Forum of Mathematics, Sigma (2022), **Vol. 10**:e31 1–27
doi:10.1017/fms.2022.7

CAMBRIDGE UNIVERSITY PRESS

RESEARCH ARTICLE

Point Degree Spectra of Represented Spaces

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Received: 22 October 2019; **Revised:** 25 March 2021; **Accepted:** 23 November 2021

2020 Mathematics Subject Classification: *Primary* – 03D30, 54H05; *Secondary* – 03D78, 54F45, 46J10

Abstract
We introduce the point degree spectrum of a represented space as a substructure of the Medvedev degrees, which integrates the notion of Turing degrees, enumeration degrees, continuous degrees and so on. The notion of point degree spectrum creates a connection among various areas of mathematics, including computability theory, descriptive set theory, infinite-dimensional topology and Banach space theory. Through this new connection, for instance, we construct a family of continuum many infinite-dimensional Cantor manifolds with property C whose Borel structures at an arbitrary finite rank are mutually nonisomorphic. This resolves a long-standing question by Jayne and strengthens various theorems in infinite-dimensional topology such as Pol's solution to Alexandrov's old problem.

Contents

1. Introduction

Computability Theory

In computable analysis [51, 65], there has for a long time been an interest in how complicated the set of codes of some element in a suitable space may be. Pour-El and Richards [51] observed that any real number and, more generally, any point in a Euclidean space has a Turing degree. They subsequently raised the question of whether the same holds true for any computable metric space. Miller [37] later proved that various infinite-dimensional metric spaces such as the Hilbert cube and the space of continuous functions on the unit interval contain points which lack Turing degrees; that is, have no simplest code w.r.t. Turing reducibility. A similar phenomenon was also observed in algorithmic randomness theory. Day and Miller [12] showed that no neutral measure has Turing degree by understanding each measure as a point in the infinite-dimensional space consisting of probability measures on an underlying space.

These previous works convince us of the need for a reasonable theory of degrees of unsolvability of points in an arbitrary represented space. To establish such a theory, we associate a substructure of the Medvedev degrees with a represented space, which we call its *point degree spectrum*. A wide variety of classical degree structures are realised in this way; for example, Turing degrees [61], enumeration degrees [16], continuous degrees [37] and degrees of continuous functionals [20]. What is more noteworthy is that the concept of a point degree spectrum is closely linked to infinite-dimensional topology. For instance, we shall see that for a Polish space all points have Turing degrees if and only if the small transfinite inductive dimension of the space exists.

In a broader context, there are various instances of smallness properties (i.e., σ-ideals) of spaces and sets that start making sense for points in an effective treatment; for example, arithmetical (Cohen) genericity [14, 41], Martin–Löf randomness [14] and effective Hausdorff dimension [35]. In all of these cases, individual points can carry some amount of complexity; for example, a Martin–Löf random point is in some sense too complicated to be included in a computable G_δ set having effectively measure zero. A recent important example [50, 66] from forcing theory is genericity with respect to the σ-ideal generated by finite-dimensional compact metrisable spaces. Our work provides an effective notion corresponding to topological invariants such as small inductive dimension or metrisability and, for example, allows us to say that certain points are too complicated to be (computably) a member of a (finite-dimensional) Polish space.

Additionally, the actual importance of point degree spectrum is not merely conceptual but also applicative. Indeed, unexpectedly, our notion of point degree spectrum turned out to be a powerful tool in descriptive set theory and infinite-dimensional topology, in particular in the study of restricted Borel isomorphism problems, as explained in more depth below.

Descriptive Set Theory

A *Borel isomorphism problem* (see [6, 10, 19, 62]) asks to find a nontrivial isomorphism type in a certain class of Borel spaces (i.e., topological spaces together with their Borel σ-algebras). As is well known, Kuratowski's theorem tells us that every uncountable Polish space is Borel isomorphic to the real line $\mathbb{R}$. It is lesser known that what Kuratowski really showed is that an uncountable Polish space is unique up to *ωth-level* Borel isomorphism (cf. [33, Remark (ii) in p. 451]). Here, an *αth-level Borel/Baire isomorphism* between $\mathbf{X}$ and $\mathbf{Y}$ is a bijection f such that $E \subseteq \mathbf{X}$ is of additive Borel/Baire class α (i.e., $\boldsymbol{\Sigma}^0_{1+\alpha}$) if and only if $f[E] \subseteq \mathbf{Y}$ is of additive Borel/Baire class α. These restricted Borel isomorphisms were also considered by Jayne [23] in Banach space theory, in order to obtain certain variants of the

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Banach–Stone theorem and the Gelfand–Kolmogorov theorem for Banach algebras of the forms $\mathcal{B}^*_\alpha(\mathbf{X})$ for realcompact spaces $\mathbf{X}$. Here, $\mathcal{B}^*_\alpha(\mathbf{X})$ is the Banach algebra of bounded real-valued Baire class α functions on a space $\mathbf{X}$ with respect to the supremum norm and the pointwise operation [5, 11, 23]. The first- and second-level Borel/Baire isomorphic classifications have been studied by several authors (see [24, 25]). For instance, it is proved that there are at least two second-level Borel isomorphism types of uncountable Polish spaces; that is, types of finite-dimensional Euclidean spaces $\mathbb{R}^n$ and the Hilbert cube $[0,1]^{\mathbb{N}}$. However, it is not certain even whether more than two second-level Borel isomorphism types exist:

Problem 1.1 (The Second-Level Borel Isomorphism Problem). *Are all uncountable Polish spaces second-level Borel isomorphic either to* $\mathbb{R}$ *or to* $\mathbb{R}^{\mathbb{N}}$*?*

Problem 1.1 and the nth-level analogues were recently highlighted by Motto Ros [54, Question 8.5] and Motto Ros et al. [55, Question 4.29]. As already pointed out by Motto Ros [54], Jayne's work [23] mentioned above shows that Problem 1.1 is closely related to asking the following problem on Banach algebras.

Problem 1.2. *Is the Banach space* $\mathcal{B}^*_2(\mathbf{X})$ *of the Baire class* 2 *functions on an uncountable Polish space* $\mathbf{X}$ *linearly isometric (or ring isomorphic) either to* $\mathcal{B}^*_2([0,1])$ *or to* $\mathcal{B}^*_2([0,1]^{\mathbb{N}})$*?*

The very recent successful attempts to generalise the Jayne–Rogers theorem and the Solecki dichotomy (see [54, 47] and also [17] for a computability theoretic proof) have revealed the close connection between second-level Borel isomorphism and σ-homeomorphism for Polish spaces (see Subsection 2.2.1). Here, a topological space $\mathbf{X}$ is σ-homeomorphic to $\mathbf{Y}$ (written as $\mathbf{X} \cong^{\mathfrak{T}}_{\sigma} \mathbf{Y}$) if there are a bijection $f : \mathbf{X} \to \mathbf{Y}$ and countable covers $\{\mathbf{X}_i\}_{i\in\omega}$ and $\{\mathbf{Y}_i\}_{i\in\omega}$ of $\mathbf{X}$ and $\mathbf{Y}$ such that $f \upharpoonright \mathbf{X}_i$ gives a homeomorphism between $\mathbf{X}_i$ and $\mathbf{Y}_i$ for every $i \in \omega$.

Therefore, the second-level Borel isomorphism problem is closely related to the following problem.

Problem 1.3 (Motto Ros et al. [55]). *Is any Polish space* $\mathbf{X}$ *either* σ*-embedded into* $\mathbb{R}$ *or* σ*-homeomorphic to* $\mathbb{R}^{\mathbb{N}}$*?*

Unlike the classical Borel isomorphism problem, which is reducible to the same problem on zero-dimensional Souslin spaces, the second-level Borel isomorphism problem is inescapably tied to *infinite-dimensional topology* [64], since all transfinite-dimensional uncountable Polish spaces are mutually second-level Borel isomorphic.

The study of σ-homeomorphic maps in topological dimension theory dates back to a classical work by Hurewicz–Wallman [22] characterising transfinite dimensionality. Alexandrov [2] asked whether there exists a weakly infinite-dimensional compactum which is not σ-homeomorphic to the real line. Roman Pol [49] solved this problem by constructing such a compactum. Roman Pol's compactum is known to satisfy a slightly stronger covering property, called property C [1].

Our notion of *degree spectrum* on Polish spaces serves as an invariant under second-level Borel isomorphism. Indeed, an invariant which we call *degree co-spectrum*, a collection of Turing ideals realised as lower Turing cones of points of a Polish space, plays a key role in solving the second-level Borel isomorphism problem. By utilising these computability-theoretic concepts, we will construct a continuum many pairwise incomparable σ-homeomorphism types of compact metrisable C-spaces; that is:

> There is a collection $(\mathbf{X}_\alpha)_{\alpha<2^{\aleph_0}}$ of continuum many compact metrisable C-spaces such that, whenever $\alpha \neq \beta$, $\mathbf{X}_\alpha$ cannot be written as a countable union of homeomorphic copies of subspaces of $\mathbf{X}_\beta$.

This also shows that there are continuum many second-level Borel isomorphism types of compact metric spaces. More generally, a *finite-level Borel embedding of* $\mathbf{X}$ *into* $\mathbf{Y}$ is an nth-level Borel isomorphism between $\mathbf{X}$ and a subset of $\mathbf{Y}$ of finite Borel rank for some $n \in \mathbb{N}$. Then, our result entails the following as a corollary:

> There is a collection $(\mathbf{X}_\alpha)_{\alpha<2^{\aleph_0}}$ of continuum many compact metrisable C-spaces such that, whenever $\alpha \neq \beta$, $\mathbf{X}_\alpha$ cannot be finite-level Borel embedded into $\mathbf{X}_\beta$.

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The key idea is measuring the quantity of all possible Scott ideals realised within the degree co-spectrum of a given space. Our spaces are completely described in the terminology of computability theory (based on Miller's work on the continuous degrees [37]). Nevertheless, the first of our examples turns out to be second-level Borel isomorphic to (the sum of countably many copies of) Roman Pol's compactum (but, of course, our remaining continuum many examples cannot be second-level Borel isomorphic to Pol's compactum). Hence, our solution can also be viewed as a refinement of Roman Pol's solution to Alexandrov's problem.

Summary of Results

In Section 3, we introduce the notion of point degree spectrum and clarify the relationship with σ-continuity. In Section 4, we introduce the notion of an ω-left-computably enumerable in-and-above (CEA) operator (see Section 4.2) in the Hilbert cube as an infinite-dimensional analogue of an ω-CEA operator (in the sense of classical computability theory) and show that the graph of a universal ω-left-CEA operator is an individual counterexample to Problems 1.1, 1.2 and 1.3. In Section 5, we describe a general procedure to construct uncountably many mutually different compacta under σ-homeomorphism. In Section 6, we clarify the relationship between a universal ω-left-CEA operator and Roman Pol's compactum.

Future work

The methods introduced in this article, in particular the notion of the point degree spectrum and the associated connection between topology and computability theory (recursion theory), have already inspired and enabled several other studies. Some additional results are found in the extended arXiv version [31]. In [17], Gregoriades, Kihara and Ng made significant progress on the decomposability conjecture from descriptive set theory. A core aspect of this work is whether certain degree-theoretic results like the Shore–Slaman join theorem and the cone avoidance theorem for Π^0_1 classes hold for the point degree spectra of Polish spaces.

Building upon our work, Andrews et al. [3] used an effective metrisation argument to show that the point degree spectrum of the Hilbert cube coincides with the almost total enumeration degrees, which in turn is used to show the purely computability-theoretic consequence that *PA above* is definable in the enumeration degrees. Our idea was also utilised by Kihara [29] to explain the relationship between non-total continuous degrees and PA degrees in the context of reverse mathematics.

Kihara, Ng and Pauly [30] have embarked on the systematic endeavour to classify the point degree spectra of second-countable spaces from *Counterexample in Topology* [63]. This has already proven to be a rich source for the fine-grained study of the enumeration degrees, as both previously studied substructures as well as new ones of interest to computability theorists appear in this fashion. Kihara explored the truth-table reducibility variant of our generalised Turing degrees in [28].

Based on the results both in the present article and in the extension mentioned here, we are confident that both directions of the link between topology and computability theory established here have significant potential for applications. This work can also be considered as part of a general development to study the descriptive theory of represented spaces [43], together with approaches such as synthetic descriptive set theory proposed in [45, 46].

2. Preliminaries

2.1. Computability Theory

2.1.1. Basic Notations

We use the standard notations from modern computability theory and computable analysis. We refer the reader to [41, 42, 61] for the basics on computability theory and to [51, 65, 44] for the basics on computable analysis.

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By $f :\subseteq X \to Y$, we mean a function from a subset of X into Y. Such a function is called a partial function. We fix a pairing function $(m, n) \mapsto \langle m, n\rangle$, which is a computable bijection from $\mathbb{N}^2$ onto $\mathbb{N}$ such that $\langle m, n\rangle \mapsto m$ and $\langle m, n\rangle \mapsto n$ are also computable. For $x, y \in \mathbb{N}^{\mathbb{N}}$, the join $x \oplus y \in \mathbb{N}^{\mathbb{N}}$ is defined by $(x \oplus y)(2n) = x(n)$ and $(x \oplus y)(2n + 1) = y(n)$. An oracle is an element of $\{0, 1\}^{\mathbb{N}}$ or $\mathbb{N}^{\mathbb{N}}$. By the notation Φ^z_e we denote the computation of the eth Turing machine with oracle z. We often view Φ^z_e as a partial function on $\{0, 1\}^{\mathbb{N}}$ or $\mathbb{N}^{\mathbb{N}}$. More precisely, $\Phi^z_e(x) = y$ if and only if given an input $n \in \mathbb{N}$ with oracle $x \oplus z$, the eth Turing machine computation halts and outputs $y(n)$. The terminology 'c.e.' stands for 'computably enumerable.' For an oracle z, by 'z-computable' and 'z-c.e.', we mean 'computable relative to z' and 'c.e. relative to z'. For an oracle x, we write x' for the Turing jump of x; that is, the halting problem relative to x. Generally, for a computable ordinal α, we use $x^{(\alpha)}$ to denote the αth Turing jump of x. Here, regarding the basics on computable ordinals and transfinite Turing jumps, see [9, 58].

One of the most fundamental observations in computable analysis is that a partial function on the space $\{0, 1\}^{\mathbb{N}}$ or $\mathbb{N}^{\mathbb{N}}$ (topologised as the product of the discrete space $\{0, 1\}$ or $\mathbb{N}$) is continuous if and only if it is computable relative to an oracle (cf. [65]). This fundamental 'relativisation argument' will be repeatedly utilised.

We will also use the following fact, known as the *Kleene recursion theorem* or the *Kleene fixed point theorem*.

Fact 2.1 (The Kleene Recursion Theorem; see [41, Theorem II.2.10]). *Given a computable function $f : \mathbb{N} \to \mathbb{N}$, one can effectively find an index $e \in \mathbb{N}$ such that for all oracles $z \in \{0, 1\}^{\mathbb{N}}$ the partial functions Φ^z_e and $\Phi^z_{f(e)}$ are identical.*

2.1.2. Represented spaces

A *represented space* is a pair $\mathbf{X} = (X, \delta_X)$ of a set X and a partial surjection $\delta_X :\subseteq \mathbb{N}^{\mathbb{N}} \to X$. Informally speaking, δ_X (called a representation) gives names of elements in X by using infinite words. It enables tracking of a function f on abstract sets by a function on infinite words (called a realiser of f). This is crucial for introducing the notion of computability on abstract sets because we already have the notion of computability on infinite words.

Formally, a δ_X*-name* or simply a *name* of $x \in \mathbf{X}$ is any $p \in \mathbb{N}^{\mathbb{N}}$ such that $\delta_X(p) = x$. A function between represented spaces is a function between the underlying sets. For $f : \mathbf{X} \to \mathbf{Y}$ and $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$, we call F a *realiser* of f, iff $\delta_Y(F(p)) = f(\delta_X(p))$ for all $p \in \operatorname{dom}(f\delta_X)$; that is, if the following diagram commutes:

$$\begin{array}{ccc} \mathbb{N}^{\mathbb{N}} & \xrightarrow{F} & \mathbb{N}^{\mathbb{N}} \\ {\scriptstyle\delta_{\mathbf{X}}}\downarrow & & \downarrow{\scriptstyle\delta_{\mathbf{Y}}} \\ \mathbf{X} & \xrightarrow{f} & \mathbf{Y} \end{array}$$

A map between represented spaces is called *computable* (*continuous*), iff it has a computable (continuous) realiser. In other words, a function f is computable (continuous) iff there is a computable (continuous) function F on infinite words such that, given a name p of a point x, $F(p)$ returns a name of $f(x)$. We also use the same notation Φ^z_e to denote a function on represented spaces realised by the eth partial z-computable function. Similarly, we call a point $x \in \mathbf{X}$ *computable*, iff there is some computable $p \in \mathbb{N}^{\mathbb{N}}$ with $\delta_X(p) = x$; that is, x has a computable name. In this way, we think of a represented space as a kind of space equipped with the notion of computability.

If a set X is already topologised, the above notion of continuity (= relative computability, by the fundamental 'relativisation argument' mentioned in Subsection 2.1.1) can be inconsistent with topological continuity. To eliminate such an undesired situation, we shall consider a restricted class of representations which are consistent with a given topological structure, so-called *admissible representations*. An admissible representation is a partial continuous surjection $\delta :\subseteq \mathbb{N}^{\mathbb{N}} \to X$ such that for any partial continuous function $f :\subseteq \mathbb{N}^{\mathbb{N}} \to X$ there is a partial continuous function θ on $\mathbb{N}^{\mathbb{N}}$ such that $f = \delta \circ \theta$. We will not go into the details of admissibility here but just mention that if a T_0-space has a countable cs-network (a.k.a. a countable sequential pseudo-base), then it always has an admissible representation (see Schröder [59]). Hence, in this article, we can assume that the 'relativisation argument' always works.

A particularly relevant subclass of represented spaces are the computable Polish spaces, which are derived from complete computable metric spaces by forgetting the details of the metric and just retaining the representation (or, rather, the equivalence class of representations under computable translations). Forgetting the metric is relevant when it comes to compatibility with definitions in effective descriptive set theory as shown in [18].

Example 2.2. The following are examples of admissible representations:

1. The representation of $\mathbb{N}$ is given by $\delta_{\mathbb{N}}(0^n10^{\mathbb{N}}) = n$. It is straightforward to verify that the computability notion for the represented space $\mathbb{N}$ coincides with classical computability over the natural numbers.
2. A *computable metric space* is a tuple $\mathbf{M} = (M, d, (a_n)_{n\in\mathbb{N}})$ such that (M, d) is a metric space and $(a_n)_{n\in\mathbb{N}}$ is a dense sequence in (M, d) such that the relation
$$\{(t, u, v, w) \mid \nu_{\mathbb{Q}}(t) < d(a_u, a_v) < \nu_{\mathbb{Q}}(w)\}$$
is recursively enumerable, where $\nu_{\mathbb{Q}}$ is the standard numbering of the rationals. The *Cauchy representation* $\delta_{\mathbf{M}} :\subseteq \mathbb{N}^{\mathbb{N}} \to M$ associated with the computable metric space $\mathbf{M} = (M, d, (a_n)_{n\in\mathbb{N}})$ is defined by
$$\delta_{\mathbf{M}}(p) = x \;:\Longleftrightarrow \begin{cases} d(a_{p(i)}, a_{p(k)}) \le 2^{-i} \text{ for } i < k \\ \text{and } x = \lim_{i\to\infty} a_{p(i)}. \end{cases}$$
3. Another, more general, subclass is the quasi-Polish spaces introduced by de Brecht [13]. A space X is *quasi-Polish* if it is countably based and has a total admissible representation $\delta_{\mathbf{X}} : \mathbb{N}^{\mathbb{N}} \to X$. These include the computable Polish spaces as well as ω-continuous domains.
4. Generally, a topological T_0-space $\mathbf{X}$ with a countable base $\mathcal{B} = \langle B_n \rangle_{n\in\mathbb{N}}$ is naturally represented by defining $\delta_{(\mathbf{X},\mathcal{B})}(p) = x$ iff p enumerates the code of a neighbourhood basis for x; that is, $\mathrm{range}(p) = \{n \in \mathbb{N} : x \in B_n\}$. One can also use a network to give a representation of a space as suggested above.

We always assume that $\{0,1\}^{\mathbb{N}}$, $\mathbb{R}^n$ and $[0,1]^{\mathbb{N}}$ are admissibly represented by the Cauchy representations obtained from their standard metrics.

A real $x \in \mathbb{R}$ is *left-c.e.* if there is a computable sequence $(q_n)_{n\in\mathbb{N}}$ of rationals such that $x = \sup_n q_n$ (cf. [14]). Generally, a real $x \in \mathbb{R}$ is *left-c.e. relative to* $y \in \mathbf{X}$ if there is a partial computable function $f :\subseteq \mathbf{X} \to \mathbb{Q}^{\mathbb{N}}$ such that $x = \sup_n f(y)(n)$. If $(M, d, (a_n)_{n\in\mathbb{N}})$ is a computable metric space, there is a computable list $(B_e)_{e\in\mathbb{N}}$ of open balls of the form $B(a_n; q)$, where $B(a_n; q)$ is the open ball of radius q centred at a_n. We say that a set U is *c.e. open* if there is a c.e. set $W \subseteq \mathbb{N}$ such that $U = \bigcup_{e\in W} B_e$. The complement of a c.e. open set is called Π^0_1. By $\Pi^0_1(z)$, we mean Π^0_1 relative to an oracle z, whose complement is defined using a z-c.e. set W instead of a c.e. set.

2.1.3. Degree structures

The Medvedev degrees $\mathfrak{M}$ [36] are a cornerstone of our framework. These are obtained by taking equivalence classes from Medvedev reducibility $\le_M$, defined on subsets A, B of Baire space $\mathbb{N}^{\mathbb{N}}$ via $A \le_M B$ iff there is a partial computable function $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that $B \subseteq \mathrm{dom}(F)$ and $F[B] \subseteq A$. Important substructures of $\mathfrak{M}$ also relevant to us are the Turing degrees $\mathcal{D}_T$, the continuous degrees $\mathcal{D}_r$ and the enumeration degrees $\mathcal{D}_e$; these satisfy $\mathcal{D}_T \subsetneq \mathcal{D}_r \subsetneq \mathcal{D}_e \subsetneq \mathfrak{M}$.

Turing degrees are obtained from the usual Turing reducibility $\le_{\mathrm{T}}$ defined on points $p, q \in \mathbb{N}^{\mathbb{N}}$ with $p \le_{\mathrm{T}} q$ iff there is a computable function $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ with $F(q) = p$. We thus see $p \le_{\mathrm{T}} q \Leftrightarrow \{p\} \le_M \{q\}$ and can indeed understand the Turing degrees to be a subset of the Medvedev

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degrees. The continuous degrees were introduced by Miller in [37]. Enumeration degrees have received a lot of attention in computability theory and were originally introduced by Friedberg and Rogers [16] (see also [42, Chapter XIV]). In both cases, we can provide a simple definition directly as a substructure of the Medvedev degrees later on.

A further reducibility notion is relevant, although we are not particularly interested in its degree structure. This is Muchnik reducibility $\leq_w$ [40], defined again for sets $A, B \subseteq \mathbb{N}^{\mathbb{N}}$ via $A \leq_w B$ iff, for any $p \in B$, there is $q \in A$ such that $q \leq_T p$. Clearly, $A \leq_M B$ implies $A \leq_w B$, but the converse is false in general.

2.2. Topology and Dimension

2.2.1. Isomorphism and Classification

We are now interested in isomorphisms of a particular kind; this always means a bijection in that function class, such that the inverse is also in that function class. For instance, consider the following morphisms. For a function $f : \mathbf{X} \to \mathbf{Y}$,

1. f is *σ-computable* (*σ-continuous*, respectively) if there are sets $(X_n)_{n\in\mathbb{N}}$ such that $\mathbf{X} = \bigcup_{n\in\mathbb{N}} X_n$ and each $f|_{X_n}$ is computable (continuous, respectively)
2. f is *$\mathbf{\Gamma}$-piecewise continuous* if there are $\mathbf{\Gamma}$-sets $(X_n)_{n\in\mathbb{N}}$ such that $\mathbf{X} = \bigcup_{n\in\mathbb{N}} X_n$ and each $f|_{X_n}$ is continuous.
3. f is *nth-level Borel measurable* if $f^{-1}[A]$ is $\mathbf{\Sigma}^0_{n+1}$ for every $\mathbf{\Sigma}^0_{n+1}$ set $A \subseteq \mathbf{Y}$.

In particular, f is *second-level Borel measurable* iff $f^{-1}[A]$ is $G_{\delta\sigma}$ for every $G_{\delta\sigma}$ set $A \subseteq \mathbf{Y}$. We also say that f is *finite-level Borel measurable* if it is nth-level Borel measurable for some $n \in \mathbb{N}$. Note that σ-continuity is also known as *countable continuity*. A *σ-homeomorphism* is a bijection $f : \mathbf{X} \to \mathbf{Y}$ such that both f and f^{-1} are σ-continuous. Similarly, a *σ-embedding* of $\mathbf{X}$ into $\mathbf{Y}$ is a σ-homeomorphism between $\mathbf{X}$ and a subspace of $\mathbf{Y}$.

Remark 2.3. Note that if $f : \mathbf{X} \to \mathbf{Y}$ is a σ-homeomorphism, then f is a countable union of partial homeomorphisms: By definition, we can write f as the union of continuous injections $f_i : X_i \to \mathbf{Y}$ and, similarly, f^{-1} as the union of $g_j : Y_j \to \mathbf{X}$. Then, the restriction f_{ij} of f_i up to $X_i \cap g_j[Y_j]$ is a homeomorphism between $X_i \cap g_j[Y_j]$ and $f_i[X_i] \cap Y_j$. Clearly, f is the union of f_{ij}s. It is clear that the converse is also true.

By recent results from descriptive set theory (cf. [17, 27, 54, 47]), we have the following implication for functions on Polish spaces:

$$\begin{aligned} \mathbf{\Sigma}^0_{n+1}\text{-piecewise continuous} &\iff \mathbf{\Pi}^0_n\text{-piecewise continuous} \\ &\implies n\text{th-level Borel measurable} \implies \sigma\text{-continuous.} \end{aligned}$$

The last implication was recently proved by [54, 47] and, more recently, an alternative computability theoretic proof was given by [17] using our framework of point degree spectra of Polish spaces.

Observation 2.4. *Let $\mathbf{X}$ and $\mathbf{Y}$ be Polish spaces. Then, $\mathbb{N} \times \mathbf{X}$ and $\mathbb{N} \times \mathbf{Y}$ are σ-homeomorphic if and only if $\mathbb{N} \times \mathbf{X}$ and $\mathbb{N} \times \mathbf{Y}$ are second-level Borel isomorphic.*

Proof. For the ‘if’ direction, assume that $\mathbf{X}$ and $\mathbf{Y}$ are second-level Borel isomorphic; that is, there is a bijection $f : \mathbf{X} \to \mathbf{Y}$ such that both f and f^{-1} are second-level Borel measurable. From the above argument, both f and f^{-1} are σ-continuous and, therefore, f is a σ-homeomorphism.

To show the ‘only if’ direction, recall (from Remark 2.3) that a σ-homeomorphism of $\mathbf{X}$ into $\mathbf{Y}$ is a countable union of partial homeomorphisms. Then, note that, by the Lavrentiev theorem (cf. [26, Theorem 3.9]), every homeomorphism between subsets of Polish spaces can be extended to a homeomorphism between G_δ sets. Therefore, we have homeomorphisms h_n between G_δ sets $X_n \subseteq \mathbf{X}$ and $Y_n \subseteq \mathbf{Y}$ such that $\bigcup_n X_n = \mathbf{X}$ and $\bigcup_n Y_n = \mathbf{Y}$. Then, by defining $h^*_n : X_n \setminus \bigcup_{m<n} X_m \to \{n\} \times \mathbf{Y}$

with $h_n^*(x) = (n, h_n(x))$, we obtain a $\mathbf{\Delta}^0_3$-piecewise embedding of $\mathbf{X}$ into $\mathbb{N} \times \mathbf{Y}$ whose image is $\mathbf{\Delta}^0_3$. Hence, whenever Polish spaces $\mathbf{X}$ and $\mathbf{Y}$ are σ-homeomorphic, we get second-level Borel embeddings $f : \mathbf{X} \to \mathbb{N} \times \mathbf{Y}$ and $g : \mathbf{Y} \to \mathbb{N} \times \mathbf{X}$ with $\mathbf{\Delta}^0_3$ images. Then, using a finer version (see [25, Lemma 5.2]) of the Cantor–Bernstein argument, one can construct a second-level Borel isomorphism between $\mathbb{N} \times \mathbf{X}$ and $\mathbb{N} \times \mathbf{Y}$. This verifies our assertion since $\mathbb{N} \simeq \mathbb{N}^2$. □

Consequently, the second-level Borel isomorphic classification and the σ-homeomorphic classification of Polish spaces are almost the same. Hence, three classification problems, Problems 1.1, 1.2 and 1.3 in Section 1, are almost equivalent.

Hereafter, for notation, let $\cong$ be computable isomorphism, $\cong^{\mathfrak{T}}$ continuous isomorphism (i.e., homeomorphism), $\cong_\sigma$ isomorphism by σ-computable functions and $\cong^{\mathfrak{T}}_\sigma$ σ-continuous isomorphism (i.e., σ-homeomorphism).

For any of these notions, we write $\mathbf{X} \leq \mathbf{Y}$ with the same decorations on $\leq$ if $\mathbf{X}$ is isomorphic to a subspace of $\mathbf{Y}$ (i.e., $\mathbf{X}$ is embedded into $\mathbf{Y}$) in that way. If $\mathbf{X} \leq \mathbf{Y}$ holds, but $\mathbf{Y} \leq \mathbf{X}$ does not, then we also write $\mathbf{X} < \mathbf{Y}$, again with the suitable decorations on $<$. If neither $\mathbf{X} \leq \mathbf{Y}$ nor $\mathbf{Y} \leq \mathbf{X}$, we write $\mathbf{X} \mid \mathbf{Y}$ (again, with the same decorations). Again, the Cantor–Bernstein argument shows the following.

Observation 2.5. *Let $\mathbf{X}$ and $\mathbf{Y}$ be represented spaces. Then, $\mathbf{X} \cong_\sigma \mathbf{Y}$ if and only if $\mathbf{X} \leq_\sigma \mathbf{Y}$ and $\mathbf{Y} \leq_\sigma \mathbf{X}$.*

2.2.2. Topological Dimension theory

As a general source for topological dimension theory, we point to Engelking [15]. See also van Mill [64] for infinite-dimensional topology. A topological space $\mathbf{X}$ is *countable dimensional* if it can be written as a countable union of finite-dimensional subspaces. Recall that a Polish space is countable dimensional if and only if it is *transfinite dimensional*; that is, its transfinite small inductive dimension is less than ω_1 (see [22, pp. 50–51]). One can see that a Polish space $\mathbf{X}$ is countable dimensional if and only if $\mathbf{X} \leq^{\mathfrak{T}}_\sigma \{0,1\}^{\mathbb{N}}$.

To investigate the structure of uncountable dimensional spaces, Alexandrov introduced the notion of weakly/strongly infinite-dimensional space. We say that C is a *separator* (usually called a *partition* in dimension theory) of a pair (A, B) in a space $\mathbf{X}$ if there are two pairwise disjoint open sets $A' \supseteq A$ and $B' \supseteq B$ such that $A' \sqcup B' = \mathbf{X} \setminus C$. A family $\{(A_i, B_i)\}_{i\in\Lambda}$ of pairwise disjoint closed sets in $\mathbf{X}$ is *essential* if whenever C_i is a separator of (A_i, B_i) in $\mathbf{X}$ for every $i \in \mathbb{N}$, $\bigcap_{i\in\mathbb{N}} C_i$ is nonempty. An infinite-dimensional space X is said to be *strongly infinite-dimensional* if it has an essential family of infinite length. Otherwise, X is said to be *weakly infinite-dimensional*.

We also consider the following covering property for topological spaces. Let $\mathcal{O}[\mathbf{X}]$ be the collection of all open covers of a topological space $\mathbf{X}$ and $\mathcal{O}_2[\mathbf{X}] = \{\mathcal{U} \in \mathcal{O}[X] : |\mathcal{U}| = 2\}$; that is, the collection of all covers by two open sets. For $\mathcal{A}, \mathcal{B} \in \{\mathcal{O}_2, \mathcal{O}\}$, we write $\mathbf{X} \in \mathcal{S}_c(\mathcal{A}, \mathcal{B})$ if and only if for any sequence $(\mathcal{U}_n)_{n\in\mathbb{N}} \in \mathcal{A}[\mathbf{X}]^{\mathbb{N}}$, there is a sequence $(\mathcal{V}_n)_{n\in\mathbb{N}}$ of pairwise disjoint open sets such that $\mathcal{V}_n$ refines $\mathcal{U}_n$ for each $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n \in \mathcal{B}[\mathbf{X}]$.

Note that a topological space $\mathbf{X}$ is weakly infinite-dimensional if and only if $\mathbf{X} \in \mathcal{S}_c(\mathcal{O}_2, \mathcal{O})$. We say that $\mathbf{X}$ is a *C-space* [1] or *selectively screenable* [4] if $\mathbf{X} \in \mathcal{S}_c(\mathcal{O}, \mathcal{O})$. For a topological property $\mathcal{P}$, we say that $\mathbf{X}$ is *hereditarily $\mathcal{P}$* if every subspace of $\mathbf{X}$ is $\mathcal{P}$. We have the following implications:

$$\text{countable-dimensional} \;\Rightarrow\; C\text{-space} \;\Rightarrow\; \text{weakly infinite-dimensional}.$$

Alexandrov's old problem was whether there exists a weakly infinite-dimensional compactum which is not countable dimensional; that is, $\mathbf{X} >^{\mathfrak{T}}_\sigma \{0,1\}^{\mathbb{N}}$. This problem was solved by R. Pol [49] by constructing a compact metrisable space of the form $R \cup L$ for a strongly infinite-dimensional totally disconnected subspace R and a countable dimensional subspace L. Such a compactum is a C-space but not countable dimensional. Namely, R. Pol's theorem says that there are at least two σ-homeomorphism types of compact metrisable C-spaces.

There are previous studies on the structure of continuous isomorphism types (Fréchet dimension types) of various kinds of infinite-dimensional compacta; for example, strongly infinite-dimensional

Cantor manifolds (see [7, 8]). For instance, by combining the Baire category theorem and the result by Chatyrko-Pol [8], one can show that there are continuum many *first-level* Borel isomorphism types of strongly infinite-dimensional Cantor manifolds. However, there is an enormous gap between first and second level and, hence, such an argument never tells us anything about second-level Borel isomorphism types. Concerning weakly infinite-dimensional Cantor manifolds, Elżbieta Pol [48] (see also [7]) constructed a compact metrisable C-space in which no separator of nonempty subspaces can be hereditarily weakly infinite-dimensional. We call such a space a *Pol-type Cantor manifold*.

3. Point Degree Spectra

3.1. Generalised Turing Reducibility

Recall that the notion of a represented space involves the notion of computability. More precisely, every point in a represented space is coded by an infinite word, called a *name*. Then, we estimate how complicated a given point is by considering the *degree of difficulty of calling a name* of the point. Of course, it is possible for each point to have many names, and this feature yields the phenomenon that there is a point with no easiest names with respect to Turing degree.

Formally, we associate analogies of Turing reducibility and Turing degrees with an arbitrary represented space in the following manner.

Definition 3.1. Let $\mathbf{X}$ and $\mathbf{Y}$ be represented spaces. We say that $y \in \mathbf{Y}$ is *point-Turing reducible* to $x \in \mathbf{X}$ if there is a partial computable function $f :\subseteq \mathbf{X} \to \mathbf{Y}$ such that $f(x) = y$. In other words, the set $\delta_Y^{-1}(y)$ of names of y is Medvedev reducible to the set $\delta_X^{-1}(x)$ of names of x. In this case, we write $y^{\mathbf{Y}} \leq_{\mathrm{T}} x^{\mathbf{X}}$, or simply $y \leq_{\mathrm{T}} x$.

Roughly speaking, by the condition $y \leq_{\mathrm{T}} x$ we mean that if one knows a name of x, one can compute a name of y, in a uniform manner. This pre-ordering relation $\leq_{\mathrm{T}}$ clearly yields an equivalence relation $\equiv_{\mathrm{T}}$ on points $x^{\mathbf{X}}$ of represented spaces, and we then call each equivalence class $[x^{\mathbf{X}}]_{\equiv_{\mathrm{T}}}$ the *point-Turing degree* of $x \in \mathbf{X}$, denoted by $\deg(x^{\mathbf{X}})$. In other words,

$$\deg(x^{\mathbf{X}}) = [\delta_X^{-1}(x)]_{\equiv_M} = \text{‘the Medvedev degree of the set of all } \delta_X\text{-names of } x\text{’}.$$

Then, we introduce the notion of point degree spectrum of a represented space as follows.

Definition 3.2. For a represented space $\mathbf{X}$, define

$$\mathrm{Spec}(\mathbf{X}) = \{\deg(x^{\mathbf{X}}) \mid x \in \mathbf{X}\} \subseteq \mathfrak{M}.$$

We call $\mathrm{Spec}(\mathbf{X})$ the *point degree spectrum* of $\mathbf{X}$. Given an oracle p, we also define the p-relativised point degree spectrum by replacing a partial computable function in Definition 3.1 with a partial p-computable function. Equivalently, define $\deg^p(x^{\mathbf{X}}) = [\{p\} \times \delta_{\mathbf{X}}^{-1}(x)]_{\equiv_M}$ and $\mathrm{Spec}^p(\mathbf{X}) = \{\deg^p(x^{\mathbf{X}}) : x \in \mathbf{X}\}$.

Clearly, one can identify the Turing degrees $\mathcal{D}_T$, the continuous degrees $\mathcal{D}_r$ and the enumeration degrees $\mathcal{D}_e$ with degree spectra of some spaces as follows:

- ◦ $\mathrm{Spec}(\{0,1\}^{\mathbb{N}}) = \mathrm{Spec}(\mathbb{N}^{\mathbb{N}}) = \mathrm{Spec}(\mathbb{R}) = \mathcal{D}_T$,
- ◦ (Miller [37]) $\mathrm{Spec}([0,1]^{\mathbb{N}}) = \mathrm{Spec}(\mathcal{C}([0,1],[0,1])) = \mathcal{D}_r$,
- ◦ $\mathrm{Spec}(\mathcal{O}(\mathbb{N})) = \mathcal{D}_e$, where $\mathcal{O}(\mathbb{N})$ is the space of all subsets of $\mathbb{N}$ where a basic open set is the set of all supersets of a finite subset of $\mathbb{N}$. Note that $\mathcal{O}(\mathbb{N})$ is (computably) homeomorphic to $\mathbb{S}^{\mathbb{N}}$, where $\mathbb{S}$ is the Sierpiński space.

As any separable metric space embeds into the Hilbert cube $[0,1]^{\mathbb{N}}$, we find in particular that $\mathrm{Spec}(\mathbf{X}) \subseteq \mathcal{D}_r$ for any computable metric space $\mathbf{X}$. As any second-countable T_0 space embeds into the Scott domain $\mathcal{O}(\mathbb{N})$, we also have that $\mathrm{Spec}(\mathbf{X}) \subseteq \mathcal{D}_e$ for any computable second-countable T_0 space $\mathbf{X}$. In the latter case, the point degree of $x \in \mathbf{X}$ corresponds to the enumeration degree of neighbourhood basis as in Example 2.2 (4). The Turing degrees will be characterised in Subsection 3.2 in the context of topological dimension theory.

In computable model theory, the *degree spectrum* of a countable structure S is defined as the collection of Turing degrees of isomorphic copies of S coded in $\mathbb{N}$ (see [21, 53]). The notion of *degree spectrum on a cone* (i.e., degree spectrum relative to an oracle) also plays an important role in (computable) model theory (see [38, 39]). One can define the space of countable structures as done in invariant descriptive set theory; however, from this perspective, a countable structure is a point and, therefore, the degree spectrum of a structure corresponds to the degree spectrum of a point rather than that of a space.

Given a point $x \in \mathbf{X}$, we define $\mathrm{Spec}(x^{\mathbf{X}})$ as the set of all oracles $z \in \{0,1\}^{\mathbb{N}}$ which can compute a name of x and $\mathrm{Spec}^p(x^{\mathbf{X}})$ as its relativisation by an oracle $p \in \{0,1\}^{\mathbb{N}}$. Then, the *weak point degree spectrum* $\mathrm{Spec}_w(\mathbf{X})$ is the collection of all degree spectra of points of $x \in \mathbf{X}$ and $\mathrm{Spec}^p_w(\mathbf{X})$ is its relativisation by an oracle p; that is,

$$\mathrm{Spec}(x^{\mathbf{X}}) = \{z \in \{0,1\}^{\mathbb{N}} : x \leq_{\mathrm{T}} z\}, \qquad \mathrm{Spec}^p(x^{\mathbf{X}}) = \{z \in \{0,1\}^{\mathbb{N}} : x \leq_{\mathrm{T}} (z,p)\},$$
$$\mathrm{Spec}_w(\mathbf{X}) = \{\mathrm{Spec}(x^{\mathbf{X}}) : x \in \mathbf{X}\}, \qquad \mathrm{Spec}^p_w(\mathbf{X}) = \{\mathrm{Spec}^p(x^{\mathbf{X}}) : x \in \mathbf{X}\}.$$

Note that this notion can be described in terms of Muchnik reducibility [40]; that is, we can think of the degree spectrum of $x \in \mathbf{X}$ as

$$\mathrm{Spec}(x^{\mathbf{X}}) \approx [\delta_X^{-1}(x)]_{\equiv_w} = \text{'the Muchnik degree of the set of all } \delta_X\text{-names of } x\text{'}.$$

Observation 3.3. *If* $\mathbf{X}$ *and* $\mathbf{Y}$ *are admissibly represented second-countable* T_0*-spaces, then there is an oracle* p *such that for all* $q \geq_T p$*,*

$$\mathrm{Spec}^q(\mathbf{X}) \subseteq \mathrm{Spec}^q(\mathbf{Y}) \iff \mathrm{Spec}^q_w(\mathbf{X}) \subseteq \mathrm{Spec}^q_w(\mathbf{Y}).$$

Proof. It is known that enumeration reducibility coincides with its nonuniform version (see [60] or [37, Theorem 4.2]); that is, for $A, B \subseteq \mathbb{N}$, the condition $A \leq_e B$ is equivalent to the following: For any $Z \in \{0,1\}^{\mathbb{N}}$, if B is Z-c.e., then A is also Z-c.e. In our terminology, for $\mathbf{C} = \{0,1\}^{\mathbb{N}}$ and $\mathbf{D} = \mathcal{O}(\mathbb{N})$, the abovementioned equivalence says that for any $a, b \in \mathbf{D}$,

$$\begin{aligned} a^{\mathbf{D}} \leq_{\mathrm{T}} b^{\mathbf{D}} &\iff (\forall z \in \mathbf{C})\ [b^{\mathbf{D}} \leq_{\mathrm{T}} z^{\mathbf{C}} \implies a^{\mathbf{D}} \leq_{\mathrm{T}} z^{\mathbf{C}}] \\ &\iff \mathrm{Spec}(b^{\mathbf{D}}) \subseteq \mathrm{Spec}(a^{\mathbf{D}}). \end{aligned}$$

In particular, $a^{\mathbf{D}} \equiv_{\mathrm{T}} b^{\mathbf{D}}$ if and only if $\mathrm{Spec}(b^{\mathbf{D}}) = \mathrm{Spec}(a^{\mathbf{D}})$. Let $\mathbf{X}$ and $\mathbf{Y}$ be subspaces of $\mathbf{D}$. Note that $\mathrm{Spec}(\mathbf{X}) \subseteq \mathrm{Spec}(\mathbf{Y})$ iff for any $x \in \mathbf{X}$ there is $y \in \mathbf{Y}$ such that $x^{\mathbf{X}} \equiv_{\mathrm{T}} y^{\mathbf{Y}}$. Since $x^{\mathbf{X}} \equiv_{\mathrm{T}} y^{\mathbf{Y}}$ is equivalent to $\mathrm{Spec}(x^{\mathbf{X}}) = \mathrm{Spec}(y^{\mathbf{Y}})$, we get that $\mathrm{Spec}_w(\mathbf{X}) \subseteq \mathrm{Spec}_w(\mathbf{Y})$.

As in Example 2.2 (4), every second-countable T_0-space can be embedded into the Scott domain $\mathcal{O}(\mathbb{N})$. Use the relativisation argument to get an oracle p such that there are p-computable embeddings of $\mathbf{X}$ and $\mathbf{Y}$ into $\mathcal{O}(\mathbb{N})$. Then, the desired assertion can be verified by relativising the above argument to any oracle $q \geq_T p$. □

3.2. Degree Spectra and Dimension Theory

One of the main tools in our work is the following characterisation of the point degree spectra of represented spaces.

Theorem 3.4. *The following are equivalent for admissibly represented spaces* $\mathbf{X}$ *and* $\mathbf{Y}$*:*

1. $\mathrm{Spec}^r(\mathbf{X}) = \mathrm{Spec}^r(\mathbf{Y})$ *for some oracle* $r \in \{0,1\}^{\mathbb{N}}$.
2. $\mathbb{N} \times \mathbf{X}$ *is* σ*-homeomorphic to* $\mathbb{N} \times \mathbf{Y}$*; that is,* $\mathbb{N} \times \mathbf{X} \cong^{\mathfrak{T}}_{\sigma} \mathbb{N} \times \mathbf{Y}$.

Moreover, if $\mathbf{X}$ *and* $\mathbf{Y}$ *are Polish, then the following assertions (3) and (4) are also equivalent to the above assertions (1) and (2).*

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3. $\mathbb{N} \times \mathbf{X}$ *is second-level Borel isomorphic to* $\mathbb{N} \times \mathbf{Y}$.
4. *The Banach algebra* $\mathcal{B}_2^*(\mathbb{N} \times \mathbf{X})$ *is linearly isometric (ring isomorphic and so on) to* $\mathcal{B}_2^*(\mathbb{N} \times \mathbf{Y})$.

One can also see that the following assertions are equivalent:

2′. $\mathbb{N} \times \mathbf{X}$ is G_δ-piecewise homeomorphic to $\mathbb{N} \times \mathbf{Y}$.
3′. $\mathbb{N} \times \mathbf{X}$ is nth-level Borel isomorphic to $\mathbb{N} \times \mathbf{Y}$ for some $n \geq 2$.
4′. The Banach algebra $\mathcal{B}_n^*(\mathbb{N} \times \mathbf{X})$ is linearly isometric (ring isomorphic and so on) to $\mathcal{B}_n^*(\mathbb{N} \times \mathbf{Y})$ for some $n \geq 2$.

By Observation 2.4 and its proof, the assertions (2), (2′) and (3) are equivalent. Obviously, the assertions (3) and (4) imply (3′) and (4′), respectively. The equivalence between (3) and (4) (and the equivalence between (3′) and (4′)) has already been shown by Jayne [23] for second-countable (or, more generally, real compact) spaces $\mathbf{X}$ and $\mathbf{Y}$. Consequently, all assertions from (2) to (4′) are equivalent.

To see the equivalence between (1) and (2), we characterise the point degree spectra of represented spaces in the context of computable σ-embedding.

Lemma 3.5. *The following are equivalent for represented spaces* $\mathbf{X}$ *and* $\mathbf{Y}$*:*

1. $\mathrm{Spec}(\mathbf{X}) \subseteq \mathrm{Spec}(\mathbf{Y})$
2. $\mathbf{X} \leq_\sigma \mathbb{N} \times \mathbf{Y}$*; that is,* $\mathbf{X}$ *is a countable union of subspaces that are computably isomorphic to subspaces of* $\mathbf{Y}$.

Proof. We first show that the assertion (1) implies (2). By assumption, for any $x \in \mathbf{X}$ we find $x^{\mathbf{X}} \equiv_M y_x^{\mathbf{Y}}$ for some $y_x \in \mathbf{Y}$. For any $i, j \in \mathbb{N}$, let $\mathbf{X}_{ij}$ be the set of all points where the reductions are witnessed by Φ_i and Φ_j. More precisely, put $\mathbf{X}_{ij} = \{x \in \mathbf{X} : \Phi_j\Phi_i(x) = x\}$, where we recall that Φ_e is a partial function on represented spaces realised by the eth partial computable function. Let $\mathbf{Y}_{ij} = \{\Phi_i(x) \mid x \in \mathbf{X}_{ij}\} \subseteq \mathbf{Y}$. Then Φ_i and Φ_j also witness $\mathbf{X}_{ij} \cong \mathbf{Y}_{ij}$. Obviously, $\mathbf{X} = \bigcup_{\langle i,j\rangle\in\mathbb{N}} \mathbf{X}_{ij}$ since $x^{\mathbf{X}} \equiv_M y_x^{\mathbf{Y}}$ is witnessed by some Φ_i and Φ_j; that is, $\Phi_i(x) = y_x$ and $\Phi_j(y_x) = x$. Then, the union of computable homeomorphisms $\mathbf{X}_{ij} \simeq \{\langle i, j\rangle\} \times \mathbf{Y}_{ij}$ gives a σ-computable embedding of $\mathbf{X}$ into $\mathbb{N} \times \mathbf{Y}$.

Conversely, the point degree spectrum is preserved by computable isomorphism and, clearly, $\mathrm{Spec}(\bigcup_{n\in\mathbb{N}} \mathbf{X}_n) = \bigcup_{n\in\mathbb{N}} \mathrm{Spec}(\mathbf{X}_n)$, so the claim follows. □

Proof of Theorem 3.4 (1) ⇔ (2). It follows from relativisations of Lemma 3.5 and Observation 2.5. Here, it is easy to see that the assertion (2) is equivalent to $\mathbb{N} \times \mathbf{X} \leq_\sigma \mathbb{N} \times \mathbf{Y}$. □

This simple argument completely solves a mystery about the occurrence of non-Turing degrees in proper infinite-dimensional spaces. Concretely speaking, by relativising Lemma 3.5, we can characterise the Turing degrees in terms of topological dimension theory as follows.[1]

Corollary 3.6. *The following are equivalent for a separable metrisable space* $\mathbf{X}$ *endowed with an admissible representation:*

1. $\mathrm{Spec}^p(\mathbf{X}) \subseteq \mathcal{D}_T$ *for some oracle* $p \in \{0,1\}^{\mathbb{N}}$
2. $\mathbf{X}$ *is countable dimensional.*

By a dimension-theoretic fact (see Subsection 2.2.2), if $\mathbf{X}$ is Polish, transfinite dimensionality is also equivalent to the condition for $\mathbf{X}$ in which any point has a Turing degree relative to some oracle.

By definition, $\mathrm{Spec}(\mathbf{X})$ can be considered as a degree structure (i.e., a substructure of the enumeration degrees or the Medvedev degrees). Hence, by Theorem 3.4, σ-homeomorphic classification can be viewed as a kind of degree theory dealing with the *order structure on degree structures* (on a cone). Thus, from the viewpoint of degree theory, it is natural to ask whether Post's problem (of whether there is an intermediate degree structure strictly between the bottom $\{0,1\}^{\mathbb{N}}$ and the top $[0,1]^{\mathbb{N}}$), the Friedberg–Muchnik theorem (there is a pair of incomparable degree structures), the Sacks density

[1]The same observation was independently made by Hoyrup. Brattka and Miller had conjectured that dimension would be the crucial demarkation line for spaces with only Turing degrees (all personal communication).

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theorem (given two comparable, but different, degree structures, there is an intermediate degree structure strictly between them) and so on are true for the order of degree structures of uncountable Polish spaces.

More details of the structure of degree spectra of Polish spaces will be investigated in Sections 4 and 5.

4. Intermediate Point Degree Spectra

4.1. Intermediate Polish Spaces

In this section, we investigate the structure of σ-homeomorphic types or, (almost) equivalently, point degree spectra (up to relativisation) of uncountable Polish spaces.

It is well-known that for every uncountable Polish space X:

$$\{0,1\}^{\mathbb{N}} \leq_c^{\mathfrak{T}} X \leq_c^{\mathfrak{T}} [0,1]^{\mathbb{N}},$$

where, recall that $\leq_c^{\mathfrak{T}}$ is the topological embeddability relation (i.e., the ordering of Fréchet dimension types). In this section, we focus on Problem 1.3 asking whether there exists a Polish space $\mathbf{X}$ satisfying the following:

$$\{0,1\}^{\mathbb{N}} <_\sigma^{\mathfrak{T}} \mathbf{X} <_\sigma^{\mathfrak{T}} [0,1]^{\mathbb{N}}.$$

One can see that there is no difference between the structures of σ-homeomorphism types of uncountable Polish spaces and uncountable compact metric spaces.

Fact 4.1. *Every Polish space is σ-homeomorphic to a compact metrisable space.*

Proof. If a pair of countable spaces has the same cardinality, then they are clearly σ-homeomorphic. Moreover, there are compact metrisable spaces of all countable cardinalities.

So let $\mathbf{X}$ be an uncountable Polish space. Lelek [34] showed that every Polish space $\mathbf{X}$ has a compactification $\gamma\mathbf{X}$ such that $\gamma\mathbf{X} \setminus \mathbf{X}$ is countable dimensional. Clearly, $\mathbf{X} \leq_c \gamma\mathbf{X}$. Then, we have $\gamma\mathbf{X} \setminus \mathbf{X} \leq_\sigma^{\mathfrak{T}} \{0,1\}^{\mathbb{N}} \leq_\sigma^{\mathfrak{T}} \mathbf{X}$, since $\mathbf{X}$ is uncountable Polish and $\gamma\mathbf{X} \setminus \mathbf{X}$ is countable dimensional. Consequently, $\mathbf{X}, \gamma\mathbf{X} \setminus \mathbf{X} \leq_\sigma^{\mathfrak{T}} \mathbf{X}$, and this implies $\gamma\mathbf{X} = \mathbf{X} \cup (\gamma\mathbf{X} \setminus \mathbf{X}) \leq_\sigma^{\mathfrak{T}} \mathbf{X}$. □

4.2. The Graph Space of a Universal ω-Left-CEA Operator

Now, we provide a concrete example having an intermediate degree spectrum. We say that a point $(r_n)_{n\in\mathbb{N}} \in [0,1]^{\mathbb{N}}$ is *ω-left-CEA in $x \in \mathbb{N}^{\mathbb{N}}$* if r_{n+1} is left-c.e. in $\langle x, r_0, r_1, \dots, r_n\rangle$ uniformly in $n \in \mathbb{N}$. In other words, there is a computable function $\Psi : \mathbb{N}^{\mathbb{N}} \times [0,1]^{<\omega} \times \mathbb{N}^2 \to \mathbb{Q}_{\geq 0}$ such that

$$r_n = \sup_{s\in\mathbb{N}} \Psi(x, r_0, \dots, r_{n-1}, n, s)$$

for every x, n, s, where $\mathbb{Q}_{\geq 0}$ denotes the set of all nonnegative rationals. If, moreover, we have $r_0 \geq_M x$, then we say that $(r_n)_{n\in\mathbb{N}}$ is *ω-left-CEA over $x \in \mathbb{N}^{\mathbb{N}}$*.

Whenever $r_n \in [0,1]$ for all $n \in \mathbb{N}$, such a computable function Ψ generates an operator $J_\Psi^\omega : \mathbb{N}^{\mathbb{N}} \to [0,1]^{\mathbb{N}}$ with $J_\Psi^\omega(x) = (x, r_0, r_1, \dots)$, which is called an *$\omega$-left-CEA operator*. An ω-left-CEA operator J^ω is *universal* if for any ω-left-CEA operator J, there is $e \in \mathbb{N}$ such that $J^\omega(\langle e, x\rangle) = J(x)$.

Proposition 4.2. *A universal ω-left-CEA operator exists.*

Proof. We first give an effective enumeration $(J_e^\omega)_{e\in\mathbb{N}}$ of all ω-left-CEA operators. It is not hard to see that $y \in [0,1]$ is left-c.e. in $x \in \mathbb{N}^{\mathbb{N}} \times [0,1]^k$ if and only if there is a c.e. set $W \subseteq \mathbb{N} \times \mathbb{Q}$ such that

$$y = J_W^k(x) := \sup\{\min\{|p|, 1\} : x \in B_i^k \text{ for some } (i,p) \in W\},$$

where B_i^k is the ith rational open ball in $\mathbb{N}^\mathbb{N} \times [0,1]^k$. Thus, we have an effective enumeration of all left-c.e. operators $J : \mathbb{N}^\mathbb{N} \times [0,1]^k \to [0,1]$ by putting $J_e^k = J_{W_e}^k$, where W_e is the eth c.e. subset of $\mathbb{N} \times \mathbb{Q}$. Then, we define

$$J_e^\omega(x) = (x, J_{\langle e,0\rangle}^0(x), J_{\langle e,1\rangle}^1(x, J_{\langle e,0\rangle}^0(x)), \dots);$$

that is, J_e^ω is the ω-left-CEA operator generated by the uniform sequence $(J_{\langle e,k\rangle}^k)_{k\in\mathbb{N}}$ of left-c.e. operators. Clearly, $(J_e^\omega)_{e\in\mathbb{N}}$ is an effective enumeration of all ω-left-CEA operators. Then, define $J^\omega(\langle e,x\rangle) = J_e^\omega(x)$. It is not hard to check that J^ω is a universal ω-left-CEA operator. □

Definition 4.3. The *ω-left-computably enumerable in-and-above space* $\omega\mathbf{CEA}$ is a subspace of $\mathbb{N} \times \{0,1\}^\mathbb{N} \times [0,1]^\mathbb{N}$ defined by

$$\begin{aligned}\omega\mathbf{CEA} &= \{(e,x,r) \in \mathbb{N} \times \mathbb{N}^\mathbb{N} \times [0,1]^\mathbb{N} : J_e^\omega(x) = (x,r)\}\\ &\simeq \text{`the graph of a universal } \omega\text{-left-CEA operator'}.\end{aligned}$$

Note that in classical recursion theory, an operator Ψ is called a *CEA-operator* (also known as an *REA-operator*, a *pseudojump*, or a *hop*) if there is a c.e. procedure W such that $\Psi(A) = \langle A, W(A)\rangle$ for any $A \subseteq \mathbb{N}$ (see Odifreddi [42, Chapters XII and XIII]). An ω-CEA operator (also called an ω-hop) is the ωth iteration of a uniform sequence of CEA-operators. In general, computability theorists have studied α-CEA operators for computable ordinals α in the theory of Π_2^0 singletons. We will also use a generalisation of the notion of a Π_2^0 singleton in Section 5.

We say that a continuous degree is *ω-left-CEA* if it contains a point $r \in [0,1]^\mathbb{N}$ which is ω-left-CEA over an oracle $z \in \{0,1\}^\mathbb{N}$. The point degree spectrum of the space $\omega\mathbf{CEA}$ (as a subspace of $[0,1]^\mathbb{N}$) can be described as follows:

$$\mathrm{Spec}(\omega\mathbf{CEA}) = \{\mathbf{a} \in \mathcal{D}_r : \mathbf{a} \text{ is } \omega\text{-left-CEA}\}.$$

This is because $J_e^\omega(x)$ is always ω-left-CEA over x and, conversely, if r is ω-left-CEA over x, then by universality of J^ω (Proposition 4.2) there is e such that $J_e^\omega(x) = (x,r)$, which is equivalent to r as $x \leq_\mathrm{T} r$. Clearly,

$$\mathrm{Spec}(\{0,1\}^\mathbb{N}) \subseteq \mathrm{Spec}(\omega\mathbf{CEA}) \subseteq \mathrm{Spec}([0,1]^\mathbb{N}).$$

The following is an analog of the well-known fact from classical computability theory that every ω-CEA set is a Π_2^0-singleton (see Odifreddi [42, Proposition XIII.2.7]).

Lemma 4.4. *The ω-left-CEA space $\omega\mathbf{CEA}$ is Polish.*

Proof. It suffices to show that $\omega\mathbf{CEA}$ is Π_2^0 (hence G_δ) in $\mathbb{N}^\mathbb{N} \times [0,1]^\mathbb{N}$ since a G_δ subset of a Polish space is Polish. The stage s approximation to J_e^k is denoted by $J_{e,s}^k$; that is, $J_{e,s}^k(z) = \max\{\min\{|p|,1\} : (\exists\langle i,p\rangle \in W_{e,s})\ z \in B_i^k\}$, where $W_{e,s}$ is the stage s approximation to the eth computably enumerable set W_e. Note that the function $(e,s,k,z) \mapsto J_{e,s}^k(z)$ is computable. We can easily see that $(e,x,r) \in \omega\mathbf{CEA}$ if and only if

$$(\forall n,k \in \mathbb{N})(\exists s > n)\ d\left(\pi_k(r), J_{e,s}^k(x, \pi_0(r), \pi_1(r), \dots, \pi_{k-1}(r))\right) < 2^{-n},$$

where d is the Euclidean metric on $[0,1]$ and π_i is the ith projection (i.e., $\pi_i(r) = r_i$ for $r = (r_j)_{j\in\mathbb{N}}$). The above formula is clearly Π_2^0. □

We devote the rest of this section to a proof of the following theorem.

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Theorem 4.5. *The space* $\omega\mathbf{CEA}$ *has an intermediate* σ*-homeomorphism type; that is,*

$$\{0,1\}^{\mathbb{N}} <^{\mathfrak{T}}_{\sigma} \omega\mathbf{CEA} <^{\mathfrak{T}}_{\sigma} [0,1]^{\mathbb{N}}.$$

Consequently, the space $\omega\mathbf{CEA}$ is a concrete counterexample to Problem 1.3.

4.3. Proof of Theorem 4.5

The key idea is to measure *how similar the space* $\mathbf{X}$ *is to a zero-dimensional space* by approximating each point in a space $\mathbf{X}$ by a zero-dimensional space. Recall from (the proof of) Observation 3.3 that, for points in represented spaces which computably embed into $\mathcal{O}(\mathbb{N})$, there is a one-to-one correspondence between the point-Turing degree $\deg(x) = [x]_{\equiv_M}$ and the spectrum $\mathrm{Spec}(x)$. Via this correspondence, the point-Turing degree $\deg(x)$ of a point $x \in \mathbf{X}$ can be identified with its Turing upper cone; that is,

$$\deg(x) \approx \mathrm{Spec}(x) = \{z \in \{0,1\}^{\mathbb{N}} : x \leq_{\mathrm{T}} z\}.$$

We think of the spectrum $\mathrm{Spec}(x)$ as *the upper approximation of* $x \in \mathbf{X}$ *by the zero-dimensional space* $\{0,1\}^{\mathbb{N}}$. Now, we need the notion of *the lower approximation of* $x \in \mathbf{X}$ *by the zero-dimensional space* $\{0,1\}^{\mathbb{N}}$. We introduce the *co-spectrum* of a point $x \in \mathbf{X}$ as its Turing lower cone

$$\mathrm{coSpec}(x) = \{z \in \{0,1\}^{\mathbb{N}} : z \leq_{\mathrm{T}} x\},$$

and, moreover, we define the *degree co-spectrum* of a represented space $\mathbf{X}$ as follows:

$$\mathrm{coSpec}(\mathbf{X}) = \{\mathrm{coSpec}(x) : x \in \mathbf{X}\}.$$

As we will see below, the degree spectrum of a represented space fully determines its co-spectrum, while the converse is not true. For every oracle $p \in \{0,1\}^{\mathbb{N}}$, we may also introduce relativised co-spectra $\mathrm{coSpec}^p(x) = \{z \in \{0,1\}^{\mathbb{N}} : z \leq_{\mathrm{T}} (x,p)\}$ and the relativised degree co-spectra $\mathrm{coSpec}^p(\mathbf{X})$ in the same manner.

Observation 4.6. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be admissibly represented spaces. If* $\mathrm{Spec}^p(\mathbf{X}) \subseteq \mathrm{Spec}^p(\mathbf{Y})$*, then we also have* $\mathrm{coSpec}^p(\mathbf{X}) \subseteq \mathrm{coSpec}^p(\mathbf{Y})$.

Therefore, by Theorem 3.4, the cospectrum of an admissibly represented space up to an oracle is invariant under σ*-homeomorphism. Indeed, by relativising Lemma 3.5, one can see that* $\mathbf{X} \leq^{\mathfrak{T}}_{\sigma} \mathbf{Y}$ *implies* $\mathrm{coSpec}^p(\mathbf{X}) \subseteq \mathrm{coSpec}^p(\mathbf{Y})$ *for some* p.

Proof. Clearly, $[x^{\mathbf{X}}]_{\equiv_{\mathrm{T}}} = [y^{\mathbf{Y}}]_{\equiv_{\mathrm{T}}}$ implies that $\{z \in \{0,1\}^{\mathbb{N}} : z \leq_{\mathrm{T}} x^{\mathbf{X}}\} = \{z \in \{0,1\}^{\mathbb{N}} : z \leq_{\mathrm{T}} y^{\mathbf{Y}}\}$. This observation can be relativised to any oracle p. This verifies the first assertion. □

We say that a collection $\mathcal{I}$ of subsets of $\mathbb{N}$ is *realised as the co-spectrum of* x if $\mathrm{coSpec}(x) = \mathcal{I}$. A countable set $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ is a *Scott ideal* if it is the standard system of a countable nonstandard model of Peano arithmetic or, equivalently, a countable ω-model of the theory WKL_0. We will not go into the details of a Scott ideal (see Miller [37, Section 9] for a more explicit definition); we will only use the fact that every jump ideal is a Scott ideal. Here, a *jump ideal* $\mathcal{I}$ is a collection of subsets of natural numbers which is closed under the join $\oplus$, downward Turing reducibility $\leq_{\mathrm{T}}$ and the Turing jump; that is, $p, q \in \mathcal{I}$ implies $p \oplus q \in \mathcal{I}$; $p \leq_{\mathrm{T}} q \in \mathcal{I}$ implies $p \in \mathcal{I}$; and $p \in \mathcal{I}$ implies $p' \in \mathcal{I}$. Miller [37, Theorem 9.3] showed that every countable Scott ideal (hence, every countable jump ideal) is realised as a co-spectrum in $[0,1]^{\mathbb{N}}$.

Example 4.7. The spectra and co-spectra of Cantor space $\{0,1\}^{\mathbb{N}}$, the space ωCEA and the Hilbert cube $[0,1]^{\mathbb{N}}$ are illustrated as follows (see also Figure 1):

1. The co-spectrum $\mathrm{coSpec}(x)$ of any point $x \in \{0,1\}^{\mathbb{N}}$ is principal and *meets with* $\mathrm{Spec}(x)$ exactly at $\deg_T(x)$. The same is true up to some oracle for an arbitrary Polish spaces $\mathbf{X}$ such that $\mathbf{X} \cong^{\mathfrak{T}}_{\sigma} \{0,1\}^{\mathbb{N}}$.

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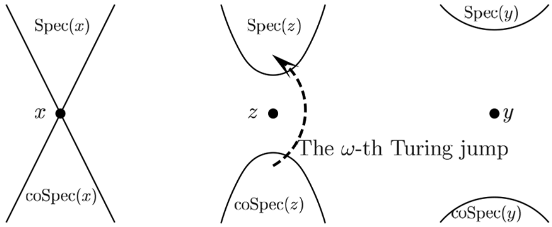

Figure 1. *The upper and lower approximations of* $\{0,1\}^{\mathbb{N}}$, ω**CEA** *and* $[0,1]^{\mathbb{N}}$

2. For any point $z \in \omega\mathbf{CEA}$, the '*distance*' between $\mathrm{Spec}(z)$ and $\mathrm{coSpec}(z)$ has to be at most the ωth Turing jump (see the proof of Theorem 4.5 below).
3. (Miller [37, Theorem 9.3]) An arbitrary countable Scott ideal is realised as $\mathrm{coSpec}(y)$ of some point $y \in [0,1]^{\mathbb{N}}$. Hence, $\mathrm{Spec}(y)$ and $\mathrm{coSpec}(y)$ can be separated by *an arbitrary distance*. (Consider countable Scott ideals closed under the αth Turing jump, the hyperjump, the Δ^1_n-jump, etc.)

This upper/lower approximation method clarifies the differences of σ-homeomorphism types of spaces because both relativised point-degree spectra and co-spectra are invariant under σ-homeomorphism by Theorem 3.4 and Observation 4.6.

Proof of Theorem 4.5. We first show that $\omega\mathbf{CEA} <^{\mathfrak{T}}_{\sigma} [0,1]^{\mathbb{N}}$. This follows from the following claim: For any oracle $p \in \{0,1\}^{\mathbb{N}}$, there is a countable Scott ideal which cannot be realised as a p-co-spectrum of an ω-left-CEA continuous degree.

To see this, let $y = (e,x,r) \in \omega\mathbf{CEA}$ be an arbitrary point. Clearly, $x \leq_T (e,x,r)$, and this means that $x \in \mathrm{coSpec}(y)$ since $x \in \{0,1\}^{\mathbb{N}}$. However, as $r = (r_n)_{n\in\mathbb{N}}$ is ω-left-CEA in x, we know that r_0 is c.e. in x (so computable in the Turing jump of x) and r_{n+1} is c.e. in $(x, r_0, \dots, r_n)$. By induction, this implies that r_n is computable in the $(n+1)$th jump of x uniformly in n and, therefore, r is computable in the ωth jump of x; hence, $y = (e,x,r) \leq_T x^{(\omega)}$; that is, $x^{(\omega)} \in \mathrm{Spec}(y)$. In particular, $\mathrm{coSpec}(y)$ does not contain the $(\omega+1)$-st Turing jump of the second entry x of given any $y \in \omega\mathbf{CEA}$. Thus, for any oracle p, the jump ideal $\mathcal{A}^p = \{x \in \{0,1\}^{\mathbb{N}} : (\exists n \in \mathbb{N})\ x \leq_T p^{(\omega\cdot n)}\}$ cannot be realised as a co-spectrum in $\omega\mathbf{CEA}$. This verifies the claim.

By the above claim and Miller's result on Scott ideals mentioned in Example 4.7 (3), we have $\mathrm{coSpec}^p(\omega\mathbf{CEA}) \subsetneq \mathrm{coSpec}^p([0,1]^{\mathbb{N}})$ for any oracle p. Therefore, by Theorem 3.4 and Observation 4.6, we conclude that the ω-left-CEA space is not σ-homeomorphic to the Hilbert cube; that is, $\omega\mathbf{CEA} <^{\mathfrak{T}}_{\sigma} [0,1]^{\mathbb{N}}$.

We next show $\{0,1\}^{\mathbb{N}} <^{\mathfrak{T}}_{\sigma} \omega\mathbf{CEA}$. In other words, we have to show that the ω-left-CEA space is not countable-dimensional. For a compact set $P \subseteq [0,1]^{\mathbb{N}}$, we inductively define $\min P \in P$ as follows:

$$\pi_n(\min P) = \min \pi_n[\{z \in P : (\forall i < n)\ \pi_i(z) = \pi_i(\min P)\}],$$

where $\pi_n : [0,1]^{\mathbb{N}} \to [0,1]$ is the projection onto the nth coordinate. We call the point $\min P$ the *leftmost point of* P. Kreisel's basis theorem (see [41, Proposition V.5.31]) in classical computability theory says that the leftmost element of a Π^0_1 subset of $\{0,1\}^{\mathbb{N}}$ or $[0,1]$ is always left-c.e. We consider the following infinite-dimensional version of Kreisel's basis theorem: For any oracle $p \in \{0,1\}^{\mathbb{N}}$, the leftmost point of a $\Pi^0_1(p)$ subset of $[0,1]^{\mathbb{N}}$ is ω-left-CEA in p.

To see this, one can easily check that the Hilbert cube $[0,1]^{\mathbb{N}}$ is computably compact in the sense that there is a computable enumeration of all finite collections $\mathcal{D}$ of basic open sets which covers the whole space; that is, $\bigcup \mathcal{D} = [0,1]^{\mathbb{N}}$. In particular, given a Π^0_1 set $P \subseteq [0,1]^{\mathbb{N}}$, the predicate $P = \emptyset$ is Σ^0_1 uniformly in a Π^0_1 code of P.

Fix a $\Pi^0_1(p)$ set $P \subseteq [0,1]^{\mathbb{N}}$. It suffices to show that $\pi_{n+1}(\min P)$ is left-c.e. in $\langle \pi_i(\min P)\rangle_{i\leq n}$ uniformly in n relative to p. Given a finite sequence $\mathbf{a} = (a_0, a_1, \dots, a_n)$ of reals and a real q, we denote

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by $C(\mathbf{a},q)$ the set of all points in P of the form $(a_0,a_1,\dots,a_n,r,\dots)$ for some $r\le q$; that is,

$$C(\mathbf{a},q)=P\cap\bigcap_{i\le n}\pi_i^{-1}\{a_i\}\cap\pi_{n+1}^{-1}[0,q].$$

It is easy to check that $C(\mathbf{a},q)$ is a Π^0_1 subspace of $[0,1]$ relative to $\mathbf{a}$ and q. By computable compactness of the Hilbert cube, one can see that $C^*(\mathbf{a}):=\{q\in[0,1]:C(\mathbf{a},q)=\emptyset\}$ is p-c.e. open uniformly relative to $\mathbf{a}$ (since $C(\mathbf{a},q)=\emptyset$ is Σ^0_1 uniformly relative to $\mathbf{a}$ and q). Therefore, $\sup C^*(\mathbf{a})$ is p-left-c.e. uniformly relative to $\mathbf{a}$. Finally, we claim that $\pi_{n+1}(\min P)$ is exactly $\sup C^*(\langle\pi_i(\min P)\rangle_{i\le n})$. By definition, $\pi_{n+1}(\min P)$ is the least $q\in[0,1]$ such that there exists $(r_m)_{m\ge n+2}$ such that $(\pi_0(\min P),\dots,\pi_n(\min P),q,r_{n+2},r_{n+3},\dots)\in P$. This is equal to the least $q\in[0,1]$ such that $C(\langle\pi_i(\min P)\rangle_{i\le n},q)$ is nonempty. This is exactly the same as $\sup C^*(\langle\pi_i(\min P)\rangle_{i\le n})$.

We use the following relativised versions of Miller's lemmas.

Lemma 4.8 (Miller [37, Lemma 6.2]). *For every $p\in\{0,1\}^{\mathbb{N}}$, there is a multivalued function $\Psi^p:[0,1]^{\mathbb{N}}\rightrightarrows[0,1]^{\mathbb{N}}$ with a $\Pi^0_1(p)$ graph and nonempty, convex images such that, for all $e\in\mathbb{N}$, $\alpha\in[0,1]^{\mathbb{N}}$ and $\beta\in\Psi^p(\alpha)$, if for every name λ of α, $\varphi_e^{\lambda\oplus p}$ is a name of $x\in[0,1]$, then $\beta(e)=x$.*

Note that Kakutani's fixed point theorem ensures the existence of a fixed point of Ψ^p. If α is a fixed point of Ψ^p – that is – $\alpha\in\Psi^p(\alpha)$, then $\mathrm{coSpec}^p(\alpha)=\{\alpha(n):n\in\mathbb{N}\}$ (to be more precise, $\mathrm{coSpec}^p(\alpha)$ is the set of all binary expansions of reals in $\{\alpha(n):n\in\mathbb{N}\}$ or, equivalently, $\mathrm{coSpec}^p(\alpha)=\{\alpha(n):n\in\mathbb{N}\}\cap\{0,1\}^{\mathbb{N}}$ when $\{0,1\}^{\mathbb{N}}$ is regarded as the canonical Cantor set in $[0,1]$). Therefore, such an α has no Turing degree relative to p (see [37, Proposition 5.3]).

Lemma 4.9 (Miller [37, Lemma 9.2]). *For every $p\in\{0,1\}^{\mathbb{N}}$, there is an index $e\in\mathbb{N}$ such that for any $x\in[0,1]$, there is a fixed point α of Ψ^p such that $\alpha(e)=x$.*

We show the following: For any oracle $p\in\{0,1\}^{\mathbb{N}}$, there is an ω-left-CEA continuous degree which is not contained in $\mathrm{Spec}^p(\{0,1\}^{\mathbb{N}})$.

Let $\mathrm{Fix}(\Psi^p)$ be the set of all fixed points of Ψ^p. Then, $\mathrm{Fix}(\Psi^p)$ is $\Pi^0_1(p)$ since it is the intersection of the graph of Ψ^p (which is a $\Pi^0_1(p)$ set) and the diagonal set. Let e be an index as in Lemma 4.9. Clearly, $A=\{\alpha\in\mathrm{Fix}(\Psi^p):\alpha(e)=p\}$ is again a $\Pi^0_1(p)$ subset of $[0,1]^{\mathbb{N}}$ and A is nonempty by Lemma 4.9. Given $\alpha\in[0,1]^{\mathbb{N}}$, define α^* as the result of swapping the values of the 0th and eth entries of α; that is, $\alpha^*(0)=\alpha(e)$, $\alpha^*(e)=\alpha(0)$ and $\alpha^*(n)=\alpha(n)$ for $n\notin\{0,e\}$. It is clear that $\alpha\mapsto\alpha^*$ is a computable homeomorphism. Thus, $A^*=\{\alpha\in[0,1]:\alpha^*\in A\}$ is computably homeomorphic to A; hence, A^* is also a nonempty $\Pi^0_1(p)$ set. By our infinite-dimensional version of Kreisel's basis theorem, A^* contains an element α which is ω-left-CEA in p. Indeed, α is ω-left-CEA over p since $\alpha(0)=\alpha^*(e)=p$. By the property of an element of $A\subseteq\mathrm{Fix}(\Psi^p)$ discussed above, $\alpha^*\in A$ has no Turing degree relative to p. Moreover, since moving the eth entry of α to the first entry does not affect the degree, the degree of α is equal to that of α^*. Hence, α has a ω-left-CEA continuous degree but has no Turing degree relative to p.

By this claim, $\mathrm{Spec}^p(\{0,1\}^{\mathbb{N}})\subsetneq\mathrm{Spec}^p(\omega\mathbf{CEA})$ for any oracle p. Again by Theorem 3.5 and Observation 4.6, we conclude $\{0,1\}^{\mathbb{N}}<^{\mathfrak{T}}_{\sigma}\omega\mathbf{CEA}$. □

5. Structure of σ-Homeomorphism Types

In this section, we will show that there are continuum many σ-homeomorphism types of compact metrisable spaces.

Theorem 5.1. *There exists a collection $(\mathbf{X}_\alpha)_{\alpha<2^{\aleph_0}}$ of continuum many compact metric spaces such that if $\alpha\neq\beta$, $\mathbf{X}_\alpha$ cannot be σ-embedded into $\mathbf{X}_\beta$.*

We devote the rest of this section to prove Theorem 5.1. Actually, we will show the following:

There is an embedding of the inclusion ordering $([\omega_1]^{\le\omega},\subseteq)$ of countable subsets of the smallest uncountable ordinal ω_1 into the σ-embeddability ordering of compact metric spaces.

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As a corollary, there are an uncountable chain and a continuum antichain of σ-homeomorphism types of compact metric spaces.

5.1. Almost Arithmetical Degrees

In Section 4, we used the co-spectrum as a σ-topological invariant. More explicitly, in our proof, it was essential to examine closure properties of co-spectra to obtain an intermediate σ-homeomorphism type of Polish spaces. In this section, we will develop a method for controlling closure properties of co-spectra. As a result, we will construct a compact metrisable space whose co-spectra realise a given well-behaved family of *'almost' arithmetical degrees*.

First, we introduce a notion which estimates the strength of closure properties of functions up to the arithmetical equivalence.

Definition 5.2. Let g and h be total Borel measurable functions from $\{0,1\}^{\mathbb{N}}$ into $\{0,1\}^{\mathbb{N}}$.

1. We inductively define $g^0(x) = x$ and $g^{n+1}(x) = g^n(x) \oplus g(g^n(x))$.
2. For every oracle $r \in \{0,1\}^{\mathbb{N}}$, consider the following jump ideal defined as
$$\mathcal{J}_a(g,r) = \{z \in \{0,1\}^{\mathbb{N}} : (\exists n \in \mathbb{N})\ z \leq_a g^n(r)\},$$
where $\leq_a$ denotes the arithmetical reducibility; that is, $p \leq_a q$ is defined by $p \leq_{\mathrm{T}} q^{(m)}$ for some $m \in \mathbb{N}$ (see Odifreddi [42, Section XII.2 and Chapter XIII]).
3. A function g is *almost arithmetical reducible to* a function h (written as $g \leq_{aa} h$) if for any $r \in \{0,1\}^{\mathbb{N}}$ there is $x \in \{0,1\}^{\mathbb{N}}$ with $x \geq_T r$ such that
$$\mathcal{J}_a(g,x) \subseteq \mathcal{J}_a(h,x).$$
4. Let $\mathcal{G}$ and $\mathcal{H}$ be countable sets of total functions. We say that $\mathcal{G}$ is *aa-included in* $\mathcal{H}$ (written as $\mathcal{G} \subseteq_{aa} \mathcal{H}$) if for all $g \in \mathcal{G}$, there is $h \in \mathcal{H}$ such that $g \equiv_{aa} h$ (i.e., $g \leq_{aa} h$ and $h \leq_{aa} g$).

A function $g : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ is said to be *monotone* if $x \leq_{\mathrm{T}} y$ implies $g(x) \leq_{\mathrm{T}} g(y)$.

Remark 5.3. Although it will not be used later, one can show that $\leq_{aa}$ is transitive on monotone Borel measurable functions using Borel determinacy: First note that the condition $\mathcal{J}_a(g,x) \subseteq \mathcal{J}_a(h,x)$ is equivalent to saying that for any i there is j such that $g^i(x) \leq_a h^j(x)$. Thus, this is a Borel property. Given Borel measurable functions g and h, consider the following game: Player I plays r (bit by bit), Player II responds with x and Player II wins this game if $x \geq_T r$ and $\mathcal{J}_a(g,x) \subseteq \mathcal{J}_a(h,x)$. If $g \leq_{aa} h$, then Player I cannot have a winning strategy, so by Borel determinacy, II has a winning strategy α. This strategy yields an α-computable transformation $r \mapsto x$, which implies $x \leq_{\mathrm{T}} r \oplus \alpha$. In particular, if $r \geq_T \alpha$, then there is $x \equiv_{\mathrm{T}} r$ such that $\mathcal{J}_a(g,x) \subseteq \mathcal{J}_a(h,x)$. By monotonicity of g, if $z \equiv_{\mathrm{T}} x$, then $\mathcal{J}_a(g,x) = \mathcal{J}_a(g,z)$ and the same property holds for h. Thus, using monotonicity of g and h, for any $x \geq_T \alpha$ we get $\mathcal{J}_a(g,x) \subseteq \mathcal{J}_a(h,x)$. Using this characterisation, it is now easy to show that $\leq_{aa}$ is transitive.

An *oracle* $\mathbf{\Pi}^0_2$*-singleton* is a total function $g : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ whose graph is G_δ. Clearly, every oracle $\mathbf{\Pi}^0_2$-singleton is Borel measurable, whereas there is no upper bound of Borel ranks of oracle $\mathbf{\Pi}^0_2$-singletons. For instance, if α is a computable ordinal, then the αth Turing jump $j_\alpha(x) = x^{(\alpha)}$ is a monotone oracle $\mathbf{\Pi}^0_2$-singleton for every computable ordinal α (see Odifreddi [42, Proposition XII.2.19], Sacks [58, Corollary II.4.3] and Chong-Yu [9, Theorem 2.1.4]). The following is the key lemma in our proof, which will be proved in Subsection 5.2.

Lemma 5.4 (Realisation Lemma). *There is a map* **Rea** *transforming each countable set of monotone oracle* $\mathbf{\Pi}^0_2$*-singletons into a Polish space such that*
$$\mathbf{Rea}(\mathcal{G}) \leq^{\mathfrak{T}}_{\sigma} \mathbf{Rea}(\mathcal{H}) \implies \mathcal{G} \subseteq_{aa} \mathcal{H}.$$

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5.2. Construction

We construct a Polish space whose co-spectrum codes almost arithmetical degrees contained in a given countable set $\mathcal{G}$ of oracle $\mathbf{\Pi}^0_2$-singletons. For notational simplicity, given $x \in [0,1]^{\mathbb{N}}$, we write x_n for the nth coordinate of x and, moreover, $x_{<n}$ and $x_{\leq n}$ for $(x_i)_{i<n}$ and $(x_i)_{i\leq n}$, respectively. We also consider a sequence like $(r, x_{<i}, x_\ell)$ and, in this case, for sake of simplicity, we assume that any name of $(r, x_{<i}, x_\ell)$ codes information for i and ℓ.

Our idea comes from the construction by Miller [37, Lemma 9.2]. Our purpose is constructing a Polish space such that given $g \in \mathcal{G}$ and oracle r the space has a point $x = (x_i)_{i\in\mathbb{N}}$ whose co-spectrum is not very different from $\mathcal{J}_a(g, r)$. Then, at least, such a point should compute $g^i(r)$ for all $i \in \mathbb{N}$. We can achieve this by requiring $x_i = g^i(r)$ for infinitely many $i \in \mathbb{N}$; however, we need to control the co-spectrum simultaneously and, therefore, we have to choose such coding locations i very carefully. The actual construction is that, from r and $x_{<v}$, we will find a finite set $(\ell(u))_{u\leq v}$ of candidates of safe coding locations and then we define $x_{\ell(u)} = g^{\ell(u)}(r)$ at a genuine safe coding location $\ell(u)$. Then, for each i with $v \leq i < \ell(u)$, we define x_i from $(r, x_{<i}, x_{\ell(u)})$ in a left-c.e. manner. This idea yields the following definition.

Definition 5.5. Let $\mathcal{G} = (g_n)_{n\in\mathbb{N}}$ be a countable collection of oracle $\mathbf{\Pi}^0_2$-singletons. The space $\omega\mathbf{CEA}(\mathcal{G})$ consists of $(n, d, e, r, x) \in \mathbb{N}^3 \times \{0,1\}^{\mathbb{N}} \times [0,1]^{\mathbb{N}}$ such that for every i,

1. either $x_i = g^i_n(r)$ or
2. there are $u \leq v \leq i$ such that $x_i \in [0,1]$ is the eth left-c.e. real relative to $\langle r, x_{<i}, x_{\ell(u)}\rangle$ and $x_{\ell(u)} = g^{\ell(u)}_n(r)$, where $\ell(u) = \Phi_d(u, r, x_{<v}) \geq i$ (recall that Φ_d is the dth partial computable function).

Moreover, for a set $P \subseteq \{0,1\}^{\mathbb{N}} \times [0,1]^{\mathbb{N}}$, define $\omega\mathbf{CEA}(\mathcal{G}, P)$ to be the set of all points $(n, d, e, r, x) \in \omega\mathbf{CEA}(\mathcal{G})$ with $(r, x) \in P$.

Lemma 5.6. *Suppose that $\mathcal{G}$ is a countable collection of oracle $\mathbf{\Pi}^0_2$-singletons and P is a $\mathbf{\Pi}^0_2$ subset of $\{0,1\}^{\mathbb{N}} \times [0,1]^{\mathbb{N}}$. Then, $\omega\mathbf{CEA}(\mathcal{G}, P)$ is Polish.*

Proof. It suffices to show that $\omega\mathbf{CEA}(\mathcal{G})$ is $\mathbf{\Pi}^0_2$. The condition (1) in Definition 5.5 is clearly $\mathbf{\Pi}^0_2$. Let $\forall a \exists b > a\ G(a, b, n, \ell, r, x)$ be a $\mathbf{\Pi}^0_2$ condition representing $x = g^\ell_n(r)$, where G is open and let $\ell(u)[s]$ be the stage s approximation of $\Phi_d(u, r, x_{<v})$. The condition (2) is equivalent to the statement that there are $u \leq v \leq i$ such that

$$(\forall t \in \mathbb{N})(\exists s > t)\ \ell(u)[s] \downarrow \geq i,\ \ d(x_i, J^{i+1}_{e,s}(r, x_{<i}, x_{\ell(u)[s]})) < 2^{-t},$$
$$\text{and } G(t, s, n, \ell(u)[s], r, x_{\ell(u)[s]}).$$

Clearly, this condition is $\mathbf{\Pi}^0_2$. □

Remark 5.7. The space $\omega\mathbf{CEA}(\mathcal{G})$ is totally disconnected for any countable set $\mathcal{G}$ of oracle $\mathbf{\Pi}^0_2$ singletons, since for any fixed $(n, d, e, r) \in \mathbb{N}^3 \times \{0,1\}^{\mathbb{N}}$, its extensions form a finite-branching infinite tree $T \subseteq [0,1]^{<\omega}$.

Recall from Lemma 4.8 that Miller [37, Lemma 6.2] constructed a Π^0_1 set $\mathrm{Fix}(\Psi) \subseteq [0,1]^{\mathbb{N}} = [0,1] \times [0,1]^{\mathbb{N}}$ such that $\mathrm{coSpec}(x) = \{x_i : i \in \mathbb{N}\}$ for every $x = (x_i)_{i\in\mathbb{N}} \in \mathrm{Fix}(\Psi)$. By Lemma 4.9, without loss of generality, we may assume that $\mathrm{Fix}(\Psi) \cap \pi_0^{-1}\{r\} \neq \emptyset$ for every $r \in [0,1]$. Define $\mathrm{Fix}^*(\Psi) = \mathrm{Fix}(\Psi) \cap \pi_0^{-1}[\{0,1\}^{\mathbb{N}}] = \mathrm{Fix}(\Psi) \cap (\{0,1\}^{\mathbb{N}} \times [0,1]^{\mathbb{N}})$, where $\{0,1\}^{\mathbb{N}}$ is always thought of as a subset of $[0,1]$ (as a Cantor set). Now, consider the space $\mathbf{Rea}(\mathcal{G}) = \omega\mathbf{CEA}(\mathcal{G}, \mathrm{Fix}^*(\Psi))$. To state properties of $\mathbf{Rea}(\mathcal{G})$, for an oracle $\mathbf{\Pi}^0_2$-singleton g and an oracle $r \in \{0,1\}^{\mathbb{N}}$, we use the following Turing ideal:

$$\mathcal{J}_T(g, r) = \{z \in \{0,1\}^{\mathbb{N}} : (\exists n \in \mathbb{N})\ z \leq_T g^n(r)\}.$$

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The following is the key lemma, which states that any collection of jump ideals generated by countably many oracle $\mathbf{\Pi}^0_2$-singletons has to be the degree co-spectrum of a Polish space up to the almost arithmetical equivalence!

Lemma 5.8. *Suppose that $\mathcal{G} = (g_n)_{n\in\mathbb{N}}$ is a countable set of oracle $\mathbf{\Pi}^0_2$-singletons.*

1. *For every $x \in \mathbf{Rea}(\mathcal{G})$, there are $r \in \{0,1\}^{\mathbb{N}}$ and $n \in \mathbb{N}$ such that*

$$\mathcal{J}_T(g_n, r) \subseteq \mathrm{coSpec}(x) \subseteq \mathcal{J}_a(g_n, r).$$

2. *For every $r \in \{0,1\}^{\mathbb{N}}$ and $n \in \mathbb{N}$, there is $x \in \mathbf{Rea}(\mathcal{G})$ such that*

$$\mathcal{J}_T(g_n, r) \subseteq \mathrm{coSpec}(x) \subseteq \mathcal{J}_a(g_n, r).$$

Proof of Lemma 5.8 (1). We have $(r, x) \in \mathrm{Fix}(\Psi)$ for every $y = (n, d, e, r, x) \in \mathbf{Rea}(\mathcal{G})$. For every $i \in \mathbb{N}$, we inductively assume that for every $j < i$, x_j is arithmetical in $g^k_n(r)$ for some $k \in \mathbb{N}$. Now, either $x_i = g^i_n(r)$ or x_i is left-c.e. in $(r, x_{<i}, g^\ell_n(r))$ for some ℓ. In both cases, x_i is arithmetical in $g^k_n(r)$ for some k. Since $(r, x) \in \mathrm{Fix}(\Psi)$, by Lemma 4.8, $\mathrm{coSpec}(y) = \{r\} \cup \{x_i : i \in \mathbb{N}\}$. This shows that $\mathrm{coSpec}(y) \subseteq \mathcal{J}_a(g_n, r)$. Moreover, $x_i = g^i_n(r)$ for infinitely many $i \in \mathbb{N}$, since either $x_i = g^i_n(r)$ holds or there is $\ell \geq i$ such that $x_\ell = g^\ell_n(r)$ by the condition (2) in Definition 5.5. Therefore, $g^k_n(r) \leq_{\mathrm{T}} x$ for all $k \in \mathbb{N}$; that is, $\mathcal{J}_T(g_n, r) \subseteq \mathrm{coSpec}(y)$. □

To verify the assertion (2) in Lemma 5.8 – indeed, for any $n \in \mathbb{N}$ – we will construct indices d and e such that for every $r \in \{0,1\}^{\mathbb{N}}$, there is $x = (x_i)_{i\in\mathbb{N}}$ with $(n, d, e, r, x) \in \mathbf{Rea}(\mathcal{G})$, where $x_i = g^i_n(r)$ for infinitely many $i \in \mathbb{N}$. Let e be an index of a left-c.e. procedure $J^{i+1}_e(r, x_{<i}, x_{\ell(u)})$ which is a simple procedure extending $r, x_{<i}, x_{\ell(u)}$ to the leftmost $r, x_{\leq i}, x_{\ell(u)}$ which is extendable to a fixed point of Ψ (as in Kreisel's basis theorem in the proof of Theorem 4.5). The function Φ_d searches for a *safe coding location* $c(n)$ from a given *name* of $x_{\leq c(n-1)}$, where $c(n-1)$ is the previous coding location.

To make sure the search of the next coding location is bounded, as in Definition 5.5, we have to restrict the set of names of a v-tuple $x_{<v}$ to at most $v+1$ candidates. It is known that a separable metrisable space is at most n-dimensional if and only if it is the union of $n+1$ many zero-dimensional subspaces (see [15, Theorem 1.5.8] or [64, Corollary 3.1.7]). We say that an admissibly represented Polish space is *computably at most n-dimensional* if it is the union of $n+1$ many subspaces that are computably homeomorphic to subspaces of $\mathbb{N}^{\mathbb{N}}$.

Lemma 5.9. *Suppose that $(\mathbf{X}, \rho_X)$ is a computably at most n-dimensional admissibly represented space. Then, there is a partial computable injection $\nu_X :\subseteq (n+1) \times \mathbf{X} \to \mathbb{N}^{\mathbb{N}}$ such that for every $x \in \mathbf{X}$, there is $k \leq n$ such that $(k, x) \in \mathrm{dom}(\nu_X)$ and $\rho_X \circ \nu_X(k, x) = x$.*

Proof. By definition, $\mathbf{X}$ is divided into $n+1$ many subspaces $S_0, \dots, S_n$ such that S_k is homeomorphic to $N_k \subseteq \mathbb{N}^{\mathbb{N}}$ via computable maps $\tau_k : S_k \to N_k$ and $\tau_k^{-1} : N_k \to S_k$. Then, τ_k^{-1} can also be viewed as a partial computable injection $\tau_k^{-1} :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbf{X}$ and then it has a computable realiser τ^*_k; that is, $\tau_k^{-1} = \rho_X \circ \tau^*_k$. Define $\nu_X(k, x) = \tau^*_k \circ \tau_k(x)$ for $x \in S_k$. Then, we have $\rho_X \circ \nu_X(k, x) = \tau_k^{-1} \circ \tau_k(x) = x$ for $x \in S_k$. □

The Euclidean n-space $\mathbb{R}^n$ is clearly computably n-dimensional; for example, for $k \leq n$, let S_k be the set of all points $x \in \mathbb{R}^n$ such that exactly k many coordinates are irrationals. Furthermore, one can effectively find an index of $\nu_n := \nu_{\mathbb{R}^n}$ in Lemma 5.9 uniformly in n. Hereafter, let ρ_i be the usual Euclidean admissible representation of $\mathbb{R}^i$ (cf. [65]). One can use Miller's argument [37, Lemma 9.2] to ensure the existence of a safe coding location $c(n)$ as a fixed point in the sense of Kleene's recursion theorem. Therefore, as Kleene's recursion theorem is uniform (see Fact 2.1), one can effectively find such a location in the following sense:

Lemma 5.10 (Miller [37, Lemma 9.2]). *Suppose that $(r, x_{<i})$ can be extended to a fixed point of Ψ and fix a partial computable function ν which sends $x_{<i}$ to its name; that is, $\rho_i \circ \nu(x_{<i}) = (x_{<i})$. From an*

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index t of v and the sequence $x_{<i}$, one can effectively find a location $p = \Gamma(t, r, x_{<i})$ such that for every real y, the sequence $(r, x_{<i})$ can be extended to a fixed point (r, x) of Ψ such that $x_p = y$.

Note that, in Lemma 5.10, we always have $p \geq i$ since an arbitrary x_p is allowed. Let $t(n,k)$ be an index of the partial computable function $x \mapsto v_n(k, x)$ for $k \leq n$ and let d be an index such that $\Phi_d(u, r, x_{<v})$ is equal to $\Gamma(t(v,u), r, x_{<v})$ for every $u \leq v$. Note that indices d and e do not depend on g_n (where e is an index chosen in the paragraph after the proof of Lemma 5.8 (1)).

Proof of Lemma 5.8 (2). Now, we claim that for every $r \in \{0,1\}^{\mathbb{N}}$ and $n \in \mathbb{N}$, there is x with $(n, d, e, r, x) \in \mathbf{Rea}(\mathcal{G})$, where $x_i = g_n^i(r)$ for infinitely many $i \in \mathbb{N}$. We follow the argument by Miller [37, Lemma 9.2]. Suppose that i is a coding location of $g_n^i(r)$ and $(r, x_{\leq i})$ is extendible to a fixed point of Ψ. Here, for the base case, consider $i = -1$ and x_{-1} is the empty sequence. Then, there is a genuine $k \leq i+1$; that is, $v_{i+1}(k, x_{\leq i})$ returns a name of $x_{\leq i}$. For such a k, $p = \Phi_d(k, r, x_{\leq i})$ is defined and then we set $x_p = g_n^p(r)$, where we have $p \geq i+1$ as mentioned in the paragraph below Lemma 5.10. By the property of Φ_d (Lemma 5.10), $(r, x_{\leq i}, x_p)$ can be extended to a fixed point of Ψ. Then, the eth left-c.e. procedure automatically produces $x_{\leq p}$ which extends $x_{\leq i}$ and is extendible to a fixed point of Ψ. Note that the condition (2) in Definition 5.5 is ensured via $u = k$, $v = i+1$ and $\ell(u) = p$. Eventually, we obtain $(r, x) \in \mathrm{Fix}(\Psi)$ such that $z = (n, d, e, r, x) \in \mathbf{Rea}(\mathcal{G})$.

Clearly, $g_n^k(r) \in \mathrm{coSpec}(z)$ for every $k \in \mathbb{N}$, since $\mathrm{coSpec}(z)$ is a Turing ideal and $g_n^k(r) \leq_{\mathrm{T}} g_n^{k+1}(r)$. Consequently, $\mathcal{J}_T(g_n, r) \subseteq \mathrm{coSpec}(z)$. The inclusion $\mathrm{coSpec}(z) \subseteq \mathcal{J}_a(g_n, r)$ can be shown as in the proof of Lemma 5.8 (1). □

Proof of Lemma 5.4. Suppose that $\mathbf{Rea}(\mathcal{G}) \leq_{\sigma}^{\mathfrak{T}} \mathbf{Rea}(\mathcal{H})$. Then, $\mathbb{N} \times \mathbf{Rea}(\mathcal{G}) \leq_{\sigma}^{\mathfrak{T}} \mathbb{N} \times \mathbf{Rea}(\mathcal{H})$ and by Lemma 3.5 and Observation 4.6, the degree cospectrum of $\mathbf{Rea}(\mathcal{G})$ is a sub-cospectrum of that of $\mathbf{Rea}(\mathcal{H})$ up to an oracle p. Fix enumerations $\mathcal{G} = (g_n)_{n \in \mathbb{N}}$ and $\mathcal{H} = (h_n)_{n \in \mathbb{N}}$.

Claim. *For any n and $u \geq_T p$, there are m and v such that $\mathcal{J}_a(g_n, u) = \mathcal{J}_a(h_m, v)$.*

By Lemma 5.8 (2), for any n and $u \geq_T p$, there is $x \in \mathbf{Rea}(\mathcal{G})$ such that $\mathcal{J}_T(g_n, u) \subseteq \mathrm{coSpec}(x) \subseteq \mathcal{J}_a(g_n, u)$. Note that $p \leq_{\mathrm{T}} u$ implies $p \in \mathcal{J}_T(g_n, u) \subseteq \mathrm{coSpec}(x)$; that is, p is x-computable and, therefore, $\mathrm{coSpec}(x) = \mathrm{coSpec}^p(x)$. By our assumption, there is $y \in \mathbf{Rea}(\mathcal{H})$ such that $\mathrm{coSpec}^p(x) = \mathrm{coSpec}^p(y)$.

We claim that $p \leq_{\mathrm{T}} y$: Otherwise, (y, p) has Turing degree by *almost totality* of continuous degrees (cf. [3]); that is, if $\mathbf{q} = \deg(q) \in \mathrm{Spec}([0,1]^{\mathbb{N}}) \setminus \mathrm{Spec}(\mathbb{N}^{\mathbb{N}})$, then, for any $p \in \mathbb{N}^{\mathbb{N}}$, the condition $\deg(p, q) \in \mathrm{Spec}(\mathbb{N}^{\mathbb{N}})$ is equivalent to $p \not\leq_{\mathrm{T}} q$. However, that (y, p) has a Turing degree means that $\mathrm{coSpec}^p(y)$ forms a principal ideal. Then, $\mathrm{coSpec}^p(y) = \mathrm{coSpec}(x)$ implies that $\mathrm{coSpec}(x)$ is principal, which is equivalent to saying that x has a Turing degree. However, it is impossible since $x \in \mathbf{Rea}(\mathcal{G}) \subseteq \mathbb{N}^3 \times \mathrm{Fix}(\Psi)$ implies that x has no Turing degree by Lemma 4.8.

Thus, we have $\mathrm{coSpec}(y) = \mathrm{coSpec}^p(y)$. Then, by Lemma 5.8 (1), there exist m and v such that $\mathcal{J}_T(h_m, v) \subseteq \mathrm{coSpec}(y) \subseteq \mathcal{J}_a(h_m, v)$. Now, $\mathrm{coSpec}(x) = \mathrm{coSpec}(y)$ holds and note that $\mathcal{J}_T(h_m, v) \subseteq \mathcal{J}_a(g_n, u)$ implies $\mathcal{J}_a(h_m, v) \subseteq \mathcal{J}_a(g_n, u)$. This verifies the claim.

For a fixed n, $\beta_n(u)$ chooses m fulfilling the above claim for some v. It is not hard to see that there is $m(n)$ such that $\beta_n(u) = m(n)$ for cofinally many u.

We will show that, for cofinally many u, there is v such that $\mathcal{J}_a(g_n, u \oplus v) = \mathcal{J}_a(h_{m(n)}, u \oplus v)$: By our proof of the above claim, such u and v involve some x and y such that $u \in \mathrm{coSpec}(x) = \mathrm{coSpec}(y) \ni v$. This x also has the property $\mathrm{coSpec}(x) \subseteq \mathcal{J}_a(g_n, u)$. Thus, $v \in \mathcal{J}_a(g_n, u)$; that is, $v \leq_a g_n^i(u)$ for some i and, therefore, by monotonicity of g_n, we get $g_n^j(u \oplus v) \leq_a g_n^{i+j}(u)$. Thus, $\mathcal{J}_a(g_n, u \oplus v) = \mathcal{J}_a(g_n, u)$. Similarly, we have $\mathcal{J}_a(h_{m(n)}, u \oplus v) = \mathcal{J}_a(h_{m(n)}, v)$.

Therefore, $g_n \equiv_{aa} h_{m(n)}$. Consequently, $\mathcal{G} \subseteq_{aa} \mathcal{H}$. □

Proof of Theorem 5.1. Let S be a countable subset of ω_1. Note that $\sup S$ is countable by regularity of ω_1. Then, there is an oracle p such that $\sup S < \omega_1^{\mathrm{CK},p}$, where $\omega_1^{\mathrm{CK},p}$ is the smallest noncomputable ordinal relative to p. Now, the αth Turing jump operator j_α^p for $\alpha < \omega_1^{\mathrm{CK},p}$ is defined via a p-computable coding of α. By Spector's uniqueness theorem (see Sacks [58, Corollary II.4.6] or Chong-Yu [9, Section 2.3]), the Turing degree of $j^p_\alpha(x)$ for $x \geq_T p$ is independent of the choice of coding of α and so is $\mathcal{J}_a(j^p_\alpha, x)$. Therefore, we simply write j_α for j^p_α.

Define $\mathcal{G}_S = \{j_{\omega^{1+\alpha}} : \alpha \in S\}$. We show that $S \subseteq T$ if and only if $\mathcal{G}_S \subseteq_{aa} \mathcal{G}_T$. Suppose $\alpha \neq \beta$, say, $\alpha < \beta$. Clearly, $j_{\omega^\alpha} \leq_{aa} j_{\omega^\beta}$. Suppose for the sake of contradiction that $j_{\omega^\beta} \leq_{aa} j_{\omega^\alpha}$. Then, in particular, for every $x \leq_a \emptyset^{(\omega^\beta \cdot t)}$ with $t \in \mathbb{N}$, we must have $\emptyset^{(\omega^\beta \cdot (t+1))} \leq_a x^{(\omega^\alpha \cdot m)}$ for some $m \in \mathbb{N}$. Thus, there is n such that $\emptyset^{(\omega^\beta \cdot t + \omega^\beta)} \leq_{\mathrm{T}} \emptyset^{(\omega^\beta \cdot t + \omega^\alpha \cdot m + n)} <_T \emptyset^{(\omega^\beta \cdot t + \omega^{\alpha+1})}$. This is a contradiction.

Now, given countable sets $S, T \subseteq \omega_1$, if $S \subseteq T$, then $\mathbf{Rea}(\mathcal{G}_S)$ clearly embeds into $\mathbf{Rea}(\mathcal{G}_T)$. If $S \not\subseteq T$, then the above argument shows that $\mathcal{G}_S \not\subseteq_{aa} \mathcal{G}_T$ and, therefore, by Lemma 5.4, we have $\mathbf{Rea}(\mathcal{G}_S) \not\leq^{\mathfrak{T}}_\sigma \mathbf{Rea}(\mathcal{G}_T)$. Consequently, $S \mapsto \gamma\mathbf{Rea}(\mathcal{G}_S)$ is an order-preserving embedding of $([\omega_1]^{\leq\omega}, \subseteq)$ into the σ-embeddability order $\leq^{\mathfrak{T}}_\sigma$ on compact metrisable spaces, where $\gamma\mathbf{X}$ is Lelek's compactification of $\mathbf{X}$ in Fact 4.1. □

Corollary 5.11. *There exists a collection $(\mathbf{X}_\alpha)_{\alpha<2^{\aleph_0}}$ of continuum many compact metrisable spaces satisfying the following conditions:*

1. *If $\alpha \neq \beta$, then $\mathbf{X}_\alpha$ does not finite-level Borel embed into $\mathbf{X}_\beta$.*
2. *If $\alpha \neq \beta$, then the Banach algebra $\mathcal{B}^*_n(\mathbf{X}_\alpha)$ is not linearly isometric (not ring isomorphic etc.) to $\mathcal{B}^*_n(\mathbf{X}_\beta)$ for all $n \in \mathbb{N}$.*

Proof. By Theorems 3.4 and 5.1. Here, we note that if $\mathbf{X}$ is nth-level Borel isomorphic to $\mathbf{Y}$, then $\mathbb{N} \times \mathbf{X}$ is again nth-level Borel isomorphic to $\mathbb{N} \times \mathbf{Y}$. □

6. Infinite-Dimensional Topology

6.1. Pol's Compactum

In this section, we will shed light on dimension-theoretic perspectives of the ω-left-CEA space. Note that $\omega\mathbf{CEA}$ is a totally disconnected infinite-dimensional space. We first compare our space $\omega\mathbf{CEA}$ and a totally disconnected infinite-dimensional space **RSW** which is constructed by Rubin, Schori and Walsh [57]. A *continuum* is a connected compact metric space and a continuum is *nondegenerated* if it contains at least two points.

It is known that the hyperspace $\mathsf{CK}(\mathbf{X})$ of continua in a compact metrisable space $\mathbf{X}$ equipped with the Vietoris topology forms a Polish space. Hence, we may think of $\mathsf{CK}(\mathbf{X})$ as a represented space, which corresponds to a positive and negative representation of the hyperspace in computable analysis. We consider the closed subspace S of $\mathsf{CK}([0,1]^{\mathbb{N}})$ consisting of all continua connecting opposite faces $\pi_0^{-1}\{0\}$ and $\pi_0^{-1}\{1\}$. Then, fix a total Cantor representation of S; that is, a continuous surjection δ_{CK} from the Cantor set $C \subseteq [0,1]$ onto S.

Remark 6.1. If we are interested in effectivity, we can give a more explicit construction of δ_{CK}: Let δ be the standard positive-and-negative representation of the hyperspace $\mathcal{A}([0,1]^{\mathbb{N}})$ of all closed subsets of the Hilbert cube. Then, one can see that S is a Π^0_1 subspace of $\mathcal{A}([0,1]^{\mathbb{N}})$ as follows.

A δ-name of $A \in \mathcal{A}([0,1]^{\mathbb{N}})$ can be thought of as an enumeration of all finite open covers $\mathcal{U} = \{U_i\}_{i<n}$ of A such that $A \cap U_i \neq \emptyset$ for each $i < n$. If A is disconnected, then there are disjoint open sets U, V in $[0,1]^{\mathbb{N}}$ such that $A \subseteq U \cup V$. Thus, by compactness of A, such disjoint open sets U and V can be replaced with finite unions U_s, V_s of basic open sets. Note that, if U and V are finite unions of basic open sets, one can effectively decide whether U and V are disjoint. This means that, if p is a name of a disconnected set, then some $p(s)$ witnesses this fact.

Next, assume that A does not connect $\pi_0^{-1}\{0\}$ and $\pi_0^{-1}\{1\}$. First consider the case that $A \cap \pi_0^{-1}\{y\}$ is empty for some $y \in \{0,1\}$. By compactness, $A \cap \pi_0^{-1}\{y\} = \emptyset$ is a Σ^0_1 property relative to a name of A. Thus, this fact is witnessed after reading a finite amount of information of the name. Now, assume that $A \cap \pi_0^{-1}\{y\}$ is nonempty for each $y \in \{0,1\}$. Then, that A does not connect $\pi_0^{-1}\{0\}$ and $\pi_0^{-1}\{1\}$ means that there are disjoint open sets U and V such that $A \cap \pi_0^{-1}\{0\} \subseteq U$ and $A \cap \pi_0^{-1}\{1\} \subseteq V$. As in the previous argument, this fact is also witnessed by some $p(s)$.

Consequently, S is Π^0_1 – that is, if p is a name of some $A \notin S$ – then one can obtain this information from p by some finite stage. As usual, one can consider a partial name $(p(0), p(1), \dots, p(s-1))$ as a closed set A^p_s and it converges to $A^p := \delta(p)$. By the above argument, if $A^p \notin S$, then $A^p_s \notin S$ for some s. If s is the least such number, then $A^p_{s-1} \in S$ and we define $\delta_{\mathsf{CK}}(p) = A^p_{s-1}$. If it does not happen, then $\delta_{\mathsf{CK}}(p) = \delta(p)$. Then, δ_{CK} is a total representation of S.

We define the *Rubin–Schori–Walsh space* **RSW** [57] (see also [64, Theorem 3.9.3]) as follows:

$$\begin{aligned}\mathbf{RSW} &= \{\min(\delta_{\mathsf{CK}}(p)^{[p]}) : p \in C\},\\ &= \{\min A^{[p]} : A \text{ is the } p\text{th continuum of } [0,1]^{\mathbb{N}} \text{ with } [0,1] \subseteq \pi_0[A]\},\end{aligned}$$

where $A^{[p]} = A \cap \pi_0^{-1}\{p\} = \{z \in A : \pi_0(z) = p\}$ and recall that $\min P$ is the leftmost point of P defined in the proof of Theorem 4.5. For notational convenience, without loss of generality, we may assume that the eth z-computable continuum is equal to the $\langle e, z\rangle$th continuum, where recall that $\langle \cdot, \cdot \rangle$ is a pairing function.

A compactification of **RSW** is well-known in the context of Alexandrov's old problem in dimension theory. Pol's compactum **RP** is given as a compactification in the sense of Lelek of the space **RSW**. Hence, we can see that **RP** and **RSW** have the same point degree spectra (modulo an oracle) as in the proof of Fact 4.1. Surprisingly, these spaces have the same degree spectra as the space ω**CEA** up to an oracle.[2]

Theorem 6.2. *All of the following spaces have the same point degree spectra relative to some oracle:*

1. *The ω-left-CEA space ω***CEA***.*
2. *Rubin–Schori–Walsh's totally disconnected strongly infinite-dimensional space* **RSW***.*
3. *Roman Pol's counterexample* **RP** *to Alexandrov's problem.*

Indeed, we will show that $\mathbf{RP} \simeq^{\mathfrak{T}}_{\sigma} \mathbf{RSW} \leq_c \omega\mathbf{CEA} \leq_{\sigma\mathbf{RSW}}$. As a corollary, $\mathbb{N} \times \omega\mathbf{CEA}$, $\mathbb{N} \times \mathbf{RSW}$ and $\mathbb{N} \times \mathbf{RP}$ are all σ-homeomorphic (hence second-level Borel isomorphic) to each other. To prove Theorem 6.2, we show two lemmata.

Lemma 6.3. *Every point of* **RSW** *is ω-left-CEA.*

Proof. As we have seen in the proof of Theorem 4.5, $\min A^{[p]}$ is ω-left-CEA in p, since $A^{[p]}$ is $\Pi^0_1(p)$ (see also Remark 6.1 on δ_{CK}). Moreover, clearly, $p \leq_{\mathrm{T}} \min A^{[p]}$. Thus, $\min A^{[p]}$ is ω-left-CEA. □

For $\omega\mathbf{CEA} \leq^{\mathfrak{T}}_{\sigma} \mathbf{RSW}$, we need to show that every ω-left-CEA point is realised as a leftmost point of a computable continuum in a uniform manner. Indeed, we will show the following.

Lemma 6.4. *Suppose that $x \in [0,1]^{\mathbb{N}}$ is ω-left-CEA in a point $z \in \{0,1\}^{\mathbb{N}}$. Then, there is a nondegenerated z-computable continuum $A \subseteq [0,1]^{\mathbb{N}}$ such that $[0,1] \subseteq \pi_0[A]$ and $\min A^{[p]} = (p, x)$ for a name p of A.*

Proof. Given $p = \langle e, z\rangle$, we will effectively construct a name $\Psi(p)$ of a continuum A. We can view this construction as defining a computable function $f : \mathbb{N} \to \mathbb{N}$ such that $\Phi^z_{f(e)} = \Psi(p)$. By Kleene's recursion theorem (Fact 2.1), we may fix e such that, for $p = \langle e, z\rangle$, the pth continuum is equal to the $\Psi(p)$th continuum.

We first describe how to obtain the negative information about $\Psi(p)$. Fix an ω-left-CEA operator J generated by $\langle W_n\rangle_{n\in\mathbb{N}}$ such that $J(z) = x$. Here, as in the proof of Proposition 4.2, each W_n is a c.e. list of pairs (i, q), which indicates that 'if a given n-tuple $(z_0, \dots, z_{n-1})$ is in the ith ball $B^n_i \subseteq [0,1]^n$, then $J^n_{W_n}(z_0, \dots, z_{n-1}) \geq q$'. Since $p = \langle e, z\rangle$ for some $e \in \mathbb{N}$, we have a computable function π with $\pi(p) = z$ and then redefine W_0 to be $W_0 \circ \pi$. In this way, we may assume that $J(p) = x$.

[2]According to an anonymous referee, Motto Ros had independently conjectured that Roman Pol's compactum would have an intermediate σ-homeomorphism type.

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At stage 0, Ψ constructs $A_0 = [0,1] \times [0,1]^{\mathbb{N}}$. At stage $s+1$, if we find some rational open ball $B_i^n \subseteq [0,1]^n$ and a rational $q \in \mathbb{Q}$ such that $W_{n,s}$ declares that 'if a given n-tuple $(z_0, \ldots, z_{n-1})$ is in the ith ball B_i^n, then $J^n_{W_n}(z_0, \ldots, z_{n-1}) \geq q$', by enumerating (i,q), then Ψ removes $\pi_0^{-1}[B(p; 2^{-s})] \cap (B_i^n \times [0,q) \times [0,1]^{\mathbb{N}})$ from the previous continuum A_{s-1}, where $B(p; 2^{-s})$ is the rational open ball with centre p and radius 2^{-s}.

Now, we show $\min A^{[p]} = x := (x_0, x_1, \ldots)$. Assume that $x_0, \ldots, x_{n-1}$ is an initial segment of $\min A^{[p]}$. We will show that $x_n = \pi_n(\min A^{[p]}) = \min \pi_n[\{z \in A^{[p]} : (\forall i < n)\ \pi_i(z) = x_i\}]$. Since $J^n_{W_n}(p, x_0, \ldots, x_{n-1}) = x_n$, W_n declares this fact at some point; that is, for any rational $q < x_n$, there is i such that $(i,q) \in W_n$ and $(p, x_0, \ldots, x_{n-1}) \in B_i^n$. Therefore, $A \cap (\pi_0^{-1}[B(p; 2^{-s})] \cap (B_i^n \times [0,q) \times [0,1]^{\mathbb{N}})) = \emptyset$. Hence, if $y < x_n$, then no extension of $(p, x_0, \ldots, x_{n-1}, y)$ is contained in A. Moreover, if $(p, x_0, \ldots, x_{n-1}) \in B_i^n$ and $q < x_n$, then $(i,q) \notin W_n$. Hence, $x_n = \pi_n(\min A^{[p]})$ as desired.

Now, clearly, $\min A^{[p]} = (p, x)$. Note that Ψ defines a z-computable continuum A in a uniform manner. We can obtain the positive information, too, as we remove only subsets of $\pi_0^{-1}[B(p; 2^{-s})]$ after stage s. Thus, to ascertain that a ball of radius greater than 2^{-s} intersects A, we only need to perform the construction up to stage s. For the connectivity, assume that $A \subseteq U \cup V$ for some open sets $U, V \subseteq [0,1]^{\mathbb{N}}$. By compactness, one can assume that U and V mention only finitely many coordinates; that is, there is n_0 such that if $y = (y_n)_{n \in \mathbb{N}} \in U$ (V, respectively) then $(y_0, \ldots, y_{n_0}, z_{n_0+1}, z_{n_0+2}, \ldots) \in U$ (V, respectively) for any $(z_{n_0+m})_{m \in \mathbb{N}}$. Given $y = (y_n)_{n \in \mathbb{N}}$, define $y^* = (y_0, \ldots, y_{n_0}, \vec{1})$. By our choice of n_0 and our definition of A, $y \in A \cap U$ implies $y^* \in A \cap U$. By our construction of A, if $k \leq n_0$, then any $(y_0, \ldots, y_k, \vec{1}) \in A \cap U$ is connected to $(y_0, \ldots, y_{k-1}, 1, \vec{1}) \in A \cap U$ by a line segment inside $A \cap U$. Therefore, for any point $y \in A \cap U$, y^* is connected to $\vec{1}$ by a polygonal line inside $A \cap U$. The same holds true for V. Hence, if $A \cap U$ and $A \cap V$ are nonempty, $y \in A \cap U$ and $z \in A \cap V$ say, $y^* \in A \cap U$ and $z^* \in A \cap V$ and they are connected to $\overline{1}$ and, therefore, there is a path from y^* to z^* in $A \cap (U \cup V)$. By connectivity of the path, $A \cap U$ and $A \cap V$ have an intersection in the path. This shows that A cannot be written as a union of disjoint open subsets. Consequently, A is connected. □

Proof of Theorem 6.2. By Theorem 3.4 and Lemmata 6.3 and 6.4. □

The properness of **RSW** $<^{\mathfrak{T}}_{\sigma}$ $[0,1]^{\mathbb{N}}$ can also be obtained by some relatively recent work on infinite-dimensional topology: the Hilbert cube (indeed, any strongly infinite-dimensional compactum) is not σ-hereditary-disconnected (see [52]). However, such an argument does not go any further for constructing second-level Borel isomorphism types and, indeed, to the best of our knowledge, no known topological technique provides us four or more second-level Borel isomorphism types.

On a side note, one can also define the graph n**CEA** $\subseteq \mathbb{N} \times \{0,1\}^{\mathbb{N}} \times [0,1]^n$ of a universal n-left-CEA operator (as an analogy of an n-REA operator) in a straightforward manner. The space n**CEA** is an example of a finite-dimensional Polish space whose infinite product has again the same dimension. The first such examples were constructed by Kulesza in [32].

Proposition 6.5. *The space* n**CEA** *is a totally disconnected n-dimensional Polish space. Moreover, the countable product* n**CEA**$^{\mathbb{N}}$ *is again n-dimensional.*

Proof. Clearly, n**CEA** is totally disconnected and Polish. To check the n-dimensionality, we think of n**CEA** as a subspace of $[0,1]^{n+1}$ by identifying $(e,x) \in \mathbb{N} \times \{0,1\}^{\mathbb{N}}$ with $\iota(0^e 1x) \in [0,1]$, where ι is a computable embedding of $\{0,1\}^{\mathbb{N}}$ into $[0,1]$. We claim that n**CEA** intersects with all continua $A \subseteq [0,1]^{n+1}$ such that $[0,1] \subseteq \pi_0[A]$. We have a computable function d such that the $d(e)$th n-left-CEA procedure $J^n_{d(e)}(x)$ for a given input $x \in \{0,1\}^{\mathbb{N}}$ outputs the value $y \in [0,1]^n$ such that $(\iota(0^e 1x), y) = \min A_{e,x}^{[\iota(0^e 1x)]}$, where $A_{e,x}$ is the eth x-computable continuum in $[0,1]^{n+1}$ such that $[0,1] \subseteq \pi_0[A_{e,x}]$. By Kleene's recursion theorem (Fact 2.1), there is r such that $J^n_{d(r)} = J^n_r$. Hence, $(\iota(0^r 1x), J^n_r(x)) \in n$**CEA** $\cap A_{e,x}$, which verifies the claim. The claim implies that n**CEA** is n-dimensional (see van Mill [64, Corollary 3.7.5]).

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To verify the second assertion, consider the (computably) continuous map g from the square $n\mathbf{CEA}^2$ into $\{0,1\}^{\mathbb{N}} \times [0,1]^n$ such that for two points $\mathbf{x} = (e, r, x_0, \dots, x_{n-1})$ and $\mathbf{y} = (d, s, y_0, \dots, y_{n-1})$ in $n\mathbf{CEA}$,

$$g(\mathbf{x}, \mathbf{y}) = (\langle e, d \rangle, r \oplus s, (x_0 + y_0)/2, \dots, (x_{n-1} + y_{n-1})/2).$$

To verify that g^{-1} is also (computably) continuous, using given left r- and s-computable approximations of x_0 and y_0, one can compute x_0 and y_0 from $(x_0 + y_0)/2$. By induction, one can computably recover $\mathbf{x}$ and $\mathbf{y}$ from $g(\mathbf{x}, \mathbf{y})$. Hence, $n\mathbf{CEA}^2$ is computably embedded into $\{0,1\}^{\mathbb{N}} \times [0,1]^n$. In particular, it is n-dimensional. The same argument shows that $n\mathbf{CEA}^k$ is n-dimensional for any $k \in \mathbb{N}$. Then, we can conclude that $n\mathbf{CEA}^{\mathbb{N}}$ is also n-dimensional (by the same argument as in van Mill [64, Theorem 3.9.5]). □

6.2. Nondegenerated Continua and ωCEA *Degrees*

We may extract computability-theoretic contents from the construction of Rubin–Schori–Walsh's strongly infinite-dimensional totally disconnected space **RSW**. The standard proof of noncountable dimensionality of **RSW** (hence, the existence of a non-Turing degree in **RSW**) indeed implies the following computability theoretic result.

Proposition 6.6. *There exists a nondegenerated continuum $A \subseteq [0,1]^{\mathbb{N}}$ in which no point has Turing degree.*

Proof. Define $\mathbf{H}_{\langle i,j \rangle} \subseteq [0,1]^{\mathbb{N}}$ to be the set of all points which can be identified with an element in $\{0,1\}^{\mathbb{N}}$ via the witnesses Φ_i and Φ_j (as in the proof of Lemma 3.5). Then, $\bigcup_n \mathbf{H}_n$ is the set of all points in $[0,1]^{\mathbb{N}}$ having Turing degrees. Note that each $\mathbf{H}_n$ is zero-dimensional since it is homeomorphic to a subspace of $\{0,1\}^{\mathbb{N}}$.

Consider the hyperplane $P_n^i = [0,1]^n \times \{i\} \times [0,1]^{\mathbb{N}}$ for each $n \in \mathbb{N}$ and $i \in \{0,1\}$. It is well known that $\{(P_n^0, P_n^1)\}_{n \in \mathbb{N}}$ is essential in $[0,1]^{\mathbb{N}}$. Then, by using the dimension-theoretic fact (see van Mill [64, Corollary 3.1.6]), we can find a separator L_n of (P_{n+1}^0, P_{n+1}^1) in $[0,1]^{\mathbb{N}}$ such that $L_n \cap \mathbf{H}_n = \emptyset$ since $\mathbf{H}_n$ is zero-dimensional.

Put $L = \bigcap_n L_n$. Then, L contains no point having Turing degree, since $L \cap \mathbf{H}_n = \emptyset$ for every $n \in \mathbb{N}$. Moreover, L contains a continuum A from P_0^0 to P_0^1 (see van Mill [64, Proposition 3.7.4]). □

Recall that our infinite-dimensional version of Kreisel's basis theorem (shown in the proof of Theorem 4.5) says that every Π_1^0 subset P of the Hilbert cube has a point of an ω-left-CEA continuous degree. Surprisingly, we do not need any effectivity assumption on P to prove this if P is a nontrivial connected compact set.

Proposition 6.7. *Every nondegenerated continuum $A \subseteq [0,1]^{\mathbb{N}}$ contains a point of an ω-left-CEA continuous degree.*

Proof. Note that there is $n \in \omega$ such that $P_n^{[0,p]}$ and $P_n^{[q,1]}$ with some rationals $p < q \in \mathbb{Q}$ intersect with A, since A is nondegenerated, where $P_n^{[a,b]} = [0,1]^n \times [a,b] \times [0,1]^{\mathbb{N}}$. Clearly, there is no separator C of $P_n^{[0,p]}$ and $P_n^{[q,1]}$ with $C \cap A = \emptyset$ (i.e., the pair $(P_n^{[0,p]}, P_n^{[q,1]})$ is essential in A), since A is not zero-dimensional. Therefore, the pair (P_n^p, P_n^q) is essential in the compact subspace $A \cap P_n^{[p,q]}$. Hence, $A \cap P_n^{[p,q]}$ contains a continuum B intersecting with P_n^p and P_n^q (see van Mill [64, Proposition 3.7.4]). Consider a computable homeomorphism $h : P_n^{[p,q]} \cong [0,1]^{\mathbb{N}}$ mapping P_n^p and P_n^q to $P_0^0 = \pi_0^{-1}(0)$ and $P_0^1 = \pi_1^{-1}(1)$, respectively. Then $h[B]$ is a continuum intersecting with $\pi_0^{-1}(0)$ and $\pi_0^{-1}(0)$ and therefore $[0,1] \subseteq \pi_0[h[B]]$. Let s be a name of $h[B]$. Then, by definition, $\min h[B]^{[s]} \in \mathbf{RSW}$, which has an ω-left-CEA continuous degree by Lemma 6.3. In particular, $h[B]$ contains a point of an ω-left-CEA continuous degree and so does A since h is a computable homeomorphism and $B \subseteq A$. □

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As a corollary, we can see that every compactum $A \subseteq [0,1]^{\mathbb{N}}$ of positive dimension contains a point of an ω-left-CEA continuous degree. Our proof of Theorem 6.6 is essentially based on the fact that for any sequence of zero-dimensional spaces $\{X_i\}_{i\in\mathbb{N}}$, there exists a continuum avoiding all X_is. Contrary to this fact, Theorem 6.7 says that $\{X_i\}_{i\in\mathbb{N}}$ cannot be replaced with a sequence of totally disconnected spaces. We say that a space is σ-totally disconnected if it is a countable union of totally disconnected subspaces. Note that the complement of a σ-totally-disconnected subset of the Hilbert cube is infinite-dimensional.

Corollary 6.8. *There exists a σ-totally-disconnected set $X \subseteq [0,1]^{\mathbb{N}}$ such that any compact subspace of the complement $Y = [0,1]^{\mathbb{N}} \setminus X$ is zero-dimensional.*

Proof. Define $X_{\langle i,j\rangle}$ to be the set of all points which can be identified with an element in $\omega\mathbf{CEA}$ via the witnesses Φ_i and Φ_j. Then, $X_{\langle i,j\rangle}$ is totally disconnected since it is homeomorphic to a subspace of $\omega\mathbf{CEA}$. Clearly, no point $Y = [0,1]^{\mathbb{N}} \setminus \bigcup_{i,j\in\mathbb{N}} X_{\langle i,j\rangle}$ has an ω-left-CEA continuous degree. Assume that Z is a compact subspace of Y of positive dimension. Then Z has a nondegenerated subcontinuum A. However, by Theorem 6.7, A contains a point of an ω-left-CEA continuous degree. □

6.3. Weakly Infinite-Dimensional Cantor Manifolds

Recall that a Pol-type Cantor manifold is a compact metrisable C-space which cannot be disconnected by a hereditarily weakly infinite-dimensional compact subspaces. By combining a known construction in infinite-dimensional topology, we can slightly extend Theorem 5.1 as follows.

Proposition 6.9. *There exists a collection $(\mathbf{X}_\alpha)_{\alpha<2^{\aleph_0}}$ of continuum many Pol-type Cantor manifolds satisfying the following conditions:*

1. *if $\alpha \neq \beta$, $\mathbf{X}_\alpha$ does not σ-embed into $\mathbf{X}_\beta$.*
2. *If $\alpha \neq \beta$, then $\mathbf{X}_\alpha$ does not finite-level Borel embed into $\mathbf{X}_\beta$.*
3. *If $\alpha \neq \beta$, then the Banach algebra $\mathcal{B}^*_n(\mathbf{X}_\alpha)$ is not linearly isometric (not ring isomorphic etc.) to $\mathcal{B}^*_n(\mathbf{X}_\beta)$ for all $n \in \mathbb{N}$.*

Lemma 6.10. *For any $\mathcal{G}$, there exists a Pol-type Cantor manifold $\mathbf{Z}(\mathcal{G})$ such that $\omega\mathbf{CEA} \oplus \mathbf{Rea}(\mathcal{G}) \equiv^{\mathfrak{T}}_{\sigma} \mathbf{Z}(\mathcal{G})$.*

Proof. Recall from Theorem 6.2 that $\omega\mathbf{REA}$ is σ-homeomorphic to a strongly infinite-dimensional space $\mathbf{RSW}$. Let $\mathbf{R}_0$ and $\mathbf{R}_1$ be homeomorphic copies of $\mathbf{RSW}$ and let $\mathbf{X}$ be a compactification of $\mathbf{R}_0 \oplus \mathbf{R}_1 \oplus \mathbf{Rea}(\mathcal{G})$ in the sense of Lelek (recall from Fact 4.1). Then, $\mathbf{X}$ is σ-homeomorphic to $\omega\mathbf{CEA} \oplus \mathbf{Rea}(\mathcal{G})$.

We follow the construction of Elżbieta Pol [48, Example 4.1]. Now, $\mathbf{R}_0$ has a hereditarily strongly infinite-dimensional subspace $\mathbf{Y}$ [56]. Choose a point $p \in \mathbf{Y}$ and a closed set $F \subseteq \mathbf{Y}$ containing p such that every separator between p and $\mathrm{cl}_{\mathbf{X}}F$ is strongly infinite-dimensional as in [48, Example 4.1 (A)].

Define $\mathbf{K} = \mathbf{X}/\mathrm{cl}_{\mathbf{X}}F$ as in [48, Example 4.1 (A)]. To see that $\mathbf{K}$ is σ-homeomorphic to $\mathbf{X}$, we note that $\mathrm{cl}_{\mathbf{X}}F \cap (\mathbf{R}_1 \cup \mathbf{Rea}(\mathcal{G})) = \emptyset$ since $\mathbf{R}_0$, $\mathbf{R}_1$ and $\mathbf{Rea}(\mathcal{G})$ are separated in $\mathbf{X}$. Therefore, $\mathrm{cl}_{\mathbf{X}}F$ is covered by the union of $\mathbf{R}_0$ (which is homeomorphic to $\mathbf{R}_1$) and a countable dimensional space. Define $\mathbf{Z}$ as a Pol-type Cantor manifold in [48, Example 4.1 (C)]. Then, $\mathbf{Z}(\mathcal{G}) := \mathbf{Z}$ is the union of a finite-dimensional space and countably many copies of $\mathbf{K}$. Consequently, $\mathbf{Z}(\mathcal{G})$ is σ-homeomorphic to $\mathbf{Rea}(\mathcal{G})$. □

Proof of Proposition 6.9. Combine Theorem 5.1, Corollary 5.11 and Lemma 6.10. □

Acknowledgements. The authors are grateful to Masahiro Kumabe, Joseph Miller, Luca Motto Ros, Philipp Schlicht and Takamitsu Yamauchi for their insightful comments and discussions. The work has benefitted from the Marie Curie International Research Staff Exchange Scheme *Computable Analysis*, PIRSES-GA-2011-294962. For the duration of this research, the first author was partially supported by a Grant-in-Aid for JSPS fellows (FY2012–2014) and for JSPS overseas research fellows (FY2015–2016; Host: University of California, Berkeley).

Conflicts of Interest. None.

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