various S_α are precisely the $\mathcal{J}$-classes of S, so that $a \mathcal{J} b$ and $e \mathcal{J} f$, we can transcribe the condition that $\phi_{\alpha,\beta}$ be one-to-one by the implication in part (ii) of the proposition.

(ii) *implies* (i). In view of [**14**, IV.4.3], the above argument also shows that conversely (ii) implies (i).

(i) *implies* (iii). This follows easily from [**14**, IV.5.1].

(iii) *implies* (iv). Let $S \subseteq Y \times C$ be a regular semigroup subdirect product of a semilattice Y and a completely simple semigroup C. Let

$$(\alpha, x)(\beta, a)(\gamma, y) \mathcal{H} (\gamma, y)(\beta, a)(\alpha, x)$$

in S. Then $xay \mathcal{H} yax$ in C, which clearly implies that $xy \mathcal{H} yx$. Since $\alpha\gamma \mathcal{H} \gamma\alpha$ in Y, and according to [**14**, IV.4.1], the restriction of the $\mathcal{H}$-relation in $Y \times C$ coincides with the $\mathcal{H}$-relation in S, we deduce that $(\alpha, x)(\gamma, y) \mathcal{H} (\gamma, y)(\alpha, x)$.
Next let $(\alpha, x)(\beta, y) = (\alpha, x)^2(\beta, y)$ in S. Then $xy = x^2y$ in C, which in view of the Rees theorem evidently implies that $x = x^2$. Consequently $(\alpha, x) = (\alpha, x)^2$.

(iv) *implies* (ii). We assume the hypotheses of the implication in part (ii). It follows that $aeb = (bf)b = bea$ which by hypothesis yields $ab \mathcal{H} ba$. The assumption $a = xby$ and $b = waz$ shows that $a,b \in S_\alpha$ for some completely simple component of S. But then $ab \mathcal{H} ba$ evidently implies that $a \mathcal{H} b$. Further,

$$(a^{-1}b)^2e = a^{-1}ba^{-1}(be) = a^{-1}b(a^{-1}ae) = a^{-1}be,$$

so that $a^{-1}b = f$, the identity of the maximal subgroup of S containing a and b. Consequently $a = b$, as required.

Now Propositions 4.4 and 7.1 directly imply that semigroups in $\mathcal{S}\mathcal{S}\mathcal{C}\mathcal{S}$ are characterized by the implications

$$(9) \quad \begin{array}{l} (xy)(xy)^{-1} = (yx)(yx)^{-1} \Leftrightarrow (xay)(xay)^{-1} = (yax)(yax)^{-1} \\ xy = x^2y \Rightarrow x = x^2. \end{array}$$

7.2 *Notation.* Let $\mathcal{I}$ be a family of implications on $\mathcal{C}\mathcal{S}$. For each implication $\{u_i = v_i\}_{i=1}^n \Rightarrow u = v$ in $\mathcal{I}$, construct a new implication

$$\{u_i = v_i\}_{i=1}^n \Rightarrow uv = vu, \; uvu = vuv$$

and let $\mathscr{I}^*$ be the family of all implications which are so derived from implications in $\mathscr{I}$.

Compare the above with Notation 4.5, and note that any semilattice satisfies any $\mathscr{I}^*$.

7.3 PROPOSITION. *If* $\mathscr{Q} = [\mathscr{I}] \in \mathscr{Q}(\mathscr{C}\mathscr{S})$, *then*

$$\mathscr{Y} \vee \mathscr{Q} = [\mathscr{I}^*, (5), (9)] = \{S \in \mathscr{R} | S \lhd Y \times C, \; Y \in \mathscr{Y}, C \in \mathscr{C}\mathscr{S}\}.$$

Proof. Denote by A_1, A_2, A_3 the sets in the statement of the proposition in the given order. Both $\mathscr{Y}$ and $\mathscr{Q}$ are clearly contained in A_2 and hence $A_2 \subseteq A_3$. Since a quasivariety is closed under subdirect products, we have that $A_3 \subseteq A_1$ (note that the semigroups in A_3 are automatically completely regular).

Let $S \in A_2$. We may assume that $S \subseteq Y \times C$ is a subdirect product, where $Y \in \mathscr{Y}$ and $C \in \mathscr{C}\mathscr{S}$ since S satisfies (5) and (9). Let $\{u_i = v_i\}_{i=1}^n \Rightarrow u = v$ be an implication in $\mathscr{I}$ and $a_1, a_2, \ldots, a_m$ be elements of C satisfying the equations $\{u_i = v_i\}_{i=1}^n$. There exist $\alpha_j \in Y$ such that $(\alpha_j, a_j) \in S$ for $j = 1, 2, \ldots, m$. Let $\alpha = \alpha_1\alpha_2 \ldots \alpha_m$. Then $\alpha_j \geqq \alpha$ and [**15**, Theorem 4.3] implies that $(\alpha, a_j) \in S$ for $j = 1, 2, \ldots, m$. Hence the elements (α, a_j), $j = 1, 2, \ldots, m$, satisfy the equations $\{u_i = v_i\}_{i=1}^n$. By hypothesis, these elements also satisfy the equations $uv = vu$ and $uvu = vuv$. But then the elements $a_1, a_2, \ldots, a_m$ themselves satisfy the equations $uv = vu$ and $uvu = vuv$, which, as in the proof of Proposition 4.6, implies that they satisfy $u = v$. Consequently $S \in A_3$ and thus $A_2 \subseteq A_3$, which completes the proof.

We can now prove the main result of this section.

7.4 THEOREM. *The function*

$$\chi : \mathscr{Q} \to (\mathscr{Q} \cap \mathscr{Y}, \mathscr{Q} \cap \mathscr{C}\mathscr{S}) \quad (\mathscr{Q} \in \mathscr{Q}(\mathscr{S}\mathscr{S}\mathscr{C}\mathscr{S}))$$

is an isomorphism of $\mathscr{Q}(\mathscr{S}\mathscr{S}\mathscr{C}\mathscr{S})$ *onto* $\mathscr{Q}(\mathscr{Y}) \times \mathscr{Q}(\mathscr{C}\mathscr{S})$.

Proof. The function χ clearly maps $\mathscr{Q}(\mathscr{S}\mathscr{S}\mathscr{C}\mathscr{S})$ into $\mathscr{Q}(\mathscr{Y}) \times \mathscr{Q}(\mathscr{C}\mathscr{S})$ and is order preserving. We assert next that

$$\mathscr{Q} = (\mathscr{Q} \cap \mathscr{Y}) \vee (\mathscr{Q} \cap \mathscr{C}\mathscr{S}) \quad (\mathscr{Q} \in \mathscr{Q}(\mathscr{S}\mathscr{S}\mathscr{C}\mathscr{S})).$$

Indeed, the only nontrivial part consists in assuming that $\mathscr{Y} \subseteq \mathscr{Q}$, letting $S \in \mathscr{Q}$, and proving that $S \in \mathscr{Y} \vee (\mathscr{Q} \cap \mathscr{C}\mathscr{S})$. To this end, we first note that such a semigroup S can be assumed to be a subdirect product, in fact a subsemigroup, of $Y \times C$ where $Y \in \mathscr{Y}$, $C \in \mathscr{C}\mathscr{S}$. An argument similar to that in the preceding proof can be used to show that $C \in \mathscr{Q}$. Consequently $S \in \mathscr{Y} \vee (\mathscr{Q} \cap \mathscr{C}\mathscr{S})$. This establishes the above assertion, which in its turn implies that χ is one-to-one.

Using Proposition 7.3 and an argument closely similar to that of the second part of the proof of Theorem 4.4, one shows without difficulty that χ maps $\mathscr{Q}(\mathscr{S}\mathscr{S}\mathscr{C}\mathscr{S})$ onto $\mathscr{Q}(\mathscr{Y}) \times \mathscr{Q}(\mathscr{C}\mathscr{S})$.

The lattice $\mathcal{Q}(\mathcal{Y})$ has only two elements, as it is easy to see. As contrasted to this, the lattice $\mathcal{Q}(\mathcal{CS})$ is entirely unknown.

8. Quasivarieties of sturdy semilattices of rectangular groups. These are evidently semigroups which are orthodox sturdy semilattices of completely simple semigroups. They are thus characterized by implications (3), (5) and (9). The conjunction of (3) and (5) is clearly equivalent to identity (7). Consequently $\mathcal{SSRG}$ – sturdy semilattices of rectangular groups – are characterized, within $\mathcal{CR}$, by identity (7) and implications (9). For normal bands, implications (9) take on the form

(10) $\quad xay = yax \Rightarrow xy = yx,$

so that on $\mathcal{ONBG}$, it has the form

(11) $\quad e = e^2, \quad f = f^2, \quad g = g^2, \quad egf = fge \Rightarrow ef = fe.$

8.1 LEMMA. *On* $\mathcal{ONBG}$, *implications* (9) *are equivalent to the conjunction of implications* (8) *and* (11).

Proof. Any $S \in \mathcal{ONBG}$ can be assumed to be of the form $S = [Y; S_\alpha, \chi_{\alpha,\beta}]$, where $S_\alpha = L_\alpha \times R_\alpha \times G_\alpha$ with L_α, R_α, G_α being a left zero semigroup, a right zero semigroup, and a group, respectively. Here $\chi_{\alpha,\beta} : L_\alpha \times R_\alpha \times G_\alpha \rightarrow L_\beta \times R_\beta \times G_\beta$ is a homomorphism if $\alpha \geqq \beta$. All such homomorphisms were computed in [**14**, IV.4.4] as follows

$$\chi_{\alpha,\beta} : (l, r, g) \rightarrow (l\phi_{\alpha,\beta}, r\psi_{\alpha,\beta}, g\omega_{\alpha,\beta})$$

where $\phi_{\alpha,\beta} : L_\alpha \rightarrow L_\beta$, $\psi_{\alpha,\beta} : R_\alpha \rightarrow R_\beta$, $\omega_{\alpha,\beta} : G_\alpha \rightarrow G_\beta$ are homomorphisms. It is easy to see that $\chi_{\alpha,\beta}$ is one-to-one if and only if $\phi_{\alpha,\beta}$, $\psi_{\alpha,\beta}$ and $\omega_{\alpha,\beta}$ are one-to-one. Furthermore,

(9) is equivalent to all $\chi_{\alpha,\beta}$ being one-to-one,

(8) is equivalent to all $\omega_{\alpha,\beta}$ being one-to-one,

(11) is equivalent to all $\phi_{\alpha,\beta}$ and all $\psi_{\alpha,\beta}$ being one-to-one,

which can be easily verified. The assertion of the lemma now follows directly.

8.2 PROPOSITION. $\mathcal{SSRG} = \mathcal{SSRB} \vee \mathcal{G} = \mathcal{Y} \vee \mathcal{RG} = \mathcal{UBG} \cap \mathcal{SSCS} = \{\mathcal{S} \in \mathcal{CR} | E_S$ *unitary*, $E_S \in \mathcal{SSRB}\} = \{S \in \mathcal{R} | S \lhd Y \times R \times G,\ Y \in \mathcal{Y}, R \in \mathcal{RB}, G \in \mathcal{G}\}$.

Proof. Denote the sets in the statement of the proposition by $A_1, A_2, \ldots, A_6$ in the given order.

We have remarked above that semigroups in $\mathcal{SSRG}$ are characterized by identities (7) and (9), and in view of Lemma 8.1 alternatively by (7), (8) and (11). In particular (11) on a normal band B is equivalent to $B \in \mathcal{SSRB}$. The equality $A_1 = A_5$ now follows without difficulty.

Let $S \in A_5$. Then $S \cong [Y; S_\alpha, \chi_{\alpha,\beta}]$ where $S_\alpha = R_\alpha \times G_\alpha$ with $R_\alpha \in \mathcal{RB}$, $G_\alpha \in \mathcal{G}$, and all $\chi_{\alpha,\beta}$ are one-to-one, see the preceding proof. According to

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[**14**, III.7.11], S is a subdirect product of Y and S/τ, where the latter is a rectangular group, say $S/\tau \cong R \times G$ with $R \in \mathscr{RB}$ and $G \in \mathscr{G}$. Consequently $S \lhd Y \times (R \times G)$ which evidently implies that $S \lhd Y \times R \times G$. Hence $A_5 \subseteq A_6$. If $S \in A_6$, then a straightforward verification shows that $S \in A_5$. Thus $A_6 \subseteq A_5$ and therefore $A_1 = A_5 = A_6$.

It is clear that $A_6 \subseteq A_2 \subseteq A_1$, and $A_6 \subseteq A_4 \subseteq A_1$ which proves the equalities $A_1 = A_2 = A_3$. Also, it is easily seen that $A_1 \subseteq A_4 \subseteq A_5$ which establishes the equality $A_1 = A_4$.

8.3 Corollary. $\mathscr{SSRB} = \mathscr{Y} \vee \mathscr{RB} = \mathscr{Y} \vee \mathscr{LZ} \vee \mathscr{RZ}$.

Similarly as in Corollary 5.3, we also have $\mathscr{Q}(\mathscr{SSRG}) \cong Y_2{}^3 \times \mathscr{Q}(\mathscr{G})$, $\mathscr{Q}(\mathscr{SSRB}) \cong Y_2{}^3$, etc.

9. Construction of orthodox bands of groups. We have characterized these semigroups, within $\mathscr{CR}$, in Section 3 as those satisfying identities (2) and (3) or, alternatively, identity (4). According to [**15**, Theorem 3.1], they can be characterized, within $\mathscr{R}$, as subdirect (respectively spined) products of a band and a semilattice of groups. The following symbolism will prove convenient.

9.1 *Notation.* For any class $\mathscr{C}$ of semigroups, $\mathscr{H}(\mathscr{C})$ denotes the *homomorphic closure* of $\mathscr{C}$, i.e., the class of all semigroups which are homomorphic images of semigroups in $\mathscr{C}$.

We will need a theorem several times (see e.g. [**8**, § 23, Theorem 3]) which indicates that for two varities of universal algebras $\mathscr{A}$ and $\mathscr{B}$, of the same kind, their join $\mathscr{A} \vee \mathscr{B}$ consists of homomorphic images of subdirect products of A in $\mathscr{A}$ and B in $\mathscr{B}$.

9.2 Proposition. *If* $\mathscr{V}' \in \mathscr{V}(\mathscr{B})$ *and* $\mathscr{V}'' \in \mathscr{V}(\mathscr{G})$, *then*

$$\mathscr{V}' \vee \mathscr{V}'' = \mathscr{H}(\mathscr{V}' \bigvee \mathscr{V}'').$$

Proof. This follows directly from [**8**, § 23, Theorem 3], and Proposition 6.2.

9.3 Corollary. $\mathscr{OBG} = \mathscr{H}(\mathscr{UBG})$.

As an alternative to a construction of orthodox bands of groups in terms of subdirect or spined products mentioned above, Corollary 9.3 provides an opportunity to construct them as homomorphic images of unitary bands of groups. According to Proposition 6.2, the latter can be thought of as regular semigroups which are subdirect products of a band and a group. In order to construct all congruences on these subdirect products, we first present a construction of subdirect products, and their congruences, for a more general situation, which encompasses the hypotheses on completely regular semigroups considered in this and the next two sections.

9.4 *Notation.* Recall that η stands for the least semilattice congruences on any semigroup. For any regular semigroup S, we denote by $\mathscr{S}(S)$ the lattice, under inclusion, of all regular subsemigroups of S.

Let B be a band and C be a completely simple semigroup. Let $\phi : B/\eta \to \mathscr{S}(C)$ be an order inverting full mapping in the sense that $e\eta^{\#} \geqq f\eta^{\#}$ implies $e\eta^{\#}\phi \subseteq f\eta^{\#}\phi$ and $\bigcup_{e \in B} e\eta^{\#}\phi = C$. Denote the set

$$\{(e, c) \in B \times C | c \in e\eta^{\#}\phi\},$$

together with the multiplication induced on it by the direct product, by $(B, C; \phi)$. It is easy to verify that $(B, C; \phi)$ is a regular semigroup which is a subdirect product of B and C.

Further, let $S \subseteq B \times C$ be a subdirect product of B and C. Let ξ be a congruence on B, and for each $x \in (B/\xi)/\eta$ let

$$\Phi_x = \{c \in C | (e, c) \in S \text{ where } x = e\xi^{\#}\eta^{\#}\},$$

and θ_x be a congruence on Φ_x. Assume that if $x \geqq y$, then $\theta_x \subseteq \theta_y$ (the inclusion of binary relations). Define a relation $\rho = \rho_{(\xi, \theta_x)}$ on S by

$$(e, a)\rho(f, b) \quad \text{if} \quad e\xi f \text{ and } a\theta_{e\xi^{\#}\eta^{\#}}b.$$

First note that for any $x \in (B/\xi)/\eta$, Φ_x is indeed a subsemigroup of C. The inclusion $\theta_x \subseteq \theta_y$ is in terms of θ_x and θ_y as binary relations since their domains must only satisfy $\Phi_x \subseteq \Phi_y$ and need not be equal. If $S = (B, C; \phi)$, then the condition "$x \geqq y$ implies $\Phi_x \subseteq \Phi_y$" can be written as

$$e\xi^{\#}\eta^{\#} \geqq f\xi^{\#}\eta^{\#} \Rightarrow e\eta^{\#}\phi \subseteq f\eta^{\#}\phi$$

which reduces to the condition in the second paragraph of Notation 9.4 if ξ is the equality relation. For any $e \in B$, we have

$$\Phi_{e\xi^{\#}\eta^{\#}} = \bigcup\{f\eta^{\#}\phi | e\xi^{\#}\eta^{\#} = f\xi^{\#}\eta^{\#}\}.$$

With the notation as in 9.4, we prove

9.5 Lemma. *For $S \subseteq B \times C$ a subdirect product, the relation ρ is a congruence.*

Proof. It is easy to see that ρ is an equivalence relation on S. Let $(e, a)\ \rho\ (f, b)$ and $(g, c) \in S$. By hypothesis $e\ \xi\ f$ and $a\theta_{e\xi^{\#}\eta^{\#}}b$. Since ξ is a congruence, we have $eg\ \xi\ fg$. The relation $e\xi^{\#}\eta^{\#} \geqq (eg)\xi^{\#}\eta^{\#}$ by hypothesis implies $\theta_{e\xi^{\#}\eta^{\#}} \subseteq \theta_{(eg)\xi^{\#}\eta^{\#}}$ so that $a\theta_{e\xi^{\#}\eta^{\#}}b$ implies $a\theta_{(eg)\xi^{\#}\eta^{\#}}b$. Further, $g\xi^{\#}\eta^{\#} \geqq (eg)\xi^{\#}\eta^{\#}$ implies $\theta_{g\xi^{\#}\eta^{\#}} \subseteq \theta_{(eg)\xi^{\#}\eta^{\#}}$ which together with $c\theta_{g\xi^{\#}\eta^{\#}}c$ yields $c\theta_{(eg)\xi^{\#}\eta^{\#}}c$. The latter means that $c \in \Phi_{(eg)\xi^{\#}\eta^{\#}}$. We now have $a, b, c \in \Phi_{(eg)\xi^{\#}\eta^{\#}}$ and $a\theta_{(eg)\xi^{\#}\eta^{\#}}b$. Since $\theta_{(eg)\xi^{\#}\eta^{\#}}$ is a congruence on $\Phi_{(eg)\xi^{\#}\eta^{\#}}$, it follows that $ac\theta_{(eg)\xi^{\#}\eta^{\#}}bc$. Consequently $ac\ \rho\ bc$; one shows analogously that $ca\ \rho\ cb$. Therefore ρ is a congruence.

9.6 Lemma. *Let $\rho_{(\xi, \theta_x)}$ and $\rho_{(\xi', \theta'_{x'})}$ be congruences on a subdirect product $S \subseteq B \times C$. Then $\rho_{(\xi, \theta_x)} \subseteq \rho_{(\xi', \theta'_{x'})}$ if and only if $\xi \subseteq \xi'$ and $\theta_{e\xi^{\#}\eta^{\#}} \subseteq \theta_{e\xi'^{\#}\eta^{\#}}$ for all $e \in B$.*

Proof. Necessity. Let $e\,\xi\, f$. Then $\theta_{e\xi^*\eta^*} = \theta_{f\xi^*\eta^*}$ and thus $\Phi_{e\xi^*\eta^*} = \Phi_{f\xi^*\eta^*}$. Further, $(e, c) \in S$ for some $c \in C$, and hence $(f, c) \in S$. But then $(e, c)\rho_{(\xi,\theta_x)}(f, c)$ which by hypothesis implies $(e, c)\ \rho_{(\xi',\theta'_{x'})}(f, c)$ and thus $e\ \xi'\ f$. Consequently $\xi \subseteq \xi'$. Next let $a\theta_{e\xi^*\eta^*}b$. Then $a, b \in \Phi_{e\xi^*\eta^*}$ and $(e, a), (e, b) \in S$. Hence $(e, a)\rho_{(\xi,\theta_x)}(e, b)$ which by hypothesis implies $(e, a)\rho_{(\xi',\theta'_{x'})}(e, b)$. But then $a\theta_{e\xi'^*\eta^*}b$, which proves that $\theta_{e\xi^*\eta^*} \subseteq \theta_{e\xi'^*\eta^*}$.

Sufficiency. The argument here is straightforward and may be omitted.

So far, we have constructed some subdirect products, viz. those of the form $(B, C; \phi)$, of a band B and a completely simple semigroup C (second paragraph of Notation 9.4). In the case that $C = G$ is a group, [**15**, Theorem 4.3] asserts that all regular semigroups which are subdirect products of B and G, contained in $B \times G$, can be so constructed. Hence we restrict our attention to the subdirect products of the form $(B, G; \phi)$. We have also constructed certain kind of congruence on a subdirect product S, which we may now take to be equal to $(B, G; \phi)$. The next result asserts that *all* congruences on $(B, G; \phi)$ can be so constructed. For the special case at hand, we may rephrase the definition of $\rho_{(\xi,\theta_x)}$ as follows.

9.7 *Notation.* Let $S = (B, G; \phi)$ be a regular semigroup subdirect product of a band B and a group G. Let ξ be a congruence on B. For every $x \in (B/\xi)/\eta$, let $x\theta$ be a normal subgroup of Φ_x and assume that $x \geqq y$ implies $x\theta \subseteq y\theta$ (see Notation 9.4). Define a relation $\rho = \rho_{(\xi,\theta)}$ on S by

$$(e, g)\ \rho\ (f, h) \quad \text{if } e\,\xi\, f \text{ and } gh^{-1} \in e\xi^*\eta^*\theta.$$

Note that the equivalence of definitions in Notations 9.4 and 9.7 is a consequence of the fact that all semigroups in $\mathcal{S}(\mathcal{G})$ and all Φ_x are subgroups of G, and that in a group, we may speak of normal subgroups instead of congruences. The main result of this section can now be established.

9.8 Theorem. *Every congruence on a regular semigroup $(B, G; \phi)$ which is a subdirect product of a band B and a group G can be uniquely expressed in the form $\rho_{(\xi,\theta)}$.*

Proof. Let ρ be a congruence on $S = (B, G; \phi)$. Define a relation ξ on B by

$$e\,\xi\, f \quad \text{if } (e, 1)\ \rho\ (f, 1)$$

where 1 denotes the identity of G. It is clear that ξ is a congruence on B.

Let $e, f \in B$ be such that $e\xi^*\ \eta\ f\xi^*$, and $(e, g), (f, g) \in S$. Then $f\ \xi\ fef$ and thus $(f, 1)\ \rho\ (fef, 1)$. Assume next that $(e, g)\ \rho\ (e, 1)$. Using $(f, 1)\ \rho\ (fef, 1)$, we obtain

$$\begin{aligned}(f, g)\ \rho\ (f, 1)(f, g)\ \rho\ (fef, 1)(f, g) &= (fef, g)\\ &= (f, 1)(e, g)(f, 1)\ \rho\ (f, 1)(e, 1)(f, 1)\\ &= (fef, 1)\ \rho\ (f, 1),\end{aligned}$$

that is $(f, g)\ \rho\ (f, 1)$. By symmetry, we conclude that

$$e\xi^{\#}\ \eta\ f\xi^{\#} \text{ implies } ((e, g)\ \rho\ (e, 1) \quad \Leftrightarrow \quad (f, g)\ \rho\ (f, 1)).$$

We may thus define a function θ on $(B/\xi)/\eta$ by

$$e\xi^{\#}\eta^{\#}\theta = \{g \in G | (e, g)\ \rho\ (e, 1)\}.$$

It is easy to verify that $e\xi^{\#}\eta^{\#}\theta$ is a normal subgroup of $\Phi_{e\xi^{\#}\eta^{\#}}$ (see Notation 9.7). For $e, f \in B$, assume that $e\xi^{\#}\eta^{\#} \geqq f\xi^{\#}\eta^{\#}$, and let $g \in e\xi^{\#}\eta^{\#}\theta$. Then $(e, g)\ \rho\ (e, 1)$ so that $(ef, g)\ \rho\ (ef, 1)$, where

$$(ef)\xi^{\#}\eta^{\#} = (e\xi^{\#}\eta^{\#})(f\xi^{\#}\eta^{\#}) = f\xi^{\#}\eta^{\#}$$

and thus $g \in f\xi^{\#}\eta^{\#}\theta$. This shows that θ inverts order.

Next let $(e, g)\ \rho\ (f, h)$. Then $(ef, gh^{-1})\ \rho\ (f, 1)$ so that $(ef, gh^{-1})\ \rho\ (ef, 1)$ and thus $(f, 1)\ \rho\ (ef, 1)$. Further, $(e, 1)\ \rho\ (ef, g^{-1}h)$ which implies $(ef, 1)\ \rho\ (ef, g^{-1}h)$, and hence $(e, 1)\ \rho\ (ef, 1)$. Consequently $(e, 1)\ \rho\ (f, 1)$ which says that $e\ \xi\ f$. Also $(e, gh^{-1})\ \rho\ (ef, 1)$ with $e\xi^{\#}\eta^{\#} = f\xi^{\#}\eta^{\#} = (ef)\xi^{\#}\eta^{\#}$ so that $gh^{-1} \in e\xi^{\#}\eta^{\#}\theta$.

Conversely, let $e\ \xi\ f$ and $gh^{-1} \in e\xi^{\#}\eta^{\#}\theta$, where $(e, g), (f, h) \in S$. Then $(e, 1)\ \rho\ (f, 1)$ and $(e, gh^{-1})\ \rho\ (f, 1)$ which evidently yields $(e, g)\ \rho\ (f, h)$.

Therefore $\rho = \rho_{(\xi, \theta)}$ as required. Uniqueness of this representation is a direct consequence of Lemma 9.6.

10. Construction of normal bands of groups. We have characterized these semigroups, within $\mathscr{C}\mathscr{R}$, in Section 4 as those satisfying identity (5). According to [**14**, IV.4.3], they can be alternatively characterized as strong semilattices of completely simple semigroups. We will adopt here a construction analogous to that discussed in the preceding section.

10.1 Proposition. *If* $\mathscr{V}' \in \mathscr{V}(\mathscr{Y})$ *and* $\mathscr{V}'' \in \mathscr{V}(\mathscr{C}\mathscr{S})$, *then*

$$\mathscr{V}' \vee \mathscr{V}'' = \mathscr{H}(\mathscr{V}' \bigvee \mathscr{V}'').$$

Proof. This follows directly from [**8**, §23, Theorem 3] and Proposition 7.3.

10.2 Corollary. $\mathscr{N}\mathscr{B}\mathscr{G} = \mathscr{H}(\mathscr{S}\mathscr{S}\mathscr{C}\mathscr{S})$.

According to Proposition 7.3, we can think of a semigroup in $\mathscr{S}\mathscr{S}\mathscr{C}\mathscr{S}$ as a subdirect product of a semilattice and a completely simple semigroup. Now Corollary 10.2 indicates that we can obtain an isomorphic copy of each semigroup in $\mathscr{N}\mathscr{B}\mathscr{G}$ by constructing all congruences on regular semigroups which are subdirect product of a semilattice and a completely simple semigroup. Lemma 9.5 provides some congruences on such subdirect products; we will see below that there are no others. In this case, we can simplify somewhat the notation introduced in 9.4 as follows.

10.3 *Notation.* Let $S = (Y, C; \phi)$ be a regular semigroup subdirect product of a semilattice Y and a completely simple semigroup C. Let ξ be a congruence on Y and let

$$\Phi_x = \{c \in C | (\alpha, c) \in S \text{ where } x = \alpha\xi^{\#}\}$$

and θ_x be a congruence on Φ_x. Assume that if $x \geqq y$ then $\theta_x \subseteq \theta_y$. Then $\rho = \rho_{(\xi, \theta_x)}$ is given by

$$(\alpha, a) \; \rho \; (\mu, b) \quad \text{if } e \, \xi \, f \text{ and } a \, \theta_{\alpha\xi^{\#}} \, b.$$

For the main result of this section, we will need the following auxiliary statement.

10.4 LEMMA. *Let ρ be a congruence on a Rees matrix semigroup $S = \mathscr{M}(I, G, M; P)$. If $a, b \in S$ are such that $a \, \rho \, b$, then $aa^{-1} \, \rho \, bb^{-1}$.*

Proof. We let $a = (i, g, \mu)$ and $b = (j, h, \nu)$ and start with the special case $i = j$. It is well known that we may assume that the sandwich matrix P is normalized at $i = j = 1 \in M \cap I$. Consequently, our hypothesis is that $(1, g, \mu) \; \rho \; (1, h, \nu)$. Let the identity of G be denoted by e. Multiplying on the left by $(1, g^{-1}, 1)$, we obtain $(1, e, \mu) \; \rho \; (1, g^{-1}h, \nu)$. Now multiplying on the right by $(1, e, \mu)$, we get $(1, e, \mu) \; \rho \; (1, g^{-1}h, \mu)$. It follows that $(1, g^{-1}h, \mu) \; \rho \; (1, g^{-1}h, \nu)$. Multiplying this on the left by $(1, h^{-1}g, 1)$, we have $(1, e, \mu) \; \rho \; (1, e, \nu)$.

Now ignoring the normalization, we have proved that if $i = j$, then the assertion of the lemma is valid, i.e., $(i, g, \mu) \; \rho \; (i, h, \nu)$ implies $(i, p_{\mu i}^{-1}, \mu) \; \rho \; (i, p_{\nu i}^{-1}, \nu)$. Symmetrically, we have that $(i, g, \mu) \; \rho \; (j, h, \mu)$ implies $(i, p_{\mu i}^{-1}, \mu) \; \rho \; (j, p_{\mu j}^{-1}, \mu)$.

We now consider the general case, i.e., we assume that $(i, g, \mu) \; \rho \; (j, h, \nu)$. Multiplying this on the left by $(i, p_{\mu i}^{-1}, \mu)$, we obtain $(i, g, \mu) \; \rho \; (i, p_{\mu i}^{-1}p_{\mu j}h, \nu)$. By the special case above, we have $(i, p_{\nu i}^{-1}, \mu) \; \rho \; (i, p_{\nu i}^{-1}, \nu)$, and by the symmetric special case, we must have $(i, p_{\nu i}^{-1}, \nu) \; \rho \; (j, p_{\nu j}^{-1}, \nu)$. From the last two relations, it follows that $(i, p_{\mu i}^{-1}, \mu) \; \rho \; (j, p_{\nu j}^{-1}, \nu)$. This is equivalent to the assertion of the lemma.

10.5 THEOREM. *Every congruence on a regular semigroup $(Y, C; \phi)$ which is a subdirect product of a semilattice Y and a completely simple semigroup C can be uniquely expressed in the form $\rho_{(\xi, \theta_x)}$.*

Proof. Let ρ be a congruence on $S = (Y, C; \phi)$. Define a relation ξ on Y by

$$\alpha \, \xi \, \beta \quad \text{if} \quad (\alpha, a) \; \rho \; (\alpha\beta, a), \; (\beta, b) \; \rho \; (\alpha\beta, b)$$

for some $a, b \in C$ such that $(\alpha, a), (\beta, b) \in S$. It is clear that ξ is reflexive and symmetric. With this notation, assume that $\alpha \, \xi \, \beta$ and that $\beta \, \xi \, \gamma$ with $(\beta, b') \; \rho \; (\beta\gamma, b')$ and $(\gamma, c) \; \rho \; (\beta\gamma, c)$. It follows that $(\alpha\beta, ab') \; \rho \; (\alpha\beta\gamma, ab')$ which together with $(\alpha, a) \; \rho \; (\alpha\beta, a)$ yields $(\alpha, a) \; \rho \; (\alpha\beta\gamma, a)$ by [**13**, Theorem 3]. Multiplying the last relation by (γ, c) on the right, we obtain $(\alpha\gamma, ac) \; \rho \; (\alpha\beta\gamma, ac)$. In view of the construction of $S = (Y, C; \phi)$ (see Notation 9.4), we have that $(\alpha, a) \in S$ implies $(\alpha\gamma, a) \in S$. Now $(\alpha\gamma, ac) \; \rho \; (\alpha\beta\gamma, ac)$ by [**13**, Theorem 3] implies that $(\alpha\gamma, a) \; \rho \; (\alpha\beta\gamma, a)$. The latter together with $(\alpha, a) \; \rho \; (\alpha\beta\gamma, a)$ yields $(\alpha, a) \; \rho \; (\alpha\gamma, a)$. By symmetry, we conclude that also $(\gamma, c) \; \rho \; (\alpha\gamma, c)$, which shows that ξ is transitive, and is thus an equivalence relation.

Continuing with the same notation, assume that $\alpha \xi \beta$ and let $\gamma \in Y$. Then $(\gamma, c) \in S$ for some $c \in S$. We obtain $(\alpha\gamma, ac) \rho (\alpha\beta\gamma, ac)$ and $(\beta\gamma, bc) \rho (\alpha\beta\gamma, bc)$ which means that $\alpha\gamma \xi \beta\gamma$. Consequently ξ is a congruence on Y. Following Notation 10.3, we easily verify that $\Phi_{\alpha\xi^+}$ is a regular subsemigroup of C for every $\alpha \in Y$. On each $\Phi_{\alpha\xi^+}$, we define a relation $\theta_{\alpha\xi^+}$ by

$$a\theta_{\alpha\xi^+}b \quad \text{if} \quad (\gamma, a) \rho (\gamma, b) \text{ for some } \gamma \xi \alpha.$$

Let $x = \alpha\xi^+$. It is clear that θ_x is reflexive and symmetric. Let $a\theta_x b$ and $b\theta_x c$ with $(\gamma, a) \rho (\gamma, b)$ and $(\delta, b) \rho (\delta, c)$, where $\gamma \xi \alpha$, $\delta \xi \alpha$. It follows that $\gamma \xi \delta$ so that by the definition of ξ, there exists $d \in C$ such that $(\gamma, d) \rho (\gamma\delta, d)$. But then [**13**, Theorem 3] implies that $(\gamma, a) \rho (\gamma\delta, a)$. Analogously, we must have $(\gamma, b) \rho (\gamma\delta, b)$, $(\delta, b) \rho (\gamma\delta, b)$ and $(\delta, c) \rho (\gamma\delta, c)$. Consequently

$$(\gamma\delta, a) \rho (\gamma, a) \rho (\gamma, b) \rho (\gamma\delta, b) \rho (\delta, b) \rho (\delta, c) \rho (\gamma\delta, c)$$

which proves that θ_x is transitive.

Next let $a \theta_x b$ with $(\beta, a) \rho (\beta, b)$, $\beta \xi \alpha$, and $(\gamma, c) \in S$. Hence $(\beta\gamma, ac) \rho (\beta\gamma, bc)$ where $\beta\gamma \xi \alpha\gamma$ so that $ac\theta_{(\alpha\gamma)\xi^+}bc$. One shows analogously that also $ca\theta_{(\alpha\gamma)\xi^+}cb$. Since this is valid for all $c \in \Phi_x$, we have shown that θ_x is a congruence on Φ_x.

Now let $\alpha, \beta \in Y$ be such that $\alpha \geqq \beta$, and let $a\theta_{\alpha\xi^+}b$. Lemma 10.4 implies that $aa^{-1}\theta_{\alpha\xi^+}bb^{-1}$ since $\theta_{\alpha\xi^+}$ is a congruence on $\Phi_{\alpha\xi^+}$, which being a regular subsemigroup of C must be completely simple. Multiplying $aa^{-1}\theta_{\alpha\xi^+}bb^{-1}$ by a on the left, we have by the preceding paragraph that $a\theta_{\beta\xi^+}abb^{-1}$ since $\alpha \geqq \beta$. Similarly, multiplying $a\theta_{\alpha\xi^+}b$ by bb^{-1} on the right yields $abb^{-1}\theta_{\beta\xi^+}b$. By transitivity, the conjunction of $a\theta_{\beta\xi^+}abb^{-1}$ and $abb^{-1}\theta_{\beta\xi^+}b$ gives $a\theta_{\beta\xi^+}b$. Consequently $\alpha \geqq \beta$ implies that $\theta_{\alpha\xi^+} \subseteq \theta_{\beta\xi^+}$.

Let $(\alpha, a) \rho (\beta, b)$. Then $(\alpha, a^2) \rho (\alpha\beta, ab)$ which in view of [**13**, Theorem 3] implies that $(\alpha, a) \rho (\alpha\beta, a)$. We analogously have $(\beta, b) \rho (\alpha\beta, b)$ which implies that $\alpha \xi \beta$. Furthermore,

$$(\alpha\beta, a) \rho (\alpha, a) \rho (\beta, b) \rho (\alpha\beta, b)$$

which shows that $a \theta_{\alpha\xi^+} b$. Consequently $(\alpha, a) \rho_{(\xi,\theta_x)} (\beta, b)$. Conversely, let $(\alpha, a) \rho_{(\xi,\theta_x)} (\beta, b)$. Then $\alpha \xi \beta$ and $a\theta_{\alpha\xi^+} b$. It follows that $(\alpha, c) \rho (\alpha\beta, c)$, $(\beta, d) \rho (\alpha\beta, d)$, $(\gamma, a) \rho (\gamma, b)$ for some $c, d \in C$ and $\gamma \xi \alpha$. Since then also $\beta \xi \alpha$, it follows easily from [**13**, Theorem 3] that

$$(\alpha, a) \rho (\alpha\beta, a) \rho (\alpha\beta\gamma, a) \rho (\gamma, a) \rho (\gamma, b) \rho (\alpha\beta\gamma, b) \rho (\alpha\beta, b) \rho (\beta, b),$$

i.e., $(\alpha, a) \rho (\beta, b)$.

We have proved that $\rho = \rho_{(\xi,\theta_x)}$; the uniqueness of this representation is a direct consequence of Lemma 9.6.

11. Construction of orthodox normal bands of groups. We have characterized these semigroups in Section 5, within $\mathscr{C}\mathscr{R}$, as those satisfying identity (7). We have seen in Section 8 that sturdy semilattices of rectangular groups can be characterized, again within $\mathscr{C}\mathscr{R}$, as those semigroups satisfying implications (7) and (9), or alternatively (7), (8) and (11). A further relationship between these two quasivarieties is provided by the following statement.

11.1 PROPOSITION. $\mathscr{ONBG} = \mathscr{H}(\mathscr{SSRG})$.

Proof. By Corollary 5.2, we have $\mathscr{ONBG} = \mathscr{Y} \vee \mathscr{RG}$, and by Proposition 8.2, $\mathscr{SSRG} = \mathscr{Y} \bigvee \mathscr{RG}$. The assertion of the proposition now follows from Proposition 10.1.

11.2 COROLLARY. $\mathscr{Y} \vee \mathscr{RB} \vee \mathscr{G} = \mathscr{H}(\mathscr{Y} \bigvee \mathscr{RB} \bigvee \mathscr{G})$.

Proof. This follows from the proof of Proposition 11.1 and the fact that $\mathscr{RB} \vee \mathscr{G} = \mathscr{RB} \bigvee \mathscr{G} = \mathscr{RG}$.

Similarly as in the preceding section, we can use Proposition 11.1 to construct all semigroups in $\mathscr{ONBG}$ by constructing all congruences on semigroups in $\mathscr{SSRG}$. The latter can be thought of as regular semigroups subdirect products of a semilattice and a rectangular group. This amounts to a special case of the one considered in the preceding section. However, we can be much more explicit here. We start with a construction of subdirect products under consideration given in [**14**, IV.5.5].

11.3 PROPOSITION. *Let Y, L, R, G be a semilattice, a left zero semigroup, a right zero semigroup, and a group, respectively. Let $\phi_X : Y \to \mathscr{S}(X)$ be a full inclusion inverting mapping for $X = L, R, G$. Then*

$$S = \{(\alpha, l, r, g) \in Y \times L \times R \times G | l \in \alpha\phi_L, r \in \alpha\phi_R, g \in \alpha\phi_G\}$$

is a regular semigroup which is a subdirect of $Y \times L \times R \times G$. Conversely, every regular semigroup which is a subdirect product of $Y \times L \times R \times G$ can be so constructed.

In fact, according to [**14**, IV.5.4], $L \times R \times G$ has no proper subdirect products, so that subdirect products in Proposition 11.3 coincide with the subdirect products of $Y \times (L \times R \times G)$, where we consider only regular semigroups.

11.4 THEOREM. *Let S be as in 11.3. Let ξ be a congruence on Y and let*

$$\Phi_x = \{(l, r, g) \in L \times R \times G | (\alpha, l, r, g) \in S \text{ where } x = \alpha\xi^{\#}\}.$$

Then $\Phi_x = L_x \times R_x \times G_x$ for some $\emptyset \neq L_x \subseteq L$, $\emptyset \neq R_x \subseteq R$ and some subgroups G_x of G. For every $x \in Y/\xi$, let $\theta_x{}^L$ and $\theta_x{}^R$ be equivalence relations on L_x and R_x, respectively, and let N_x be a normal subgroup of G_x. Assume that $x \geqq y$ implies $\theta_x{}^L \subseteq \theta_y{}^L$, $\theta_x{}^R \subseteq \theta_y{}^R$, $N_x \subseteq N_y$. Define a relation $\rho = \rho_{(\xi;\theta_x{}^L,\theta_x{}^R,N_x)}$ by

$$(\alpha, l, r, g)\, \rho\, (\alpha', l', r', g') \quad \text{if} \quad \alpha\,\xi\,\beta,\; l\,\theta^L{}_{\alpha\xi^{\#}}\, l',\; r\,\theta^R{}_{\alpha\xi^{\#}}\, r',\; gg'^{-1} \in N_{\alpha\xi^{\#}}.$$

Then ρ is a congruence on S. Conversely, every congruence on S admits a unique representation of the form $\rho_{(\xi;\theta_x{}^L,\theta_x{}^R,N_x)}$.

Proof. Adopting the notation of the first part of the theorem, we first remark that Φ_x is a regular subsemigroup of $L \times R \times G$. Hence Φ_x is a subdirect product of its projections L_x, R_x and G_x in L, R and G, respectively. According to [**14**, IV.5.4], $L_x \times R_x \times G_x$ has no proper subdirect products, so we must have $\Phi_x = L_x \times R_x \times G_x$ where G_x is a subgroup of G. In L_x and R_x, every equivalence relation is a congruence; and in G_x, a normal subgroup induces a congruence in the usual way. It is now clear that θ_x defined on $L_x \times R_x \times G_x$ by

$$(l, r, g)\ \theta_x\ (l', r', g') \quad \text{if} \quad l\ \theta_x{}^L\ l',\ r\ \theta_x{}^R\ r',\ gg'^{-1} \in N_x$$

is a congruence on $L_x \times R_x \times G_x$. Furthermore, if $x \geqq y$ then clearly $\theta_x \subseteq \theta_y$. From this and the form of ρ in the statement of the theorem, we conclude that ρ has the general form introduced in Notation 9.4 and is thus a congruence on S by Lemma 9.5.

In order to prove the converse, we first denote the congruence θ_x defined above by $(\theta_x{}^L, \theta_x{}^R, N_x)$. Next let ρ by any congruence on S. Then $\rho = \rho_{(\xi,\theta_x)}$ for unique ξ and θ_x according to Theorem 10.5. It is proved in [**12**] that every θ_x can uniquely written in the form $(\theta_x{}^L, \theta_x{}^R, N_x)$ where $\theta_x{}^L$, $\theta_x{}^R$ and N_x are as in the statement of the theorem. This establishes the converse.

12. Properties of congruences. We have defined the congruences $\rho_{(\xi,\theta_x)}$ on $(B, C; \phi)$ and congruences $\rho_{(\xi,\theta)}$ on $(B, G; \phi)$ in Section 9. We have then shown in Sections 9–11 that in the special cases of subdirect products of a band and a group, a semilattice and a completely simple semigroup, and a combination thereof, all congruences are of this type. In view of the results proved in the first part of the paper, the homomorphic images give all orthodox bands of groups, all normal bands of groups, and all orthodox normal bands of groups, respectively. We will show below that for this latter purpose, we may take ξ to be the equality relation, thereby simplifying the construction of semigroups in $\mathscr{OBG}$, $\mathscr{NBG}$ and $\mathscr{ONBG}$. It is convenient to first introduce the following

12.1. *Notation.* With the symbolism of Notations 9.4 and 9.7, let

$$(B, C; \phi, \xi, \theta_x) = (B, C; \phi)/\rho_{(\xi,\theta_x)},$$

$$(B, G; \phi, \xi, \theta) = (B, G; \phi)/\rho_{(\xi,\theta)}.$$

Let ι denote the equality relation on any set.

12.2 PROPOSITION. $(B, C, \phi, \xi, \theta_x) \cong (B/\xi, C, \phi', \iota, \theta_x)$ *where* $e\xi^{\mp}\eta^{\mp}\phi' = \Phi_{e\xi^{\mp}\eta^{\mp}}$ *for all* $e \in B$.

Proof. First recall from Notation 9.4 that

$$\Phi_{e\xi^{\mp}\eta^{\mp}} = \bigcup\{f\eta^{\mp}\phi \mid e\xi^{\mp}\eta^{\mp} = f\xi^{\mp}\eta^{\mp}\}.$$

Let $a,b \in e\xi^{\mp}\eta^{\mp}\phi'$. Then $a \in e'\eta^{\mp}\phi$, $b \in e''\eta^{\mp}\phi$ for some e', e'' such that $e\xi^{\mp}\eta^{\mp} = e'\xi^{\mp}\eta^{\mp} = e''\xi^{\mp}\eta^{\mp}$. Since $(e'e'')\eta^{\mp} \leqq e'\eta^{\mp}$ and $(e'e'')\eta^{\mp} \leqq e''\eta^{\mp}$, we have $(e'e'')\eta^{\mp}\phi \supseteq e'\eta^{\mp}\phi$ and $(e'e'')\eta^{\mp}\phi \supseteq e''\eta^{\mp}\phi$. Consequently $a, b \in (e'e'')\eta^{\mp}\phi$ which implies $ab \in (e'e'')\eta^+\phi$, where $(e'e'')\xi^+\eta^+ = e\xi^+\eta^+$. But then $ab \in e\xi^+\eta^+\phi'$, which proves that $e\xi^+\eta^+\phi'$ is a subsemigroup of C.

A similar argument can be used to prove that $e\xi^+\eta^+\phi'$ is a regular semigroup, so that $e\xi^+\eta^+\phi' \in \mathscr{S}(C)$. Again, a similar type of argument shows that ϕ' is an inclusion inverting function and is full. Therefore $(B/\xi, C; \phi')$ is a regular semigroup subdirect product of B/ξ and C.

Recall from Notation 9.4 that θ_x is a congruence on Φ_x for all $x \in (B/\xi)/\eta$. Hence the quintuple $(B/\xi, C; \phi, \iota, \theta_x)$ is meaningful. Letting $\rho = \rho_{(\xi,\theta_x)}$ and $\rho' = \rho_{(\iota,\theta_x)}$, we define a function χ on $(B, C; \phi, \xi, \theta_x)$ by

$$\chi : (e, c)\rho^+ \to (e\xi^+, c)\rho'^+.$$

First note that

$$(e, c)\rho^+ = (f, d)\rho^+ \Leftrightarrow (e, c)\rho(f, d) \Leftrightarrow e\, \xi\, f,\ c\, \theta_{e\xi^+\eta^+}\, d$$
$$\Leftrightarrow e\xi^+ = f\xi^+,\ c\, \theta_{e\xi^+\eta^+}\, d$$

which shows that χ is an isomorphism of $(B, C; \phi, \xi, \theta_x)$ onto $(B/\xi, C; \phi', \iota, \theta_x)$.

This proposition shows that in order to construct all semigroups in $\mathscr{OBG}$, $\mathscr{NBG}$ and $\mathscr{ONBG}$ from the corresponding subdirect products, we can take $\xi = \iota$. Such congruences have a simpler form as follows:

$$(e, a)\ \rho_{(\iota,\theta_x)}\ (f, b) \Leftrightarrow e = f,\ a\, \theta_{e\eta^+}\, b$$

and for $C = G$, a group,

$$(e, g)\ \rho_{(\iota,\theta)}\ (f, h) \Leftrightarrow e = f,\ gh^{-1} \in e\eta^+\theta.$$

A further reduction concerns C instead of B, and can be formulated thus

12.3 Proposition. $(B, C; \phi, \iota, \theta_x) \cong (B, C/\pi; \phi', \iota, \theta_x')$ *where π is any congruence on C contained in*

$$\bigcap_{x \in B/\eta} \theta_x \cup \iota, \quad e\eta^+\phi' = e\eta^+\phi\pi^+, \quad a\pi^+\, \theta_{e\eta^+}\, b\pi^+ \Leftrightarrow a\, \theta_{e\eta^+}\, b.$$

Proof. It is routine to verify that ϕ' satisfies all the requirements for determining a regular semigroup $(B, C/\pi; \phi')$ which is a subdirect product of the band B and the completely simple semigroup C/π. Letting $\rho = \rho_{(\iota,\theta_x)}$ and $\rho' = \rho_{(\iota,\theta_x')}$, we define χ on $(B, C; \phi, \iota, \theta_x)$ by

$$\chi : (e, c)\rho^+ \to (e, c\pi^+)\rho'^+.$$

Noting that

$$(e, c)\rho^+ = (f, d)\rho^+ \Leftrightarrow (e, c)\rho(f, d) \Leftrightarrow e = f,\ c\, \theta_{e\eta^+}\, d$$
$$\Leftrightarrow e = f,\ c\pi^+\, \theta'_{e\eta^+}\, d\pi^+ \Leftrightarrow (e, c\pi^+)\rho'(f, d\pi^+)$$
$$\Leftrightarrow (e, c\pi^+)\rho'^+ = (f, d\pi^+)\rho'^+,$$

we deduce that χ is the required isomorphism.

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Combining Propositions 12.2 and 12.3, we obtain

12.4 COROLLARY. $(B, C; \phi, \xi, \theta_x) \cong (B/\xi, C/\pi; \phi, \iota, \theta_x')$ *where* $\pi \subseteq \bigcap\{\theta_x | x \in (B/\xi)/\eta\} \cup \iota$ *is a congruence on* C, *for suitable* ϕ' *and* θ_x'.

12.5 COROLLARY. $(B, G; \phi, \xi, \theta) \cong (B/\xi, G/N; \phi', \iota, \theta')$ *where* $N \subseteq \bigcap_{e \in B} e\xi^{+}\eta^{+}\theta$ *is a normal subgroup of* G, *for suitable* ϕ' *and* θ'.

Even with $(B, C; \phi, \xi, \theta_x)$ or $(B, G; \phi, \xi, \theta)$ reduced as in Propositions 12.4 and 12.5, isomorphisms of $(B, C; \phi, \xi, \theta_x)$ and $(B', C'; \phi', \xi', \theta_{x'}')$ and of $(B, G; \phi, \xi, \phi)$ and $(B', G'; \phi', \xi', \theta')$, respectively, still do not seem to admit a sufficiently simple form. Recall that a congruence ρ on any semigroup S is *idempotent-separating* if for any $e, f \in E_S$, $e \rho f$ implies $e = f$. We show next

12.6 PROPOSITION. *The congruence* $\rho_{(\xi, \theta_x)}$ *is idempotent separating if and only if* ξ *and all* θ_x *are idempotent separating.*

Proof. Let $\rho_{(\xi,\theta_x)}$ be idempotent separating. Let $e, f \in B$ be such that $e \,\xi\, f$. Then (e, a), $(f, b) \in S$ for some $a, b \in C$, where $S = (B, C; \phi)$. Since $ef \leqq e$, we have $e\eta^{+}\phi \subseteq (ef)\eta^{+}\phi$ and thus $(ef, a) \in S$. Further $(ef, aa^{-1}) \in S$ since S is completely regular. It follows that $(e, aa^{-1}) \,\rho_{(\xi,\theta_x)}\, (ef, aa^{-1})$ which by hypothesis implies that $e = ef$. By symmetry, we conclude that $f = ef$ and hence $e = f$. Thus $\xi = \iota$ and is thus, trivially, idempotent separating.

Next let $u, v \in E_C$ be such that $u \,\theta_x\, v$ for some $x \in B/\eta$. It follows that (e, u), $(f, v) \in S$ for some $e, f \in B$ such that $x = e\eta^{+} = f\eta^{+}$. Similarly as above, we conclude that (ef, u), $(ef, v) \in S$ and thus $(ef, u) \,\rho_{(\xi,\theta_x)}\, (ef, v)$ since $u\theta_{(ef)\eta^{+}}v$. The hypothesis implies that $u = v$, which proves that θ_x is idempotent separating.

The proof of the converse is trivial.

It is clear that a congruence ξ on a band is idempotent separating if and only if it is the equality relation. On the other hand, each θ_x being defined on a completely simple semigroup Φ_x, we have that θ_x is idempotent-separating if and only if θ_x is contained in the $\mathscr{H}$-relation on Φ_x, by a well-known result concerning regular semigroups.

12.7 COROLLARY. *Every orthodox band of groups is an idempotent-separating homomorphic image of a subdirect product of a band and a group* (*i.e., the congruence induced by this homomorphism is idempotent-separating*).

Proof. This follows from Corollary 9.3, Theorem 9.8, Proposition 12.2 and Proposition 12.6.

As a special case, we have the following result of McAlister [**10**].

12.8 COROLLARY. *Every semilattice of groups is an idempotent-separating homomorphic image of a subdirect product of a band and a group.*

13. Problems. A solution of the following problems would further clarify the structure of the lattices as well of the completely regular semigroups under study.

1. What is the join of the varities $\mathscr{OBG} \vee \mathscr{NBG}$? It coincides with the join $\mathscr{B} \vee \mathscr{CS}$. In terms of identities

$$[x^2 = x] \vee [aa^{-1} = (aba)(aba)^{-1}] = [?].$$

An upper bound for these varieties is $\mathscr{BG}$-bands of groups.

2. The same question for the join of the quasivarieties $\mathscr{UBG} \bigvee \mathscr{SSCS}$. It coincides with the join $\mathscr{B} \bigvee \mathscr{CS}$.

3. Find the lattice $\mathscr{V}(\mathscr{CS})$ (resp. $\mathscr{Q}(\mathscr{CS})$) of varieties (resp. quasivarieties) of completely simple semigroups.

4. Construct the lattice $\mathscr{Q}(\mathscr{B})$ of all quasivarieties of bands.

5. Construct the lattices $\mathscr{V}(\mathscr{B} \vee \mathscr{CS})$ and $\mathscr{Q}(\mathscr{B} \bigvee \mathscr{CS})$.

6. Can the quintuples $(B, C; \phi, \xi, \theta_x)$, defined in Notation 12.1, be so chosen that a simple isomorphism criterion can be established for them?

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https://doi.org/10.4153/CJM-1977-118-7 Published online by Cambridge University Press