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NONOSCILLATORY SOLUTIONS OF NEUTRAL DELAY DIFFERENTIAL EQUATIONS

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Consider the following neutral delay differential equation

$$(*) \qquad \frac{d}{dt}[x(t)+px(t-\tau)]+Q(t)x(t-\delta)=0,\ t\geqslant t_0$$

where $p\in R$, $\tau\in(0,\infty)$, $\delta\in R^+=[0,\infty)$ and $Q\in C([t_0,\infty),R)$. We show that if

$$(**) \qquad \int_0^\infty |Q(s)|\,ds<\infty$$

then Equation (*) has a nonoscillatory solution when $p\neq -1$. We also deal in detail with a conjecture of Chuanxi, Kulenovic and Ladas, and Györi and Ladas.

1. INTRODUCTION

Consider the following neutral delay differential equation

$$(1) \qquad \frac{d}{dt}[x(t)+px(t-\tau)]+Q(t)x(t-\delta)=0,\ t\geqslant t_0$$

where

$$(2) \qquad p\in R,\ \tau\in(0,\infty),\ \delta\in R^+=[0,\infty)\ \text{ and } Q\in C([t_0,\infty),R).$$

Recently, the oscillation and asymptotic behaviour of Equation (1) have been investigated by many authors, see for example [**1, 2, 3, 5, 6, 7, 8**]. For a recent survey, see [**4**]. All the papers mentioned above, however, assume that $Q(t)$ is nonnegative. Considerably less is known about the behaviour of the solutions of Equation (1) when the coefficient $Q(t)$ is oscillatory. In particular, by combining the result in [**5, 7**, and **8**] we know that if

$$(3) \qquad Q(t)\geqslant 0 \text{ and } \int_{t_0}^\infty Q(s)ds<\infty$$

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then Equation (1) has a nonoscillatory solution when $p \neq -1$. For the critical case $p = -1$, Yu, Wang and Qian [5] found a sufficient condition for the oscillation of all solutions of Equation (1) under the assumption (3). One of our aims in this paper is to study the existence of a nonoscillatory solution of Equation (1) when $Q(t)$ is oscillatory. In section 2, we show that

$$\int_0^\infty |Q(s)|\,ds < \infty \tag{4}$$

implies that Equation (1) has a nonoscillatory solution when $p \neq -1$.

In addition, the following result on the asymptotic behaviour of a nonoscillatory solution of Equation (1) has been established by Chuanxi, Kulenovic and Ladas [2], see also [4].

Theorem A. [2] *Assume that (2) holds and that*

$$Q(t) \geqslant 0 \text{ and } \int_{t_0}^\infty Q(s)ds = \infty. \tag{5}$$

Let $x(t)$ be nonoscillatory solution of Equation (1). Then the following statements hold:

(a) *If $p < -1$, then $\lim_{t\to\infty} |x(t)| = \infty$*
(b) *If $p > -1$, and $p \neq 1$, then $\lim_{t\to\infty} x(t) = 0$.*

Remark 1. As was shown in [1] and [2], the assumption tht $p = -1$, as well as (2) and (5) hold, implies that every solution of Equation (1) oscillates. Thus the assumption in the above Theorem A that $p \neq -1$ is harmless. But, the case $p = 1$ has not yet been handled. Therefore, Chuanxi, Kulenovic and Ladas [2] posed the following conjecture. See also [4, Problem 6.12.9 (Conjecture)].

Conjecture B. [2, 4] *Assume that (2) and (5) hold. Let $x(t)$ be a nonoscillatory solution of the neutral differential equation*

$$\frac{d}{dt}[x(t) + x(t-\tau)] + Q(t)x(t-\delta) = 0,\ t \geqslant t_0. \tag{6}$$

Then

$$\lim_{t\to\infty} x(t) = 0. \tag{7}$$

The second aim in this paper is to answer in detail the above Conjecture B. In section 3, we first give an existence result of a nonoscillatory solution not satisfying (7) of Equation (6), and then by using this result we answer Conjecture B in the negative.

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Finally, we also show that, under appropriate additional hypothesis on $Q(t)$, Conjecture B is also true.

Let $t_1 \geqslant t_0$ and let $\phi \in C([t_1 - m], R)$, where $m = \max\{\tau, \delta\}$. By a solution of Equation (1) with initial function ϕ at t_1 we mean a function $x \in C([t_1 - m], \infty, R)$ such that $x(t) = \phi(t)$ for $t \in [t_1 - m, t_1]$, $x(t) + px(t-\tau)$ is continuously differentiable for $t \geqslant t_1$ and $x(t)$ satisfies Equation (1) for all $t \geqslant t_1$.

As usual, a solution of Equation (1) is called nonoscillatory if it is eventually positive or eventually negative and oscillatory if it has arbitrarily large zeros.

2. Nonoscillatory solutions of equation (1)

In this section we study the existence of a nonoscillatory solution of Equation (1) with $p \neq -1$. The main result in this section is the following theorem.

Theorem 1. *Assume that (2) and (4) hold with $p \neq -1$. Then Equation (1) has a nonoscillatory solution.*

Proof: The proof of this theorem is rather long and will be divided into five claims. Let X be the set of all continuous and bounded functions on $[t_0, \infty)$ with the sup-norm. Then X is a Banach space. ∎

Claim 1. *For the case $-1 < p \leqslant 0$, choose a $t_1 > t_0$ sufficiently large such that $t_1 - \tau \geqslant t_0$, $t_1 - \delta \geqslant t_0$ and*

$$\int_{t_1}^{\infty} |Q(s)|\, ds \leqslant \frac{1+p}{4}.$$

Define a bounded, closed and convex subset of X as follows

$$A = \{x \in X;\ \frac{2(1+p)}{3} \leqslant x(t) \leqslant \frac{4}{3} \text{ for } t \geqslant t_0\}.$$

Now we define a mapping $T : A \to X$ as follows

$$(Tx)(t) = \begin{cases} 1 + p - px(t-\tau) + \int_t^{\infty} Q(s)x(s-\delta)ds, & t \geqslant t_1 \\ (Tx)(t_1), & t_0 \leqslant t \leqslant t_1. \end{cases}$$

Clearly, T is continuous. For every $x \in A$ and $t \geqslant t_1$, we see that

$$(Tx)(t) \leqslant 1 + p - \frac{4}{3}p + \frac{4}{3}\frac{1+p}{4} = \frac{4}{3}$$

and

$$(Tx)(t) \geqslant 1 + p - \frac{4}{3}\frac{1+p}{4} = \frac{2(1+p)}{3}.$$

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Hence, $(2(1+p))/3 \leqslant (Tx)(t) \leqslant 4/3$ *for* $t \geqslant t_0$ *and so* $TA \subset A$.

Now we shall shows that T is a contraction mapping on A. In fact, for any $x_1, x_2 \in A$ and $t \geqslant t_1$ we have

$$\begin{aligned} |(Tx_1)(t) - (Tx_2)(t)| &\leqslant -p\,|x_1(t-\tau) - x_2(t-\tau)| \\ &\qquad + \int_t^\infty |Q(s)|\,|x_1(s-\delta) - x_2(s-\delta)|\,ds \\ &\leqslant \frac{1-3p}{4}\,\|x_1 - x_2\|. \end{aligned}$$

Then it follows that

$$\|Tx_1 - Tx_2\| \leqslant \frac{1-3p}{4}\,\|x_1 - x_2\|.$$

Since $0 < (1-3p)/4 < 1$, we see that T is a contraction. Therefore, by the Banach contraction principle, T has a fixed point $x \in A$, that is, $Tx = x$. Clearly, $x(t)$ is a positive solution of Equation (1) on $[t_1, \infty)$ and so the proof of Claim 1 is complete.

CLAIM 2. *For the case* $p < -1$, *let* $t_1 > t_0$ *be such that* $t_1 + \tau - \delta \geqslant t_0$ *and*

$$\int_{t+\tau}^\infty |Q(s)|\,ds \leqslant -\frac{1+p}{4}.$$

Set

$$A = \{x \in X;\ -\frac{p}{2} \leqslant x(t) \leqslant -2p \text{ for } t \geqslant t_0\}.$$

Then A *is a bounded, closed and convex subset of* X. *Define a mapping* $T : A \to X$ *as follows*

$$(Tx)(t) = \begin{cases} -p - 1 - \dfrac{1}{p}x(t+\tau) + \dfrac{1}{p}\displaystyle\int_{t+\tau}^\infty Q(s)x(s-\delta)ds, & t \geqslant t_1 \\ (Tx)(t_1), & t_0 \leqslant t \leqslant t_1. \end{cases}$$

In is easy to show that T *maps* A *into itself, and by a fashion similar to that in the proof of Claim 1 we see that for any* $x_1, x_2 \in A$

$$\|Tx_1 - Tx_2\| \leqslant \frac{p-3}{4p}\,\|x_1 - x_2\|.$$

This means that T *is a contraction, since* $0 < (p-3)/(4p) < 1$. *Therefore, by the Banach contraction principle,* T *has a fixed point* $x \in A$. *It is easy to see that this* x *is a positive solution of Equation (1) and the proof of Claim 2 is finished.*

CLAIM 3. *For the case* $0 < p < 1$, *let* $t_1 > t_0$ *be such that* $t_1 - \tau \geqslant t_0$, $t_1 - \delta \geqslant t_0$ *and*

$$\int_{t_1}^\infty |Q(s)|\,ds \leqslant \frac{1-p}{4}.$$

Set

$$A = \{x \in X :\ 2(1-p) \leqslant x(t) \leqslant 4 \text{ for } t \geqslant t_0\}$$

which is a bounded, closed and convex subset of X. *Define* $T : A \to X$ *as follows:*

$$(Tx)(t) = \begin{cases} 3 + p - px(t-\tau) + \int_t^\infty Q(s)x(s-\delta)ds, & t \geqslant t_1 \\ (Tx)(t_1), & t_0 \leqslant t \leqslant t_1. \end{cases}$$

It is easy to show that T *maps* A *into* A *and for any* $x_1, x_2 \in A$,

$$\|Tx_1 - Tx_2\| \leqslant \frac{1+3p}{4} \|x_1 - x_2\|.$$

As $0 < (1+3p)/4 < 1$, *the Banach contraction principle can be applied to obtain a fixed point* $x \in A$. *We can see easily that this* x *is a positive solution of Equation (1). This completes the proof of Claim 3.*

Claim 4. *For the case* $p = 1$, *let* $t_1 > t_0$ *be such that* $t_1 + \tau - \delta \geqslant t_0$ *and*

$$\int_{t_1+\tau}^\infty |Q(s)|\, ds \leqslant \frac{1}{4}.$$

Clearly, the set $A = \{x \in X : 2 \leqslant x(t) \leqslant 4,\ \text{for } t \geqslant t_0\}$

is a bounded, closed and convex subset of X. *Define a mapping* T *on* A *as follows*

$$(Tx)(t) = \begin{cases} 3 + \sum_{i=1}^{\infty} \int_{t+(2i-1)\tau}^{t+2i\tau} Q(s)x(s-\delta)ds, & t \geqslant t_1 \\ (Tx)(t_1), & t_0 \leqslant t \leqslant t_1. \end{cases}$$

Clearly T *is continuous. It is easy to show that* T *maps* A *into* A *and for any* $x_1, x_2 \in A$,

$$\|Tx_1 - Tx_2\| \leqslant \frac{1}{4} \|x_1 - x_2\|.$$

Thus, by the Banach contraction principle, T *has a fixed point* $x \in A$, *that is,*

$$x(t) = \begin{cases} 3 + \sum_{i=1}^{\infty} \int_{t+(2i-1)\tau}^{t+2i\tau} Q(s)x(s-\delta)ds, & t \geqslant t_1, \\ x(t_1), & t_0 \leqslant t \leqslant t_1. \end{cases}$$

It follows that $x(t) + x(t-\tau) = 6 + \int_t^\infty Q(s)x(s-\delta)dx,\ t \geqslant t_1 + \tau.$

From this we see that $x(t)$ *is a positive solution of Equation (1) with* $p = 1$ *on* $[t_1 + \tau, \infty)$ *and so the proof for the case* $p = 1$ *is complete.*

Claim 5. *Finally we consider the last case when* $p > 1$. *Let* $t_1 > t_0$ *be such that* $t_1 + \tau - \delta \geqslant t_0$ *and*

$$\int_{t+\tau}^\infty |Q(s)|\, ds \leqslant \frac{p-1}{4}.$$

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Consider the bounded, closed and convex subset of X;

$$A = \{x \in X : \ 2(p-1) \leqslant x(t) \leqslant 4p \text{ for } t \geqslant t_0\},$$

and define a mapping T *on* A *as follows:*

$$(Tx)(t) = \begin{cases} 3p + 1 - \dfrac{1}{p}x(t+\tau) + \dfrac{1}{p}\displaystyle\int_{t+\tau}^{\infty} Q(s)x(s-\delta)ds, & t \geqslant t_1 \\ (Tx)(t_1), & t_0 \leqslant t \leqslant t_1. \end{cases}$$

By an argument similar to that in the previous four cases, we can easily show that all assumptions of the Banach contraction principle are satisfied. Therefore, T *has a fixed point* $x \in A$. *It is easy to see that this* x *is a positive solution of Equation (1) on* $[t_1 + \tau, \infty)$, *and the proof of Theorem 1 is complete.*

3. On the Conjecture B

Consider the neutral delay differential equation

$$\frac{d}{dt}[x(t) + x(t-\tau)] + Q(t)x(t-\delta) = 0 \tag{8}$$

where

$$\tau > 0, \ \delta \geqslant 0 \text{ and } Q \in C([t_0,\infty),(0,\infty)). \tag{9}$$

First we establish the following result on the existence of a nonoscillatory solution of Equation (8).

Theorem 2. *Assume that (9) holds and there exists a nonnegative continuous function* $B(t)$ *on* $[t_0, \infty)$ *such that*

$$B(t) + B(t-\tau) = Constant, \ t \geqslant t_0 + \tau. \tag{10}$$

Also suppose that there exists a positive number λ *such that*

$$\lambda(1 + e^{\lambda\tau}) \geqslant Q(t)[B(t-\delta)e^{\lambda t} + e^{\lambda\delta}], \ t \geqslant t_0 + \delta. \tag{11}$$

Then Equation (8) has a positive solution $x(t)$ *satisfying* $x(t) \geqslant B(t)$ *for* $t \geqslant t_0 + \tau + \delta$ *and* $x(t) - B(t) \to 0$ *as* $t \to \infty$.

Proof: Set

$$y(t) = e^{-\lambda t}$$

Then by (11) we have

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$$[y(t)+y(t-\tau)]'+Q(t)[B(t-\delta)+y(t-\delta)]\leqslant 0,\ t\geqslant t_0+\delta.$$

Also let
$$Q^*(t)=-[y(t)+y(t-\tau)]'/[B(t-\delta)+y(t-\delta)].$$

Then $Q^*(t)\geqslant Q(t)$ and $y(t)$ satisfies

$$[y(t)+y(t-\tau)]'+Q^*(t)[B(t-\delta)+y(t-\delta)]=0,\ t\geqslant t_0+\delta.$$

From this we have

$$y(t)+y(t-\tau)=\int_t^\infty Q^*(t)[B(s-\delta)+y(s-\delta)]ds,\ t\geqslant t_0+\delta$$

which yields

$$y(t)=\sum_{i=1}^{\infty}\int_{t+(2i-1)\tau}^{t+2i\tau}Q^*(s)[B(s-\delta)+y(s-\delta)]ds,\ t\geqslant t_0+\delta-\tau. \tag{12}$$

By a slight modification of Lemma 10.5.2 in [4] we can easily see that the corresponding integral equation

$$z(t)=\sum_{i=1}^{\infty}\int_{t+(2i-1)\tau}^{t+2i\tau}Q(s)[B(s-\delta)+z(s-\delta)]ds$$

has a positive solution $z:[t_0+\delta-\tau,\infty)\to(0,\infty)$ and $0<z(t)\leqslant y(t)$. From (13) we have

$$z(t)+z(t-\tau)=\int_t^\infty Q(s)[B(s-\delta)+z(s-\delta)]ds,\ t\geqslant t_0+\delta.$$

That is
$$[z(t)+z(t-\tau)]'+Q(t)[B(t-\delta)+z(t-\delta)]=0,\ t\geqslant t_0+\delta,$$

which implies that $x(t)=B(t)+z(t)$ is a positive solution of Equation (8) satisfying $x(t)>B(t)$ and $x(t)-B(t)\to 0$ as $t\to\infty$. The proof of Theorem 2 is complete. $\square$

Note that by using Theorem 2, we can easily construct many examples which show that the answer to Conjective B is negative. In fact for any constant $c>0$, we define

$$B(t)=\begin{cases}0, & t\in[2i\tau,\left(2i+\frac{1}{2}\right)\tau]\\ \frac{2c}{\tau}[t-\left(2i+\frac{1}{2}\right)\tau], & t\in[\left(2i+\frac{1}{2}\right)\tau,(2i+1)\tau]\\ c, & t\in[(2i+1)\tau,\left(2i+\frac{3}{2}\right)\tau]\\ \frac{2c}{\tau}[-t+(2i+1)\tau], & t\in[\left(2i+\frac{3}{2}\right)\tau,2(i+1)\tau]\end{cases}$$

where $i=0,1,2,3,\ldots$. Clearly

$$B(t)+B(t-\tau)=c\text{ for }t\geqslant\tau$$

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and $B(t)$ is a nonnegative continuous function on $[0,\infty)$. Thus, if $Q(t)$ satisfies for some $\lambda > 0$

$$\frac{\lambda(1+e^{\lambda\tau})}{2(e^{\lambda\delta}+B(t-\delta)e^{\lambda t})} \leqslant Q(t) \leqslant \frac{\lambda(1+e^{\lambda\tau})}{e^{\lambda\delta}+B(t-\delta)e^{\lambda t}}, \text{ for } t \geqslant \delta,$$

then (5) is satisfied and by Theorem 2 Equation (8) has a positive solution $x(t)$ satisfying $x(t) - B(t) \to 0$ as $t \to \infty$. But $x(t) \nrightarrow 0$ as $t \to \infty$, since $\limsup_{t\to\infty} B(t) = c$.

The above example indeed shows that Conjecture B is not true. But, on the other hand, the following theorem shows that, under appropriate additional hypothesis on $Q(t)$, Conjecture B is true.

Theorem 3. *Assume that all the assumptions of Conjecture B hold. Further assume that there exists a positive constant β such that $\beta \neq 1$, and*

$$Q(t) \leqslant \beta Q(t-\tau), \text{ for } t \geqslant t_0 + \tau. \tag{14}$$

Then every nonoscillatory solution of Equation (6) goes to zero as $t \to \infty$.

Proof: Let $x(t)$ be a nonoscillatory solution of Equation (6). As $-x(t)$ is also a solution of Equation (6), we may assume that $x(t)$ is eventually positive. Thus there exists a $t_1 \geqslant t_0 + \tau$ such that $x(t-\tau) > 0$, $x(t-\delta) > 0$ for $t \geqslant t_1$. Set

$$y(t) = x(t) + x(t-\tau).$$

Then by (6) we have $$y'(t) \leqslant 0, \quad y(t) > 0 \text{ for } t \geqslant t_1$$

and
$$\begin{aligned} y'(t) &= -Q(t)x(t-\delta) = -Q(t)y(t-\delta) + Q(t)x(t-\tau-\delta) \\ &\leqslant -Q(t)y(t-\delta) + \beta Q(t-\tau)x(t-\tau-\delta) \\ &= -Q(t)y(t-\delta) - \beta y'(t-\tau). \end{aligned}$$

That is $$y'(t) + \beta y'(t-\tau) + Q(t)y(t-\delta) \leqslant 0 \text{ for } t \geqslant t_1.$$

It follows by (5) that $$\lim_{t\to\infty} y(t) = 0.$$

Consequently, $$\lim_{t\to\infty} x(t) = 0.$$

The proof of Theorem 3 is complete. ∎

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