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PSEUDO-UMBILICAL SUBMANIFOLDS OF CONSTANT CURVATURE RIEMANNIAN MANIFOLDS

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Abstract. Bai [**1**] gave an intrinsic integral inequality for compact minimal submanifolds of constant curvature Riemannian manifolds. In this paper, we extend Bai's result to the case of pseudo-umbilical submanifolds.

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1. Introduction. Let V^{n+p} be an $(n+p)$-dimensional Riemannian manifold of constant curvature a. Let M^n be an n-dimensional Riemannian manifold immersed in V^{n+p}. Let h be the second fundamental form of the immersion, and ξ the mean curvature vector. Denote by $\langle\cdot,\cdot\rangle$ the scalar product of V^{n+p}. If there exists a function λ on M^n such that

$$\langle h(X, Y), \xi\rangle = \lambda\langle X, Y\rangle, \tag{1.1}$$

for any tangent vector X, Y on M^n, then M^n is called a *pseudo-umbilical submanifold of* V^{n+p}. It is clear that $\lambda \geq 0$. If the mean curvature vector $\xi = 0$ identically, then M^n is called a *minimal submanifold of* V^{n+p}. Every minimal submanifold of V^{n+p} is itself a pseudo-umbilical submanifold of V^{n+p}. Z. G. Bai [**1**] gave an intrinsic integral inequality for compact minimal submanifolds of V^{n+p} as follows.

Theorem A. *Let M^n be an n-dimensional compact minimal submanifold of $(n+p)$-dimensional constant curvature Riemannian manifold V^{n+p}. Then*

$$\int_{M^n}\{p\sum R_{ijkl}^2 + 2p\sum R_{ij}^2 - R^2 - n(3p-2n+2)aR\} * 1$$
$$\geq n^2(n-1)(n-p-1)a^2\mathrm{Vol}(M^n), \tag{1.2}$$

where $\sum R_{ijkl}^2$ is the square length of the Riemannian curvature tensor, $\sum R_{ij}^2$ is the square length of the Ricci curvature tensor and R is the scalar curvature.

In this paper, we extend Theorem A to the case in which M^n is pseudo-umbilical. We shall prove the following result.

Theorem. *Let M^n be an n-dimensional compact pseudo-umbilical submanifold of V^{n+p}. Then*

$$\int_{M^n}\{p\sum R_{ijkl}^2+2p\sum R_{ij}^2-R^2-n(3p-2n+2)aR-n(3p-2n)H^2R$$
$$+2n^2(n-1)paH^2+2n^3(n-1)aH^2+n^3pH^4-n^4H^4\}*1$$
$$\geq n^2(n-1)(n-p-1)a^2\mathrm{Vol}(M^n), \tag{1.3}$$

where H is the mean curvature of M^n.

If $H\equiv 0$, i.e. M^n is minimal, then (1.3) becomes (1.2).

2. Local formulae. In this paper, we shall make use of the following convention on the ranges of indices:

$$1\leq A,B,C,\cdots\leq n+p;\quad 1\leq i,j,k,\cdots\leq n,\quad n+1\leq\alpha,\beta,\gamma,\cdots\leq n+p,$$

and we shall agree that repeated indices are summed over the respective ranges.

We choose a local field of orthonormal frames $e_1,\cdots,e_{n+p}$ in V^{n+p} such that, restricted to M^n, $e_1,\cdots,e_n$ are tangent to M^n. Let $\omega_1,\cdots,\omega_{n+p}$ be its dual frame field. Then the structure equations of V^{n+p} are given by

$$d\omega_A=-\sum\omega_{AB}\wedge\omega_B,\quad \omega_{AB}+\omega_{BA}=0,$$
$$d\omega_{AB}=-\sum\omega_{AC}\wedge\omega_{CB}+\frac{1}{2}\sum K_{ABCD}\omega_C\wedge\omega_D, \tag{2.1}$$
$$K_{ABCD}=a(\delta_{AC}\delta_{BD}-\delta_{AD}\delta_{BC}).$$

We restrict these forms to M^n; then we have

$$\omega_\alpha=0,\quad \omega_{\alpha i}=\sum h_{ij}^\alpha\omega_j,\quad h_{ij}^\alpha=h_{ji}^\alpha,$$
$$d\omega_{ij}=-\sum\omega_{ik}\wedge\omega_{kj}+\frac{1}{2}\sum R_{ijkl}\omega_k\wedge\omega_l,$$
$$R_{ijkl}=a(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk})+\sum(h_{ik}^\alpha h_{jl}^\alpha-h_{il}^\alpha h_{jk}^\alpha), \tag{2.2}$$
$$d\omega_{\alpha\beta}=-\sum\omega_{\alpha\gamma}\wedge\omega_{\gamma\beta}+\frac{1}{2}\sum R_{\alpha\beta kl}\omega_k\wedge\omega_l,$$
$$R_{\alpha\beta kl}=\sum(h_{ik}^\alpha h_{il}^\beta-h_{ik}^\beta h_{il}^\alpha). \tag{2.3}$$

We call $h=\sum h_{ij}^\alpha\omega_i\omega_j e_\alpha$ the *second fundamental form of the immersed manifold M^n*. Denote by $S=\sum(h_{ij}^\alpha)^2$ the square length of h , $\xi=\frac{1}{n}\sum\mathrm{tr}H_\alpha e_\alpha$ the mean curvature vector and $H=\frac{1}{n}\sqrt{\sum(\mathrm{tr}H_\alpha)^2}$ the mean curvature of M^n respectively. Here tr is the trace of the matrix $H_\alpha=(h_{ij}^\alpha)$. Now let e_{n+1} be parallel to ξ. Then we have

$$\mathrm{tr}H_{n+1}=nH,\quad \mathrm{tr}H_\alpha=0,\alpha\neq n+1. \tag{2.4}$$

Let h_{ijk}^α and h_{ijkl}^α denote the covariant derivative and the second covariant derivative of h_{ij}^α respectively, defined by

$$\sum h^\alpha_{ijk}\omega_k = dh^\alpha_{ij} - \sum h^\alpha_{ik}\omega_{kj} - \sum h^\alpha_{jk}\omega_{ki} - \sum h^\beta_{ij}\omega_{\beta\alpha}, \tag{2.5}$$

$$\sum h^\alpha_{ijkl}\omega_l = dh^\alpha_{ijk} - \sum h^\alpha_{ijl}\omega_{lk} - \sum h^\alpha_{ilk}\omega_{lj} - \sum h^\alpha_{ljk}\omega_{li} - \sum h^\beta_{ijk}\omega_{\beta\alpha}. \tag{2.6}$$

Then we have

$$h^\alpha_{ijk} - h^\alpha_{ikj} = 0, \tag{2.7}$$

$$h^\alpha_{ijkl} - h^\alpha_{ijlk} = \sum h^\alpha_{im}R_{mjkl} + \sum h^\alpha_{mj}R_{mikl} - \sum h^\beta_{ij}R_{\alpha\beta kl}. \tag{2.8}$$

The Laplacian $\triangle h^\alpha_{ij}$ of h^α_{ij} is defined by $\triangle h^\alpha_{ij} = \sum h^\alpha_{ijkk}$. By a direct calculation we have (cf. [**2**,**3**])

$$\begin{aligned}\frac{1}{2}\triangle S &= \sum (h^\alpha_{ijk})^2 + \sum h^\alpha_{ij}\triangle h^\alpha_{ij}\\ &= \sum (h^\alpha_{ijk})^2 + \sum h^\alpha_{ij}h^\alpha_{kkij}\\ &\quad + \sum h^\alpha_{ij}h^\alpha_{mk}R_{mijk} + \sum h^\alpha_{ij}h^\alpha_{mi}R_{mkjk} - \sum h^\alpha_{ij}h^\beta_{ki}R_{\alpha\beta jk}.\end{aligned} \tag{2.9}$$

3. Proof of Theorem. From (1.1) and (2.4) we have

$$\langle h(e_i, e_j), He_{n+1}\rangle = H^2\delta_{ij}; \tag{3.1}$$

therefore

$$\sum h^\alpha_{ij}h^\alpha_{kkij} = nH\triangle H, \tag{3.2}$$

and

$$\sum (h^\alpha_{ijk})^2 \geq \sum (h^{n+1}_{iik})^2 = n\sum (\nabla_i H)^2 = n|\nabla H|^2. \tag{3.3}$$

It is obvious that

$$\frac{1}{2}\triangle H^2 = H\triangle H + |\nabla H|^2 \tag{3.4}$$

and therefore

$$\sum (h^\alpha_{ijk})^2 + \sum h^\alpha_{ij}h^\alpha_{kkij} \geq n|\nabla H|^2 + nH\triangle H = \frac{1}{2}n\triangle H^2. \tag{3.5}$$

On the other hand, from (2.2)

$$\begin{aligned}\sum h^\alpha_{ij}h^\alpha_{mk}R_{mijk} &= \frac{1}{2}\sum (h^\alpha_{ij}h^\alpha_{mk} - h^\alpha_{mj}h^\alpha_{ik})R_{mijk}\\ &= \frac{1}{2}\sum \{R_{imjk} - a(\delta_{ij}\delta_{mk} - \delta_{mj}\delta_{ik})\}R_{mijk} = -\frac{1}{2}\sum R^2_{mijk} + aR\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}\sum h^\alpha_{ij}h^\alpha_{mi}R_{mkjk} &= \sum \{(n-1)a\delta_{mj} + nH^2\delta_{mj} - R_{mj}\}R_{mj}\\ &= -\sum R^2_{mj} + (n-1)aR + nH^2R.\end{aligned} \tag{3.7}$$

From (2.3) we have

$$\sum h_{ij}^{\alpha} h_{ki}^{\beta} R_{\alpha\beta jk} = \sum h_{ij}^{\alpha} h_{lj}^{\alpha} h_{ki}^{\beta} h_{lk}^{\beta} - \sum h_{ij}^{\alpha} h_{lk}^{\alpha} h_{ik}^{\beta} h_{lj}^{\beta},$$

while

$$\begin{aligned} &\sum h_{ij}^{\alpha} h_{lj}^{\alpha} h_{ki}^{\beta} h_{lk}^{\beta} = \sum \{(n-1)a\delta_{il} + nH^2\delta_{il} - R_{il}\}^2 \\ =&n(n-1)^2a^2 - 2n^2(n-1)aH^2 - 2(n-1)aR - 2nH^2R + n^3H^4 + \sum R_{il}^2. \end{aligned} \tag{3.8}$$

Let

$$S_{\alpha} = \sum_{i,j} (h_{ij}^{\alpha})^2;$$

then we have

$$S = \sum_{\alpha} S_{\alpha}.$$

Since

$$S^2 = \left(\sum_{\alpha} S_{\alpha}\right)^2 = \sum_{\alpha} S_{\alpha}^2 + 2\sum_{\alpha<\beta} S_{\alpha} S_{\beta}$$

and

$$\sum_{\alpha<\beta} (S_{\alpha} - S_{\beta})^2 = (p-1)\sum_{\alpha} S_{\alpha}^2 - 2\sum_{\alpha<\beta} S_{\alpha} S_{\beta} \geq 0$$

it follows that

$$(p-1)\sum_{\alpha} S_{\alpha}^2 \geq 2\sum_{\alpha<\beta} S_{\alpha} S_{\beta} = S^2 - \sum_{\alpha} S_{\alpha}^2;$$

that is

$$\sum_{\alpha} S_{\alpha}^2 \geq \frac{1}{p} S^2.$$

Since

$$\begin{aligned} &\sum_{i,j,l,k} \left(\sum_{\alpha} h_{ij}^{\alpha} h_{lk}^{\alpha}\right)^2 = \sum_{i,j,l,k} \left(\sum_{\beta} h_{ik}^{\beta} h_{lj}^{\beta}\right)^2 = \sum h_{ij}^{\alpha} h_{ij}^{\beta} h_{lk}^{\alpha} h_{lk}^{\beta} \\ =&\sum_{\alpha,\beta} \left(\sum_{i,j} h_{ij}^{\alpha} h_{ij}^{\beta}\right)^2 \geq \sum_{\alpha} (\sum_{i,j} h_{ij}^{\alpha} h_{ij}^{\alpha})^2 = \sum_{\alpha} S_{\alpha}^2, \end{aligned}$$

we have

$$\begin{aligned}
&\sum h_{ij}^{\alpha} h_{lk}^{\alpha} h_{ik}^{\beta} h_{lj}^{\beta} \\
&\geq \sum h_{ij}^{\alpha} h_{lk}^{\alpha} h_{ik}^{\beta} h_{lj}^{\beta} - \frac{1}{2}\sum_{i,j,l,k}\left(\sum_{\alpha} h_{ij}^{\alpha} h_{lk}^{\alpha}\right)^2 - \frac{1}{2}\sum_{i,j,l,k}\left(\sum_{\beta} h_{ik}^{\beta} h_{lj}^{\beta}\right)^2 + \sum_{\alpha} S_{\alpha}^2 \\
&= -\frac{1}{2}\sum_{i,j,l,k}\left\{\sum_{\alpha}(h_{ij}^{\alpha} h_{lk}^{\alpha} - h_{ik}^{\alpha} h_{lj}^{\alpha})\right\}^2 + \sum_{\alpha} S_{\alpha}^2 \\
&= -\frac{1}{2}\sum \{R_{iljk} - a(\delta_{ij}\delta_{lk} - \delta_{ik}\delta_{lj})\}^2 + \sum_{\alpha} S_{\alpha}^2 \\
&\geq -\frac{1}{2}\sum R_{iljk}^2 + 2aR - n(n-1)a^2 + \frac{1}{p}S^2. \qquad (3.9)
\end{aligned}$$

From (2.9), (3.5)–(3.9) we have

$$\begin{aligned}
\frac{1}{2}\triangle S \geq \frac{1}{2}n\triangle H^2 - \sum R_{ijkl}^2 - 2\sum R_{ij}^2 + 3naR + 3nH^2R \\
- n^2(n-1)a^2 - 2n^2(n-1)aH^2 - n^3H^4 + \frac{1}{p}S^2. \qquad (3.10)
\end{aligned}$$

Since M^n is compact and

$$S = n(n-1)a + n^2H^2 - R,$$

we have

$$\begin{aligned}
\int_{M^n} \{p\sum R_{ijkl}^2 + 2p\sum R_{ij}^2 - R^2 - n(3p-2n+2)aR - n(3p-2n)H^2R \\
+2n^2(n-1)paH^2 + 2n^3(n-1)aH^2 + n^3pH^4 - n^4H^4\} * 1 \\
\geq n^2(n-1)(n-p-1)a^2\mathrm{Vol}(M^n).
\end{aligned}$$

Hence the Theorem is proved. □

REMARK. From (2.2), if M^n is totally geodesic, i.e. $S \equiv 0$, then (1.3) becomes the equality. It is interesting to study the geometrical property of M^n in this case.

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