THE BULLETIN OF SYMBOLIC LOGIC
Volume 28, Number 2, June 2022

ORDER TYPES OF MODELS OF FRAGMENTS OF PEANO ARITHMETIC

LORENZO GALEOTTI AND BENEDIKT LÖWE

Abstract. The complete characterisation of order types of non-standard models of Peano arithmetic and its extensions is a famous open problem. In this paper, we consider subtheories of Peano arithmetic (both with and without induction), in particular, theories formulated in proper fragments of the full language of arithmetic. We study the order types of their non-standard models and separate all considered theories via their possible order types. We compare the theories with and without induction and observe that the theories without induction tend to have an algebraic character that allows model constructions by closing a model under the relevant algebraic operations.

§1. Introduction.

1.1. Background I: order types of models of Peano arithmetic and its extensions. It is well-known that non-standard models of Peano arithmetic have order type $\mathbb{N} + \mathbb{Z} \cdot D$ where D is a dense linear order without first or last element (cf. [10, Theorem 6.4]). For countable models, this determines their order type up to isomorphism: it is necessarily $\mathbb{N} + \mathbb{Z} \cdot \mathbb{Q}$. In general, not every order of the form $\mathbb{N} + \mathbb{Z} \cdot D$ is the order type of a model of Peano arithmetic and it is not known how to characterise those dense orders D for which this is the case.

QUESTION 1. *For which dense orders D is there a model of Peano arithmetic with order type* $\mathbb{N} + \mathbb{Z} \cdot D$?

If T is any consistent extension of Peano arithmetic, we write $\mathcal{O}_T$ for the class of order types of models of T. Friedman asked the "especially vexing question" [12, p. 281] whether this class depends on the choice of T:[1]

QUESTION 2 (Friedman's Question). *Are there any consistent extensions T and T′ of Peano arithmetic such that* $\mathcal{O}_T \neq \mathcal{O}_{T'}$?

Received April 10, 2020.
2020 *Mathematics Subject Classification*. 03C62, 03H15, 03C64, 06A05, 08A99.
Key words and phrases. non-standard models of Peano arithmetic, order types, arithmetical theories without induction.

[1] Friedman's original question was formulated as follows: "Is there any extension T of Peano arithmetic such that $\mathcal{O}_T \neq \mathcal{O}_{\mathrm{Th}(\mathbb{N})}$?" [7, Problem 14]. Shelah has another (not quite equivalent) variant of the question: "Is there any non-standard model M of Peano arithmetic such that all models with the same order type must be elementarily equivalent to M?" [24, Question 1.4]. All variants of Friedman's Question are open.

1079-8986/22/2802-0002
DOI :10.1017/bsl.2021.48

While Questions 1 and 2 are wide open, many results have been obtained that give us some information about the relationship between models of arithmetic and their order types. We refer the reader to the expertly written survey paper [2] that outlines the state of knowledge.

Let λ be a regular uncountable cardinal with $2^{<\lambda} = \lambda$. Then there is a unique dense saturated order Q_λ of size λ [14, Corollary 4.3.14 and Theorem 4.3.20] and it follows from [10, Theorem 6.4] that any saturated model of Peano arithmetic of cardinality λ has order type $\mathbb{N} + \mathbb{Z} \cdot Q_\lambda$. However, Pabion's Theorem shows that more is true:

Theorem 3 (Pabion; [15, Proposition 2]). *Let κ be an uncountable cardinal. Then a model of Peano arithmetic is κ-saturated if and only if its order type is κ-saturated.*

In his doctoral dissertation [1], Bovykin constructed many examples of additional order types of models of Peano arithmetic (under the appropriate assumptions that guarantee that the dense saturated orders exist), e.g., $\mathbb{N} + \mathbb{Z} \cdot Q_{\aleph_1} \cdot \mathbb{N}$, $\mathbb{N} + \mathbb{Z} \cdot Q_{\aleph_1} \cdot \mathbb{Z}$, $\mathbb{N} + \mathbb{Z} \cdot Q_{\aleph_1} \cdot (\omega_1^* + \omega_1)$, $\mathbb{N} + \mathbb{Z} \cdot Q_{\aleph_2} \cdot (\omega_2^* + \omega_2)$, $\mathbb{N} + \mathbb{Z} \cdot (Q_{\mathfrak{c}^+} + Q_{\mathfrak{c}^+} \cdot Q_{\mathfrak{c}^{++}})$, and many others [2, Section 5] (here, as usual, $\mathfrak{c} := 2^{\aleph_0}$). He also gave examples of order types that have different non-isomorphic models of Peano arithmetic living on them, e.g., $\mathbb{N} + \mathbb{Z} \cdot Q_{\aleph_1} \cdot \mathbb{N}$ [2, Proposition 5.3]. However, for many similar orders, we do not know whether they are order types of models of Peano arithmetic, e.g., $\mathbb{N} + \mathbb{Z} \cdot (\mathbb{Q} + \mathbb{Q} \cdot Q_{\aleph_1})$ [2, Question 3].

More is known if we restrict our attention to classes of particularly well-structured models of Peano arithmetic, e.g., *order self-similar* models [2, Section 8] or *resplendent* models [2, Sections 8 and 9] (cf. also [11, 22]). The most recent results in this direction are Shelah's investigation of almost isomorphism of order rigid models in [24]: we shall discuss these briefly in Section 8.

1.2. Background II: fragments of Peano arithmetic. All of the results mentioned in Section 1.1 are about models of Peano arithmetic or its extensions. In this paper, we shall go in the other direction and consider fragments of Peano arithmetic where the natural analogue of Friedman's Question 2 would be:

Question 4. *Given two fragments T and T' of Peano arithmetic, when is $\mathcal{O}_T = \mathcal{O}_{T'}$?*

The standard language of Peano arithmetic contains the symbols $<$, s, $+$, and $\cdot$; the fragments we consider are those axiomatised in languages lacking some of these symbols, in particular, *Successor arithmetic* without $+$ and $\cdot$ and *Presburger arithmetic* without $\cdot$ (cf. [17]). We shall not be considering the fragment containing only multiplication, known as *Skolem arithmetic* (cf. [26]) for reasons discussed at the end of Section 2.1.[2]

[2]We note that in contrast to Peano arithmetic, both Presburger and Skolem arithmetic are complete and decidable [19, Section 1.2.3]. It is the combination of addition and multiplication that makes theories *sequential*, i.e., they can encode the notion of finite sequence; this in turn paves the path to Gödel's incompleteness argument.

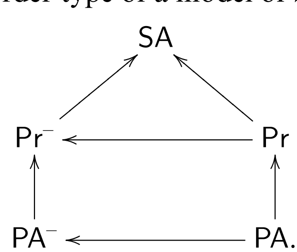
The axioms of our systems of arithmetic consist of a list of *algebraic axioms* governing the meaning of the symbols $<$, s, $+$, and $\cdot$ and the *axiom scheme of induction*. Almost all of the algebraic axioms are universal sentences[3] and allow us to think of models as generated by the operations. Models of theories defined by the algebraic axioms alone therefore have an algebraic and constructive character. In contrast, theories with the full induction scheme (or sufficiently large fragments of the induction scheme) require model theoretic methods to analyse their models.[4] We shall discuss and illustrate this notable difference in Section 6 with some elementary examples.

1.3. Overview. We consider three operations, the unary successor operation and the binary operations of addition and multiplication, as well as their associated languages: $\mathcal{L}_{<,\mathrm{s}} := \{0, <, \mathrm{s}\}$, the language with an order relation and the successor operation, $\mathcal{L}_{<,\mathrm{s},+} := \{0, <, \mathrm{s}, +\}$, the language augmented with addition, and $\mathcal{L}_{<,\mathrm{s},+,\cdot} := \{0, <, \mathrm{s}, +, \cdot\}$, the full language of arithmetic. For each of the languages, we shall define the appropriate arithmetical axiom systems and the corresponding axiom schemes of induction, resulting in a total of six theories,

$$\begin{array}{ccc} \mathsf{SA}^- & \subseteq & \mathsf{SA} \\ \cap\!\! \mid & & \cap\!\!\mid \\ \mathsf{Pr}^- & \subseteq & \mathsf{Pr} \\ \cap\!\!\mid & & \cap\!\!\mid \\ \mathsf{PA}^- & \subseteq & \mathsf{PA}, \end{array}$$

where the theories in the left column are without induction and the theories in the right column are with the axiom scheme of induction (for definitions, cf. Section 2.1).

It is a folklore result that SA^- proves the axiom scheme of induction for $\mathcal{L}_{<,\mathrm{s}}$ (Theorem 11) and hence SA^- and SA are the same theory, reducing our diagram to five theories. We solve Question 4 for these theories by showing that for any two theories T and T' from this list, we have $\mathcal{O}_T \neq \mathcal{O}_{T'}$. In the following diagram, an arrow from a theory T to a theory S means "every order type that occurs in a model of T occurs in a model of S." In Section 9, we shall show that the diagram is complete in the sense that if there is no arrow from T to S, then there is an order that is the order type of a model of T that cannot be the order type of a model of S.



[3] The exceptions are axioms S1 and $\divideontimes$; cf. the discussion in Section 6.

[4] Some research on models of fragments of Peano arithmetic deals with weakenings of the induction scheme to subclasses of formulae (cf., e.g., [3, 4, 13, 29]). These theories have infinitely many instances of the axiom scheme of induction and consequently do not have the algebraic character that our induction-free theories have.

https://doi.org/10.1017/bsl.2021.48 Published online by Cambridge University Press

§2. Definitions and basic results.

2.1. Definitions. In this section, we shall introduce the axiomatic systems studied in this paper. The axioms come in four groups corresponding to the order relation, the successor function, addition, and multiplication. As usual, we use the following syntactic abbreviations: for $n \in \mathbb{N}$ and a variable x, we write

$$\mathrm{s}^n(x) := \underbrace{\mathrm{s}(\cdots(\mathrm{s}}_{n \text{ times.}}(x))\cdots) \text{ and}$$

$$nx := \underbrace{x + \cdots + x}_{n \text{ times.}}.$$

The order axioms O1 to O4 express that $<$ describes a linear order with least element 0 (O1 is trichotomy, O2 is transitivity, and O3 is irreflexivity):

$$x < y \vee x = y \vee x > y, \tag{O1}$$

$$(x < y \wedge y < z) \rightarrow x < z, \tag{O2}$$

$$\neg(x < x), \tag{O3}$$

$$x = 0 \vee 0 < x. \tag{O4}$$

The successor axioms S1 to S4 express that $<$ is discrete and that s is the successor operation with respect to $<$:

$$x = 0 \leftrightarrow \neg\exists y x = \mathrm{s}(y), \tag{S1}$$

$$x < y \rightarrow y = \mathrm{s}(x) \vee \mathrm{s}(x) < y, \tag{S2}$$

$$x < y \rightarrow \mathrm{s}(x) < \mathrm{s}(y), \tag{S3}$$

$$x < \mathrm{s}(x). \tag{S4}$$

Taken together, the axioms O1 to O4 and S1 to S4 (later called SA^-) constitute the theory of discrete linear orders with a minimum and a strictly increasing successor function. Note that all the order and successor axioms with the exception of S1 are universal sentences. The axiom S1 is a particular instance of the induction scheme and is in $\forall\exists$ form.

The addition axioms P1 to P5 express the fact that $+$ and $<$ satisfy the axioms of ordered abelian monoids:

$$(x + y) + z = x + (y + z), \tag{P1}$$

$$x + y = y + x, \tag{P2}$$

$$x + 0 = x, \tag{P3}$$

$$x < y \rightarrow x + z < y + z, \tag{P4}$$

$$x + \mathrm{s}(y) = \mathrm{s}(x + y). \tag{P5}$$

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Axiom ⋇ expresses the fact that if $x < y$, then the difference between them exists:

$$x < y \to \exists z x + z = y. \tag{⋇}$$

Like S1, axiom ⋇ is an instance of the induction axiom scheme and not a universal sentence but in $\forall\exists$ form; we shall comment on S1 and ⋇ in Section 6.

The multiplication axioms M1 to M6 express that $\cdot$ and $+$ are commutative semiring operations respecting $<$:

$$(x \cdot y) \cdot z = x \cdot (y \cdot z), \tag{M1}$$

$$x \cdot y = y \cdot x, \tag{M2}$$

$$(x + y) \cdot z = x \cdot z + y \cdot z, \tag{M3}$$

$$x \cdot \mathrm{s}(0) = x, \tag{M4}$$

$$x \cdot \mathrm{s}(y) = (x \cdot y) + x, \tag{M5}$$

$$(x < y \wedge z \neq 0) \to x \cdot z < y \cdot z. \tag{M6}$$

Finally we have a schema of induction axioms.

$$(\varphi(0, \bar{y}) \wedge \forall x(\varphi(x, \bar{y}) \to \varphi(\mathrm{s}(x), \bar{y})) \to \forall x \varphi((x, \bar{y}). \tag{Ind$_\varphi$}$$

When considering subsystems of these axioms, we denote the axiom schema of induction restricted to the formulas of a language $\mathcal{L}$ by $\mathrm{Ind}(\mathcal{L})$. We shall consider the following systems of axioms:

$$\begin{aligned}
\mathsf{SA}^- &= \mathrm{O1} + \mathrm{O2} + \mathrm{O3} + \mathrm{O4} + \mathrm{S1} + \mathrm{S2} + \mathrm{S3} + \mathrm{S4},\\
\mathsf{SA} &= \mathsf{SA}^- + \mathrm{Ind}(\mathcal{L}_{<,\mathrm{s}}),\\
\mathsf{Pr}^- &= \mathsf{SA}^- + ⋇ + \mathrm{P1} + \mathrm{P2} + \mathrm{P3} + \mathrm{P4} + \mathrm{P5},\\
\mathsf{Pr} &= \mathsf{Pr}^- + \mathrm{Ind}(\mathcal{L}_{<,\mathrm{s},+}),\\
\mathsf{PA}^- &= \mathsf{Pr}^- + \mathrm{M1} + \mathrm{M2} + \mathrm{M3} + \mathrm{M4} + \mathrm{M5} + \mathrm{M6},\\
\mathsf{PA} &= \mathsf{PA}^- + \mathrm{Ind}(\mathcal{L}_{<,\mathrm{s},+,\cdot});
\end{aligned}$$

standing for 'Successor Arithmetic', 'Presburger Arithmetic', and 'Peano Arithmetic', respectively.[5]

In his original paper, Presburger uses a different axiomatisation of Presburger Arithmetic that we shall call Pr^{D} [17]. The axioms of Pr^{D} are those of Pr^- plus the following axiom schema:

$$\forall x \exists y x = ny \vee x = \mathrm{s}(ny) \vee \cdots \vee x = \mathrm{s}^{n-1}(ny), \tag{D$_n$}$$

for $0 < n \in \mathbb{N}$. Presburger's famous theorem shows that Pr^{D} axiomatises the complete theory $\mathrm{Th}(\mathbb{N}, +)$. Since our Pr clearly implies Pr^{D}, it also axiomatises $\mathrm{Th}(\mathbb{N}, +)$.

[5]Note that SA should not be confused with the theory $\mathrm{Th}(\mathbb{Q}, +)$ called SA in [9, 27] (the 'S' there stands for 'Skolem').

In this paper we do not take into consideration Skolem arithmetic Sk, i.e., the multiplicative fragment of PA. This is due to the fact that Sk, usually defined as $\mathrm{Th}(\mathbb{N}, \cdot)$, does not carry an order structure, i.e., the order is not definable in $\mathcal{L}_{\cdot}$. Moreover, adding the order to Skolem arithmetic makes it much more expressive: Robinson showed that addition is definable in $\mathrm{Th}(\mathbb{N}, <, \cdot)$, and thus $\mathrm{Th}(\mathbb{N}, <, \cdot)$ is essentially full arithmetic [20, Theorem 1.1]. Therefore, an analysis of Skolem arithmetic in terms of order types is not fruitful.

2.2. Order types. As usual, order types are the isomorphism classes of partial orders. If $\mathcal{L}$ is any language containing $<$ and M is an $\mathcal{L}$-structure, by a slight abuse of language, we refer to the $\{<\}$-reduct of M as its *order type*. In situations where the order structure is clear from the context, we do not explicitly include it in the notation: e.g., the notation $\mathbb{Z}$ refers to both the set of integers and the ordered structure $(\mathbb{Z}, <)$ with the natural order $<$ on $\mathbb{Z}$.

If A and B are two linear orders, then A^* is the inverse order of A, $A + B$ is the order sum, and $A \cdot B$ is the (anti-lexicographic) product order, i.e., $(a, b) \leq (a', b')$ if and only if $b < b'$ or $b = b'$ and $a \leq a'$. If B has a least element 0 then B^A is the set of functions f from A to B with finite support (i.e., the set $\{a \in A\, ;\, f(a) \neq 0\}$ is finite) ordered anti-lexicographically, i.e., $f < g$ if and only if $f(a) < g(a)$ for the largest $a \in A$ such that $f(a) \neq g(a)$. Note that in the case that A and B are ordinal numbers, then the above operations of sum, product, and exponentiation correspond to the classical ordinal operations.

If $a \in A$, we denote the *initial segment defined by a* as $\mathrm{IS}(a) := \{b \in A\, ;\, b < a\}$ and the *final segment defined by a* as $\mathrm{FS}(a) := \{b \in A\, ;\, a < b\}$.

If $(G, 0, <, +)$ is an ordered abelian group (i.e., satisfies the axioms O1 to O4 and P1 to P4), then we define $G^+ := \{g \in G\, ;\, 0 < g\} = \mathrm{FS}(0)$ to be the *positive part of G*. We call linear orders *groupable* if and only if there is an ordered abelian group with the same order type.

Let $(G, <, +)$ be an ordered group. We define the *standard monoid over G* as the ordered monoid $(\mathbb{N} + \mathbb{Z} \cdot G^+, <, +)$ where $<$ is the order relation of $\mathbb{N} + \mathbb{Z} \cdot G^+$ and $+$ is defined point-wise, i.e.,

$$x + y = \begin{cases} n + m, & \text{if } x = n, y = m \text{ and } m, n \in \mathbb{N}, \\ (z + x, g), & \text{if } x \in \mathbb{N} \text{ and } y = (z, g) \in \mathbb{Z} \cdot G^+, \\ (z + y, g), & \text{if } y \in \mathbb{N} \text{ and } x = (z, g) \in \mathbb{Z} \cdot G^+, \\ (z_x + z_y, g_x + g_y), & \text{if } x = (z_x, g_x) \in \mathbb{Z} \cdot G^+ \text{ and} \\ & \quad y = (z_y, g_y) \in \mathbb{Z} \cdot G^+. \end{cases}$$

It is easy to see that for each ordered group G the standard monoid over G is indeed a monoid with least element 0.

If $(B, <, +)$ is any ordered group and X is a variable, we can consider the set $B[X]$ of polynomials in the variable X over B, consisting of terms

$$f = b_n X^n + \cdots + b_1 X + b_0$$

https://doi.org/10.1017/bsl.2021.48 Published online by Cambridge University Press

with $b_n \neq 0$ (unless $n = 0$); the *degree* of a polynomial is the highest occurring exponent, i.e., $\deg(f) = n$ in the term given above. If $n = b_0 = 0$ (i.e., $f = 0$), we call f the *zero polynomial*. We order polynomials anti-lexicographically: if $f = b_n X^n + \cdots + b_1 X + b_0$ and $g = c_m X^m + \cdots c_1 X + c_0$ are different, find the largest natural number k such that $b_k \neq c_k$; then

$$f < g \text{ if and only if } b_k < c_k.$$

This order respects addition and multiplication of polynomials in the sense of axioms P4 and M6, respectively. A polynomial is called *positive* if it is larger than the zero polynomial in this order. If we define

$$\begin{aligned} O_0 &= \varnothing, \\ O_{\gamma+1} &= O_\gamma + \mathbb{Z}^\gamma \cdot \mathbb{N}, \\ O_\lambda &= \bigcup_{\gamma \in \lambda} O_\gamma \text{ for } \lambda \text{ limit}, \end{aligned}$$

then for every natural number $n > 0$, the linear order O_n is the order type of non-negative polynomials with integer coefficients of degree at most $n - 1$ and thus O_ω is the order type of all non-negative polynomials with integer coefficients.

2.3. Basic properties. In this section, we shall remind the reader about basic tools of model theory of PA. We refer the reader to [10] for a comprehensive introduction to the theory of non-standard models of PA. One of the main tools in studying the order types of models of PA is the concept of *Archimedean class*.

DEFINITION 5. Let M be a model of SA⁻. Given $x, y \in M$ we say that x *and* y *are of the same magnitude*, in symbols $x \sim y$, if there are $m, n \in \mathbb{N}$ such that $\mathrm{s}^n(y) \geq x$ and $y \leq \mathrm{s}^m(x)$. The relation $\sim$ is an equivalence relation. For every $x \in M$, we shall denote by $[x]$ the equivalence class of x with respect to $\sim$ called the *Archimedean class of* x.

The Archimedean classes of a model of SA⁻ partition the model into convex blocks: if $y, w \in [x]$ and $y < z < w$, then $z \in [x]$ (the reader can check that only the axioms of SA⁻ are needed for this). Therefore, the quotient structure $M/{\sim}$ of Archimedean classes is linearly ordered by the relation $<$ defined by $[x] < [y]$ if and only if $x < y$ and $[x] \neq [y]$. Furthermore, [0] is the least element of the quotient structure. We refer to the classes that are different from [0] as the *non-zero Archimedean classes*. In particular, if A is the order type of the non-zero Archimedean classes of M, then the order type of M is $\mathbb{N} + \mathbb{Z} \cdot A$.

So far, we worked entirely in the language $\mathcal{L}_{<,\mathrm{s}}$ with just the axioms of SA⁻. If we also have addition in our language, we observe:

LEMMA 6. *Let M be a non-standard model of* Pr⁻ *and $a \in M$ be a non-standard element of M. Then for every $n, m \in \mathbb{N}$ such that $n < m$ we have $[na] < [ma]$. In particular, if $\mathbb{N} + \mathbb{Z} \cdot A$ is the order type of M, then A does not have a largest element.*

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Proof. Assume that $n < m$. We want to prove that $[na] < [ma]$. Let $n' > 0$ be such that $m = n + n'$. Let $i \in \mathbb{N}$ we want to show that $na + \mathrm{s}^i(0) < ma$. By definition, $ma = (n + n')a = na + n'a$. Now by P4 and by the fact that a is non-standard and $n' > 0$ we have $na + \mathrm{s}^i(0) < na + a = (n+1)a \leq (n + n')a = ma$. Therefore $[na] < [ma]$ as desired. ⊣

Another important tool in the classical study of order types of models of PA is the *overspill* property:

Definition 7. Let M be a model of SA^-. Then $I \subseteq M$ is a *cut* of M if it is an initial segment of M with respect to $<$ and it is closed under s, i.e., for every $i \in I$ we have $\mathrm{s}(i) \in I$. A cut of M is *proper* if it is neither empty nor M itself.

Definition 8. Let $\mathcal{L} \supseteq \mathcal{L}_{<,\mathrm{s}}$ be a language. A theory $T \supseteq \mathsf{SA}^-$ has the $\mathcal{L}$-*overspill property* if for every model $M \models T$ there are no $\mathcal{L}$-definable proper cuts of M.

Overspill is essentially a notational variant of induction:

Theorem 9. *Let $\mathcal{L} \supseteq \mathcal{L}_{<,\mathrm{s}}$ be a language and $T \supseteq \mathsf{SA}^-$ be any theory. Then the following are equivalent*:

(i) $\mathrm{Ind}(\mathcal{L}) \subseteq T$ *and*
(ii) *T has the $\mathcal{L}$-overspill property.*

Proof. "(i)⇒(ii)." Let $M \models T$ and I be a proper cut of M. Then $0 \in I$. Suppose towards a contradiction that I is definable by an $\mathcal{L}$-formula φ. Then Ind_φ implies that $I = M$, so I was not proper.

"(ii)⇒(i)." Assume that $\mathrm{Ind}_\varphi \notin T$ for some $\mathcal{L}$-formula φ and find $M \models T$ such that $M \models \neg\mathrm{Ind}_\varphi$. Define the formula $\varphi'(x) := \varphi(x) \wedge \forall y(y < x \rightarrow \varphi(y))$. Then φ' defines a proper cut in M, and thus, T does not have the $\mathcal{L}$-overspill property. ⊣

In particular, SA, Pr, and PA have the overspill property for their respective languages $\mathcal{L}_{<,\mathrm{s}}$, $\mathcal{L}_{<,\mathrm{s},+}$, and $\mathcal{L}_{<,\mathrm{s},+,\cdot}$.

§3. Successor arithmetic. We begin our study by considering the two subsystems obtained by restricting our language to $\mathcal{L}_{<,\mathrm{s}}$, viz. SA^- and SA. The theory SA^- is the theory of discrete linear orders with a least element and a strictly increasing successor function. Model theoretic properties of SA^- are discussed in [5, Example 3.4.4].

It is a folklore result that the theory of the structure $(\mathbb{N}, 0, <, \mathrm{s})$ has quantifier elimination (cf., e.g., [14, Exercise 3.4.4]) and the argument works for the theory SA^-. The standard reference for the axiomatic version is [6, Theorem 32A] where Enderton shows quantifier elimination for a theory he calls A_L which is essentially the conjunction of our O1 to O4, S1, S3, and S4. Enderton claims that $A_L = \mathrm{Th}(\mathbb{N}, <, \mathrm{s}, 0)$ [6, Corollary 32B(b)], but his theory cannot prove our axiom S2 (the discreteness of the order). For the sake of completeness, we give a proof of quantifier elimination in this paper.

LEMMA 10. *The theory* SA^- *satisfies quantifier elimination.*

PROOF. It is enough to prove that for every quantifier free formula $\chi(\overline{x}, y)$ there is a quantifier free formula φ such that

$$\mathsf{SA}^- \models \exists y \chi(\overline{x}, y) \leftrightarrow \varphi(\overline{x}),$$

where y does not appear in φ. We prove this claim by induction over χ. The only interesting cases are the atomic formulas.

If $\chi(x, y) \equiv \mathrm{s}^n(x) < \mathrm{s}^m(y)$: let $\varphi \equiv x = x$. Let $M \models \mathsf{SA}^-$, we want to show $M \models \exists y \chi(x, y)$. First assume $m \geq n$. Since $\mathsf{SA}^- \vdash \forall x \mathrm{s}^n(x) < \mathrm{s}^{m+1}(x)$ we have $M \models \exists y \mathrm{s}^n(x) < \mathrm{s}^m(y)$ as desired. Otherwise if $n > m$ since $\mathsf{SA}^- \vdash \forall x x < \mathrm{s}^{(n-m)+1}(x)$ then $M \models \exists y \chi(\overline{x}, y)$. Hence:

$$\mathsf{SA}^- \models \exists y \chi(\overline{x}, y) \leftrightarrow \varphi(\overline{x})$$

as desired.

If $\chi(x, y) \equiv \mathrm{s}^n(y) < \mathrm{s}^m(x)$: first assume $m > n$ then since $\mathsf{SA}^- \vdash \forall x \mathrm{s}^n(x) < \mathrm{s}^m(x)$ we have $\mathsf{SA}^- \vdash \exists y \chi(x, y) \leftrightarrow x = x$. If $m \leq n$ then $\mathsf{SA}^- \vdash \exists y \chi(x, y) \leftrightarrow \mathrm{s}^n(0) < \mathrm{s}^m(x)$. Indeed, let $M \models \mathsf{SA}^-$ be a model such that there is a $y \in M$ such that $M \models \mathrm{s}^n(y) < \mathrm{s}^m(x)$ and $M \models \neg \mathrm{s}^n(0) < \mathrm{s}^m(x)$. We have two cases: if $M \models \mathrm{s}^n(0) = \mathrm{s}^m(x)$ then we would have $M \models \mathrm{s}^n(y) < \mathrm{s}^m(x) = \mathrm{s}^n(0)$ but since $M \models \forall x \mathrm{s}^n(x) < \mathrm{s}^n(y) \rightarrow x < y$ then we would have $M \models y < 0$. If $M \models \mathrm{s}^m(x) < \mathrm{s}^n(0)$ again we would have $M \models \mathrm{s}^n(y) < \mathrm{s}^m(x) < \mathrm{s}^n(0)$ which implies $M \models y < 0$. On the other hand if $M \models \mathrm{s}^n(0) < \mathrm{s}^m(x)$ then trivially $M \models \exists y \chi(\overline{x}, y)$ as desired.

If $\chi(\overline{x}, y)$ does not have occurrences of y: then $\exists y \chi(\overline{x}, y)$ is either equivalent to $0 = 0$ or $\neg(0 = 0)$.

If $\chi(x, y) \equiv \mathrm{s}^n(x) = \mathrm{s}^m(y)$: similar to the second case. ⊣

By using quantifier elimination, it is not hard to see that SA^- proves the induction schema.[6]

THEOREM 11. *For every formula* φ *in the language* $\mathcal{L}_{<,\mathrm{s}}$ *we have*

$$\mathsf{SA}^- \vdash \mathrm{Ind}_\varphi.$$

PROOF. We shall prove that for every model M of SA^-, the only definable set which contains 0 and is closed under s is M itself. This proves the claim by Theorem 9.

We say that $I \subseteq M$ is an *open interval* if there are $a, b \in M \cup \{\infty\}$ such that $I = \{x \in M\, ;\, a < x < b\}$ and a set $X \subseteq M$ is called *basic* if it is a finite union of open intervals and singletons. As usual, an $\mathcal{L}$-theory T is called *o-minimal* or *order-minimal* if every $\mathcal{L}$-definable subset is basic.

We claim that SA^- is an o-minimal theory: Let $(M, 0, <, \mathrm{s}) \models \mathsf{SA}^-$ and $X \subseteq M$ be $\mathcal{L}_{<,\mathrm{s}}$-definable; by Lemma 10, SA^- has quantifier elimination and therefore, X is definable by a quantifier-free $\mathcal{L}_{<,\mathrm{s}}$-formula. We observe that

[6] An alternative route to proving $\mathsf{SA} = \mathsf{SA}^-$ without going through quantifier elimination is the following: [5, Example 3.4.4] proves that the theory SA^- is model complete and complete; thus all theorems true in the standard model (including induction) are true in every model, so $\mathsf{SA}^- \models \mathsf{SA}$.

sets definable by atomic formulae are either open intervals or points, hence basic; we furthermore observe that the basic sets are closed under finite intersections and complements. Thus all sets definable by quantifier-free formulae are basic.

Now suppose X is an $\mathcal{L}_{<,\mathrm{s}}$-definable cut in M. By o-minimality, we have that $X = I_0 \cup \cdots \cup I_n$ where for every $0 \leq j \leq n$, the set I_j is either a non-empty open interval (a_j, b_j) or a singleton $\{b_j\}$. Towards a contradiction, let $y \in M$ be such that $y \notin X$. Let $m := \max\{b_j\,;\, 0 \leq j \leq n\}$. Note that for all $x \in X$ we have $x \leq m$.

Case 1: $m \in X$. Then, since X is closed under successors, and so $\mathrm{s}(m) \in X$. But then $m < \mathrm{s}(m)$ which is a contradiction.

Case 2: $m \notin X$. Then there is some $0 \leq j \leq n$ with $I_j = (a_j, m)$. By axiom S1, we find $m' \in I_j \subseteq X$ such that $\mathrm{s}(m') = m$. Once more, since X is closed under successors, $m \in X$, but this yields a contradiction as we have seen before. ⊣

In particular, this means that SA and SA⁻ axiomatize the same theory:

Corollary 12. *Let M be a structure in the language $\mathcal{L}_{<,\mathrm{s}}$. Then $M \models$ SA if and only if $M \models$ SA⁻.*

Visser asked whether there is a reasonable finitely axiomatised theory that satisfies full induction (preferably in the full language of arithmetic); it is known that such a theory cannot be sequential (cf. [18, 28] for more on sequentiality). By Corollary 12, SA is a finitely axiomatised theory that satisfies full induction (and is not sequential).

Corollary 13. *A linear order L is the order type of a model of* SA *if and only if there is a linear order A such that $L \cong \mathbb{N} + \mathbb{Z} \cdot A$.*

Proof. By Corollary 12, it is enough to show that a model satisfies SA⁻ in order to get full SA. We already observed that the forward direction holds in Section 2.3 (the linear order A is the quotient structure $M/{\sim}$ with the least element removed). For the other direction, if A is a linear order then $\mathbb{N} + \mathbb{Z} \cdot A$ can be easily made into an SA⁻ model by defining $\mathrm{s}(n) := n + 1$ and $\mathrm{s}(z, a) := (z + 1, a)$. ⊣

§4. Models based on generalised formal power series.

4.1. Definitions. *Generalised formal power series*, introduced by Levi-Civita, are a generalisation of polynomials over a ring: while polynomials only have natural number exponents, generalised formal power series allow more general formal exponents.[7] In this section, we shall adapt the classical theory of generalised formal power series to our context. In particular, we shall show how generalised power series can be used as a tool in building non-standard models of Pr⁻, PA⁻, and Pr.

A linear order $(\Gamma, 0, <)$ with a least element 0 will be called an *exponent order*; an ordered abelian group $(B, 0, <, +)$ with $\mathbb{Z} \subseteq B$ will be called the

[7] For an introduction to the theory of generalised formal power series, cf. [8].

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base group and its elements will be called *coefficients*. The generalised formal power series will generalise the idea of a polynomial with coefficients in B, using formal terms of the form

$$\sum f(a)X^a,$$

where a ranges over elements of Γ.

If $f : \Gamma \to B$, we shall call the set $\mathrm{supp}(f) = \{a \in \Gamma ; f(a) \neq 0\} \cup \{0\}$ the *support of f*. As usual, we say that a subset $S \subseteq \Gamma$ is *reverse well-founded* if it has no strictly increasing infinite sequences. A function $f : \Gamma \to B$ is called a *formal power series with base B and exponent* Γ if $\mathrm{supp}(f)$ is reverse well-founded and $f(0) \in \mathbb{Z}$.[8] Since non-empty reverse well-founded sets have a maximal element, we can define the *leading term of a formal power series f*, denoted by $\mathrm{LT}(f)$, as the maximal element of $\mathrm{supp}(f)$.

We say that f is *non-negative* if $f(\mathrm{LT}(f)) > 0$ or f is the function that is constant and equal to 0 everywhere. The set of non-negative formal power series with base B and exponent Γ is denoted by $B(X^\Gamma)$.

4.2. Addition of formal power series. We think of $f \in B(X^\Gamma)$ as the formal sum $\sum_{a \in \mathrm{supp}(f)} f(a)X^a$ and define the order and additive structure on $B(X^\Gamma)$ according to this intuition:

1. The order $<$ on $B(X^\Gamma)$ is the anti-lexicographic order: if $f \neq g$, then the reverse well-foundedness of $\mathrm{supp}(f)$ and $\mathrm{supp}(g)$ implies that there is a largest $a \in \Gamma$ such that $f(a) \neq g(a)$; we define $f < g$ if and only if $f(a) < g(a)$ for that largest such a.
2. The constant function that is equal to 0 everywhere is clearly the minimal non-negative formal power series with respect to the order $<$ and will be denoted by 0.
3. If $f \in B(X^\Gamma)$, we define its successor by
$$\mathrm{s}(f)(a) := \begin{cases} f(a), & \text{if } a \neq 0 \text{ and} \\ f(a)+1, & \text{if } a = 0. \end{cases}$$
4. Given $f, g \in B(X^\Gamma)$, we define $f + g$ pointwise by $(f+g)(a) := f(a) + g(a)$. Note that $\mathrm{supp}(f+g) \subseteq \mathrm{supp}(f) \cup \mathrm{supp}(g)$, so $\mathrm{supp}(f+g)$ is reverse well-founded. Furthermore, if f and g are both non-negative, then $f + g$ is non-negative. Thus, $+$ is a binary operation on $B(X^\Gamma)$.

THEOREM 14. *If* Γ *is an exponent order and* B *is a base group, then* $(B(X^\Gamma), 0, <, \mathrm{s}, +)$ *is a model of* Pr^-.

PROOF. It is routine to check that the axioms of Pr^- are satisfied. Note that it is axiom S2 (the fact that $\mathrm{s}(x)$ is the order successor of x) that uses our additional requirement that $f(0) \in \mathbb{Z}$. ⊣

[8] Cf. the proof of Theorem 14 to understand the last requirement.

Let us consider a few instructive examples:

1. If $\Gamma = \{0\} = 1$ and $B = \mathbb{Z}$ then $B(X^\Gamma) = \mathbb{Z}(X^1)$ and $(\mathbb{Z}(X^1), 0, <, \mathrm{s}, +)$ is isomorphic to the natural numbers.
2. If $\Gamma = \{0, 1\} = 2$ and $B = \mathbb{Z}$, then $B(X^\Gamma) = \mathbb{Z}(X^2)$ and $(\mathbb{Z}(X^2), 0, <, \mathrm{s}, +)$ is isomorphic to the non-negative polynomials of degree at most one on $\mathbb{Z}$ with the standard order and operations, and, more generally for every $0 < n \in \mathbb{N}$, if $\Gamma = n$ and $B = \mathbb{Z}$, then $(\mathbb{Z}(X^n), 0, <, \mathrm{s}, +)$ is isomorphic to the non-negative polynomials of degree at most $n - 1$ over $\mathbb{Z}$ with the standard order and operations.
3. Finally, by taking $\Gamma = \mathbb{N}$ and $B = \mathbb{Z}$ we have that $(\mathbb{Z}(X^\mathbb{N}), 0, <, \mathrm{s}, +)$ is isomorphic to the non-negative polynomials over $\mathbb{Z}$ with the standard order and operations. As mentioned in Section 2.2, this means that the order type of $\mathbb{Z}(X^n)$ is O_n and the order type of $\mathbb{Z}(X^\mathbb{N})$ is O_ω.[9]

If we require in addition that B is a divisible group, then Presburger's theorem implies that the formal power series construction gives a model of Pr. This fits with Theorem 17(ii) (due to Llewellyn-Jones) discussed in the next section.

Theorem 15. *Let* $(\Gamma, 0, <)$ *be a linearly ordered set with least element* 0 *and* $(B, 0, <, +)$ *be a ordered divisible abelian group. Then* $(B(X^\Gamma), 0, <, \mathrm{s}, +)$ *is a model of* Pr.

Proof. By Theorem 14 and Presburger's characterisation of Pr by Pr^D, we only need to show that that for every natural number $n > 0$, the axiom D_n holds.

Let $f \in B(X^\Gamma)$ and $0 < n \in \mathbb{N}$; we shall define g such that $f = \mathrm{s}^m(g \cdot n)$. First find $z \in \mathbb{Z}$ and $0 < m < n$ such that $f(0) = zn + m$. Use the divisibility of B to find for every $a \in \Gamma$ an element $b_a \in B$ such that $f(a) = b_a \cdot n$. Now, define

$$g(a) = \begin{cases} z, & \text{if } a = 0, \\ b_a, & \text{if } a > 0. \end{cases}$$

Then $f = \mathrm{s}^m(g \cdot n)$ as desired. ⊣

4.3. Multiplication of formal power series. In order to define multiplication on formal power series, we need an additive structure on the exponents and a multiplicative structure on the coefficients. So, we now assume that we have an addition $+$ on our exponent order such that $(\Gamma, 0, <, +)$ is an ordered abelian monoid with least element 0 and a multiplication $\cdot$ on the base group such that $(B, 0, 1, <, +, \cdot)$ is an ordered ring.

If $f, g \in B(X^\Gamma)$, we define $f \cdot g$ by

$$(f \cdot g)(a) := \sum_{b+c=a} f(b) \cdot g(c).$$

[9]Cf. the proof of Theorem 33 for further generalisations of this example.

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In order to see that this defines a formal power series, we need to check that for every $a \in \Gamma$, there are only finitely many pairs $c, b \in \Gamma$ such that $c + b = a$ and $f(b) \neq 0$ and $g(c) \neq 0$. Assume towards a contradiction that there are infinite sets $\{b_n ; n \in \mathbb{N}\}$ and $\{c_n ; n \in \mathbb{N}\}$ such that for all $n \in \mathbb{N}$, we have $b_n + c_n = a$, $f(b_n) \neq 0$, and $g(c_n) \neq 0$. From this, we now build either a strictly increasing sequence in $\mathrm{supp}(f)$ or in $\mathrm{supp}(g)$, contradicting their reverse well-foundedness.

Given a sequence $s : \mathbb{N} \to \Gamma$ we call an element $s(n)$ of the sequence a *spike* if for all $m > n$ we have $s(n) > s(m)$. Consider $b : \mathbb{N} \to \Gamma$ and $c : \mathbb{N} \to \Gamma$ as sequences defined by $b(n) := b_n$ and $c(n) := c_n$. Each of them either has infinitely many spikes or some n such that there are no spikes after n.

If b has only finitely many spikes, then we can easily define a strictly increasing subsequence, contradicting the reverse well-foundedness of $\mathrm{supp}(f)$. On the other hand, if b has infinitely many spikes, then the subsequence of spikes forms a strictly decreasing sequence, but since $b_n + c_n = a$, the corresponding subsequence of c must form a strictly increasing sequence, contradicting the reverse well-foundedness of g.

A minor modification of that argument shows that if $f, g \in B(X^\Gamma)$, then

$$Z := \{b + c ; b \in \mathrm{supp}(f) \text{ and } c \in \mathrm{supp}(g)\}$$

is reverse well-founded. Since $\mathrm{supp}(f \cdot g) \subseteq Z$, we have that $\mathrm{supp}(f \cdot g)$ is reverse well-founded. Furthermore, $\mathrm{LT}(f \cdot g) = \mathrm{LT}(f) + \mathrm{LT}(g)$ and $(f \cdot g)(\mathrm{LT}(f \cdot g)) = f(\mathrm{LT}(f)) \cdot g(\mathrm{LT}(g)) > 0$, so $f \cdot g$ is non-negative. Together, $f \cdot g \in B(X^\Gamma)$.

THEOREM 16. *Let* $(\Gamma, 0, <, +)$ *be an ordered abelian monoid with least element* 0 *and* $(B, 0, 1, <, +, \cdot)$ *be an ordered commutative ring. Then the structure* $(B(X^\Gamma), 0, <, \mathrm{s}, +, \cdot)$ *is a model of* PA^-.

PROOF. It is routine to check the axioms M1 to M6. ⊣

Note that by Theorem 15, if $B = \mathbb{Q}$ and $\Gamma = 2$, then $\mathbb{Q}(X^2)$ is a model of Pr of order type $\mathbb{N} + \mathbb{Z} \cdot \mathbb{Q}$, but it is not closed under multiplication and so cannot be a model of PA^-. This model is well-known in the literature; cf., e.g., [29].

§5. Presburger arithmetic. Presburger arithmetic, the additive fragment of arithmetic, is closely related to ordered abelian groups. Llewellyn-Jones considered an integer version of Presburger arithmetic, allowing for additive inverses and gives an axiomatisation for this theory that we shall call $\mathsf{Pr}^\mathbb{Z}$ [13]. If $(M, 0, <, \mathrm{s}, +) \models \mathsf{Pr}^\mathbb{Z}$, then $(M, 0, <, +)$ is an ordered abelian group; Llewellyn-Jones calls these groups *Presburger groups* and proves in his integer setting that G is a Presburger group if and only if G is isomorphic to $\mathbb{Z} \cdot H$ where H is an ordered divisible abelian group [13, Sections 3.1 and 3.2]. In the following, we reformulate Llewellyn-Jones's approach in the standard setting of arithmetic (i.e., without additive inverses).

THEOREM 17. *Let M be an $\mathcal{L}_{<,s,+}$-structure.*

(i) *The structure M is a model of* Pr^- *if and only if there is an ordered abelian group G such that M is isomorphic to the standard monoid over G, and*

(ii) *the structure M is a model of* Pr *if and only if there is an ordered divisible abelian group G such that M is isomorphic to the standard monoid over G.*

PROOF. This proof is a reformulation of the characterisation of Presburger groups as in [13] to the standard setting.

For the forward direction of (i), it is enough to see that in $\mathbb{N} + \mathbb{Z} \cdot G^+$ all the axioms of Pr^- are trivially satisfied. For the other direction, if $M \models \mathsf{Pr}^-$ then by (the proof of) Corollary 13, the order type of M is $\mathbb{N} + \mathbb{Z} \cdot A$ for a linear order A consisting of the non-zero Archimedean classes of M. For each $a \in A$, we define a formal *negative element* $-a$ such that the negative elements are all distinct from the elements of A and pairwise distinct. Then we define $-A := \{-a\,;\, a \in A\}$ and $G := -A \cup \{[0]\} \cup A$. For notational convenience, we define $-[0] := [0]$. We define an abelian group structure on G as follows:

1. For any $g \in G$, $g + [0] := [0] + g := g$.
2. If $a, b \in A$ are non-zero Archimedean classes of M, then there is a unique $c \in A$ such that for all $x \in a$ and $y \in b$, we have that $x + y \in c$; define $a + b := b + a := c$ and $(-a) + (-b) := (-b) + (-a) := -c$.
3. If $a, b \in A$, $x \in a$, and $y \in b$ with $x < y$, then by ⋇, we find z such that $x + z = y$. Let c be the Archimedean class of z, i.e., $c \in A \cup \{[0]\}$. Then $(-a) + b := b + (-a) := c$ and $a + (-b) := (-b) + a := -c$.

It is routine to check that $(G, 0, <, +)$ is an ordered abelian group and that M isomorphic to $\mathbb{N} + \mathbb{Z} \cdot G^+$. For (ii), all that is left to show is that divisibility of the group corresponds to the additional axioms D_n of Pr^{D}. ⊣

COROLLARY 18 (Folklore). *There is a model of* Pr *with order type* $\mathbb{N} + \mathbb{Z} \cdot \mathbb{R}$.

PROOF. The real numbers $\mathbb{R}$ are an ordered divisible abelian group, so by Theorem 17 (ii), there is a model of Pr with order type $\mathbb{N} + \mathbb{Z} \cdot \mathbb{R}^+$. The claim follows from the fact that $\mathbb{R}^+$ and $\mathbb{R}$ have the same order type. ⊣

COROLLARY 19. *Let M be a non-standard model of* Pr. *Then M has order type $\mathbb{N} + \mathbb{Z} \cdot A$ where A is a dense linear order without endpoints.*

PROOF. It is enough to observe that divisibility implies density and use Theorem 17. ⊣

We can use Theorem 17 and the general theory of groupable linear orders to get a characterisation theorem for the order types of models of Pr^-. First let us recall a classical result about groupable linear orders; cf., e.g., [21, Theorem 8.14]:

THEOREM 20. *A linear order $(L, <)$ is groupable if and only if there is an ordinal α and a densely ordered abelian group $(D, 0, <, +)$ such that L has order type $\mathbb{Z}^\alpha \cdot D$.*

COROLLARY 21. *A structure M is a model of* Pr^- *if and only if there is an ordinal* α *and a densely ordered abelian group* $(D, 0, <, +)$ *such that* M *has order type* $\mathbb{N} + \mathbb{Z} \cdot (\mathbb{Z}^\alpha \cdot D)^+$.

PROOF. Follows from Theorems 17 and 20. ⊣

As we have seen in Section 4, the non-negative formal power series on $\mathbb{Z}$ with exponent 2 are isomorphic to the ordered abelian monoid of polynomials of degree at most one with integer coefficients. Moreover, by Theorem 14 (or Theorem 17), $(\mathbb{Z}(X^2), 0, <, \mathrm{s}, +) \models \mathsf{Pr}^-$. The next theorem expresses that this is a lower bound for non-standard models of Pr^-.

THEOREM 22. *Let M be a non-standard model of* Pr^-. *Then M has a submodel isomorphic to* $(\mathbb{Z}(X^2), 0, <, \mathrm{s}, +)$.

PROOF. Let M be a non-standard model of Pr^- and $a \in M$ be a non-standard element of M. define the following mapping $\varphi : \mathbb{Z}(X^2) \to M$:

$$\varphi(f) = \begin{cases} \mathrm{s}^n(0), & \text{if } \mathrm{LT}(f) = 0 \text{ and } f(0) = n, \\ \mathrm{s}^m(na), & \text{if } \mathrm{LT}(f) = 1 \text{ and } f(1) = n, f(0) = m \geq 0, \\ b, & \text{if } \mathrm{LT}(f) = 1 \text{ and } f(1) = n, f(0) = m < 0 \text{ and} \\ & \quad \mathrm{s}^{-m}(b) = na. \end{cases}$$

It is easy to see that φ is an order-preserving injection. ⊣

COROLLARY 23. *Let M be a non-standard model of* Pr^- *then the order* $\mathbb{N} + \mathbb{Z} \cdot \mathbb{N}$ *can be embedded in the order type of M.*

PROOF. As mentioned, $\mathbb{Z}(X^2)$ is the set of non-negative polynomials of degree at most 1 over $\mathbb{Z}$ and clearly has order type $\mathbb{N} + \mathbb{Z} \cdot \mathbb{N}$. The result then follows from Theorem 22. ⊣

COROLLARY 24. *Every non-standard model of* Pr^- *has a proper non-standard submodel.*

PROOF. By Theorem 22, it is enough to show that $\mathbb{Z}(X^2)$ has a non-standard submodel. Consider all polynomials with degree at most 1 and even leading terms, i.e.,

$$M := \{2nX + z \in \mathbb{Z}(X^2)\,;\, n \in \mathbb{N}, z \in \mathbb{Z}\}.$$

Clearly, this set is closed under s and $+$, so it is a substructure of $\mathbb{Z}(X^2)$. Moreover, every element of M except for 0 has a predecessor. Therefore, M satisfies axiom S1. The only existential axiom of Pr^- which still needs to be checked is ⋇. Let $f, g \in M$ such that $f < g$. Define $h(a) = g(a) - f(a)$. We want to show that $h \in M$. If $\mathrm{LT}(f) = 0$, this is trivially true since $h(1) = g(1)$. If $\mathrm{LT}(f) = 1$, then $f(1) = 2n$ and $g(1) = 2n'$ for some $n, n' \in \mathbb{N}$ such that $n < n'$. Then $h(1) = 2n' - 2n = 2(n' - n)$; therefore $h \in M$. The fact that $f + h = g$ follows trivially by the definition of $+$ in $\mathbb{Z}(X^2)$. ⊣

§6. The algebraic nature of models of theories without induction. In this section, we are going to discuss the difference between the theories without induction and those with induction: since almost all of the axioms in the

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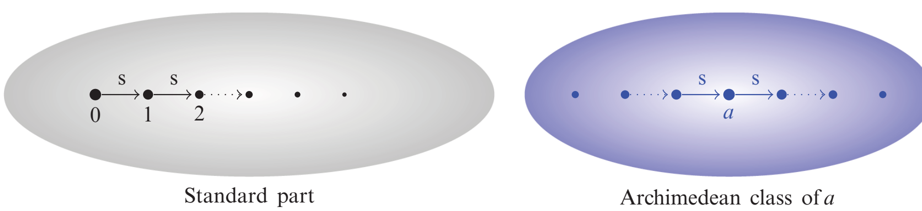
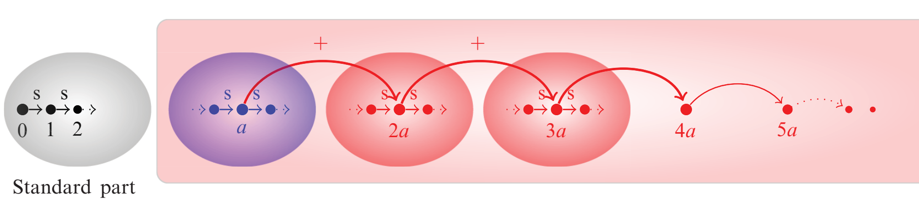
former are universal sentence, they behave essentially like algebraic theories; this means that their models can be obtained as algebraic closures. In contrast, the latter require richer constructions.

We illustrate this by considering the simplest cases of non-standard models of SA and Pr^- as discussed in Sections 3 and 5. In the case of $\mathsf{SA}^- = \mathsf{SA}$, the axioms O1 to O4 and S2 to S4 are universal, so models of these axioms are just constructed by closing under the operation s. Axiom S1 requires the existence of a s-predecessor for each element other than 0. The closure under s-successors and s-predecessors (i.e., the unique witnesses for all instances of the $\forall\exists$-axiom S1) of a given non-standard element is its Archimedean class and thus, if we take a set A of generators, it will generate the model of order type $\mathbb{N} + \mathbb{Z} \cdot A$ from Corollary 13.

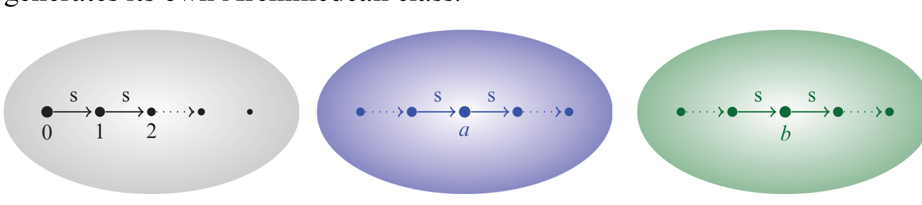
Concretely, if $A = \{a\}$ consists of just one generator, we get the standard part and a single additional Archimedean class, i.e., a model of order type $\mathbb{N} + \mathbb{Z}$:



Once we add the operation $+$ and want to extend our model to an $\mathcal{L}_{<,\mathrm{s},+}$-structure, for each non-standard element a, we need to have the elements $a + a = 2a$, $a + a + a = 3a$, etc. By Lemma 6, their Archimedean classes must be separate, so we generate an ω-sequence of Archimedean classes resulting in a model of order type $\mathbb{N} + \mathbb{Z} \cdot \mathbb{N}$, the minimal order type of a non-standard model of Pr^-. The following is a picture of the case where D is the one-element group and $\alpha = 1$ in Corollary 21. It could be considered a picture proof of Corollary 23.



Moving to the concrete case where $A = \{a, b\}$ with $a < b$ is a set of two generators, the models of SA remain purely generated: by Corollary 13, the generated model of SA has order type $\mathbb{N} + \mathbb{Z} \cdot 2$: each of the generators generates its own Archimedean class.



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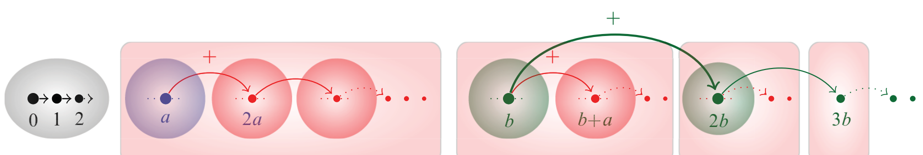
However, if we add the operation $+$, a new phenomenon occurs. The closure of this model under $+$ (taking axiom P4 into account) will produce all of the terms of the form $na + mb$ for $n, m \in \mathbb{N}$ with the order

$$na + mb < ka + \ell b \text{ if and only if } m < \ell \text{ or } m = \ell \text{ and } n < k.$$

These terms give us the generated Archimedean classes, and consequently, the generated non-standard elements of the model can be described as a pair $(z, na + mb)$ where $na + mb$ is the term determining the Archimedean class and $z \in \mathbb{Z}$ determines the position within the Archimedean class. The order type of this generated structure is $\mathbb{N} + \mathbb{Z} \cdot \mathbb{N} \cdot \mathbb{N}$ with ω many copies of the non-standard part of the model of Pr^- with one generator.



But this generated structure is not a model of Pr^-; this follows, e.g., from Theorem 17(i), since $\mathbb{N} \cdot \mathbb{N}$ is not the order type of the standard monoid of an ordered abelian group.

The reason for this failure is the is axiom $\circledast$ which –as mentioned before– is an instance of the induction scheme and the only axiom of Pr^- (other than S1 which we dealt with earlier) that is not in universal form. In this context, it guarantees the existence of an (additional) element c such that $a + c = b$, i.e., the element $b - a$. This element must be bigger than all finite products of a, i.e., above the part of the model $+$-generated from a, but smaller than b.

Yet $\circledast$ is of the syntactic form $\forall\exists$, so we can form the closure of the above structure of order type $\mathbb{N} + \mathbb{Z} \cdot \mathbb{N} \cdot \mathbb{N}$ under adding (unique) witnesses for ($\circledast$). This will produce the next case in Corollary 21 with D the one-element group and $\alpha = 2$ with order type $\mathbb{N} + \mathbb{Z} \cdot \mathbb{Z} \cdot \mathbb{N}$.

Our discussion of models generated by one and two generators illustrates the main theme of this paper: theories without induction can be handled by simple closure techniques, but theories with induction require model theoretic arguments.

§7. Peano arithmetic. Theorem 17 tells us that every model $M \models \mathsf{PA}^-$ ($M \models \mathsf{PA}$) must have the order type $\mathbb{N} + \mathbb{Z} \cdot G^+$ where G is an ordered (divisible) abelian group. However, in the case of Peano Arithmetic, this cannot be a sufficient condition since Potthoff proved that no model of PA can have the order type $\mathbb{N} + \mathbb{Z} \cdot \mathbb{R}$ [16]. It is easily checked that the proof of Potthoff's theorem given in [2, p. 5] works in PA^-:

THEOREM 25. *Let M be a non-standard model of* PA^- *with order type $\mathbb{N} + \mathbb{Z} \cdot A$. If A is dense then there are $|M|$ many non empty disjoint intervals in A. In particular, $A \neq \mathbb{R}$.*

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PROOF. Let $a \in M$ be non-standard. Consider the set $\{a_m ; m \in M\}$ where $a_m = a \cdot m$ for every $m \in M$. By M6, this set has cardinality $|M|$. We shall now show that $\{([a_m], [a_{s(m)}]) ; m \in M\}$ forms a collection of non-empty disjoint intervals of size $|M|$ in A:

By Lemma 6, $[a \cdot m] < [a \cdot s(m)]$ for every $m \in M$. By density of A, the interval $([a_m], [a_{s(m)}])$ is not empty in A. Now if $m < m'$, then by M6 we have $a \cdot s(m) \le a \cdot m'$ and $[a \cdot s(m)] \le [a \cdot m']$. Therefore $([a_m], [a_{s(m)}]) \cap ([a_{m'}], [a_{s(m')}]) = \varnothing$ as desired.

If $A = \mathbb{R}$, then the order type of M is $\mathbb{N} + \mathbb{Z} \cdot \mathbb{R}$ and hence $|M| = 2^{\aleph_0}$. Now the main claim of the theorem gives us an uncountable family of pairwise disjoint intervals in $\mathbb{R}$ which contradicts the countable chain condition of the real line. ⊣

Theorem 25 shows that the closure under multiplication adds more requirements on the order type of models of PA^-. The following is a natural requirement: we remind the reader of the definitions of initial and final segments and order exponentiation from Section 2.2. In particular, if L is a linear order and $\ell \in L$, then $\mathrm{IS}(\ell)$ is the initial segment given by ℓ and $\mathrm{IS}(\ell)^\omega$ is the set of functions from ω to $\mathrm{IS}(\ell)$ with finite support, ordered anti-lexicographically.

DEFINITION 26. Let L be a linear order. We say that L is *closed under finite products of initial segments* if for every $\ell \in L$ the order $\mathrm{IS}(\ell)^\omega$ embeds into $\mathrm{FS}(\ell)$.

THEOREM 27. *Let M be a non-standard model of PA^- with order type $\mathbb{N} + \mathbb{Z} \cdot L$. Then L is closed under finite products of initial segments.*

PROOF. As before, we assume that L is the set of non-zero Archimedean classes of M. For every $\ell \in L$ choose a representative $r_\ell \in M$ such that $r_\ell \in \ell$ and $r_\ell > 0$. Let $\ell \in L$ be an element of the linear order L. We want to define an order embedding of $\mathrm{IS}(\ell)^\omega$ into $\mathrm{FS}(\ell)$. Fix some non-standard $a \in M$ such that $\ell \le [a]$ and consider the following function:

$$\varphi(f) = \left[\sum_{i \le \mathrm{LT}(f)} r_{f(i)} \cdot a^{i+1} \right],$$

for every $f \in \mathrm{IS}(\ell)^\omega$. Note that since f has finite support, the function φ is well defined. Now we want to prove that φ is order-preserving. First we prove the following claim:

CLAIM 28. *For every $n > 0$ and every finite sequence $(\ell_0, \ldots, \ell_{n-1})$ of elements of $\mathrm{IS}(\ell)$ we have*

$$\sum_{i<n} r_{\ell_i} \cdot a^{i+1} < a^{n+1}.$$ ⊣

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Proof. By induction on n. For $n = 1$ we have $r_{\ell_0} \cdot a < a \cdot a$. For $n = n' + 1 > 1$ we have

$$\begin{aligned}\sum_{i<n'+1} r_{\ell_i} \cdot a^{i+1} &= \sum_{i<n'} r_{\ell_i} \cdot a^{i+1} + r_{\ell_{n'}} \cdot a^{n'+1} \\ &< a^{n'+1} + r_{\ell_{n'}} \cdot a^{n'+1} \\ &= a^{n'+1} \cdot (\mathrm{s}(0) + r_{\ell_{n'}}) < a^{n'+2}.\end{aligned}$$ ⊣

We want to prove that if $f < f'$ are two elements of $\mathrm{IS}(\ell)^{\omega}$ then $\varphi(f) < \varphi(f')$. Let $n \in \mathbb{N}$ be the biggest natural number such that $f(n) \neq f'(n)$. Since $f < f'$ we have $f(n) < f'(n)$, then $[r_{f(n)}] < [r_{f'(n)}]$.

Moreover since $n \leq \mathrm{LT}(f')$ we have

$$\sum_{n<i\leq \mathrm{LT}(f')} r_{f(i)} \cdot a^{i+1} = \sum_{n<i\leq \mathrm{LT}(f')} r_{f'(i)} \cdot a^{i+1}.$$

Therefore, by monotonicity of $+$ it is enough to prove that for every $n' \in \mathbb{N}$ we have

$$\sum_{i\leq n} r_{f(i)} \cdot a^{i+1} + \mathrm{s}^{n'}(0) < r_{f'(n)} \cdot a^{n+1}.$$

For $n = 0$ it is trivially true. For $n > 0$, we have

$$\begin{aligned}\sum_{i\leq n} r_{f(i)} \cdot a^{i+1} + \mathrm{s}^{n'}(0) &= \sum_{i<n} r_{f(i)} \cdot a^{i+1} + r_{f(n)} \cdot a^{n+1} + \mathrm{s}^{n'}(0) \\ &< a^{n+1} + r_{f(n)} \cdot a^{n+1} + \mathrm{s}^{n'}(0) \\ &< a^{n+1} \cdot (r_{f(n)} + \mathrm{s}^{n'+1}(0)) \\ &< a^{n+1} \cdot r_{f'(n)},\end{aligned}$$

where we used Claim 28 in the first inequality. Therefore φ is order-preserving as desired. ⊣

Theorem 16 showed that the non-negative polynomials with integer coefficients $\mathbb{Z}(X^{\mathbb{N}})$ are a model of PA⁻. In analogy to Theorem 22, we show that this provides a lower bound for the order type of non-standard models of PA⁻:

Theorem 29. *Let M be a non-standard model of* PA⁻. *Then there is a submodel of M isomorphic to* $(\mathbb{Z}(X^{\mathbb{N}}), 0, <, \mathrm{s}, +, \cdot)$.

Proof. Let M be a non-standard model of PA⁻ and $a \in M$ be a non-standard element of M. Let $f \in \mathbb{Z}(X^{\mathbb{N}})$; remember that if $\mathrm{supp}(f) \subseteq \{0, \dots, n\}$ and $\mathrm{LT}(f) = n$, then f can be thought of as a polynomial

$$f(n)X^n + f(n-1)X^{n-1} + \cdots + f(0),$$

where $f(n) > 0$ and $f(i) \in \mathbb{Z}$ (for $0 \leq i < n$). We define the function

$$\varphi : \mathbb{Z}(X^{\mathbb{N}}) \to M : f \mapsto f(n)a^n + f(n-1)a^{n-1} + \cdots + f(0),$$

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where negative terms are uniquely interpreted by the fact that we have axiom ⋇. It is routine to check that φ is an embedding of $(\mathbb{Z}(X^{\mathbb{N}}), 0, <, \mathrm{s}, +, \cdot)$ into M. ⊣

COROLLARY 30. *Let M be a non-standard model of* PA^-. *Then the order type O_ω can be embedded in the order type of M. In particular, $\mathbb{Z}(X^2)$ is not a model of* PA^-.

PROOF. Since O_ω is the order type of the non-negative polynomials on $\mathbb{Z}$, this follows directly from Theorem 29. ⊣

COROLLARY 31. *Every non-standard model of* PA^- *has a proper non-standard submodel.*

PROOF. As in the proof of Corollary 24, by Theorem 29, it is enough to check that $\mathbb{Z}(X^{\mathbb{N}})$ has a proper non-standard submodel. Consider the polynomials in which only terms with even exponent occur and observe that they are closed under addition and multiplication and that the structure satisfies ⋇. ⊣

The contrast between $\mathbb{Z}(X^2) \not\models \mathsf{PA}^-$ and $\mathbb{Z}(X^{\mathbb{N}}) \models \mathsf{PA}^-$ derives from the fact that the exponential order $\mathbb{N}$ is closed under addition and forms a monoid (allowing us to use Theorem 16) whereas the exponential order 2 is not.

This can be easily generalised: an ordinal α is called *additively indecomposable* (or a *gamma number*) if it cannot be written as $\alpha = \xi + \eta$ for ordinals $\xi, \eta < \alpha$; equivalently, it is closed under ordinal addition, i.e., for all $\xi, \eta \in \alpha$, we have $\xi + \eta \in \alpha$. If α is additively indecomposable, then it is also closed under the natural (Hessenberg) sum of ordinals, denoted by $\oplus$, which is a commutative operation (cf. [25, XIV.28]).

Clearly, all ordinals of the form $\alpha = \omega^\gamma$ are additively indecomposable, and if α is additively indecomposable, then the structure $(\alpha, <, 0, \oplus)$ is an ordered commutative positive monoid. Thus, by Theorem 16, the structure $\mathbb{Z}(X^\alpha)$ is a model of PA^-. Note that the order type of $\mathbb{Z}(X^\alpha)$ is O_α (defined in Section 2.2).

We end this section by showing that these order types give us many non-isomorphic order types of models of PA^- of a given cardinality. This yields a markedly different situation from the theories with induction, Pr and PA, in the countable case.

LEMMA 32. *Let α and β be two ordinals. Then O_α and O_β are order isomorphic if and only if $\alpha = \beta$.*

PROOF. We observe that for positive ordinals α, ω^α embeds order-preservingly into O_α. We shall show that $\omega^{\alpha+1}$ does not embed into O_α. Once we have proved this, these two statements immediately imply the claim: if $\alpha < \beta$, then O_β cannot embed order-preservingly into O_α and so they cannot be isomorphic.

In order to show our non-embedding results, we first prove the following claim by induction for positive ordinals α:

Every order-preserving embedding $\varphi : \omega^\alpha \to \mathbb{Z}^\alpha$ must be cofinal. $(*)$

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The claim is obvious for $\alpha = 1$ and the limit case follows directly from the induction hypothesis. Let now $\alpha = \beta + 1$ and $\varphi : \omega^{\beta+1} \to \mathbb{Z}^{\beta+1}$ be an order-preserving embedding where we write elements of $\omega^{\beta+1} = \omega^{\beta} \cdot \omega$ as pairs (γ, n) with $\gamma \in \omega^{\beta}$ and $n \in \omega$ and elements of $\mathbb{Z}^{\beta+1} = \mathbb{Z}^{\beta} \cdot \mathbb{Z}$ as pairs (g, z) with $g \in \mathbb{Z}^{\beta}$ and $z \in \mathbb{Z}$. If $n \in \omega$ and $z \in \mathbb{Z}$, we write $A_n := \{\varphi(\gamma, n)\,;\, \gamma \in \omega^{\beta}\} \subseteq \operatorname{ran}(\varphi)$ and $B_z := \{(g, z)\,;\, g \in \mathbb{Z}^{\beta}\}$; these sets have order type ω^{β} and $\mathbb{Z}^{\beta}$, respectively. Clearly, $\operatorname{ran}(\varphi) = \bigcup_{n\in\omega} A_n$ and $\mathbb{Z}^{\beta+1} = \bigcup_{z\in\mathbb{Z}} B_z$.

If we fix $n \in \omega$, the set A_n is non-empty and bounded in $\mathbb{Z}^{\beta+1}$. Therefore there is a minimal integer z_n such that for all $z \geq z_n$, we have $A_n \cap B_z = \varnothing$. In particular, $A_n \cap B_{z_n-1} \neq \varnothing$, and thus B_{z_n-1} contains a final segment of A_n. The order type of A_n is ω^{β}, i.e., an additively indecomposable ordinal; therefore, final segments of A_n still have order type ω^{β}. By the induction hypothesis, we know that $A_n \cap B_{z_n-1}$ must lie cofinal in B_{z_n-1}.

Now we consider elements of A_{n+1}: these are strictly bigger than all of the elements of A_n, and therefore they must lie in some B_z for $z > z_n$. In particular, $z_{n+1} > z_n$. This shows that the sequence z_n is a strictly increasing sequence of integers indexed by natural numbers, hence cofinal in $\mathbb{Z}$. This implies that $\operatorname{ran}(\varphi)$ is cofinal in $\mathbb{Z}^{\beta} \cdot \mathbb{Z}$, finishing the proof of (*).

We now use (*) to prove that every order-preserving embedding from ω^{α} into O_{α} must be cofinal. This clearly implies our desired non-embedding claim for $\omega^{\alpha+1}$ and therefore finishes the proof of the lemma.

As in the proof of (*), the case $\alpha = 1$ is obvious and the limit case follows directly from the induction hypothesis. Let $\alpha = \beta + 1$ and let $\varphi : \omega^{\beta+1} \to O_{\beta+1} = O_{\beta} + \mathbb{Z}^{\beta} \cdot \mathbb{N}$ be an order-preserving embedding. We apply the induction hypothesis to $\varphi{\restriction}\omega^{\beta}$ and obtain that its image cannot be bounded in O_{β}; thus, a final segment of $\operatorname{ran}(\varphi)$ lies in the $\mathbb{Z}^{\beta} \cdot \mathbb{N}$ part of $O_{\beta+1}$. By the fact that $\omega^{\beta+1}$ is additively indecomposable, this final segment has order type $\omega^{\beta+1}$, so we can partition it into a strictly increasing ω-sequence of sets each of order type ω^{β}. Applying (*) inductively to these fragments of the map φ, we see that the nth of these sets has to lie cofinal in the nth copy of $\mathbb{Z}^{\beta}$ or reach beyond it. In total, the image of φ has to lie cofinal in the entire $\mathbb{Z}^{\beta} \cdot \mathbb{N}$ part of $O_{\beta+1}$ and hence in $O_{\beta+1}$ itself. ⊣

THEOREM 33. *There are at least λ^{+} non-isomorphic order types of models of* PA⁻ *that are not models of* Pr (*or* PA) *of cardinality λ.*

PROOF. There are λ^{+} many additively indecomposable ordinals smaller than λ^{+}. For each such α, we have that $(\mathbb{Z}(X^{\alpha}), 0, <, \mathrm{s}, +, \cdot)$ is a model of PA⁻ of cardinality λ; by Lemma 32, they have pairwise non-isomorphic order types. It is easy to see that none of these models are models of Pr using Corollary 19. ⊣

Note that for $\lambda = \omega$, Theorem 33 gives us uncountably many non-isomorphic countable models of PA⁻ in stark contrast with the two order types of countable models of PA (by Cantor's theorem, $\mathbb{N}$ and $\mathbb{N} + \mathbb{Z} \cdot \mathbb{Q}$ are the only possible order types).

We note that for uncountable λ, Theorem 33 is the consequence of a result related to Shelah's famous *Non-structure Theorem* or *Many-Models*

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Theorem. As part of his proof of the Non-structure Theorem, Shelah considers the number of reducts in [23, Chapter VIII]:

THEOREM 34 (Shelah; [23, Theorem VIII.0.4]). *Let λ be an uncountable regular cardinal, $\mathcal{L} \subseteq \mathcal{L}^*$ be two countable languages, T an $\mathcal{L}$-theory, and $T' \supseteq T$ an $\mathcal{L}^*$-theory. If T^* has infinite models and T is not stable, then there are 2^λ many pairwise non-isomorphic models of T that are $\mathcal{L}$-reducts of models of T^*.*

Now let T be the complete theory of discrete linear orders with a least but no largest element (which is a standard example for an unstable theory) and T' to be PA^-. Then Theorem 34 yields 2^λ many non-isomorphic order types of models of PA^- of regular size $\lambda \geq \aleph_1$.

Comparing Theorems 33 and 34, we see that our construction gets λ^+ instead of 2^λ many models, but provides the additional information that they are not models of Pr and works for $\lambda = \omega$.

§8. Shelah on almost isomorphism of order rigid models. The most recent progress on Friedman's Question 2 was made by Shelah in [24]. Since this is one of the few other papers that discusses fragments of Peano arithmetic with no induction or only fragments of induction, we should like to present Shelah's results here.

Let $(M, 0, <, \mathrm{s}, +, \cdot)$ be an $\mathcal{L}_{<,\mathrm{s},+,\cdot}$-structure; as usual, without loss of generality, we assume that $\mathbb{N} \subseteq M$. For $a, b \in M$, we say that a is *exponentially small relative to* b if for all natural number n, we have that $a^n < b$. We define two equivalence relations on M: we say that a is *standardly close to* b, in symbols $a\ E^2\ b$, if there is a natural number n such that $a < b \cdot n$ and $b < a \cdot n$; we say that a is *exponentially close to* b, in symbols $a\ E^3\ b$, if there is a c which is exponentially small relative to both a and b such that $a < b \cdot c$ and $b < a \cdot c$.

The structure $(M, 0, <, \mathrm{s}, +, \cdot)$ is called *two-order rigid* if any $a, b \in M$ that define order isomorphic initial segments are standardly close; it is called 3-*order rigid* if any $a, b \in M$ that define order isomorphic initial segments are exponentially close.

If $(M, 0, <, \mathrm{s}, +)$ and $(N, 0, <, \mathrm{s}, +)$ are any $\mathcal{L}_{<,\mathrm{s},+}$-structures, we call a map $f : M \to N$ an *almost isomorphism* if it is a $\mathcal{L}_{<,\mathrm{s}}$-isomorphism and for any $a, b \in M$, the elements $f(a) + f(b)$ and $f(a + b)$ are standardly close in N.

THEOREM 35 (Shelah; [24, Theorems 2.6 and 3.5]). *If $(M, 0, <, \mathrm{s}, +, \cdot) \models$ PA is two-order rigid or three-order rigid and $(N, 0, <, \mathrm{s}, +, \cdot) \models \mathsf{PA}$ has the same order type as $(M, <)$, then the additive reducts $(M, 0, <, \mathrm{s}, +)$ and $(N, 0, <, \mathrm{s}, +)$ are almost isomorphic.*

Not much is known about how close the theorem is to being optimal: in particular, it is not known whether the additive reducts are isomorphic (rather than almost isomorphic) [24, Question 2.7].

In our context, it is interesting to note that in [24, Section 5], Shelah considers whether Theorem 35 holds for $\mathcal{L}_{<,\mathrm{s},+,\cdot}$-structures that are models

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of fragments of PA without induction. He considers four weaker theories PA_{-4}, PA_{-3}, PA_{-2}, and PA_{-1}, of which PA_{-4} is our PA^- and the others are PA^- with an increasing amount of additional instances of induction. Shelah then observes that for Theorem 35 in the case of two-order rigidity, the theory PA_{-1} is sufficient [24, Theorem 5.4].

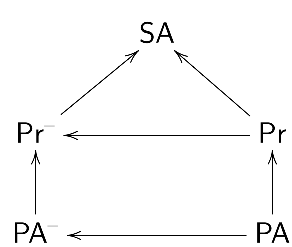
§9. Summary. We combine the various insights into possible order types of our five theories $\mathsf{SA}^- = \mathsf{SA}$, Pr^-, Pr, PA^-, and PA in order to provide the solution to Question 4 restricted to these five theories: they can all be separated by order types.

For the theories SA and Pr^-, we were able to give complete characterisations in Corollaries 13 and 21; for the theories Pr and PA^-, we were able to give necessary conditions in Corollary 19 and Theorems 25 and 27, respectively. In particular, the negative results from Sections 4 and 5 imply:

Corollary 36. *There is no model of* Pr (*and hence, no model of* PA) *with order type* O_2 *or* O_ω.

Proof. We have that $O_2 = \mathbb{N} + \mathbb{Z} \cdot \mathbb{N}$ and $O_\omega = \mathbb{N} + \mathbb{Z} \cdot O_\omega$. Clearly, $\mathbb{N}$ and O_ω are not the positive parts of a densely ordered abelian group, so by Corollary 19, no model of Pr can have these order types. ⊣

In the following diagram, an arrow from a theory T to a theory S means "every order type that occurs in a model of T occurs in a model of S". The diagram is complete in the sense that the absence of an arrow means that no arrow can be drawn, i.e., "there is an order type of a model of T that cannot be an order type of a model of S."



The non-implication $\mathsf{SA} \nrightarrow \mathsf{Pr}^-$ follows from Corollaries 13 and 23: $\mathbb{N} + \mathbb{Z}$ is an order type witnessing the separation.

The non-implication $\mathsf{Pr}^- \nrightarrow \mathsf{Pr}$ follows from Theorem 14 and Corollary 36: $\mathbb{N} + \mathbb{Z} \cdot \mathbb{N}$ is an order type witnessing the separation.

The non-implication $\mathsf{Pr}^- \nrightarrow \mathsf{PA}^-$ follows from Theorem 14 and Corollary 30: $\mathbb{N} + \mathbb{Z} \cdot \mathbb{N}$ is an order type witnessing the separation.

The non-implication $\mathsf{PA}^- \nrightarrow \mathsf{Pr}$ follows from Theorem 16 and Corollary 36: $O_\omega = \mathbb{N} + \mathbb{Z} \cdot O_\omega$ is an order type witnessing the separation.

The non-implication $\mathsf{Pr} \nrightarrow \mathsf{PA}^-$ follows from Theorem 25 and Corollary 18: $\mathbb{N} + \mathbb{Z} \cdot \mathbb{R}$ is an order type witnessing the separation.

Acknowledgements. This work was initiated during the visit of Swaraj Dalmia in Amsterdam during the summer of 2016, funded by the

https://doi.org/10.1017/bsl.2021.48 Published online by Cambridge University Press

Indo-European Research Training Network in Logic (IERTNiL); the authors would like to thank him for his involvement in the early phase of the project. Furthermore, the authors would like to thank Andrey Bovykin, Nathan Bowler, Chase Ford, Tim Henke, and Albert Visser for discussions about the material included in this paper, as well as the anonymous referee for a number of corrections and suggestions.

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AMSTERDAM UNIVERSITY COLLEGE
POSTBUS 94160
1090 GD AMSTERDAM, THE NETHERLANDS
E-mail: l.galeotti@uva.nl

INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION
UNIVERSITEIT VAN AMSTERDAM
POSTBUS 94242
1090 GE AMSTERDAM, THE NETHERLANDS
and
FACHBEREICH MATHEMATIK
UNIVERSITÄT HAMBURG
BUNDESSTRASSE 55
20146 HAMBURG, GERMANY
and
CHURCHILL COLLEGE
UNIVERSITY OF CAMBRIDGE
STOREY'S WAY, CAMBRIDGE CB3 0DS, UK
and
DEPARTMENT OF PURE MATHEMATICS AND MATHEMATICAL STATISTICS
UNIVERSITY OF CAMBRIDGE
WILBERFORCE ROAD, CAMBRIDGE CB3 0WB, UK
and
LUCY CAVENDISH COLLEGE
UNIVERSITY OF CAMBRIDGE
LADY MARGARET ROAD, CAMBRIDGE CB3 0BU, UK
E-mail: loewe@math.uni-hamburg.de