Bull. Aust. Math. Soc. **107** (2023), 196–203
doi:10.1017/S0004972722001058

APPROXIMATING NUMBERS OF THE CANTOR SET BY ALGEBRAIC NUMBERS

YUAN ZHANG, JIA LIU and SAISAI SHI

(Received 1 May 2022; accepted 18 August 2022; first published online 3 October 2022)

Abstract

We consider the set of elements in a translation of the middle-third Cantor set which can be well approximated by algebraic numbers of bounded degree. A doubling dimensional result is given, which enables one to conclude an upper bound on the dimension of the set in question for a generic translation.

2020 *Mathematics subject classification*: primary 11J83; secondary 28A78.

Keywords and phrases: Diophantine approximation, Cantor set, Hausdorff dimension, algebraic number.

1. Introduction

In 1984, Mahler [12] posed a question regarding how well the elements in the middle-third Cantor set C can be approximated by rational numbers. This is the starting point for the study of Diophantine approximation on the Cantor set.

The first result towards Mahler's question is due to Weiss [14], who showed that when restricted to the Cantor set C, almost no points (with respect to the standard Cantor measure) can be very well approximated by rational numbers. As a next step, we would like to consider the analogy of Jarník's theorem on the Cantor set. More precisely, for any $u \geq 2$, we consider the size of $\mathcal{W}(u) \cap C$, where

$$\mathcal{W}(u) = \{x \in \mathbb{R} : |x - p/q| < q^{-u}, \text{ for infinitely many rationals } p/q\}.$$

The main difficulty of Mahler's question lies in understanding the distribution of the rational numbers lying close to the Cantor set. A well-known subfamily of rational numbers is the set of left endpoints, denoted by $\mathcal{A}$. Levesley *et al.* [11] considered how well the points in C can be approximated by these specific rational numbers $\mathcal{A}$, presenting a complete metric theory including the measure and Hausdorff dimension. Subsequently, Bugeaud [3] constructed a family of points in C with any prescribed

This work is supported by the Support Plan for Outstanding Young Talents in Colleges in Anhui Province (Key project) (No. gxyqZD2020021) and the Natural Science Research Project of Anhui University of Finance and Economics (Nos. ACKYC22083, ACKYC22084).

approximation order (for partial results, see also [11]). Though not a complete answer to Mahler's question, it is widely believed that

$$\lim_{u\to 2}\dim_H(\mathcal{W}(u)\cap C)=\gamma\quad\text{where } \gamma=\frac{\log 2}{\log 3},\tag{1.1}$$

where $\dim_H$ denotes the Hausdorff dimension and γ is the dimension of the Cantor set C. For further information regarding Diophantine approximation on the Cantor set, see [1, 7, 8].

To attack the conjecture (1.1), Bugeaud and Durand [4] presented a probabilistic model of Mahler's question, that is, given a random translation of the Cantor set C, it is shown that for Lebesgue almost every $\xi\in[0,1]$,

$$\dim_H \mathcal{W}(u)\cap(\xi+C)\le\max\Big\{\frac{2}{u}+\gamma-1,0\Big\}.$$

More results about the distribution of rational numbers near the Cantor set are given by Han [9] and Schleischitz [13]. Another way to attack the conjecture (1.1) is to enlarge the collection of rational numbers, that is, consider approximation by algebraic numbers of degree less than n, instead of merely focusing on rational numbers. For $n\in\mathbb{N}$, denote by $\mathcal{A}_n$ the set of real algebraic numbers of degree less than or equal to n. For each $\alpha\in\mathcal{A}_n$, denote by $H(\alpha)$ the height of α, that is, the maximum modulus of the coefficients of the minimal polynomial of α. Define

$$\mathcal{M}_n(\psi)=\{x\in\mathbb{R}:|x-\alpha|<\psi(H(\alpha)),\ \text{for infinitely many }\alpha\in\mathcal{A}_n\}.$$

When $\psi(q)=q^{-u}$, we write $\mathcal{M}_n(\psi)$ as $\mathcal{M}_n(u)$. In this setting, it is believed that

$$\lim_{n\to\infty}\lim_{u\to 2}\dim_H(\mathcal{M}_n(u)\cap C)=\gamma.$$

Kristensen [10] generalised Weiss's result to this setting. In [10], it was shown that when $\lambda_\psi\ge\min\{2n,(n+1)/\gamma\}$,

$$\dim_H(\mathcal{M}_n(\psi)\cap C)\le\min\Big\{\frac{2n\gamma}{\lambda_\psi},\frac{n+1}{\lambda_\psi}\Big\},$$

where λ_ψ denotes the lower order at infinity of the function $1/\psi$, namely

$$\lambda_\psi=\liminf_{r\to\infty}\frac{-\log\psi(r)}{\log r}.$$

In this paper, we consider an analogue of the result of Bugeaud and Durand for the intersection of $\mathcal{M}_n(\psi)$ with a translation of the Cantor set,

$$\mathcal{M}_n(\psi)\cap(\xi+C),\quad \xi\in[0,1].$$

To begin, we define a doubling version of this set:

$$\mathcal{K}_n(\psi)=\{(x,\xi)\in\mathbb{R}^2: x\in\xi+C\text{ and }x\in\mathcal{M}_n(\psi)\}.$$

Note that for a fixed $\xi \in \mathbb{R}$, the set $\mathcal{M}_n(\psi) \cap (\xi + C)$ may be regarded as the intersection of $\mathcal{K}_n(\psi)$ with the line $L_\xi = \{(x, \xi) : x \in \mathbb{R}\}$. We now state our main result.

THEOREM 1.1. *Let $\psi : \mathbb{N} \to \mathbb{R}_{\geq 0}$ be a positive nonincreasing function. Then*

$$\dim_H(\mathcal{K}_n(\psi)) = \frac{n+1}{\lambda_\psi} + \gamma.$$

Applying the classical slicing theorem [6], we deduce the following corollary.

COROLLARY 1.2. *For Lebesgue almost every $\xi \in \mathbb{R}$,*

$$\dim_H(\mathcal{M}_n(\psi) \cap (\xi + C)) \leq \max\left\{\frac{n+1}{\lambda_\psi} + \gamma - 1, 0\right\}.$$

2. Proofs

2.1. Lower bounds. We first state some auxiliary lemmas. Let $\mathcal{H}^\gamma|_C$ be the Hausdorff measure confined to the Cantor set C and let $B(x, r)$ denote the ball with centre x and radius r. Then, for any $x \in C$ and $r > 0$,

$$c_2 r^\gamma \leq \mathcal{H}^\gamma|_C(B(x, r)) \leq c_1 r^\gamma$$

for absolute constants c_1, c_2 (see [5]). We adapt the proof of [6, Proposition 7.9] to obtain the following slicing lemma.

LEMMA 2.1. *Let A be a Borel subset of $[0, 1]^2$ and L_x denote the line parallel to the y-axis through the point $(x, 0)$. For any $s > \gamma$,*

$$\mathcal{H}^s(A) \geq c_1^{-1} \int_C \mathcal{H}^{s-\gamma}(A \cap L_x)\, \mathcal{H}^\gamma\,(dx).$$

PROOF. Given $\varepsilon > 0$, for any $\delta > 0$, there is a δ-cover $\{D_i\}_{i\geq 1}$ of A such that

$$\sum_{i\geq 1} |D_i|^s \leq \mathcal{H}^s_\delta(A) + \varepsilon.$$

Each D_i can be surrounded by a square Q_i with sides parallel to the coordinate axes and with side length $|D_i|$. This gives a δ-cover of A by cubes without increasing the s-volume, that is,

$$\sum_{i\geq 1} |Q_i|^s \leq \mathcal{H}^s_\delta(A) + \varepsilon.$$

Let $\mathbb{I}_i$ be the indicator function of Q_i, that is,

$$\mathbb{I}_i(x, y) = \begin{cases} 1 & (x, y) \in Q_i, \\ 0 & (x, y) \notin Q_i. \end{cases}$$

https://doi.org/10.1017/S0004972722001058 Published online by Cambridge University Press

For each $x \in C$, the sets $\{Q_i \cap L_x\}_{i\geq 1}$ form a δ-cover of $A \cap L_x$, so

$$\begin{aligned}\mathcal{H}^{s-\gamma}_{\delta}(A \cap L_x) &\leq \sum_{i\geq 1} |Q_i \cap L_x|^{s-\gamma}\\ &\leq \sum_{i\geq 1} |Q_i|^{s-\gamma-1}|Q_i \cap L_x|\\ &= \sum_{i\geq 1} |Q_i|^{s-\gamma-1} \int_{\mathbb{R}} \mathbb{I}_i(x,y)\,dy.\end{aligned}$$

Write $Q_i = [a, a + |Q_i|] \times [b, b + |Q_i|]$. By Fubini's theorem,

$$\begin{aligned}\int_C \int_{\mathbb{R}} \mathbb{I}_i(x,y)\,dy\,d\mathcal{H}^{\gamma}(x) &= \int_{\mathbb{R}} \int_C \mathbb{I}_i(x,y)\,d\mathcal{H}^{\gamma}(x)\,dy\\ &= \int_b^{b+|Q_i|} \int_{C\cap[a,a+|Q_i|]} 1\,d\mathcal{H}^{\gamma}(x)\,dy\\ &\leq c_1 \int_b^{b+|Q_i|} |Q_i|^{\gamma} dy = c_1 |Q_i|^{\gamma+1}.\end{aligned}$$

Hence,

$$\begin{aligned}\int_C \mathcal{H}^{s-\gamma}_{\delta}(A \cap L_x)\,\mathcal{H}^{\gamma}\,(dx) &\leq \sum_{i\geq 1} |Q_i|^{s-\gamma-1} \int_C \int_{\mathbb{R}} \mathbb{I}_i(x,y)\,dy\,\mathcal{H}^{\gamma}\,(dx)\\ &\leq c_1 \sum_{i\geq 1} |Q_i|^{s} \leq c_1 \mathcal{H}^{s}_{\delta}(A) + \varepsilon.\end{aligned}$$

Letting $\delta \to 0$ and invoking the arbitrariness of $\varepsilon > 0$,

$$\int_C \mathcal{H}^{s-\gamma}(A \cap L_x)\,\mathcal{H}^{\gamma}\,(dx) \leq c_1 \mathcal{H}^{s}(A). \qquad \square$$

The next result deals with the dimension of the set of points which are well approximated by algebraic numbers.

LEMMA 2.2 (Bugeaud [2]). *Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be a positive nonincreasing function. If $\sum_{r=1}^{\infty} r^n \psi(r) < +\infty$, then*

$$\dim_H \mathcal{M}_n(\psi) = \frac{n+1}{\lambda_{\psi}}. \tag{2.1}$$

Second, we define a transformation $\Phi : (x, \xi) \to (x - \xi, \xi)$ from $\mathbb{R}^2$ onto itself. It is obvious that Φ is bi-Lipschitz and satisfies

$$\Phi(\mathcal{K}_n(\psi)) = \{(x, \xi) \in \mathbb{R}^2 : x \in C \text{ and } x + \xi \in \mathcal{M}_n(\psi)\}.$$

Since Hausdorff dimension is invariant under bi-Lipschitz transformations, $\mathcal{K}_n(\psi)$ has the same Hausdorff dimension as $\Phi(\mathcal{K}_n(\psi))$. So we need only focus on the dimension of the latter set.

Third, by Lemma 2.1,

$$\mathcal{H}^s(\Phi(\mathcal{K}_n(\psi))) \geq c_1^{-1} \int_C \mathcal{H}^{s-\gamma}(\Phi(\mathcal{K}_n(\psi)) \cap L_{x_0})\, \mathcal{H}^\gamma(dx_0), \tag{2.2}$$

where L_{x_0} is the set of points $(x, \xi) \in \mathbb{R}^2$ such that $x = x_0$. For each $x_0 \in \mathbb{R}$,

$$\Phi(\mathcal{K}_n(\psi)) \cap L_{x_0} = \{(x_0, \xi) : \xi \in \mathcal{M}_n(\psi) - x_0\}.$$

We define a transformation $\Xi : L_{x_0} \to \mathbb{R}$ by $\Xi(x_0, y) = y + x_0$. Then Ξ is an isometry and

$$\Xi(\Phi(\mathcal{K}_n(\psi)) \cap L_{x_0}) = \mathcal{M}_n(\psi).$$

Thus, for any $s \geq 0$,

$$\mathcal{H}^s(\Phi(\mathcal{K}_n(\psi)) \cap L_{x_0}) = \mathcal{H}^s(\mathcal{M}_n(\psi)).$$

Consequently, (2.2) can be written as

$$\mathcal{H}^s(\Phi(\mathcal{K}_n(\psi))) \geq c_1^{-1} \mathcal{H}^{s-\gamma}(\mathcal{M}_n(\psi)) \mathcal{H}^\gamma(C).$$

If $s < (n+1)/\lambda_\psi + \gamma$, we deduce from (2.1) that $\mathcal{H}^{s-\gamma}(\mathcal{M}_n(\psi))$ is infinite, so that $\mathcal{H}^s(\Phi(\mathcal{K}_n(\psi)))$ is also infinite. It follows that

$$\dim_H(\mathcal{K}_n(\psi)) = \dim_H(\Phi(\mathcal{K}_n(\psi)) \geq \frac{n+1}{\lambda_\psi} + \gamma.$$

2.2. Upper bounds. It suffices to find a sequence of appropriate coverings of $\Phi(\mathcal{K}_n(\psi))$. For any $(x, \xi) \in \Phi(\mathcal{K}_n(\psi))$, we have $x + \xi \in \mathcal{M}_n(\psi)$. So, for any integer $H_0 \geq 1$, there exist $H \geq H_0$ and an algebraic number $\alpha \in \mathcal{A}_n$ with $H(\alpha) = H$ such that

$$|x + \xi - \alpha| < \psi(H).$$

Let $j(H)$ be the integer such that

$$3^{-j(H)-1} < 2\psi(H) \leq 3^{-j(H)}.$$

The set C is naturally covered by $2^{j(H)}$ closed intervals with length $3^{-j(H)}$, whose centres are denoted by $x_{j(H),1}, x_{j(H),2}, \ldots, x_{j(H),2^{j(H)}}$. For each $x \in C$, there is an integer k with $1 \leq k \leq 2^{j(H)}$ such that

$$|x - x_{j(H),k}| \leq \frac{1}{2 \cdot 3^{j(H)}}.$$

By the triangle inequality,

$$\begin{aligned} |\xi - (\alpha - x_{j(H),k})| = |\xi - \alpha + x_{j(H),k} - x + x| &\leq |\xi + x - \alpha| + |x_{j(H),k} - x| \\ &< \frac{1}{2 \cdot 3^{j(H)}} + \frac{1}{2 \cdot 3^{j(H)}} = 3^{-j(H)}. \end{aligned}$$

If $\mathbb{R}^2$ is equipped with the product distance, it follows that (x, ξ) belongs to the open ball, denoted by $B_{H,\alpha,k}$, with radius $3^{-j(H)}$ centred at $(x_{j(H),k}, \alpha - x_{j(H),k})$. As a result, for any $H_0 \geq 1$,

https://doi.org/10.1017/S0004972722001058 Published online by Cambridge University Press

$$\Phi(\mathcal{K}_n(\psi)) \subseteq \bigcup_{H=H_0}^{\infty} \bigcup_{\alpha\in\mathcal{A}_n,\ H(\alpha)=H} \bigcup_{k=1}^{2^{j(H)}} B_{H,\alpha,k}.$$

Let $\sharp$ denote the cardinality of a finite set. Then

$$\sharp\{\alpha \in \mathcal{A}_n : H(\alpha) = H\} \le 2n(n+1)(2H+1)^n \le 12^n \cdot n^2 \cdot H^n.$$

By the definition of λ_ψ, for any $\epsilon > 0$,

$$3^{-j(H)} \le 6\psi(H) \le H^{-\lambda_\psi+\epsilon},$$

for all sufficiently large H. Thus, for each $s \ge \gamma$, the s-dimensional Hausdorff measure $\mathcal{H}^s$ of $\Psi(\mathcal{K}_n(\psi))$ can be estimated as

$$\begin{aligned}
\mathcal{H}^s(\Psi(\mathcal{K}_n(\psi))) &\le \liminf_{H_0\to\infty} \sum_{H=H_0}^{\infty} \sum_{\alpha\in\mathcal{A}_n,\ H(\alpha)=H} 2^{j(H)} \cdot 3^{-sj(H)} \\
&\le \liminf_{H_0\to\infty} \sum_{H=H_0}^{\infty} 12^n \cdot n^2 \cdot H^n \cdot 3^{\gamma j(H)} \cdot 3^{-sj(H)} \\
&\le 12^n \cdot n^2 \cdot \liminf_{H_0\to\infty} \sum_{H=H_0}^{\infty} H^{n-(s-\gamma)(\lambda_\psi-\varepsilon)},
\end{aligned}$$

which converges once $s > (n+1)/\lambda_\psi - \varepsilon + \gamma$. Summing up,

$$\dim_H(\mathcal{K}_n(\psi)) = \dim_H(\Phi(\mathcal{K}_n(\psi))) \le \frac{n+1}{\lambda_\psi} + \gamma.$$

This completes the proof of Theorem 1.1.

3. Final remarks

The results of this paper are still valid for the limit set of a rational iterated function system as presented in [7].

DEFINITION 3.1. Let S be a finite set. An iterated function system (IFS) on $\mathbb{R}$ is a collection $\{\mathcal{U}_a\}_{a\in S}$ of contracting similarities $\mathcal{U}_a : \mathbb{R} \to \mathbb{R}$ satisfying the *open set condition*: there exists an open set $W \subseteq \mathbb{R}$ such that the collection $\{\mathcal{U}_a(W)\}_{a\in S}$ is a collection of disjoint subsets of W. The limit set, denoted by J, of $\{\mathcal{U}_a\}_{a\in S}$ is the image of the *coding map* $\pi : E^{\mathbb{N}} \to \mathbb{R}$ defined by $\{\mathcal{U}_a\}_{a\in S}$, that is,

$$\pi(\varepsilon) = \lim_{n\to\infty} \mathcal{U}_{\varepsilon_1} \circ \mathcal{U}_{\varepsilon_2} \circ \cdots \circ \mathcal{U}_{\varepsilon_n}(0).$$

Call the IFS *rational* if, for each $a \in S, \mathcal{U}_a$ preserves $\mathbb{Q}$, that is,

$$\mathcal{U}_a(x) = \frac{p_a}{q_a}x + \frac{r_a}{q_a}$$

with $p_a, r_a \in \mathbb{Z}, q_a \in \mathbb{N}$.

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The arguments involved in establishing Theorem 1.1 can be modified in the obvious manner to yield the following generalisation.

THEOREM 3.2. *Suppose that $\{u_a\}_{a\in S}$ is a rational IFS and J is the corresponding limit set. Let*

$$\mathcal{K}_n^*(\psi) = \{(x,\xi) \in \mathbb{R}^2 : x \in \xi + J \text{ and } x \in \mathcal{M}_n(\psi)\}.$$

Then

$$\dim_H(\mathcal{K}_n^*(\psi)) = \frac{n+1}{\lambda_\psi} + \delta,$$

where δ is the Hausdorff dimension of J and satisfies

$$\sum_{a\in S}\Big(\frac{p_a}{q_a}\Big)^{\delta} = 1.$$

COROLLARY 3.3. *For Lebesgue almost every $\xi \in \mathbb{R}$,*

$$\dim_H(\mathcal{M}_n(\psi) \cap (\xi + J)) \le \max\Big\{\frac{n+1}{\lambda_\psi} + \delta - 1, 0\Big\}.$$

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YUAN ZHANG, Institute of Statistics and Applied Mathematics,
Anhui University of Finance and Economics, Bengbu 233030, PR China
e-mail: 120210066@aufe.edu.cn

JIA LIU, Institute of Statistics and Applied Mathematics,
Anhui University of Finance and Economics, Bengbu 233030, PR China
e-mail: liujia860319@163.com

SAISAI SHI, Institute of Statistics and Applied Mathematics,
Anhui University of Finance and Economics, Bengbu 233030, PR China
e-mail: saisai_shi@126.com