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Necessary condition for the L^2 boundedness of the Riesz transform on Heisenberg groups

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Abstract

Let μ be a Radon measure on the nth Heisenberg group $\mathbb{H}^n$. In this note we prove that if the $(2n+1)$-dimensional (Heisenberg) Riesz transform on $\mathbb{H}^n$ is $L^2(\mu)$-bounded, and if $\mu(F)=0$ for all Borel sets with $\dim_H(F)\le 2$, then μ must have $(2n+1)$-polynomial growth. This is the Heisenberg counterpart of a result of Guy David from [**Dav91**].

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1. *Introduction*

The motivation behind this paper is the following question: what are the measures μ on the Heisenberg group $\mathbb{H}^n$ which guarantee that the (correct notion of) Riesz transform is bounded from $L^2(\mu)$ to itself? This question (or some variant of it) with $\mathbb{R}^n$ instead of $\mathbb{H}^n$, was one of the major starting points of the theory that came to be known as *quantitative rectifiability*. This area of geometric measure theory has seen an impressive development in the past thirty years, starting with the landmark works of Peter Jones [**Jon90**] and David and Semmes [**DS91**], [**DS93**], through the solution of fundamental questions in complex analysis, such as the Painlevé problem (see [**MMV96**], [**Dav98**], [**Tol03**]), to more recent applications to harmonic analysis, see for example [**NTV14**] and [**AHM$^+$19**].

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In the last years, there has been an increasing interest in developing such a quantitative theory in different contexts than that of Euclidean spaces; examples of these are parabolic spaces and Heisenberg groups, or, more generally, Carnot groups. The former appear in the study of caloric measure. The latter arise naturally in the study of certain hypoelliptic operators, in the sense that the natural translations and dilations for these operators are those characterising the spaces; the Heisenberg group is the most important prototypical example, and the related operator is the so-called Kohn Laplacian; see [**BLU07**] for a comprehensive study of stratified Lie groups and the corresponding operators.

We should mention that the study of Heisenberg geometry can be approached from different perspectives and with different applications in mind; for example, see [**NY18**] for a connection with theoretical computer science.

To be a little more specific: the starting motivation to develop a theory of quantitative rectifiability connected to our initial question, is to understand basic issues such as the removable sets for harmonic functions (with respect to the relevant sub-Laplacian), or to give a characterisation of those domains where the Dirichlet problem (again, for the relevant sub-Laplacian) is well-posed. We want to underline, however, that a theory of quantitative rectifiability in the Heisenberg setting has its own, purely geometric, intrinsic appeal.

In the last couple of years, there has been some progress towards an answer to our initial question; see for example [**CFO19**], [**FO18**] and [**Orp18**]. In this note we give a necessary condition to be imposed on a Radon measure μ on $\mathbb{H}^n$ for the Riesz transform to be $L^2(\mu)$ bounded. Here R_μ is the singular integral operator whose kernel is the horizontal gradient of the fundamental solution of the Heisenberg sub-Laplacian, as defined in [**CM12**]. See Section 2 for precise definitions.

THEOREM 1·1. *Let μ be a Radon measure on $\mathbb{H}^n$ such that R_μ is bounded on $L^2(\mu)$ with norm C_1, and such that $\mu(F)=0$ whenever $\dim_H(F)\leq 2$. Then there exists a constant C_2 such that for all balls $B(x,r)\subset\mathbb{H}^n$, we have*

$$\mu(B(x,r))\leq C_2 r^{2n+1}. \tag{1·1}$$

Here C_2 depends only on n and C_1, and the ball B(x,r) is defined with respect to the Korányi metric, see Section 2.

A corresponding statement holds in the Euclidean setting, and is a result of David, [**Dav91** part III, proposition 1·4]. See [**Orp17**, proposition 6·9] for a more detailed proof. Let $\mathcal{R}^d_\mu$ denote the standard d-dimensional Riesz transform in $\mathbb{R}^n$.

THEOREM 1·2. *Assume that μ is a non-atomic Radon measure on $\mathbb{R}^n$ such that $\mathcal{R}^d_\mu$ is bounded on $L^2(\mu)$ with norm C_1. Then, for all Euclidean balls $B_{\mathbb{R}^n}(x,r)\subset\mathbb{R}^n$ we have*

$$\mu(B_{\mathbb{R}^n}(x,r))\leq C_2 r^d \tag{1·2}$$

Here C_2 depends only on C_1, n, and d.

A measure satisfying (1·2) (or (1·1)) is said to have *polynomial growth*. Let us give a couple of remarks.

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Remark 1·3. Although the result itself (both in the Euclidean and Heisenberg case) is neither hard nor deep, it is nevertheless very useful. For example, most tools developed in the last two decades that take quantitative rectifiability beyond Ahlfors regular measures still need polynomial growth[1] (see for example the book by Tolsa [**Tol14**]). Thus, we expect that our result will be quite useful, too.

Remark 1·4. While the two results above look similar, there is actually a difference, in the sense that, in the Heisenberg case, there actually exist lower dimensional measures which give a bounded Riesz transform, but are not atomic.

This is *not* a byproduct of the proof, but rather a fact of the Heisenberg geometry. Indeed, the 2-dimensional t-axis (or any Heisenberg translate of it) gives a bounded $(2n+1)$-dimensional Riesz transform; this is simply because on these sets the kernel vanishes identically, see (2·4).

One can construct a more interesting example in the vertical plane of the one dimensional Heisenberg group $\mathbb{H}$, say. Consider a tube of height 1 and radius ε_1^2 around the t-axis, and take the intersection with the vertical plane. Call the resulting rectangle $R_{1,1}$. Cut out from R_1 two smaller rectangles $R_{2,1}$ and $R_{2,2}$, one in the top right corner and one in the bottom left corner, both of height ε_2 and width ε_2^2, for some $\varepsilon_2 \le \varepsilon_1/4$. We proceed in this way, so that after k steps we have 2^{k-1} disjoint rectangles $\{R_{k,i}\}_i$ of height ε_k and width ε_k^2. Consider the natural probability measure μ on the Cantor-like set $C = \bigcap_k \bigcup_i R_{k,i}$. It is not difficult to show that, if $\varepsilon_k \to 0$ are small enough, the Heisenberg Riesz transform is bounded on $L^2(\mu)$; the idea is that the set is concentrated along the t-axis, and thus the kernel is very small (see (2·4) below). Depending on the choice of (ε_k) we have $\dim_H(C) \in [0, 2]$.

Organisation of the paper. In Section 2 we briefly recall basic facts about Heisenberg groups and the Riesz transform. We also introduce a family of "dyadic cubes" suitable to our setting.

Section 3 is dedicated to Lemma 3·1, our main technical lemma. Roughly speaking, we show that if a measure μ is such that R_μ is bounded on $L^2(\mu)$, and there is some cube Q_0 with a very high concentration of μ (i.e. $\mu(Q_0) \gg \ell(Q_0)^{2n+1}$), then we can find a family $\mathsf{HD}(Q_0)$ of much smaller cubes, contained in Q_0, such that:

(a) a very large portion of measure μ on Q_0 is concentrated on the cubes from $\mathsf{HD}(Q_0)$;

(b) the family $\mathsf{HD}(Q_0)$ is relatively small, in the sense that it consists of few cubes.

In Section 4 we show that if the polynomial growth condition (1·1) is not satisfied, then we can find a cube satisfying the assumptions of our main lemma. This in turn allows us to start an iteration algorithm, consisting of using the main lemma countably many times, that results in constructing a set Z with $\mu(Z) > 0$ and $\dim_H(Z) \le 2$. This completes the proof of Theorem 1·1.

2. *Preliminaries*

In our estimates we will often use the notation $f \lesssim g$ which means that there exists some absolute constant C for which $f \le Cg$. If the constant C depends on some parameter t, we will write $f \lesssim_t g$. Notation $f \approx g$ will stand for $f \lesssim g \lesssim f$, and $f \approx_t g$ is defined analogously. For simplicity, in our estimates we will suppress the dependence on dimension n and on absolute constant λ, Λ (see (2·7)).

[1] With some exceptions, see for example [**AS18**], or [**BS15**].

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2·1. *Heisenberg group*

In this paper we consider the nth Heisenberg group with exponential coordinates (see [**CDPT07**] or [**Fas19**] for a swift introduction to the Heisenberg group in a context close to ours). In practice, we will denote a point $p \in \mathbb{H}^n$ as $(z, t) \in \mathbb{R}^{2n} \times \mathbb{R}$, and $z = (x_1, ..., x_n, y_1, ..., y_n)$. In these coordinates the group law in $\mathbb{H}^n$ takes the form

$$p \cdot q = \left(z + z', t + t' + \frac{1}{2} \sum_{i=1}^{n} (x_i y_i' - y_i x_i') \right),$$

where $p = (z, t)$ and $q = (z', t')$. The identity element is the origin (0,0) and the inverse is given by $p^{-1} = (-z, -t)$. We make $\mathbb{H}^n$ into a metric space by setting $d(p, q) := \|q^{-1} \cdot p\|_{\mathbb{H}}$, where

$$\|p\|_{\mathbb{H}}^4 := |z|^4 + 16t^2, \tag{2·1}$$

and $|z|$ denotes the Euclidean norm of $z \in \mathbb{R}^{2n}$.

Note that $\|\cdot\|_{\mathbb{H}}$ is 1-homogeneous with respect to the anisotropic dilation $p \mapsto \lambda p = (\lambda z, \lambda^2 t)$, $\lambda > 0$. The metric d is sometimes called the Korányi metric.

Given $p \in \mathbb{H}^n$ and $r > 0$ we set

$$B(p, r) = \{q \mid d(p, q) \le r\}, \quad U(p, r) = \{q \mid d(p, q) < r\}.$$

For $\alpha > 0$ we will write $\mathcal{H}^\alpha$ to denote the usual α-dimensional Hausdorff measure with respect to metric d. For $A \subset \mathbb{H}^n$ we set $\dim_H(A)$ to be the Hausdorff dimension of A.

It follows easily from the definition of the Korányi metric that for all $p \in \mathbb{H}^n$ and $r > 0$ we have

$$\mathcal{H}^{2n+2}(B(p, r)) = \mathcal{H}^{2n+2}(B(0, 1))\, r^{2n+2}. \tag{2·2}$$

Thus, even though the topological dimension of $\mathbb{H}^n$ is $2n+1$, the Hausdorff dimension of $\mathbb{H}^n$ is equal to $2n+2$. For the sake of brevity we set $D := 2n+2$. Usually one denotes the Hausdorff dimension of $\mathbb{H}^n$ by Q, but we have decided to save that letter for cubes; hence the non-standard notation.

It is also easy to check that if $\mathcal{L}^{2n+1}$ denotes the usual Lebesgue measure on $\mathbb{R}^{2n+1} \simeq \mathbb{H}^n$, then we have a constant $C > 0$ such that

$$\mathcal{L}^{2n+1} = C\mathcal{H}^D. \tag{2·3}$$

2·2. *Heisenberg Riesz transform*

Recall that, for a function $u : \mathbb{H}^n \to \mathbb{R}$, the horizontal gradient of u is given by

$$\nabla_{\mathbb{H}} u := (X_1 u, ..., X_n u, Y_1 u, ..., Y_n u),$$

where the vector fields $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ and $\partial/\partial t$ represent the left invariant translates of the canonical basis at the identity. In particular, $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ span the horizontal distribution in $\mathbb{H}^n$.

The Heisenberg sublaplacian $\Delta_{\mathbb{H}}$ is given by $\sum_{i=1}^n X_i^2 + Y_i^2$, and its fundamental solution is

$$G(p) := c_n \|p\|_{\mathbb{H}}^{2-D}.$$

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The $(D-1)$-dimensional Riesz kernel in $\mathbb{H}^n$, first considered in [CM12], is given by $K(p) = \nabla_{\mathbb{H}} G(p)$. The Riesz transform is formally defined as

$$R_\mu f(p) = \int_{\mathbb{H}^n} K(q^{-1} \cdot p) f(q)\, d\mu(q).$$

Since it is not clear whether the integral above converges, one considers the truncated Riesz transform given by the formula

$$R_{\mu,\delta} f(p) = \int_{\mathbb{H}^n \setminus B(p,\delta)} K(q^{-1} \cdot p) f(q)\, d\mu(q),$$

for $\delta > 0$. We say that R_μ is bounded on $L^2(\mu)$ if the truncated operators $R_{\mu,\delta}$ are bounded on $L^2(\mu)$ uniformly in $\delta > 0$.

One can easily check that the Riesz kernel is actually equal to

$$\begin{aligned} &K(z,t) \\ &= n\left(\frac{-2x_1|z|^2 + 8y_1 t}{\|(z,t)\|_{\mathbb{H}}^{2n+4}}, \dots, \frac{-2x_n|z|^2 + 8y_n t}{\|(z,t)\|_{\mathbb{H}}^{2n+4}}, \frac{-2y_1|z|^2 - 8x_1 t}{\|(z,t)\|_{\mathbb{H}}^{2n+4}}, \dots, \frac{-2y_n|z|^2 - 8x_n t}{\|(z,t)\|_{\mathbb{H}}^{2n+4}}\right). \end{aligned}$$

Hence,

$$|K(z,t)|^2 = n^2 \frac{4|z|^2}{(|z|^4 + 16t^2)^{n+1}}. \tag{2·4}$$

This implies the curious fact that $|K(z,t)| \leq C$ whenever

$$|z| \leq 16|t|^{n+1}, \tag{2·5}$$

which is a 'paraboloidal' double cone around t-axis with vertex at the origin. This fact will play a key role in the subsequent analysis.

Chousionis and Mattila showed in [CM12, proposition 3·11] that the Riesz kernel is a standard kernel. In particular, it satisfies the following continuity property: whenever $q_1, q_2 \neq p \in \mathbb{H}^n$, we have

$$|K(p^{-1} \cdot q_1) - K(p^{-1} \cdot q_2)| \lesssim \max\left\{\frac{d(q_1,q_2)}{d(p,q_1)^D}, \frac{d(q_1,q_2)}{d(p,q_2)^D}\right\}.$$

Taking $p = 0$ and $q_1 = \tilde{q}_1{}^{-1} \cdot \tilde{p}$, $q_2 = \tilde{q}_2{}^{-1} \cdot \tilde{p}$, one gets immediately that for all $\tilde{q}_1, \tilde{q}_2 \neq \tilde{p} \in \mathbb{H}^n$

$$|K(\tilde{q}_1{}^{-1} \cdot \tilde{p}) - K(\tilde{q}_2{}^{-1} \cdot \tilde{p})| \lesssim \max\left\{\frac{d(\tilde{q}_1,\tilde{q}_2)}{d(\tilde{p},\tilde{q}_1)^D}, \frac{d(\tilde{q}_1,\tilde{q}_2)}{d(\tilde{p},\tilde{q}_2)^D}\right\}. \tag{2·6}$$

2·3. *Dyadic cubes*

We are going to use a family of decompositions of $\mathbb{H}^n$ into subsets that share many properties with the standard dyadic cubes from $\mathbb{R}^n$. The most classical constructions of this kind are due to Christ [Chr90] and David [Dav88], but for us it will be more convenient to use the "cubes" constructed in [KRS12].

First, note that given any ball $B(p, 2r)$, one may use the $5r$-covering lemma and the property (2·2) to conclude that there exists some absolute constant m such that $B(p, 2r)$ may be

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covered by m balls $B(p_i, r)$, where $\{p_i\}_{i=1}^m$ are points in $B(p, 2r)$. That is, $\mathbb{H}^n$ is geometrically doubling. In particular, we can use [**KRS12**, theorem 2·1, remark 2·2].

LEMMA 2·1 ([**KRS12**]). *For all $k \in \mathbb{Z}$ there exists a family of subsets of $\mathbb{H}^n$, denoted by $\mathcal{D}_k$, such that:*

(i) $\mathbb{H}^n = \bigcup_{Q\in\mathcal{D}_k} Q$;

(ii) if $k \geq l$, and $Q \in \mathcal{D}_k$, $P \in \mathcal{D}_l$, then either $Q \cap P = \varnothing$ or $Q \subset P$;

(iii) for every $Q \in \mathcal{D}_k$ there exists $p_Q \in Q$ such that

$$U(p_Q, \lambda 2^{-k}) \subset Q \subset B(p_Q, \Lambda 2^{-k}) \tag{2·7}$$

for some absolute constants λ, $\Lambda > 0$.

Let us stress once more that we will not keep track of how various parameters appearing in the proof depend on λ and Λ.

We set $\mathcal{D} = \bigcup_k \mathcal{D}_k$. For $Q \in \mathcal{D}_k$ we define the sidelength of Q as $\ell(Q) = 2^{-k}$. Clearly, by (2·2) and (2·7), for $Q \in \mathcal{D}$ we have

$$\mathcal{H}^D(Q) \approx \ell(Q)^D.$$

It follows that if $Q \in \mathcal{D}$, then for $k \geq 0$

$$\#\left\{P \in \mathcal{D} \mid P \subset Q,\ \ell(P) = 2^{-k}\ell(Q)\right\} \approx 2^{kD}. \tag{2·8}$$

Given a Radon measure μ and $Q \in \mathcal{D}$ we will denote the $(D-1)$-dimensional density of μ in Q by

$$\Theta_\mu(Q) = \frac{\mu(Q)}{\ell(Q)^{D-1}}.$$

For simplicity, we will suppress the dependence on μ and simply write $\Theta(Q)$.

3. *Main lemma*

Our main tool in the proof of Theorem 1·1 is the following lemma.

LEMMA 3·1. *Let μ be a Radon measure on $\mathbb{H}^n$ such that R_μ is bounded on $L^2(\mu)$ with norm C_1. There exist constants $A = A(n) > 1$, $s = s(A, n) \in (0, 1/2)$ and $M = M(C_1, n) > 100$ such that the following holds.*

Suppose that $Q_0 \in \mathcal{D}$ satisfies $\Theta(Q_0) \geq M$. Set[2] *$N = \lfloor A^{-2}\log(\Theta(Q_0))\rfloor$. Then, the family of high density cubes*

$$\mathsf{HD}(Q_0) = \left\{Q \in \mathcal{D} \mid Q \subset Q_0,\ \ell(Q) = 2^{-N}\ell(Q_0),\ \Theta(Q) > 2\,\Theta(Q_0)\right\}$$

satisfies

$$\sum_{Q\in\mathsf{HD}(Q_0)} \mu(Q) \geq (1 - \Theta(Q_0)^{-s})\mu(Q_0). \tag{3·1}$$

[2] Log here is base 2 logarithm.

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Moreover, we have

$$\sum_{Q\in \mathsf{HD}(Q_0)} \ell(Q)^2 \leq C_p\, \ell(Q_0)^2 \tag{3·2}$$

for some dimensional constant C_p ("p" stands for "packing").

The rest of this section is dedicated to proving the lemma above. For brevity of notation, we set $\Theta_0 = \Theta(Q_0)$. Observe that the integer N was chosen in such a way that

$$2^{A^2N} \approx \Theta_0 \geq M. \tag{3·3}$$

In particular, we have $N \geq N_0$ for some very big N_0 depending on M and A.

We split the proof of Lemma 3·1 into several steps.

First, note that by the pigeonhole principle and (2·8), we can find a cube $Q_1 \in \mathcal{D}$ with sidelength $\ell(Q_1) = 2^{-AN}\ell(Q_0)$, $Q_1 \subset Q_0$, and such that

$$\mu(Q_1) \gtrsim \frac{\mu(Q_0)}{2^{AND}}. \tag{3·4}$$

Without loss of generality, by applying the appropriate translation, we can assume that Q_1 is centred at the origin, i.e. $p_{Q_1} = 0$. Set

$$T := \left\{(z,t) \in Q_0 \mid |z| \leq 2^{-N}\ell(Q_0)\right\}$$

and for any $\kappa > 0$ set

$$T_\kappa := \left\{(z,t) \in Q_0 \mid |z| \leq \kappa\, 2^{-N}\ell(Q_0)\right\}.$$

Observe that $Q_1 \subset T$. In a sense, T can be seen as a tube with vertical axis passing through $p_{Q_1} = 0$. Note also that for any cube $Q \subset Q_0 \setminus T$ we have $\operatorname{dist}(Q, Q_1) \gtrsim 2^{-N}\ell(Q_0)$.

We start by proving a few preliminary results.

LEMMA 3·2. *There are at most $C(\kappa)\, 2^{2N}$ cubes of sidelength $2^{-N}\ell(Q_0)$ contained in T_κ.*

Proof. Observe that since $0 \in Q_0$, and by (2·7) $Q_0 \subset B(p_{Q_0}, \Lambda\ell(Q_0))$, we have $Q_0 \subset B(0, 2\Lambda\ell(Q_0))$. Hence,

$$\begin{aligned} T_\kappa &\subset \left\{(z,t) \in B(0, 2\Lambda\ell(Q_0)) \mid |z| \leq \kappa\, 2^{-N}\ell(Q_0)\right\} \\ &\subset \left\{(z,t) \in \mathbb{H}^n \mid |z| \leq \kappa\, 2^{-N}\ell(Q_0),\ 16|t|^2 \leq (2\Lambda\ell(Q_0))^4\right\} =: \widetilde{T}_\kappa. \end{aligned}$$

By (2·3),

$$\mathcal{H}^D(\widetilde{T}_\kappa) = C\mathcal{L}^{2n+1}(\widetilde{T}_\kappa) \approx (\kappa 2^{-N}\ell(Q_0))^{2n}(2\Lambda\ell(Q_0))^2 \approx_\kappa 2^{-2nN}\ell(Q_0)^D.$$

It follows that $\mathcal{H}^D(T_\kappa) \lesssim_\kappa 2^{-2nN}\ell(Q_0)^D$. On the other hand, recall that for any cube Q with sidelength $\ell(Q) = 2^{-N}\ell(Q_0)$ we have $\mathcal{H}^D(Q) \approx 2^{-ND}\ell(Q_0)^D$. Since all such cubes are pairwise disjoint, we get

$$\#\left\{Q \in \mathcal{D} \mid \ell(Q) = 2^{-N}\ell(Q_0),\ Q \subset T_\kappa\right\} \lesssim \frac{\mathcal{H}^D(T_\kappa)}{2^{-ND}\ell(Q_0)^D} \lesssim_\kappa \frac{2^{-2nN}\ell(Q_0)^D}{2^{-N(2n+2)}\ell(Q_0)^D} = 2^{2N}.$$

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LEMMA 3·3. *Let $Q \in \mathcal{D}$ satisfy $Q \subset Q_0 \setminus T$ and $\ell(Q) = \ell(Q_1) = 2^{-AN}\ell(Q_0)$. Then*

$$\mu(Q) \leq \frac{\mu(Q_0)}{\Theta_0\, 2^{AND}}. \tag{3·5}$$

Proof. We argue by contradiction. To this end, let us assume that there exists a cube $Q_2 \subset Q_0 \setminus T$ with $\ell(Q_2) = 2^{-AN}\ell(Q_0)$ such that (3·5) does not hold - that is

$$\mu(Q_2) \geq \frac{\mu(Q_0)}{\Theta_0\, 2^{AND}}. \tag{3·6}$$

Let $0 < \delta < \operatorname{dist}(Q_1, Q_2)$, let $p \in Q_2$ be arbitrary, and consider

$$R_{\mu,\delta}(\mathbb{1}_{Q_1})(p) = \int_{Q_1} K(q^{-1}\cdot p)\, d\mu(q).$$

By triangle inequality,

$$|R_{\mu,\delta}(\mathbb{1}_{Q_1})(p)| \geq \left|\int_{Q_1} K(p)\, d\mu(q)\right| - \left|\int_{Q_1} K(q^{-1}\cdot p) - K(p)\, d\mu(q)\right|. \tag{3·7}$$

We estimate the first term as follows. Note that, since $p \in Q_2$ and Q_2 lies outside T, then, writing $p = (z, t)$ and using (2·4), we have

$$|K(p)|^2 \approx \frac{|z|^2}{(|z|^4 + 16t^2)^{n+1}} \gtrsim \frac{|z|^2}{\ell(Q_0)^{4(n+1)}} \geq 2^{-2N}\ell(Q_0)^{-4n-2} = 2^{-2N}\ell(Q_0)^{-2D+2}.$$

And thus we also have

$$\left|\int_{Q_1} K(p)\, d\mu(q)\right| = |K(p)|\, \mu(Q_1) \gtrsim 2^{-N}\, \frac{\mu(Q_1)}{\ell(Q_0)^{D-1}}. \tag{3·8}$$

For the second term in (3·7) we use the continuity of the kernel K (2·6) and the fact that $d(p, q) \approx \|p\|_{\mathbb{H}} \geq 2^{-N}\ell(Q_0)$ (because $p \in Q_2 \subset Q_0 \setminus T$):

$$|K(q^{-1}\cdot p) - K(p)| \lesssim \frac{\|q\|}{\min(\|p\|_{\mathbb{H}}, d(p, q))^D} \lesssim \frac{2^{-AN}\ell(Q_0)}{(2^{-N}\ell(Q_0))^D} = \frac{2^{-AN+DN}}{\ell(Q_0)^{D-1}}. \tag{3·9}$$

Taking $A \geq 2D$ we get

$$\left|\int_{Q_1} K(q^{-1}\cdot p) - K(p)\, d\mu(q)\right| \lesssim 2^{-AN/2}\frac{\mu(Q_1)}{\ell(Q_0)^{D-1}}.$$

Together with (3·8) and (3·7), assuming N_0 bigger than some absolute constant (recall that $N \geq N_0$), this gives

$$|R_{\mu,\delta}(\mathbb{1}_{Q_1})(p)| \gtrsim 2^{-N}\, \frac{\mu(Q_1)}{\ell(Q_0)^{D-1}}$$

for all $p \in Q_2$.

Now, we use the estimate above and the $L^2(\mu)$ boundedness of R_μ to get

$$2^{-N}\frac{\mu(Q_1)}{\ell(Q_0)^{D-1}}\mu(Q_2)^{\frac{1}{2}} \lesssim \left(\int |R_{\mu,\delta}(\mathbb{1}_{Q_1})(p)|^2\, d\mu(p)\right)^{\frac{1}{2}} \leq C_1\mu(Q_1)^{\frac{1}{2}}.$$

Our assumptions on Q_1 (3·4) and Q_2 (3·6) yield

$$C_1 \gtrsim 2^{-N} \frac{\mu(Q_1)^{\frac{1}{2}} \mu(Q_2)^{\frac{1}{2}}}{\ell(Q_0)^{D-1}} \gtrsim 2^{-N} \frac{\mu(Q_0)}{2^{AND} \ell(Q_0)^{D-1}} \Theta_0^{-1/2} = 2^{-AND-N} \Theta_0^{1/2}$$
$$\overset{(3\cdot 3)}{\approx} 2^{-AND-N}\, 2^{A^2N/2}.$$

Taking $A \geq 5D$ we can bound the last term from below in the following way:

$$2^{-AND-N+A^2N/2} \geq 2^{A^2N/4} \overset{(3\cdot 3)}{\gtrsim} M^{1/4}.$$

Putting together the estimates above gives $C_1 \gtrsim M^{1/4}$, which is a contradiction for $M = M(C_1, n)$ big enough.

We immediately get the following corollary.

COROLLARY 3·4. *We have*

$$\mu(T_2) \geq (1 - \Theta_0^{-1}) \mu(Q_0). \tag{3·10}$$

Proof. Observe that if $Q \in \mathcal{D}$ satisfies $\ell(Q) = \ell(Q_1) = 2^{-AN} \ell(Q_0)$ and $Q \not\subset T_2$, then we have $Q \cap T = \varnothing$ (assuming A large enough with respect to Λ). It follows that Q satisfies the assumptions of Lemma 3·3, and so

$$\mu(Q) \leq 2^{-AND} \Theta_0^{-1} \mu(Q_0).$$

Summing over all such Q and using (2·8) yields

$$\mu(Q_0 \setminus T_2) \leq \Theta_0^{-1} \mu(Q_0).$$

Recall that

$$\mathsf{HD}(Q_0) = \left\{ Q \in \mathcal{D} \mid Q \subset Q_0,\ \ell(Q) = 2^{-N} \ell(Q_0),\ \Theta(Q) > 2\Theta_0 \right\},$$

and that Λ is the absolute constant such that $Q \subset B(p_Q, \Lambda \ell(Q))$. Without loss of generality, we may assume $\Lambda > 2$.

We are ready to prove the first part of Lemma 3·1, the estimate (3·1).

LEMMA 3·5. *There exists $s = s(A, n) \in (0, 1/2)$ such that*

$$\sum_{Q \in \mathsf{HD}(Q_0)} \mu(Q) \geq (1 - \Theta_0^{-s}) \mu(Q_0). \tag{3·11}$$

Proof. We will prove (3·11) by contradiction. Suppose that

$$\sum_{Q \in \mathsf{HD}(Q_0)} \mu(Q) < (1 - \Theta_0^{-s}) \mu(Q_0). \tag{3·12}$$

Set

$$\mathsf{LD}(Q_0) = \left\{ Q \in \mathcal{D} \mid Q \subset T_{2\Lambda},\ \ell(Q) = 2^{-N} \ell(Q_0),\ \Theta(Q) \leq 2\Theta_0 \right\}.$$

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It is easy to see that the cubes from $\mathsf{HD}(Q_0)\cup\mathsf{LD}(Q_0)$ cover T_2. If we assume $\Theta_0\geq M>100$, and $s<1/2$, then $\Theta_0^{-s}/2\geq\Theta_0^{-1}$, and so by (3·10) and (3·12) we get

$$\sum_{Q\in\mathsf{LD}(Q_0)}\mu(Q)\geq\frac{\Theta_0^{-s}}{2}\mu(Q_0). \tag{3·13}$$

On the other hand, recall from Lemma 3·2 that there are at most $C2^{2N}$ cubes of sidelength $2^{-N}\ell(Q_0)$ contained in $T_{2\Lambda}$, where $C=C(\Lambda,n)$. Moreover, for any $Q\in\mathsf{LD}(Q_0)$ we have

$$\mu(Q)\leq 2\Theta_0\ell(Q)^{D-1}=2\,\mu(Q_0)\frac{\ell(Q)^{D-1}}{\ell(Q_0)^{D-1}}=2^{-N(D-1)+1}\mu(Q_0).$$

In consequence,

$$\sum_{Q\in\mathsf{LD}(Q_0)}\mu(Q)\leq C2^{2N}2^{-N(D-1)+1}\mu(Q_0).$$

This contradicts (3·13) because

$$C\,2^{-ND+3N+1}=2\,C\,(2^{-A^2N})^{(-D+3)A^{-2}}\overset{(3·3)}{\leq}\widetilde{C}(n)\Theta_0^{(-D+3)A^{-2}}\leq\frac{\Theta_0^{-s}}{2},$$

choosing $s=s(A,n)$ small enough.

We move on to the second part of Lemma 3·1, i.e. the packing estimate (3·2).

LEMMA 3·6. *We have*

$$\bigcup_{Q\in\mathsf{HD}(Q_0)}Q\subset T_{2\Lambda}. \tag{3·14}$$

In consequence,

$$\sum_{Q\in\mathsf{HD}(Q_0)}\ell(Q)^2\lesssim\ell(Q_0)^2. \tag{3·15}$$

Proof. We will prove that for $Q\in\mathsf{HD}(Q_0)$ we have $Q\cap T_2\neq\varnothing$. Then, since $\ell(Q)=2^{-N}\ell(Q_0)$, it follows easily from (2·7) that indeed $Q\subset T_{\Lambda+2}(Q_0)\subset T_{2\Lambda}(Q_0)$.

We argue by contradiction. Suppose that $Q\in\mathsf{HD}(Q_0)$ and $Q\cap T_2=\varnothing$. Consider the cubes $\{P_i\}_{i\in I}$ with $\ell(P_i)=2^{-AN}\ell(Q_0)=2^{-(A-1)N}\ell(Q)$ and $P_i\subset Q$. Then, $Q=\bigcup_i P_i$, for all $i\in I$ we have $P_i\cap T_2=\varnothing$, and $\#I\approx 2^{(A-1)ND}$ by (2·8).

We use Lemma 3·3 to conclude that for all $i\in I$

$$\mu(P_i)\leq\frac{\mu(Q_0)}{\Theta_0\,2^{AND}}.$$

Summing over $i\in I$ yields

$$\mu(Q)=\sum_{i\in I}\mu(P_i)\leq\#I\cdot\frac{\mu(Q_0)}{\Theta_0\,2^{AND}}\approx 2^{(A-1)ND}\frac{\mu(Q_0)}{\Theta_0\,2^{AND}}=\frac{\mu(Q_0)}{\Theta_0\,2^{ND}},$$

so that

$$\Theta(Q)=\frac{\mu(Q)}{(2^{-N}\ell(Q_0))^{D-1}}\lesssim\frac{\mu(Q_0)}{\Theta_0\,2^{ND}}\cdot\frac{1}{2^{-N(D-1)}\ell(Q_0)^{D-1}}=\frac{\Theta_0}{\Theta_0\,2^{N}}=2^{-N}\leq 1.$$

But this contradicts the assumption $Q \in \mathsf{HD}(Q_0)$:

$$\Theta(Q) \geq 2\Theta_0 \geq 2M > 1,$$

and so the proof of (3·14) is finished.

Concerning (3·15), note that by (3·14) and Lemma 3·2 we have

$$\#\mathsf{HD}(Q_0) \lesssim 2^{2N}. \tag{3·16}$$

Hence,

$$\sum_{Q \in \mathsf{HD}(Q_0)} \ell(Q)^2 = \ell(Q_0)^2\, 2^{-2N} \sum_{Q \in \mathsf{HD}(Q_0)} 1 \lesssim \ell(Q_0)^2.$$

4. *Iteration argument*

To complete the proof of Theorem 1·1, we assume that the measure μ does not satisfy the polynomial growth condition (1·1). Then we will use Lemma 3·1 countably many times to construct a set Z with positive μ-measure and with Hausdorff dimension at most 2.

Suppose that there exists a ball $B(x, r)$ with $\mu(B(x, r)) \geq C_2 r^{2n+1}$; if C_2 is big enough, we can find a cube $Q_0 \in \mathcal{D},\ Q \subset B(x, r)$ such that

$$\Theta(Q_0) \geq M,$$

where M is the constant from Lemma 3·1.

Let $A > 1$ be as in Lemma 3·1. Following the notation of Lemma 3·1, for an arbitrary cube $Q \in \mathcal{D}$ with $\Theta(Q) \geq M$, set

$$N(Q) := \left\lfloor A^{-2} \log(\Theta(Q)) \right\rfloor$$

and

$$\mathsf{HD}(Q) := \left\{ P \in \mathcal{D} \mid P \subset Q,\ \ell(P) = 2^{-N(Q)}\ell(Q),\ \Theta(P) > 2\Theta(Q) \right\}.$$

Put $Z_0 := Q_0$, $\mathsf{HD}_0 := \{Q_0\}$, $\mathsf{HD}_1 := \mathsf{HD}(Q_0)$, and $Z_1 := \bigcup_{Q \in \mathsf{HD}_1} Q$. Proceeding inductively, for all $j \geq 2$ we define

$$\mathsf{HD}_j := \bigcup_{Q \in \mathsf{HD}_{j-1}} \mathsf{HD}(Q),$$

$$Z_j := \bigcup_{Q \in \mathsf{HD}_j} Q.$$

Note that for each j the cubes in HD_j form a disjoint family. Moreover, $\{Z_j\}_{j \geq 0}$ form a decreasing sequence of sets, that is $Z_{j+1} \subset Z_j$. Define

$$Z := \bigcap_{j \geq 0} Z_j.$$

Claim 4·1. *We have*

$$\mu(Z) \gtrsim_{M,s} \mu(Q_0).$$

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Proof. Observe that for $Q \in \mathsf{HD}_j$ we have

$$\Theta(Q) \geq 2^j \Theta(Q_0) \geq 2^j M. \tag{4·1}$$

In particular, $\Theta(Q) \geq M$ and so we may apply Lemma 3·1 to Q. It follows that for any $j \geq 0$ we have

$$\mu(Z_{j+1}) = \sum_{Q\in\mathsf{HD}_{j+1}} \mu(Q) = \sum_{Q\in\mathsf{HD}_j} \sum_{P\in\mathsf{HD}(Q)} \mu(P) \overset{(3·1)}{\geq} \sum_{Q\in\mathsf{HD}_j} (1-\Theta(Q)^{-s})\mu(Q)$$

$$\overset{(4·1)}{\geq} \sum_{Q\in\mathsf{HD}_j} (1-2^{-js}M^{-s})\mu(Q) = (1-2^{-js}M^{-s})\mu(Z_j).$$

Using this estimate $(j+1)$ times we arrive at

$$\mu(Z_{j+1}) \geq \prod_{i=0}^{j} (1-2^{-is}M^{-s})\mu(Q_0). \tag{4·2}$$

Since Z_j form a sequence of decreasing sets, we get by the continuity of measure

$$\mu(Z) = \lim_{j\to\infty} \mu(Z_j) \geq \prod_{i=0}^{\infty} (1-2^{-is}M^{-s})\mu(Q_0) = C(s,M)\mu(Q_0),$$

where $C(s, M)$ is positive and finite because $\sum_{i=0}^{\infty} 2^{-is} < \infty$.

Claim 4·2. *We have*

$$\dim_H(Z) \leq 2.$$

Proof. Recall that $N(Q) = \lfloor A^{-2}\log(\Theta(Q))\rfloor$. It follows from (4·1) that for $Q \in \mathsf{HD}_j$ we have $N(Q) \geq C_3 jA^{-2}$ for some absolute constant $C_3 > 0$. Thus, for $Q \in \mathsf{HD}_j$ and $P \in \mathsf{HD}(Q)$

$$\ell(P) = 2^{-N(Q)}\ell(Q) \leq 2^{-C_3 jA^{-2}}\ell(Q).$$

Using this observation j times we get that for $P \in \mathsf{HD}_{j+1}$

$$\ell(P) \leq 2^{-C_4 j(j+1)A^{-2}}\ell(Q_0),$$

where $C_4 = C_3/2$. Hence, the cubes from HD_j form coverings of Z with decreasing diameters, well suited for estimating the Hausdorff measure of Z.

Let $0 < \varepsilon < 1,\ 0 < \delta < 1$ be small. Let $j \geq 0$ be so big that for $Q \in \mathsf{HD}_j$ we have $diam(Q) \leq \Lambda\ell(Q) \leq \delta$. Then,

$$\mathcal{H}^{2+\epsilon}_\delta(Z) \leq \Lambda^{2+\varepsilon} \sum_{Q\in\mathsf{HD}_j} \ell(Q)^{2+\varepsilon} \leq \Lambda^{2+\varepsilon}(2^{-C_4 j(j-1)A^{-2}}\ell(Q_0))^{\varepsilon} \sum_{Q\in\mathsf{HD}_j} \ell(Q)^2. \tag{4·3}$$

It follows by (3·2) that

$$\sum_{Q\in\mathsf{HD}_j} \ell(Q)^2 = \sum_{P\in\mathsf{HD}_{j-1}} \sum_{Q\in\mathsf{HD}(P)} \ell(Q)^2 \leq C_p \sum_{P\in\mathsf{HD}_{j-1}} \ell(P)^2.$$

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Using the estimate above j times, and putting it together with (4·3) we arrive at

$$\mathcal{H}^{2+\epsilon}_{\delta}(Z) \le \Lambda^{2+\varepsilon}(C_p)^j \; 2^{-\varepsilon C_4 j(j-1)A^{-2}} \; \ell(Q_0)^{2+\varepsilon}.$$

The right hand side above converges to 0 as $j \to \infty$ (just note that the exponent at C_p is linear in j while the exponent at 2 is quadratic in j). Hence, $\mathcal{H}^{2+\epsilon}_{\delta}(Z) = 0$. Letting $\delta \to 0$ we get $\mathcal{H}^{2+\epsilon}(Z) = 0$. Since this is true for arbitrarily small $\epsilon > 0$, it follows that

$$\dim_H(Z) = \inf\{t \ge 0 : \mathcal{H}^t(Z) = 0\} \le 2.$$

Proof of Theorem 1·1. We have found a set $Z \subset \mathbb{H}^n$ of dimension smaller than or equal to 2 (Claim 4·2) but which nevertheless has positive μ-measure (Claim 4·1). This contradicts the assumptions of Theorem 1·1. Thus, there exists $C_2 = C_2(n, C_1)$ such that $\mu(B(x, r)) \le C_2 r^{2n+1}$ for all $x \in \mathbb{H}^n$ and $r > 0$.

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