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THE DIFFERENCE OF CONSECUTIVE EIGENVALUES

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Abstract

Let M be a smooth bounded domain in R^n with smooth boundary, $n \geq 2$, and $\Delta u = -\sum_{i=1}^n \partial^2 u/\partial x_i^2$. We prove an inequality involving the first $k+1$ eigenvalues of the eigenvalue problem:

$$\begin{cases} \sum_{m=2}^{t} a_{m-1}\Delta^m u = \lambda \Delta u & \text{in } M, \\ \left(\dfrac{\partial}{\partial \nu}\right)^s u = 0 & \text{on } \partial M, \quad s = 0, 1, \dots, t-1, \end{cases}$$

where $a_{m-1} \geq 0$ are constants and $a_{t-1} = 1$. We also obtain a uniform estimate of the upper bound of the ratios of consecutive eigenvalues.

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1. Introduction

Let M be a smooth bounded domain in the Euclidean n-space R^n with smooth boundary ∂M, $n \geq 2$ and Δ be the Laplace operator defined by

$$\Delta u = -\sum_{i=1}^{n} \partial^2 u/\partial x_i^2, \qquad u \in C^2(M).$$

For fixed integers $t > 0$ and $r \geq 0$, where $r < t$, and let L be the elliptic operator of order $2t$ defined by

$$Lu = \sum_{m=r+1}^{t} a_{m-r}\Delta^m u, \quad u \in C^{2t}(M),$$

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where a_{m-r}'s are constants with $a_{m-r} \geq 0$, $r+1 \leq m \leq t$, and $a_{t-r} = 1$. Let us consider the following Dirichlet eigenvalue problem:

$$\text{(1)} \qquad \begin{cases} Lu = \lambda \Delta^r u & \text{in } M, \\ \left(\dfrac{\partial}{\partial \nu}\right)^s u = 0 & \text{on } \partial M, \quad s = 0, 1, \ldots, t-1, \end{cases}$$

where $\partial/\partial\nu$ denotes the unit outward normal to ∂M. We can order the eigenvalues of the eigenvalue problem (1) as

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \uparrow \infty$$

This eigenvalue problem is of extreme importance in the study of various branches of mathematics and mathematical physics. One basic problem is to find some good estimates of the upper bound of the difference $\lambda_{k+1} - \lambda_k$. In the case $L = \Delta$, $r = 0$ and $n = 2$, this problem was considered by Payne, Polya and Weinberger in [6]. They verified that

$$\text{(2)} \qquad \lambda_{k+1} - \lambda_k \leq \frac{4}{n^2 k} \sum_{i=1}^{k} \lambda_i, \qquad k = 1, 2, \ldots.$$

The inequalities (2) remain true if $L = \Delta$, $r = 0$ and $n \geq 2$ [2]. If $L = \Delta^t$, $r = 1$, $t > 1$, we shall show that the following inequalities hold:

$$\text{(3)} \qquad \lambda_{k+1} - \lambda_k \leq \frac{4t}{(n+2)^2 k}(n + 2t - 2) \sum_{i=1}^{k} \lambda_i, \quad k = 1, 2, \ldots.$$

More generally, we propose the following conjectures:

CONJECTURE A. *The eigenvalues λ_k of the eigenvalue problems* (1) *satisfy the inequalities*

$$\text{(4)} \qquad \lambda_{k+1} - \lambda_k \leq \frac{2}{\bar{n}^2 k} \sum_{m=1}^{t-r} (2m + \bar{r})(n + 2\bar{m}) a_m \left(\sum_{i=1}^{k} \lambda_i\right)^{m/(t-r)} k^{(t-r-m)/(t-r)},$$

for $k = 1, 2, \ldots$, where $\bar{n} = n$, $\bar{r} = r$ and $\bar{m} = m - 1$ if r is even and $\bar{n} = n + 2$, $\bar{r} = r + 1$ and $\bar{m} = m$ if r is odd.

CONJECTURE B. *If $\lambda_k \geq 1$, then*

$$\text{(5)} \qquad \frac{\lambda_{k+1}}{\lambda_k} \leq 1 + \frac{2}{\bar{n}^2} \sum_{m=1}^{t-r} (2m + \bar{r})(n + 2\bar{m}) a_m.$$

It is easy to see that Conjecture A implies Conjecture B. Notice that the upper bound estimate in (5) is independent of the domain M and k. The main results of this paper are the following:

THEOREM A. *Conjecture* A *holds for* $r = 1$, *that is,*

$$\lambda_{k+1} - \lambda_k \le \frac{4}{(n+2)^2 k} \sum_{m=1}^{t-1} (m+1)(n+2m) a_m \left(\sum_{i=1}^{k} \lambda_i \right)^{m/(t-1)} k^{(t-1-m)/(t-1)}, \tag{6}$$

for $k = 1, 2, \ldots$.

THEOREM B. *Conjecture* B *holds for* $r = 1$, *that is, if* $\lambda_k \ge 1$, *then*

$$\frac{\lambda_{k+1}}{\lambda_k} \le 1 + \frac{4}{(n+2)^2} \sum_{m=1}^{t-1} (m+1)(n+2m) a_m.$$

If $L = \Delta^t$, we obtain the inequalities (3) from (6), hence the assumption $\lambda_k \ge 1$ is not necessary in Theorem B for this case. Conjecture A was also proved for $L = \Delta^2$ and $r = 0$ in [3, Theorem 1] and for the general linear elliptic operator L and $r = 0$ in ([1], [4]). The upper bound estimates of the ratio λ_2/λ_1 for $L = \Delta^2$ and $r = 1$ were also considered in ([3], [6]).

2. Preliminaries

Let $\{u_i\}_{i=1}^k$ be a set of orthonormal eigenfunctions corresponding to the eigenvalues $\{\lambda_i\}_{i=1}^k$ of the eigenvalue problem (1) with $r = 1$. Thus, we have

$$\int_M u_i(x) \Delta u_j(x) dx = \delta_{ij} \quad \text{and} \quad \int_M u_i L u_i = \lambda_i.$$

For C^1 functions u and v on M, define

$$\langle u, v \rangle = \nabla u \cdot \nabla v,$$

and set $\|u\|^2 = \langle u, u \rangle$. Let $\overline{L}$ denote the elliptic operator on $C^{2(t-1)}(M)$ defined by

$$\overline{L} u = \sum_{m=1}^{t-1} a_m \Delta^m u.$$

Clearly, we have $L = \Delta \overline{L}$. By integration by parts, we can easily obtain the following lemma.

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LEMMA 2.1.

(i) $$\int_M \Delta^a u_i \Delta^b u_j = \int_M \Delta^c u_i \Delta^d u_j, \quad \text{for} \quad a+b=c+d \le t.$$

(ii) *If* $a+b+1 \le t-1$, *then* $$\int_M \Delta^a u_i \Delta^{b+1} u_i = \int_M \nabla\Delta^a u_i \cdot \nabla\Delta^b u_i.$$

(iii) $$\int_M (\Delta^{m-1}u_i)(x_j\Delta\partial_j u_i) = \int_M u_i \Delta^{m-1}(x_j\Delta\partial_j u_i), \quad 1 \le m \le t-1.$$

LEMMA 2.2. *For* $1 \le m \le t-1$, *we have*

$$\int_M \langle u_i, \Delta^m u_i\rangle \le \lambda_i^{m/(t-1)}.$$

PROOF. Since $\int_M \langle u_i, \Delta^{t-1}u_i\rangle \le \int_M \langle u_i, \overline{L}u_i\rangle = \lambda_i$, it suffices to verify the following:

$$\int_M \langle u_i, \Delta^m u_i\rangle \le \left(\int_M \langle u_i, \Delta^{m+1}u_i\rangle\right)^{m/(m+1)}, \quad 1 \le m \le t-1. \tag{7}$$

We shall prove this by induction. If $m=1$, by the Cauchy-Schwarz inequality and (i) we have

$$\int_M \langle u_i, \Delta u_i\rangle \le \left(\int_M \|u_i\|^2\right)^{1/2}\left(\int_M \|\Delta u_i\|^2\right)^{1/2} = \left(\int_M \langle u_i, \Delta^2 u_i\rangle\right)^{1/2}.$$

Hence (7) holds for $m=1$. Assume that (7) is true for $m = 2k-1 < t-1$. By Lemma 2.1 and the inductive hypothesis we have

$$\begin{aligned}
\int_M \langle u_i, \Delta^{2k}u_i\rangle &= \int_M u_i\Delta^{2k+1}u_i = \int_M \Delta^k u_i \Delta^{k+1}u_i \\
&\le \left(\int_M |\Delta^k u_i|^2\right)^{1/2}\left(\int_M |\Delta^{k+1}u_i|^2\right)^{1/2} \\
&= \left(\int_M \langle u_i, \Delta^{2k-1}u_i\rangle\right)^{1/2}\left(\int_M \langle u_i, \Delta^{2k+1}u_i\rangle\right)^{1/2} \\
&\le \left(\int_M \langle u_i, \Delta^{2k}u_i\rangle\right)^{(2k-1)/4k}\left(\int_M \langle u_i, \Delta^{2k+1}u_i\rangle\right)^{1/2}.
\end{aligned}$$

From this we easily conclude that

$$\int_M \langle u_i, \Delta^{2k}u_i\rangle \le \left(\int_M \langle u_i, \Delta^{2k+1}u_i\rangle\right)^{2k/(2k+1)}.$$

Similarly, if (7) holds for $m = 2k < t - 1$, we obtain

$$\begin{aligned}
\int_M \langle u_i, \Delta^{2k+1} u_i \rangle &= \int_M \nabla \Delta^k u_i \cdot \nabla \Delta^{k+1} u_i \\
&\leq \left(\int_M |\nabla \Delta^k u_i|^2 \right)^{1/2} \left(\int_M |\nabla \Delta^{k+1} u_i|^2 \right)^{1/2} \\
&= \left(\int_M \langle u_i, \Delta^{2k} u_i \rangle \right)^{1/2} \left(\int_M \langle u_i, \Delta^{2k+2} u_i \rangle \right)^{1/2} \\
&\leq \left(\int_M \langle u_i, \Delta^{2k+1} u_i \rangle \right)^{k/(2k+1)} \left(\int_M \langle u_i, \Delta^{2k+2} u_i \rangle \right)^{1/2}.
\end{aligned}$$

This proves (7) for $m = 2k + 1$, and hence Lemma 2.3.

A different proof of Lemma 2.2 can be given by using the Remark in [5].

LEMMA 2.3.

$$2 \sum_{j=1}^{n} \int_M \langle x_j u_i, \Delta^{m-1} \partial_j u_i \rangle = -(n + 2m) \int_M \langle u_i, \Delta^{m-1} u_i \rangle, \quad 1 \leq m \leq t - 1.$$

PROOF. Since $\Delta \partial_j = \partial_j \Delta$,

$$\begin{aligned}
&\sum_{j=1}^{n} \int_M \langle x_j u_i, \Delta^{m-1} \partial_j u_i \rangle \\
&\quad = \sum_{j=1}^{n} \int_M \nabla(x_j u_i) \cdot \nabla \Delta^{m-1} \partial_j u_i = \sum_{j=1}^{n} \int_M \Delta(x_j u_i) \Delta^{m-1} \partial_j u_i \\
&\quad = -\sum_{j=1}^{n} \int_M \Delta^{m-1} u_i \Delta \partial_j (x_j u_i) + \sum_{j=1}^{n} \int_M \partial_j (\Delta^{m-1} u_i \Delta(x_j u_i)) \\
&\quad = -\sum_{j=1}^{n} \int_M \Delta^{m-1} u_i \{\Delta u_i + \Delta(x_j \partial_j u_i)\} \\
&\quad = -n \int_M u_i \Delta^m u_i - \sum_{j=1}^{n} \int_M \Delta^{m-1} u_i \{x_j \Delta \partial_j u_i - 2\partial_j \partial_j u_i\} \quad \text{(by (7))} \\
&\quad = -(n + 2) \int_M u_i \Delta^m u_i - \sum_{j=1}^{n} \int_M u_i \Delta^{m-1} (x_j \Delta \partial_j u_i) \quad \text{(by Lemma 2.1 (iii))} \\
&\quad = -(n + 2) \int_M u_i \Delta^m u_i - \sum_{j=1}^{n} \int_M u_i \{x_j \Delta^m \partial_j u_i - 2(m - 1) \partial_j \Delta^{m-1} \partial_j u_i\} \\
&\quad = -(n + 2) \int_M u_i \Delta^m u_i - \sum_{j=1}^{n} \int_M x_j u_i \Delta^m \partial_j u_i - 2(m - 1) \int_M u_i \Delta^m u_i
\end{aligned}$$

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$$= -(n+2m)\int_M \langle u_i, \Delta^{m-1}u_i\rangle - \sum_{j=1}^{n}\int_M \langle x_j u_i, \Delta^{m-1}\partial_j u_i\rangle,$$

which implies the desired result.

3. The Difference of Consecutive Eigenvalues

For any $j = 1, 2, \ldots, n$ and $i = 1, 2, \ldots, k$, define the trial functions α_{ij} by

$$\alpha_{ij} = x_j u_i - \sum_{s=1}^{k} b_{ijs} u_s,$$

where the constants $\{b_{ijs}\}$ are given by

$$b_{ijs} = \int_M x_j \langle u_i, u_s\rangle = b_{sji}.$$

It follows that

$$\text{(8)} \qquad \begin{aligned} \int_M \alpha_{ji}\Delta u_s &= \int_M \langle \alpha_{ji}, u_s\rangle = \int_M \Big\langle x_j u_i - \sum_{h=1}^{k} b_{ijh}u_h, u_s\Big\rangle \\ &= \int_M x_j\langle u_i, u_s\rangle - b_{ijs} = 0, \end{aligned}$$

that is, α_{ji} is orthogonal to u_s, $s = 1, 2, \ldots, k$.

LEMMA 3.1.

$$\begin{aligned} &\sum_{j=1}^{n}\sum_{i=1}^{k}\int_M \langle \alpha_{ji}, \overline{L}\alpha_{ji}\rangle \\ &\qquad \le \sum_{j=1}^{n}\sum_{i=1}^{k}\lambda_i \int_M \|\alpha_{ji}\|^2 - 2\sum_{j=1}^{n}\sum_{i=1}^{k}\sum_{m=1}^{t-1}(m+1)a_m \int_M \langle \alpha_{ji}, \Delta^{m-1}\partial_j u_i\rangle. \end{aligned}$$

PROOF. Since

$$\int_M (u_s\partial_j u_i + u_i\partial_j u_s) = \int_M \partial_j(u_i u_s) = 0$$

and $b_{ijs} = b_{sji}$, it follows that

$$\text{(9)} \qquad \sum_{j=1}^{n}\sum_{i,s=1}^{k}\lambda_i b_{ijs}\int_M u_s\partial_j u_i = \frac{1}{2}\sum_{j=1}^{n}\sum_{i,s=1}^{k}\lambda_i b_{ijs}\int_M (u_s\partial_j u_i + u_i\partial_j u_s) = 0.$$

Notice that by integration by parts we get

$$\sum_{j=1}^{n}\int_M x_j u_i \partial_j u_i = -\sum_{j=1}^{n}\int_M u_i \partial_j (x_j u_i)$$
$$= -\sum_{j=1}^{n}\int_M u_i^2 - \sum_{j=1}^{n}\int_M x_j u_i \partial_j u_i .$$

Thus

$$(10)\qquad 2\sum_{j=1}^{n}\int_M x_j u_i \partial_j u_i = -\sum_{j=1}^{n}\int_M u_i^2 \le 0.$$

It follows from (9) and (10) that

$$(11)\qquad 2\sum_{j=1}^{n}\sum_{i=1}^{k}\lambda_i\int_M \alpha_{ij}\partial_j u_i = 2\sum_{j=1}^{n}\sum_{i=1}^{k}\lambda_i\int_M \left(x_j u_i - \sum_{s=1}^{k} b_{ijs}u_s\right)\partial_j u_i \le 0.$$

A similar computation as above gives

$$\sum_{j=1}^{n}\sum_{i=1}^{k}\lambda_i\int_M \alpha_{ij}\Delta\alpha_{ji} = \sum_{j=1}^{n}\sum_{i=1}^{k}\lambda_i\int_M \alpha_{ji}\left(x_j\Delta u_i - 2\partial_j u_i - \sum_{s=1}^{k} b_{ijs}\Delta u_s\right)$$
$$= \sum_{j=1}^{n}\sum_{i=1}^{k}\lambda_i\int_M \alpha_{ji}x_j\Delta u_i - 2\sum_{j=1}^{n}\sum_{i=1}^{k}\lambda_i\int_M \alpha_{ji}\partial_j u_i ,$$

where the terms involving $\{b_{ijs}\}$ vanish because of (8). Hence by (11) we have

$$(12)\qquad \begin{aligned}\sum_{j=1}^{n}\sum_{i=1}^{k}\lambda_i\int_M \alpha_{ij}x_j\Delta u_i &\le \sum_{j=1}^{n}\sum_{i=1}^{k}\lambda_i\int_M \alpha_{ji}\Delta\alpha_{ji}\\ &= \sum_{j=1}^{n}\sum_{i=1}^{k}\lambda_i\int_M \|\alpha_{ji}\|^2.\end{aligned}$$

Moreover

$$(13)\qquad \sum_{m=1}^{t-1} a_m\int_M \alpha_{ji}\Delta^{m+1}u_s = \int_M \alpha_{ji}Lu_s = \lambda_s\int_M \alpha_{ji}\Delta u_s = 0,$$

by (8). Thus, the combination of (12) and (13) yields

$$\sum_{j=1}^{n}\sum_{i=1}^{k}\int_M \langle\alpha_{ji}, \overline{L}\alpha_{ji}\rangle = \sum_{j=1}^{n}\sum_{i=1}^{k}\sum_{m=1}^{t-1} a_m\int_M \alpha_{ji}\Delta^{m+1}\alpha_{ji}$$
$$= \sum_{j=1}^{n}\sum_{i=1}^{k}\sum_{m=1}^{t-1} a_m\int_M \alpha_{ji}\left\{x_j\Delta^{m+1}u_i - 2(m+1)\Delta^m\partial_j u_i - \sum_{s=1}^{k} b_{ijs}\Delta^{m+1}u_s\right\}$$

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$$= \sum_{j=1}^{n}\sum_{i=1}^{k}\int_M \alpha_{ji}x_j Lu_i - 2\sum_{j=1}^{n}\sum_{i=1}^{k}\sum_{m=1}^{t-1}(m+1)a_m\int_M \alpha_{ji}\Delta^m\partial_j u_i$$
$$\leq \sum_{j=1}^{n}\sum_{i=1}^{k}\lambda_i\int_M \|\alpha_{ji}\|^2 - 2\sum_{j=1}^{n}\sum_{i=1}^{k}\sum_{m=1}^{t-1}(m+1)a_m\int_M \langle\alpha_{ji}, \Delta^{m-1}\partial_j u_i\rangle.$$

This completes the proof.

LEMMA 3.2. *Let*

$$L_t = -2\sum_{j=1}^{n}\sum_{i=1}^{k}\sum_{m=1}^{t-1}(m+1)a_m\int_M \langle\alpha_{ji}, \Delta^{m-1}\partial_j u_i\rangle.$$

Then

$$L_t \leq \sum_{m=1}^{t-1}(m+1)(n+2m)a_m\left(\sum_{i=1}^{k}\lambda_i\right)^{(m-1)/(t-1)} k^{(t-m)/(t-1)}. \tag{14}$$

PROOF. From Lemma 2.2 we have

$$\int_M \langle u_i, \Delta^{m-1}u_i\rangle \leq \lambda_i^{(m-1)/(t-1)}. \tag{15}$$

By integration by parts we can see that

$$\int_M \langle u_s, \Delta^{m-1}\partial_j u_i\rangle + \int_M \langle u_i, \Delta^{m-1}\partial_j u_s\rangle = 0.$$

Hence

$$\sum_{j=1}^{n}\sum_{i,s=1}^{k} b_{ijs}\int_M \langle u_s, \Delta^{m-1}\partial_j u_i\rangle = 0, \qquad 1 \leq m \leq t-1.$$

It follows from (15), Lemma 2.3 and the Hölder's inequality that

$$L_t = \sum_{i=1}^{k}\sum_{m=1}^{t-1}(m+1)(n+2m)a_m\int_M \langle u_i, \Delta^{m-1}u_i\rangle$$
$$\leq \sum_{m=1}^{t-1}(m+1)(n+2m)a_m\sum_{i=1}^{k}\lambda_i^{(m-1)/(t-1)}$$
$$\leq \sum_{m=1}^{t-1}(m+1)(n+2m)a_m\left(\sum_{i=1}^{k}\lambda_i\right)^{(m-1)/(t-1)} k^{(t-m)/(t-1)}.$$

The proof of Lemma 3.2 is complete.

LEMMA 3.3. *Let the functions T and φ be defined on (λ_k, ∞) by*

$$T(\alpha) = \sum_{i=1}^{k} \sum_{j=1}^{n} (\alpha - \lambda_i) \int_M \|\alpha_{ji}\|^2, \qquad \text{and} \qquad \varphi(\alpha) = \sum_{i=1}^{k} \frac{\lambda_i^{1/(t-1)}}{\alpha - \lambda_i}.$$

Then

(16) $$T(\alpha) \geq k^2(n+2)^2/4\varphi(\alpha).$$

PROOF. For any constant $\beta > 0$, we have

$$0 \leq \sum_{i=1}^{k} \int_M \left\| \left[\frac{\beta}{2}(\alpha - \lambda_i)\right]^{\frac{1}{2}} \alpha_{ji} + \left[\frac{1}{2\beta}(\alpha - \lambda_i)^{-1}\right]^{\frac{1}{2}} \partial_j u_i \right\|^2.$$

Hence by Lemma 2.3 with $m = 1$ we obtain

$$\begin{aligned} k(n+2) &= -2 \sum_{j=1}^{n} \sum_{i=1}^{k} \int_M \langle x_j u_i, \partial_j u_i \rangle \\ &= -2 \sum_{j=1}^{n} \sum_{i=1}^{k} \int_M \langle \alpha_{ji}, \partial_j u_i \rangle \qquad (\text{since } b_{ijs} = b_{sji}) \\ &\leq \beta \sum_{j=1}^{n} \sum_{i=1}^{k} (\alpha - \lambda_i) \int_M \|\alpha_{ji}\|^2 + \beta^{-1} \sum_{i=1}^{k} (\alpha - \lambda_i)^{-1} \sum_{j=1}^{n} \int_M \|\partial_j u_i\|^2. \end{aligned}$$

However, by Lemma 2.2 we have

$$\sum_{j=1}^{n} \int_M \|\partial_j u_i\|^2 = \int_M \langle u_i, \Delta u_i \rangle \leq \lambda_i^{1/(t-1)}.$$

Hence

(17) $$k(n+2) \leq \beta T(\alpha) + \beta^{-1} \varphi(\alpha).$$

The right hand side of (17) has minimum at

$$\beta = T(\alpha)^{-1/2} \{\varphi(\alpha)\}^{1/2}.$$

Substituting this value of β into (17) and solving for $T(\alpha)$ we obtain the desired inequality (16).

LEMMA 3.4. *Let $S = \sum_{i=1}^{k} \sum_{j=1}^{n} \int_M \|\alpha_{ji}\|^2$. Then for any $\alpha > \lambda_k$,*

$$(\lambda_{k+1} - \alpha) S \leq L_t - T(\alpha).$$

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PROOF. Since $\int_M \alpha_{ji}\Delta u_s = 0$, $s = 1, 2, \dots, k$ by (8) and $(\partial/\partial\nu)^s\alpha_{ji} = 0$ on ∂M for $s = 0, 1, \dots, t-1$, hence by the well-known Rayleigh Theorem we have

$$\lambda_{k+1}\int_M \alpha_{ji}\Delta\alpha_{ji} \le \int_M \alpha_{ji}L\alpha_{ji},$$

that is,

$$\lambda_{k+1}\int_M \|\alpha_{ji}\|^2 \le \int_M \langle\alpha_{ji}, \overline{L}\alpha_{ji}\rangle.$$

Therefore, from Lemma 3.1

$$\begin{aligned}\lambda_{k+1}S &\le \alpha\sum_{i=1}^{k}\sum_{j=1}^{n}\int_M \|\alpha_{ji}\|^2 + \sum_{i=1}^{k}\sum_{j=1}^{n}(\lambda_i - \alpha)\int_M \|\alpha_{ji}\|^2 + L_t \\ &= \alpha S - T(\alpha) + L_t.\end{aligned}$$

This establishes the lemma.

PROOF OF THEOREM A. Set

$$A = \sum_{m=1}^{t-1}(m+1)(n+2m)a_m\left(\sum_{i=1}^{k}\lambda_i\right)^{(m-1)/(t-1)}k^{(t-m)/(t-1)},$$

and

$$A(\alpha) = A - \frac{k^2(n+2)^2}{4\varphi(\alpha)}, \qquad \lambda_k < \alpha < \infty.$$

Then it follows from Lemmas 3.2, 3.3 and 3.4 that

$$(\lambda_{k+1} - \alpha)S \le A(\alpha). \tag{18}$$

The function φ is a decreasing function which tends to zero as $\alpha \to \infty$, and tends to ∞ as $\alpha \to \lambda_k$. By the Intermediate Value Theorem, there is a unique σ in (λ_k, ∞) such that

$$\varphi(\sigma) = k^2(n+2)^2/4A. \tag{19}$$

Hence $A(\sigma) = 0$. From (18) we obtain

$$\lambda_{k+1} \le \sigma. \tag{20}$$

Combination of (19), (20) and Hölder's inequality yields

$$\begin{aligned}\frac{k^2(n+2)^2}{4A} = \varphi(\sigma) = \sum_{i=1}^{k}\frac{\lambda_i^{1/(t-1)}}{\sigma - \lambda_i} &\le \frac{1}{\lambda_{k+1} - \lambda_k}\sum_{i=1}^{k}\lambda_i^{1/(t-1)} \\ &\le \frac{1}{\lambda_{k+1} - \lambda_k}\left(\sum_{i=1}^{k}\lambda_i\right)^{1/(t-1)}k^{(t-2)/(t-1)}.\end{aligned}$$

This implies the inequality (6).

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We shall conclude this paper with the following brief remark without proof. Under stronger assumption, we can obtain some upper bound estimates for the difference $\lambda_{k+1} - \lambda_k$, $r \geq 2$. For instance, by using Theorem A we can show that for any fixed integer $r \geq 2$ and $h \geq 1$, we have

$$\lambda_{k+1} - \lambda_k \leq \frac{4(h+1)}{(n+2)^2 k} \sum_{i=1}^{t-r} (m+1)(n+2m)a_m \left(\sum_{i=1}^{k} \lambda_i \right)^{m/(t-r)} k^{(t-r-m)/(t-r)},$$

for $k = h+1, h+2, \ldots$ if $\lambda_{1,1} \geq 1$, $\lambda_{k,1} \geq (\lambda_{k-h,1})^{(t-1)/(t-r)}$ and $\eta_1 \geq 1$ where $\lambda_{i,1}$ denotes the ith eigenvalue of (1) with $r = 1$ and η_1 is the first eigenvalue of the eigenvalue problem

$$\begin{cases} \Delta^2 = \eta \Delta u & \text{in} \quad M, \\ u = (\partial/\partial \nu)u = 0 & \text{on} \quad \partial M. \end{cases}$$

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