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On 3-manifolds with Torus or Klein Bottle Category Two

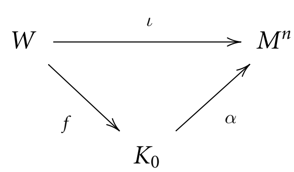
Wolfgang Heil and Dongxu Wang

Abstract. A subset W of a closed manifold M is K-contractible, where K is a torus or Klein bottle if the inclusion $W \to M$ factors homotopically through a map to K. The image of $\pi_1(W)$ (for any base point) is a subgroup of $\pi_1(M)$ that is isomorphic to a subgroup of a quotient group of $\pi_1(K)$. Subsets of M with this latter property are called $\mathcal{G}_K$-contractible. We obtain a list of the closed 3-manifolds that can be covered by two open $\mathcal{G}_K$-contractible subsets. This is applied to obtain a list of the possible closed prime 3-manifolds that can be covered by two open K-contractible subsets.

1 Introduction

The study of critical points for a smooth function on a manifold M^n led Lusternik and Schnirelmann to introduce what is today called the Lusternik–Schnirelmann category for a manifold M^n, denoted by $\text{cat}(M^n)$. It is defined as the smallest number of sets, open and contractible in M^n needed to cover M^n. It is a homotopy invariant with values between 2 and $n + 1$ and has been studied widely; many references can be found in [3].

In [2], Clapp and Puppe generalized $\text{cat}(M^n)$ as follows: For a fixed closed connected k-manifold K_0, $0 \leq k \leq n-1$, a subset W in M^n is said to be K_0-contractible if there are maps: $f\colon W \to K_0$ and $\alpha\colon K_0 \to M^n$, such that the inclusion map $\iota\colon W \to M^n$ is homotopic to $\alpha \cdot f$.



The K_0-category $\text{cat}_{K_0}(M^n)$ of M^n is the smallest number of sets, open and K_0-contractible needed to cover M^n. When K_0 is a point P, $\text{cat}_P(M^n)=\text{cat}(M^n)$, the Lusternik–Schnirelmann category. Note that a (in M) contractible set is K_0-contractible, hence $\text{cat}_{K_0}(M^n) \leq \text{cat}(M^n)$. If M^n is closed, then applying homology H_n with $\mathbb{Z}_2$-coefficients to the above homotopy commutative diagram shows that $2 \leq \text{cat}_{K_0}(M^n) \leq \text{cat}(M^n) \leq n+1$.

In particular, if M is a closed 3-manifold, we are interested in the beginning case when $\text{cat}_{K_0}(M) = 2$. If K_0 is a closed manifold, the first nontrivial choice for K_0 is S^1.

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It was shown in [4] that a closed 3-manifold M has $\mathrm{cat}_{S^1} M = 2$ if and only if $\pi_1 M$ is cyclic. From the results of Perelman (see *e.g.*, [1]), this means that $\mathrm{cat}_{S^1} M^3 = 2$ if and only if M is a lens space or the twisted S^2-bundle over S^1 (here the term "lens space" includes S^3 and $S^2 \times S^1$). We notice that, even though an S^1-contractible open set may not be homotopy equivalent to S^1 (and could be disconnected), this result shows that $\mathrm{cat}(M) = 2$ can be realized by a covering of M by two open solid tori or solid Klein bottles, each homotopy equivalent to S^1.

The next simplest choice for K_0 is S^2 or P^2. This case was considered in [6], where it was shown that for a closed 3-manifold M, $\mathrm{cat}_{S^2} M = 2$ if and only if M is S^3 or an S^2-bundle over S^1, and $\mathrm{cat}_{P^2} M = 2$ if and only if $M = S^3, P^3, P^3 \# P^3$, or $P^2 \times S^1$. We again observe that for $\mathrm{cat}_{S^2} M = 2$, M can be covered by two open $S^2 \times I$, each homotopy equivalent to S^2, and that $\mathrm{cat}_{P^2} M = 2$ can be realized by a covering of M by two simple open pieces, each an open ball or open I-bundle over P^2, *i.e.*, each homotopy equivalent to a point or P^2.

In this paper we consider the case when K_0 is a torus T or Klein bottle K.

Closely related to the K_0-category is the more algebraic $\mathcal{G}_{K_0}$-category, where $\mathcal{G}_{K_0}$ is the set of subgroups of 3-manifold quotient groups of $\pi_1(K_0)$. This is defined in Section 2. One has $1 \leq \mathrm{cat}_{\mathcal{G}_{K_0}}(M) \leq \mathrm{cat}_{K_0}(M)$.

In Theorem 4.3 we list all closed 3-manifolds for which $\mathrm{cat}_{\mathcal{G}_{\mathrm{T}}}(M) = 2$ and use this to give in Theorem 5.1 a classification of all prime closed 3-manifolds M with $\mathrm{cat}_{\mathrm{T}}(M) = 2$. It turns out that these M admit a covering by two simple T-contractible open subsets, each homeomorphic to an (open) ball, an (open) solid torus or solid Klein bottle, or an (open) I-bundle over T.

In Theorem 4.4 we list all closed 3-manifolds for which $\mathrm{cat}_{\mathcal{G}_{\mathrm{K}}}(M) = 2$. This is used to list in Theorem 5.2 the possible prime closed 3-manifolds for which $\mathrm{cat}_{\mathrm{K}}(M) = 2$. Here the obvious simple K-contractible pieces are (open) balls, (open) disk-bundles over S^1, and (open) I-bundles over K.

The proofs use techniques developed in [5, 8, 9].

2 K_0-contractible Subsets and Basic Properties

Even though the definitions can be made for any space, we assume in this section that M is a 3-manifold and K_0 is a connected complex of dimension ≤ 2.

Definition 2.1 A subset W is K_0-contractible (in M) if there are maps $f\colon W \to K_0$ and $\alpha\colon K_0 \to M$ such that the inclusion map $\iota\colon W \to M$ is homotopic to $\alpha \cdot f$. Notice we do not require W to be connected. The K_0-category $\mathrm{cat}_{K_0}(M)$ is defined to be the smallest number m such that M can be covered by m open K_0-contractible subsets.

A subset of a K_0-contractible subset is K_0-contractible.

Remark 2.2 If a complex L is a retract of a 2-complex K_0, then an L-contractible set is K_0-contractible.

This can be seen as follows. For an L-contractible set $W \subset M$, there are maps

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$f\colon W \to L$ and $\alpha\colon L \to M$ such that $\alpha \cdot f \simeq \iota$. Let $j\colon L \to K_0$ be the inclusion and $r\colon K_0 \to L$ the retraction. Then $(\alpha r) \cdot (jf) \simeq \iota$. Thus W is K_0-contractible.

Clearly for $K_0 = \mathrm{T}$ (the torus), a subset W of M that is homeomorphic to an I-bundle over T is T-contractible. By the remark, a 3-ball, a solid torus, or a solid Klein bottle in M are T-contractible.

For $K_0 = \mathrm{K}$ (the Klein bottle), subsets W of M homeomorphic to I-bundles over K, 3-balls, solid tori, or solid Klein bottles are K-contractible.

If W is K_0-contractible, then for every basepoint $* \in W$, the image: $i_*(\pi_1(W, *))$ is a subgroup of $\alpha_*(\pi_1(K_0, f(*)))$, which is isomorphic to a quotient Q of $\pi_1(K_0, f(*))$. We say that W is Q-contractible. More generally (see [9]), we have the following definition.

Definition 2.3 Let $\mathcal{G}$ be a nonempty class of groups. A subset W of M is $\mathcal{G}$-*contractible* if for any base point $* \in W$, the image $\iota_*(\pi_1(W, *)) \subseteq \pi_1(M, *)$ belongs to $\mathcal{G}$. The smallest number m such that M admits a covering by open $\mathcal{G}$-contractible subsets is the $\mathcal{G}$-category $\mathrm{cat}_{\mathcal{G}}(M)$.

If $\mathcal{G}$ is closed under subgroups, then a subset of a $\mathcal{G}$-contractible set is $\mathcal{G}$-contractible.

A quotient group Q of a group G is a 3-*manifold quotient*, if Q can be realized as the fundamental group of some 3-manifold. By considering covering spaces we see that subgroups of 3-manifold quotients are 3-manifold quotients.

With these notations, a K_0-contractible subset is also $G_{\mathcal{K}_0}$-contractible, where $G_{\mathcal{K}_0}$ is the set of subgroups of 3-manifold quotients of $\mathcal{K}_0 = \pi_1(K_0)$, and we have $1 \leq \mathrm{cat}_{\mathcal{G}_{K_0}}(M) \leq \mathrm{cat}_{K_0}(M)$.

In particular, to classify closed 3-manifold M with $\mathrm{cat}_{K_0}(M) = 2$, where K_0 is a torus or Klein bottle, we first classify those M with $\mathrm{cat}_{G_{\mathcal{K}_0}}(M) \leq 2$. The first step is to obtain in the next section a complete list of all subgroups of 3-manifold quotient groups of $\pi_1(\mathrm{T})$ and $\pi_1(\mathrm{K})$.

We use the following notation:

- $\#_k M^3$ denotes a connected sum of k copies of a 3-manifold M^3.
- L or L_i denotes a lens space different from S^3 or $S^2 \times S^1$.
- $S^2 \widetilde{\times} S^1$, $T \widetilde{\times} S^1$, $K \widetilde{\times} S^1$ denote an S^2-bundle over S^1, a torus-bundle over S^1, a Klein bottle-bundle over S^1, resp. The bundles may be product bundles.
- T_S denotes a T-semi bundle, *i.e.,* a union of two twisted I-bundles over the torus T along their (connected) torus boundary. (This terminology is due to Hatcher [10].)
- K_S denotes either a K-semi bundle, *i.e.*, a union of two twisted I-bundles over the Klein bottle K along their (connected) boundary or a union of the oriented twisted I-bundle over K and a twisted I-bundle over T along their (connected) boundary.

3 3-manifold Quotient Groups of the Klein Bottle Group

In this section we list all compact 3-manifolds whose fundamental groups are isomorphic to a subgroup of a 3-manifold quotient of the torus T or Klein bottle K. This is easy for the fundamental group $\mathcal{T}$ of the torus, since the abelian groups that occur as fundamental groups of 3-manifolds are well known (see *e.g.*, [11, Thm 9.13]).

The set of subgroups of 3-manifold quotients of $\mathcal{T} \cong \mathbb{Z}\times\mathbb{Z}$ is

$$\mathcal{G}_{\mathcal{T}} = \{1, \mathbb{Z}_m, \mathbb{Z}, \mathbb{Z}\times\mathbb{Z}_2, \mathbb{Z}\times\mathbb{Z}\}$$

(for all $m \geq 2$).

The corresponding compact 3-manifolds are listed in the next proposition. Here $\widehat{M}$ denotes the 3-manifold obtained from the 3-manifold M by capping off all 2-sphere boundary componens with 3-balls.

Proposition 3.1 *If M is a compact 3-manifold with $\pi_1(M) \in \mathcal{G}_{\mathcal{T}}$, then $\widehat{M}$ is one of the following:*

(i) *a lens space (including S^3),*
(ii) *$P^2\times I$,*
(iii) *an S^2-bundle over S^1,*
(iv) *a solid torus or solid Klein bottle,*
(v) *$P^2\times S^1$,*
(vi) *an I-bundle over the torus.*

To obtain an analogous result for 3-manifold quotients of the Klein bottle group, we begin by listing all the possible isomorphism types of subgroups of the fundamental group $\mathcal{K}$ of the Klein bottle K.

Lemma 3.2 *Let H be a nontrivial subgroup of $\mathcal{K}$. If H has infinite index, then H is isomorphic to $\mathbb{Z}$; if H has finite index, then H is isomorphic to $\mathbb{Z}\times\mathbb{Z}$ or $\mathcal{K}$.*

Proof If $H \subseteq \mathcal{K}$ has infinite index, the covering space X of the Klein bottle K corresponding to H is a noncompact surface, so H is a free group. Since $\mathcal{K}$ is solvable, it does not contain a free subgroup of rank 2, hence $H \cong \mathbb{Z}$. If $H \subseteq \mathcal{K}$ has finite index, then $H \cong \pi_1(X)$, where X is a finite sheeted covering of K. So X is a torus or a Klein bottle, and the result follows. ■

Next we list the normal subgroups of $\mathcal{K}$. We represent $\mathcal{K}$ by

$$\mathcal{K} = \langle a, b : b^{-1}ab = a^{-1}\rangle.$$

Every $g \in \mathcal{K}$ has a unique representation of the form $a^m b^n$ by using the relation

$$b^r a^s = \begin{cases} a^s b^r & \text{if } r \text{ is even,} \\ a^{-s} b^r & \text{if } r \text{ is odd.} \end{cases}$$

Then for $k \neq 0$,

$$(a^m b^n)^k = \begin{cases} a^{mk} b^{nk} & \text{if } n \text{ is even,} \\ a^m b^{nk} & \text{if } n \text{ is odd, } k \text{ is odd,} \\ b^{nk} & \text{if } n \text{ is odd, } k \text{ is even.} \end{cases}$$

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Lemma 3.3 *If H is a cyclic normal subgroup of $\mathcal{K}$ generated by $a^m b^n$, then n is even and either $n = 0$ or $m = 0$.*

Proof We have $a^{-1}a^m b^n a = (a^m b^n)^k$ for some k. If n is odd, it follows that $a^{m-2}b^n = a^m b^{nk}$ (if k is odd), or $a^{m-2}b^n = b^{nk}$ (if k is even), a contradiction.

Thus assume n is even. Then $b^{-1}a^m b^n b = (a^m b^n)^k$ for some k, or $a^{-m}b^n = a^{mk}b^{nk}$. It follows that $m = 0$ or $n = 0$. ∎

Now consider non-cyclic normal subgroups H of $\mathcal{K}$. We write $H = \langle x, y\rangle$ to mean that H is generated by x and y (not necessarily *normally* generated by x and y). By Lemma 3.2, a noncyclic subgroup of $\mathcal{K}$ can be generated by two elements. Furthermore, by [12, Proposition 3.4], we may assume that

(3.1) $$H = \langle a^r, a^m b^n\rangle, \text{ where } 0 \le m < r, \quad n > 0, \text{ with index } |\mathcal{K} : H| = rn.$$

H is a Klein bottle group (resp. torus group), if n is odd (resp. n is even).

Lemma 3.4 *Let H as in (3.1) be a normal subgroup of $\mathcal{K}$.*
If n is odd, then $H = \langle a, b^n\rangle$, $H = \langle a^2, b^n\rangle$, or $H = \langle a^2, ab^n\rangle$.
If n is even and $m \ne 0$, then $H = \langle a^{2m}, a^m b^n\rangle$.

Proof (a) Suppose n is odd. Then for each k we have $a^{-k}a^m b^n a^k = a^{m-2k}b^n \in H$. Hence, if m is even, $b^n \in H$, and if m is odd, $ab^n \in H$.

If m is even, $a^{-1}b^n a = a^{-2}b^n \in H$, hence $a^2 \in H$ and it follows that $H = \langle a^r, a^2, a^m, b^n\rangle = \langle a, b^n\rangle$ if r is odd, and $H = \langle a^2, b^n\rangle$ if r is even.

If m is odd, $b^{-1}ab^n b = a^{-1}b^n \in H$, and $ab^n(a^{-1}b^n)^{-1} = a^2 \in H$, and it follows that $H = \langle a^r, a^2, ab^n\rangle = \langle a, b^n\rangle$ if r is odd, and $H = \langle a^2, ab^n\rangle$ if r is even.

(b) Now suppose that n is even and $m \ne 0$. Then $b(a^m b^n)b^{-1} = a^{-m}b^n \in H$, so $(a^m b^n)(a^{-m}b^n)^{-1} = a^{2m} \in H$. It follows that $a^{2m} = (a^r)^k(a^m b^n)^l$, for some k, l, and since $H \cong \mathbb{Z} \times \mathbb{Z}$ with generators a^r and $a^m b^n$, that $l = 0$, $2m = rk$. But $0 < rk = 2m < 2r$, so $k = 1$, *i.e.*, $a^{2m} = a^r$. ∎

Using Lemmas 3.3 and 3.4 we can now determine which quotients $G = \mathcal{K}/H$ are fundamental groups of 3-manifolds.

A *prism manifold M_P* is a Seifert fiber space obtained from the orientable twisted I-bundle over the Klein bottle, $(K\widetilde{\times}I)_o$, and a solid torus V by identifying their torus boundaries. Its fundamental group has a presentation $\pi_1(M_P) = \langle a, b : b^{-1}ab = a^{-1}, a^m b^{2n} = 1\rangle$, where $\gcd(m, n) = 1$ and the curve $a^m b^{2n}$ on $\partial(K\widetilde{\times}I)_o$ is identified with the meridian of V.

Proposition 3.5 *Suppose $G = \mathcal{K}/H$ is the fundamental group of a 3-manifold. Then $G \in \{1, \mathbb{Z}_n, \mathbb{Z}, \mathcal{K}, \mathbb{Z} \times \mathbb{Z}_2, \mathbb{Z}_2 * \mathbb{Z}_2, \pi_1(M_P)\}$, where M_P is a prism manifold.*

Proof Suppose H is a proper normal subgroup of $\mathcal{K}$ and $G = \mathcal{K}/H = \pi_1(M)$. Since G is finitely generated, we may assume that M is compact and, by filling in boundary spheres with 3-balls, that ∂M contains no 2-spheres. In the arguments below we repeatedly use the property that every subgroup of the 3-manifold group G is a 3-manifold group. Let $A = \langle a, b^2\rangle$ be the abelian subgroup of G generated by a and b^2. If n is even, we write $2n$ instead of n (where now $n \in \mathbb{N}$). Using Lemmas 3.3 and 3.4 we consider the following cases:

(1) $H \cong \mathbb{Z}$,
(2) $H \cong \mathcal{K}$, n odd,
(3) $H \cong \mathbb{Z} \times \mathbb{Z}$, n even.

In case (1), we have two subcases:

(1a) $G = \langle a, b : b^{-1}ab = a^{-1}, b^{2n} = 1\rangle$, $\quad A \cong \mathbb{Z} \times \mathbb{Z}_n$, or
(1b) $G = \langle a, b : b^{-1}ab = a^{-1}, a^{m} = 1\rangle$, $\quad A \cong \mathbb{Z} \times \mathbb{Z}_m$.

In (1a), the subgroup A is a 3-manifold group only for $n = 1$ or 2. For $n = 1$, $G = \langle a, b : b^{-1}ab = a^{-1}, b^{2} = 1\rangle \cong \mathbb{Z}_2 * \mathbb{Z}_2$.

For $n = 2$, $G = \langle a, b : b^{-1}ab = a^{-1}, b^{4} = 1\rangle$, and we claim that this is not a 3-manifold group.

To see this, note that $x = b^2$ is the only element of order 2 in G. For if $w = a^r b^s \in G$ has order 2 and s is even, then $w \in A \cong \mathbb{Z} \times \mathbb{Z}_2 = \langle a, x : ax = xa, x^2 = 1\rangle$, and so $w = x$. If s is odd, then $w^2 = b^{2s} = x^s = 1 \in A$ implies s even, a contradiction.

Now suppose $G \cong \pi_1(M)$ and let $g \colon \widetilde{M} \to M$ be the 2-fold covering of M corresponding to A. Then $\widetilde{M}$ is a punctured $P^2 \times S^1$. Let $P^2_0 = P^2 \times \{z\}$, for a point $z \in S^1$. The generator w of $g_*(\pi_1(P^2_0))$ is the unique element $w \in G$ of order 2. If there is a 2-sphere S in $\partial\widetilde{M}$, then $g(S)$ is a projective plane $P^2_* \subset \partial M$ and the generator w_* of $\pi_1(P^2_*)$ is equal to w in G. This is a contradiction, since w lifts to the generator of $\pi_1(P^2_0)$, whereas w_* does not lift to a loop in S. So $\widetilde{M} = P^2 \times S^1$, and it follows that $M = P^2 \times S^1$, hence $\pi_1(M) \cong \mathbb{Z} \oplus \mathbb{Z}_2 \ncong G$.

In (1b), the subgroup A is a 3-manifold group only for $m = 1$ or 2. For $m = 1$, $G = \langle a, b : b^{-1}ab = a^{-1}, a = 1\rangle \cong \mathbb{Z}$. For $m = 2$, $G = \langle a, b : b^{-1}ab = a^{-1}, a^2 = 1\rangle \cong \mathbb{Z} \times \mathbb{Z}_2$.

For case (2), we have three subcases:

(2a) $G = \langle a, b : b^{-1}ab = a^{-1}, a = 1, b^{n} = 1\rangle \cong \mathbb{Z}_n$.
(2b) $G = \langle a, b : b^{-1}ab = a^{-1}, a^2 = 1, b^{n} = 1\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_n$.

Since n is odd, $G \cong \mathbb{Z}_{2n}$.

(2c) $G = \langle a, b : b^{-1}ab = a^{-1}, a^2 = 1, ab^{n} = 1\rangle \cong \mathbb{Z}_{2n}$.

In case (3), we have two subcases:

(3a) $G = \langle a, b : b^{-1}ab = a^{-1}, a^{2m} = 1, a^m b^{2n} = 1\rangle$ $(m \neq 0)$.
(3b) $G = \langle a, b : b^{-1}ab = a^{-1}, a^{r} = 1, b^{2n} = 1\rangle$, $\quad A \cong \mathbb{Z}_r \times \mathbb{Z}_n$.

In (3a), $a^{2m} = 1$ is a consequence of $b^{-1}ab = a^{-1}$ and $a^m b^{2n} = 1$, hence

$$G = \langle a, b : b^{-1}ab = a^{-1}, a^m b^{2n} = 1\rangle.$$

For $d = \gcd(m, n) = mp + nq$, $x = a^{m/d}(b^2)^{n/d}$, $y = a^q (b^2)^{-p}$, the abelian subgroup generated by a and b^2 has presentation

$$A = \langle x, y : xy = yx, x^d = 1, y^{2nm/d} = 1\rangle \cong \mathbb{Z}_d \times \mathbb{Z}_{2nm/d} = \mathbb{Z}_d \times \mathbb{Z}_{2n'm'd},$$

where $m = dm'$, $n = dn'$. This is a 3-manifold group only for $d = 1$, and it follows that G is the fundamental group of a prism manifold M_P, obtained from $(K \widetilde{\times} I)_o$,

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the orientable twisted I-bundle over K with $\pi_1(\partial(K\widetilde{\times}I)_o)$ generated by a and b^2, by attaching a solid torus V along $\partial(K\widetilde{\times}I)_o$ such that the meridian of V is identified with the curve $a^m b^{2n}$.

In (3b), if the subgroup A is a 3-manifold group, then $\gcd(r,n) = 1$. If r is even, $r = 2k$, then $(a^k)^{-1} = a^k$ and the subgroup $A_k = \langle a^k, b\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_{2n}$ is not a 3-manifold group. So assume that r is odd. By Milnor [13], every element of order 2 in a finite 3-manifold group is central, hence $(b^n)^{-1}ab^n = a$. If n is odd, then $b^{-1}ab = a^{-1}$ implies that $(b^n)^{-1}ab^n = a^{-1}$, so $a^2 = 1$. Since r is odd, then $a^r = 1$ implies $a = 1$, a contradiction (since we assumed that $r > 1$).

Hence $G = \langle a, b : b^{-1}ab = a^{-1}, a^r = 1, b^{4n'} = 1\rangle$, $n' \geq 1$, r odd. Note that there is a unique epimorphisms $\varphi\colon G \to \mathbb{Z}_4$. We now show that if M is 3-manifold different from a lens space or a prism manifold and with finite fundamental group, then there is no epimorphisms $\pi_1(M) \to \mathbb{Z}_4$.

By Perelman (see *e.g.*, [1]), M is spherical, and by Orlik [14, p. 111, Theorem 2], the fundamental groups of these M are isomorphic to $\mathbb{Z}_s \times T_m$ ($m = 3, 4, 5, 3^k 8$), where

$$T_m = \langle x, y : x^2 = (xy)^3 = y^m, x^4 = 1\rangle \text{ for } m = 3, 4, 5,$$

$$T_{3^k 8} = \langle x, y, z : x^2 = (xy)^2 = y^2, zxz^{-1} = y, zyz^{-1} = xy, z^{3^k} = 1\rangle (k \geq 1),$$

and s is coprime to 30 for $m = 5$ and coprime to 6 in all other cases.

The abelianization of T_m is $\mathbb{Z}_3$, $\mathbb{Z}_2$, 1 resp. for $m = 3, 4, 5$ and the abelianization of $T_{3^k 8}$ is $\mathbb{Z}_{3^k}$. Since s is odd, there is no epimorphism $\mathbb{Z}_s \times T_m \to \mathbb{Z}_4$. ■

Corollary 3.6 *The set of subgroups of 3-manifold quotients of $\mathcal{K}$ is*

$$\mathcal{G}_{\mathcal{K}} = \{1, \mathbb{Z}_n, \mathbb{Z}, \mathbb{Z}\times\mathbb{Z}, \mathcal{K}, \mathbb{Z} \times \mathbb{Z}_2, \mathbb{Z}_2 * \mathbb{Z}_2, \pi_1(M_P)\},$$

where M_P is a prism manifold.

Proof The nontrivial subgroups of $\mathcal{K}$ are isomorphic to $\mathbb{Z}$, $\mathbb{Z}\times\mathbb{Z}$, and $\mathcal{K}$. The proper 3-manifold quotients of $\mathcal{K}$ are listed in Proposition 3.5. In [8, Lemma 5] it is shown that the only nontrivial 3-manifold subgroups of $\mathbb{Z}_2 * \mathbb{Z}_2$ are $\mathbb{Z}_2$, $\mathbb{Z}$, and $\mathbb{Z}_2 * \mathbb{Z}_2$. If R is a subgroup of $\pi_1(M_P)$, consider the (finite sheeted) covering $p\colon \widetilde{M} \to M_P$ corresponding to R. The prism manifold M_P is a union of the orientable twisted $K\widetilde{\times}I$ and a solid torus V; since $p^{-1}(V)$ consists of solid tori, it follows that $p^{-1}(K\widetilde{\times}I) \approx K\widetilde{\times}I$ or $T\times I$, and $\widetilde{M}$ is either a prism manifold or a lens space. ■

The compact 3-manifolds whose fundamental groups are as in Corollary 3.6 are well known and are listed in the next proposition.

Proposition 3.7 *If M is a compact 3-manifold with $\pi_1(M) \in \mathcal{G}_{\mathcal{K}}$, then $\widehat{M}$ is one of the following:*

(i) *a lens space (including S^3),*
(ii) *an S^2-bundle over S^1,*
(iii) $P^2\times S^1$,
(iv) *a prism manifold,*

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(v) $P^3\#P^3$, $P^2\times I$, $P^3\#P^2\times I$, $P^2\times I\#P^2\times I$,
(vi) *an I-bundle over the torus or Klein bottle,*
(vii) *a solid torus or solid Klein bottle.*

4 Closed 3-manifolds with $\mathcal{G}$-category ≤ 2.

In this section we let $\mathcal{G} = \mathcal{G}_{\mathcal{T}}$ or $\mathcal{G}_{\mathcal{K}}$.

The following proposition is a restatement of Propositions 3.1 and 3.7 for closed 3-manifolds.

Proposition 4.1 *Let M be a closed* 3*-manifold. Then*

(i) $\operatorname{cat}_{\mathcal{G}_{\mathcal{T}}}(M) = 1$ *if and only if M is a lens space (including S^3), an S^2-bundle over S^1, or $P^2\times S^1$;*
(ii) $\operatorname{cat}_{\mathcal{G}_{\mathcal{K}}}(M) = 1$ *if and only if M is a lens space (including S^3), an S^2-bundle over S^1, $P^2\times S^1$, $P^3\#P^3$, or a prism manifold.*

We now consider the case $\operatorname{cat}_{\mathcal{G}}(M) = 2$. Then by [8, Proposition 1] we may assume that M is a union of compact (not necessarily connected) $\mathcal{G}$-contractible 3-submanifolds W_0, W_1, such that $W_0 \cap W_1 = \partial W_0 = \partial W_1$ is a surface F (not necessarily connected). For a component F' of F and W_i' of $\overline{M - N(F)}$ (where $N(F)$ is a product neighborhood of F), the images $\operatorname{im}(\pi_1(F') \to \pi_1(M))$ and $\operatorname{im}(\pi_1(W_i') \to \pi_1(M))$ are contained in $\operatorname{im}((\pi_1(W_i) \to \pi_1(M))$, for $i = 0, 1$, and since $\mathcal{G}$ is closed under subgroups, we conclude that F and the components of $\overline{M - N(F)}$ are $\mathcal{G}$-contractible. If a non-2-sphere component F' of F is compressible in $\overline{M - N(F)}$, we do surgery on a compressing disk to get a new decomposition of M with a new $\mathcal{G}$-contractible surface F of smaller complexity and such that the components of $\overline{M - N(F)}$ are $\mathcal{G}$-contractible. Here the complexity $c(F)$ is 1, if F is the sphere; if F is connected of genus $g > 0$, then $c(F) = (2g-1)\omega$ where ω is the first infinite ordinal; if F is not connected, the complexity of F is the sum of the complexities of its components (see the proof of [7, Lemma 4]). Thus we may assume that every non-sphere component of F is incompressible. If a 2-sphere component of F bounds a ball in M, we delete it from F. Then every component of F and every component $\mathcal{C}$ of $\overline{M - N(F)}$ is π_1-injective (*i.e.*, the inclusions into M induce injections of fundamental groups). To sum up, we have the following ([9, Lemma 4]).

Lemma 4.2 ([9]) *Let $\mathcal{G}$ be closed under subgroups and let M be a closed* 3*-manifold with $\operatorname{cat}_{\mathcal{G}} M \leq 2$. Then there is a closed* 2*-sided surface F in M such that F and $\overline{M - N(F)}$ are $\mathcal{G}$-contractible and every component of F is an essential* 2*-sphere or incompressible. In particular, the inclusion of each component of F and each component of $\overline{M - N(F)}$ into M is π_1-injective.*

Theorem 4.3 *Let M be a closed* 3*-manifold. Then $\operatorname{cat}_{\mathcal{G}_{\mathcal{T}}}(M) = 2$ if and only if M is not as in Proposition* 4.1 *and for some $i, j, k, m, n \geq 0$,*

$$M \in \left\{ S^3\#_i L\#_j(S^2\widetilde{\times}S^1)\#_k(P^2\times S^1)\#_m(T_S)\#_n(T\widetilde{\times}S^1)\right\}.$$

Proof We choose a 2-sided closed surface $F \subset M$ as in Lemma 4.2 with a minimal number of components. Since each component F' of F is π_1-injective, $\pi_1(F')$ is

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isomorphic to a subgroup of a quotient of $\mathbb{Z}\times\mathbb{Z}$; hence F' is a two-sphere, projective plane, or torus. Each component C of $\overline{M-N(F)}$ is $G_{\mathcal{T}}$-contractible and π_1-injective, hence C is as in Proposition 3.1. We cannot have Proposition 3.1(iv), since F' is incompressible.

Suppose F' is a torus and C_0, C_1 are adjacent components of $\overline{M-N(F)}$, such that $F'\times\{0\}$ is a component of ∂C_0 and $F'\times\{1\}$ is a component of ∂C_1 (here we identify $N(F)$ with $F\times[0,1]$). Then C_0, C_1 are punctured I-bundles over T.

If $C_0 \neq C_1$, then C_i cannot be a (punctured) product I-bundle, by the minimality of the number of components of F. Since in Proposition 3.1 there are no other compact 3-manifolds with boundary a torus, C_0 and C_1 are (punctured) twisted I-bundles and $C_0 \cup F'\times[0,1] \cup C_1'$ is a (punctured) torus semi-bundle.

If $C_0 = C_1$, then $C_0 \approx F'\times I$ and $C_0 \cup F'\times[0,1]$ is a (punctured) T-bundle over S^1.

By the same argument, if a component F' is a projective plane, we obtain a punctured $P^2\times S^1$ in M.

Hence the collection of 2-sphere components of F cuts M into punctured lens spaces, punctured $S^2\widetilde{\times}S^1$'s, punctured T-bundles over S^1, punctured T-semi bundles, and punctured $P^2\times S^1$'s, and M is as in Theorem 4.3.

Conversely, if M is as in Theorem 4.3, we can find a disjoint collection F of tori, projective planes, and 2-spheres that cuts M into punctured $T\widetilde{\times}I$'s, punctured $P^2\times S^1$'s, punctured $S^2\widetilde{\times}S^1$'s, and punctured lens spaces. Let $W_0 = F\times I$ and $W_1 = \overline{M-N(F)}$. Then W_i is $G_{\mathcal{K}}$-contractible for $i = 0, 1$, and $M = W_0 \cup W_1$. ■

Using the same technique, we obtain the following theorem.

Theorem 4.4 *Let M be a closed 3-manifold. Then $\operatorname{cat}_{\mathcal{G}_{\mathcal{K}}}(M) = 2$ if and only if M is not as in Proposition 4.1 and for some $i, j, k, m, n, s, t, r \geq 0$,*

$$M \in \left\{ S^3 \#_i L \#_j (S^2\widetilde{\times}S^1) \#_k (P^2\times S^1) \#_m (T_S) \#_n (K_S) \#_s (T\widetilde{\times}S^1) \#_t (K\widetilde{\times}S^1) \#_r (M_P) \right\}.$$

Proof Following the above proof of Theorem 4.3, we obtain a 2-sided incompressible closed surface $F \subset M$ such that every component F' of F is a two-sphere, projective plane, torus, or Klein bottle, and each component C of $\overline{M-N(F)}$ is $G_{\mathcal{K}}$-contractible, π_1-injective, with C as in Proposition 3.7, except as in Proposition 3.1(vii). If a component of $\overline{M-N(F)}$ is a (punctured) connected sum as in case (v), we cut it along the connected sum sphere and adjoin the resulting spheres to F. Also note that P^3 is a lens space. Now if a component F' is a projective plane or Klein bottle, the argument in the proof of Theorem 4.3 applies to yield a (punctured) K_S or $K\widetilde{\times}S^1$.

If F' is a torus and C_0, C_1 are adjacent components of $\overline{M-N(F)}$, then each of C_0, C_1 is a punctured I-bundle over T or the orientable punctured I-bundle over K and we obtain a (punctured) T-bundle over S^1 or a (punctured) torus semi-bundle or (punctured) K_S.

Then the collection of 2-sphere components of F cuts M into punctured lens spaces, punctured $S^2\widetilde{\times}S^1$'s, punctured T- and K-bundles over S^1, punctured T and K-semi bundles, punctured $P^2\times S^1$'s, and prism manifolds M_P, and M is as in Theorem 4.4.

The converse follows as above. ■

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5 Prime 3-manifolds with T- and K-category ≤ 2.

Let $L = T$ or $L = K$ in this section.

Since an L-contractible set is $\mathcal{G}_{\mathcal{L}}$-contractible, $1 \leq \mathrm{cat}_{\mathcal{G}_{\mathcal{L}}}(M) \leq \mathrm{cat}_L(M)$. If M is closed, $H_3(M; Z_2) \simeq Z_2$, and the identity map id: $H_3(M; Z_2) \to H_3(M; Z_2)$ does not factor through $H_3(L; Z_2) = 0$, hence $\mathrm{cat}_L(M) \geq 2$. If M is closed and $\mathrm{cat}_L(M) = 2$, it follows that M is as in Theorem 4.3 for $L = T$ and as in Theorem 4.4 for $L = K$. However, not every member of the family

$$\{S^3 \#_i L \#_j (S^2 \widetilde{\times} S^1) \#_k (P^2 \times S^1) \#_m (T_S) \#_n (T \widetilde{\times} S^1)\}, \text{ resp.} \tag{5.1}$$

$$\{S^3 \#_i L \#_j (S^2 \widetilde{\times} S^1) \#_k (P^2 \times S^1) \#_m (T_S) \#_n (K_S) \#_s (T \widetilde{\times} S^1) \#_t (K \widetilde{\times} S^1) \#_r (M_P)\} \tag{5.2}$$

is of T- (resp. K-) category 2. In [8, Lemma 2], it is shown that if M is a closed 3-manifold with $\mathrm{cat}_L(M) = 2$, (where L is a 2-dimensional complex), then (for the $\mathbb{Z}_2$-rank)

$$rk(H_1(M; \mathbb{Z}_2)) \leq rk(H_1(L; \mathbb{Z}_2)) + rk(H^2(L; \mathbb{Z}_2)).$$

So for $L = T$ or $L = K$, the number of connected sum factors is restricted by $rk(H_1(M; Z_2)) \leq 3$.

In the case that M is prime, we obtain the following theorem.

Theorem 5.1 *Let M be a prime closed 3-manifold. Then $\mathrm{cat}_T(M) = 2$ if and only if*

$$M \in \{L, S^2 \widetilde{\times} S^1, P^2 \times S^1, T_S, T \widetilde{\times} S^1\}.$$

Proof By the remarks above, it suffices to show that for every prime summand M of (5.1), $\mathrm{cat}_K(M) = 2$. Recall that solid tori, solid Klein bottles, and I-bundles over T are T-contractible, and note that L, $S^2 \widetilde{\times} S^1$ are unions of two solid tori, resp. Klein bottles along their boundary; $P^2 \times S^1$ is a union of a twisted I-bundle over T and a solid torus; T_S and $T \widetilde{\times} S^1$ are unions of two I-bundles over T. ■

In the next theorem, denote by $\widetilde{K}_S$ a semi-bundle that is the union of two twisted I-bundles over K.

Theorem 5.2 *Let M be a prime closed 3-manifold.*

(i) *If $\mathrm{cat}_K(M) = 2$, then $M \in \{L, S^2 \widetilde{\times} S^1, P^2 \times S^1, T_S, K_S, T \widetilde{\times} S^1, K \widetilde{\times} S^1, M_P\}$.*
(ii) *If $M \in \{L, S^2 \widetilde{\times} S^1, \widetilde{K}_S, K \widetilde{\times} S^1, M_P\}$, then $\mathrm{cat}_K(M) = 2$.*

Proof For part (i) we list the prime summands of (5.2). For part (ii) recall that solid tori, solid Klein bottles, and I-bundles over K are K-contractible; $\widetilde{K}_S$ and $K \widetilde{\times} S^1$ are unions of two I-bundles over K; each M_P is a union of the orientable twisted I-bundle over K and a solid torus. ■

In part (i), $P^2 \times S^1$ can be obtained as a union of three solid tori and so is of K-category at most 3. We do not know if $\mathrm{cat}_K(P^2 \times S^1) = 2$.

Some $T \widetilde{\times} S^1$ are of K-category 2 and some are of K-category 3. To see this we show the following proposition.

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Proposition 5.3 *Suppose M is prime and $\pi_1(M)$ is torsion free and does not contain a subgroup isomorphic to the Klein bottle group $\mathcal{K}$. Then $\mathrm{cat}_K(M) = 2$ if and only if $M \cong S^3$ or $M \cong S^2 \widetilde{\times} S^1$.*

Proof If $\mathrm{cat}_K(M) = 2$, then $M = W_0 \cup W_1$, and there are maps $f_i\colon W_i \to K$, $\alpha_i\colon K \to M$ such that $\alpha_i \cdot f_i \simeq j_i$, where $j_i\colon W_i \to M$ is inclusion. For each component W_i' of W_i, $j_{i*}(\pi_1(W_i')) \subset \alpha_{i*}(\pi_1(K)) \subset \pi_1(M)$ is a quotient of $\mathcal{K}$. By Proposition 3.5 and the assumptions on $\pi_1(M)$, $\alpha_{i*}(\pi_1(K))$ is trivial or $\mathbb{Z}$. Hence for $\mathcal{G} = \{1, \mathbb{Z}\}$, W_i is $\mathcal{G}$-contractible and $\mathrm{cat}_{\mathcal{G}}(M) \leq 2$. If $\mathrm{cat}_{\mathcal{G}}(M) = 1$, then $M \cong S^3$ or $S^2 \widetilde{\times} S^1$.

The case that $\mathrm{cat}_{\mathcal{G}}(M) = 2$ does not occur; otherwise, by Proposition 4.2, there is a 2-sided π_1-injective and $\mathcal{G}$- contractible surface F. Then each component of F is a 2-sphere. Since the 2-spheres of F are essential and M is prime, it follows that $M \cong S^2 \widetilde{\times} S^1$, *i.e.*, $\mathrm{cat}_{\mathcal{G}}(M) = 1$. ∎

Example 5.4 For $T_S = T \times S^1$, it follows from this proposition that $\mathrm{cat}_K(T_S) \neq 2$. Since $T \times S^1$ can be obtained as a union of three solid tori (with intersection along their boundaries), $\mathrm{cat}_K(T_S) = 3$.

For $T_S = M_2$, the T-bundle over S^1 in [11, 12.3.Examples], with

$$\pi_1(M_2) = \langle a, b, t : ab = ba, t^{-1}at = a^{-1}, t^{-1}bt = b^{-1} \rangle,$$

we have $\mathrm{cat}_K(T_S) = 2$, since M_2 is a union of two (orientable) twisted I-bundles over K.

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Department of Mathematics, Florida State University, Tallahassee, Florida 32306, USA
e-mail: heil@math.fsu.edu dxwang1981@gmail.com