SCHOTTKY UNIFORMIZATIONS OF CLOSED RIEMANN SURFACES WITH ABELIAN GROUPS OF CONFORMAL AUTOMORPHISMS

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1. Introduction Let us consider a pair (S, H) consisting of a closed Riemann surface S and an Abelian group H of conformal automorphisms of S. We are interested in finding uniformizations of S, via Schottky groups, which reflect the action of the group H. A Schottky uniformization of a closed Riemann surface S is a triple $(\Omega, G, \pi:\Omega \to S)$ where G is a Schottky group with Ω as its region of discontinuity and $\pi:\Omega \to S$ is a holomorphic covering with G as covering group. We look for a Schottky uniformization $(\Omega, G, \pi:\Omega \to S)$ of S such that for each transformation h in H there exists an automorphisms t of Ω satisfying $h \circ \pi = \pi \circ t$.

In **[7]** we have obtained necessary conditions, called the *condition* (A), to get a Schottky uniformization as desired. These necessary conditions involve only the action of H at the set of fixed points of its non-trivial elements. In particular, if H acts without fixed points, then it satisfies automatically such a condition. We show that in the case of H Abelian, condition (A) is also sufficient.

An equivalent way to describe our problem in the language of three-manifolds is the following. Let V be a handle-body of genus g and let S be the boundary of V. The surface S is a closed orientable surface of genus g. Denote by Diff(S) the group of orientation preserving diffeomorphisms of S. Let H be a finite abelian subgroup of Diff(S). We ask for the existence of an element f of Diff(S) such that the group fHf^{-1} extends to a group of orientation preserving diffeomorphisms of V. In the rest of our work we consider the Schottky group description of our problem.

In 1980 L. Keen **[11]** discussed this problem for hyperelliptic Riemann surfaces S with H being the group generated by the hyperelliptic involution. In **[5]** and **[6]** we gave a similar discussion for closed Riemann surfaces which admit a general conformal involution. In **[7]** we discuss this problem for more general groups and we obtain necessary conditions, called the condition (A), to find a Schottky group as desired. In **[8, 9, 10]** we have proved that condition (A) is sufficient if H is isomorphic to a cyclic group or to the group $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ or to the Dihedral group $\mathbb{D}_{2p}$ of order $2p$ with p a prime.

In general, if S is a closed Riemann surface of genus $g \geq 2$ and H is a group of conformal automorphisms of S satisfying the condition (A), it is not true that this condition is sufficient.

2. Necessary conditions. In this section, we recall the definition of Schottky groups (Schottky uniformizations) and we get necessary conditions to be satisfied by the group H to find a Schottky group as desired. At the end of the section we establish the main result of this work, which shows that these necessary conditions are also sufficient if the group H is Abelian. For most of the definitions, concerning Kleinian groups, **[14]** is a good reference.

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We denote by $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \cong \mathbb{CP}_1$ the Riemann Sphere. The group of conformal automorphisms of $\hat{\mathbb{C}}$ is the Möbius Group, also called the Fractional Linear Group, and denoted by $\mathbb{M}$.

Definition 1 (Schottky group of genus g). For $g \geq 1$, let C_k, C'_k, $k = 1, \ldots, g$, be $2g$ Jordan curves on the Riemann sphere, $\mathbb{C} = \mathbb{C} \cup \{\infty\}$, such that they are mutually disjoint and bound a $2g$-connected domain, say D. Suppose that for each k there exists a fractional linear transformation A_k with the following properties:
(i) $A_k(C_k) = C'_k$;
(ii) A_k maps the exterior of C_k onto the interior of C'_k.
The transformations $\{A_i : i = 1, \ldots, g\}$ generate a subgroup G of Möbius transformations, necessarily Kleinian, and D is a fundamental domain for G, called a standard fundamental domain for G. This group is called a *Schottky group of genus g*.

Observe that if G is a Schottky group, then G is a free group on g generators and all its elements, except the identity, are loxodromic **[15]**. These properties in fact define the Schottky groups of genus g, for $g \geq 1$. For our purposes, we define the Schottky group of genus zero to be the group with the identity as its only element, that is, the trivial group.

Theorem 1 **[15]**. *Let G be a Kleinian group. Then G is a Schottky group if and only if G is purely loxodromic, finitely generated and free.*

Theorem 2 **[3]**. *If G is a Schottky group, then corresponding to any set of free generators these exists a fundamental domain D, as above, whose boundary curves are identified by the given generators.*

Definition 2. If G is a Schottky group and $A_1, \ldots, A_g$ form a set of free generators, we say that $G = \langle A_1, \ldots, A_g \rangle$ is a *marked Schottky group*, and that the set of transformations $A_1, \ldots, A_g$ is a *marking* of G.

Let us remark that if G is a Schottky group of genus g, then $\Omega(G)/G$ is a closed Riemann surface of genus g. Moreover, if $A_1, \ldots, A_g$ form a set of free generators for G and D is a standard fundamental domain for these generators with boundary curves C_k, C'_k, $k = 1, \ldots, g$, then these loops project to a set of g disjoint homologically independent simple loops on S. Reciprocally, the retrosection theorem **[2]** say to us that we can reverse this situation.

Theorem 3 (Retrosection theorem). *Every closed Riemann surface S of genus g can be represented as $\Omega(G)/G$, G being a Schottky group of genus g with region of discontinuity $\Omega(G)$. More precisely, given a set of g disjoint, homologically independent, simple closed curves $\gamma_1, \ldots, \gamma_g$ on S, one can choose G, and g generators $A_1, \ldots, A_g$ of G, so that there is a standard fundamental domain D for G, bounded by curves C_1, $C'_1, \ldots, C_g$, C'_g with $A_i(C_i) = C'_i$, such that γ_i is in the free homotopy class of the image of C_i under $\Omega(G) \rightarrow \Omega(G)/G$. The marked Schottky group $G = \langle A_1, \ldots, A_g \rangle$ is determined by $(S, \gamma_1, \ldots, \gamma_g)$ except for replacing $A_1, \ldots, A_g$ by $BA_1^{n_1}B^{-1}, \ldots, BA_g^{n_g}B^{-1}$, where B is a fractional linear transformation and $n_i \in \{-1, 1\}$.*

Remark 1. This theorem was first stated by Felix Klein in 1883 **[12]** and proved rigorously by Koebe **[13]** much later. Let us remark that an easy proof of this theorem can be given using Bers ideas on quasiconformal mappings **[2]**.

Definition 3. A Schottky uniformization of a closed Riemann surfaces of genus g is a triple $(\Omega, G, \pi:\Omega \to S)$ where, G is a Schottky group (necessarily of genus g) with Ω as its region of discontinuity, and $\pi:\Omega \to S$ is a holomorphic covering, with $\mathrm{Deck}(\pi) = G$.

Definition 4. Let $(\Omega, G, \pi:\Omega \to S)$ be a Schottky uniformization of a Riemann surface S. Let H be a group of conformal automorphisms of S. We say that H *lifts* to this uniformization if each automorphism $h \in H$ lifts to a conformal automorphism of Ω under the covering $\pi:\Omega \to S$, that is, for each $h \in H$ there exists t a conformal automorphism of Ω satisfying $\pi \circ t = h \circ \pi$. Observe that such a t is necessarily a Möbius transformation [**1**].

Definition 5. For $p \in S$, the stabilizer of p with respect to H is the group

$$H(p) = \{h \in H : h(p) = p\}.$$

For the next definition, we need a classical result. Let $h \in H$ and $p \in S$ be as before such that $h(p) = p$. We can find a local coordinate system (U, ϕ) such that $\phi(p) = 0$ and $\phi \circ h \circ \phi^{-1}(z) = e^{i\alpha}z$, for all $z \in \phi(U)$. Moreover, we can assume $\phi(U) = \Delta$, where Δ denotes the unit disc in the complex plane $\mathbb{C}$. The angle $\alpha = \alpha(h, p)$ is well defined up to a multiple of 2π, independent of the local coordinate and $\alpha(h^k, p) = k\alpha(h, p)$.

Definition 6 (The rotation number). Let $h \in H$ and $p \in S$ be such that $h(p) = p$. We normalize α by assuming that $-\pi < \alpha \leq \pi$. We will call $\alpha = \alpha(h, p)$ the *rotation number* of h at p.

Assume the finite group H lifts to some Schottky uniformization $(\Omega, G, \pi:\Omega \to S)$ of S. Let K be the group obtained by the lifting of H. This group contains the Schottky group G as a normal subgroup of finite index. In particular, the region of discontinuity of K is also Ω. It is easy to see that K is a finitely generated, geometrically finite, function group without parabolic elements.

Maskit's classification of finitely generated function groups [**17**] asserts that K is constructed by use of the Klein–Maskit Combination theorems from the following basic function groups:

(i) Finite groups.
(ii) Euclidean groups.
(iii) Finite extensions of cyclic loxodromic groups.
(iv) Quasi-Fuchsian groups.
(v) Degenerated groups.

The above properties of K imply that we cannot use the groups of type (ii), (iv) or (v). So the group K is constructed from groups of type (i) and (iii).

An easy consequence of the above is the following.

Proposition 1. *Let h be any elliptic element of K with fixed points p and q. Then* (1) *Either p and q are in Ω or there is an element of G commuting with h.* (2) *If both fixed points of h are in Ω and they are equivalent under K, then there is an involution j in K permuting them.*

Proof. If K is torsion free, then there is nothing to check. Let us assume K has torsion. Let h be any elliptic element of K with x and y as its fixed points. If both points are regular points we are done. Let us assume y is a limit point. Let j be a primitive elliptic element in K fixing y.

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CLAIM (i) *x is also a fixed point of j.*

(ii) *If $g(y)=y$, some g in K, then either g is conjugate in K to a power of j or g is a loxodromic element with x and y as fixed points.*

Proof. (Claim) (i) If $j(x)\neq x$, then the commutator $[j,h]=jhj^{-1}h^{-1}$ will be a parabolic element in K with y as fixed point. Since K cannot have parabolic elements we must have $j(x)=x$ as claimed.

(ii) Let g in K be such that $g(y)=y$. The only possibility for g is to be elliptic or loxodromic. By our assumption on y, we obtain that necessarily $g(x)=x$; otherwise $[g,j]$ will be a parabolic element of K fixing the point y. At this point, g is either a power of j, or a loxodromic element with x and y as fixed points. This ends the proof of our claim.

Now we continue with the proof of Proposition 1. Let L be the geodesic in $\mathbb{H}^3$ with x and y as end points. The transformation j acts as the identity on L.

Let P be a convex fundamental polyhedron for G. Since y is a limit point, which is not a parabolic fixed point, it must be a point of approximation for G (see page 128 in [**14**]). This implies that y cannot be in the closure of P (see page 122 in [**14**]).

By the above observation, we can find a sequence of points $y_n\in L$, converging to y, all of them non-equivalent points under K, and a sequence $g_n\in K$, all of them different, such that $g_n(y_n)=z_n\in\bar{P}$, where $\bar{P}$ denotes the Euclidean closure of P.

Let us consider a subsequence such that z_n converges, say to z, $g_n(y)$ converges, say to u, and $g_n(x)$ converges, say to t. In particular, the point u and t are limit points for the group G.

Since $z_n\in\bar{P}$, we have $z\in\bar{P}$. We have two possibilities for z, that is, z is as regular point, or z is a parabolic fixed point (see page 128 in [**14**]). Since K does not have parabolic elements, z is a regular point.

It is clear that the z_n are elliptic fixed points, in fact $z_n=g_njg_n^{-1}(z_n)$. This implies that z_n must be in some edge of P. Since P has only a finite number of edges, we can assume all z_n lie on the same edge of P. Let M be the geodesic in $\mathbb{H}^3$ containing this edge. In particular, z must belong to the closure of M.

Let us consider the geodesics $L_n=g_n(L)$ through z_n, and having end points $g_n(x)$ and $g_n(y)$. Since we have supposed $g_n(x)$ and $g_n(y)$ to converge to t and u, respectively, then L_n converges either to a point or to the geodesic with end points u and t. If L_n converges to a point we necessarily have $u=t=z$, a contradiction to the fact that z is regular point and u is a limit point. The other possibility is that L_n converges to a geodesic γ, with end points u and t. In this case, since the end points of γ are limit points and z is known to be a regular point we must have z in $\gamma\cap\mathbb{H}^3$.

Any neighbourhood of z contains z_n, for n sufficiently large. Since z is a regular point, there exists a neighborhood of z which is precisely invariant by the elements of G that fixes z, which is known to be finite. We can then assume without lost of generality that $g_njg_n^{-1}(z)=z$, and $g_njg_n^{-1}=h$. In other words, $(g_m^{-1}g_n)j(g_m^{-1}g_n)^{-1}=j$. Since $g_njg_n^{-1}(z_n)=z_n$, $g_njg_n^{-1}(z)=z$, and $z_n\neq z$ for all n, we have $g_njg_n^{-1}(w)=w$, for all w in γ. In particular, $g_njg_n^{-1}(t)=t$ and $g_ntg_n^{-1}(u)=u$. It follows that $\{g_n(x),g_n(y)\}=\{t,u\}$. The facts that $t\neq u$ and $g_n(x)$ converges to t imply that $g_n(x)=t$ and $g_n(y)=u$, for n sufficiently large. We may assume it holds for every n. The last observation implies that $g_m^{-1}g_n(x)=x$ and $g_m^{-1}g_n(y)=y$, for all n, m.

The transformations $g_m^{-1}g_n$ also keep L invariant, and for $n \neq m$ this transformation cannot be the identity on L. This implies that $g_m^{-1}g_n$ is a loxodromic element of K with x and y as fixed points.

Proposition 1 implies some conditions on the set of points fixed by some non-trivial element of H that we describe below.

THE CONDITION (A). The fixed points of the non-trivial elements of H can be paired in the following way.
(A1) If (p,q) is a pair, then $p \neq q$, $H(p)=H(q)$, and $\alpha(h,p)=-\alpha(h,q)$, for all $h \in H(p)$ of order greater than two.
(A2) If (p,q) and (r,s) are two different pairs, then $\{p,q\} \cap \{r,t\} = \phi$.
(A3) If (p,q) is a pair and h is in H, then $(h(p),h(q))$ is also a pair.
(A4) If (p,q) is a pair with p and q equivalent under H, then there is an involution J in H permuting them.

REMARK 2. (1) Observe that if we have a pairing satisfying conditions (A1), (A2) and (A4), then it is easy to get another pairing satisfying (A1)–(A4). (2) If the group H is Abelian and (p,q) is a pair as above with p and q equivalent under H, then $H(p)=H(q)$ is just the cyclic group in two elements.

The following shows an example of a closed Riemann surface of genus three, non-hyperelliptic, with an automorphism h of order three with five fixed points. In particular, the cyclic group of order five generated by h does not satisfy the condition (A). Let us consider the quartic complex curve in the complex projective plane $\mathbb{C}P^2$ given by the zero locus of

$$aX^4+bY^4+cXY^3+dX^2Y^2+eX^3Y+fZ^3X+gZ^3Y=0.$$

for suitable complex numbers a, b, c, d, e, f and g (e.g. $a=b=f=1$, $c=d=e=g=0$), this quartic is non-singular and irreducible. Such a quartic is a non-hyperelliptic closed Riemann surface of genus three, admitting the automorphism of order three h induced by the linear transformation

$$h=\begin{pmatrix}1&0&0\\0&1&0\\0&0&\omega\end{pmatrix},$$

where $\omega^2+\omega+1=0$. It is easy to check that this automorphism has in fact only five fixed points on the above Riemann surface. Now it is clear that the cyclic group generated by h cannot sastisfy the condition (A).

The following is the main result of this paper.

MAIN THEOREM. *Let S be a closed Riemann surface and let H be an Abelian group of conformal automorphisms of S. Then condition (A) is necessary and sufficient to find a Schottky uniformization $(\Omega, G, \pi:\Omega \to S)$ of S for which H lifts.*

Let us recall a couple basic results of covering spaces. The proof of the two propositions below are quite simple and they are left to the reader as an exercise.

PROPOSITION 2. *Let S be a closed Riemann surface of genus g and let H be a group of conformal automorphisms of S. If there exists a set of g homologically independent disjoint* *simple loops on S, whose normalizer in $\Pi_1(S)$ is invariant under the action of H, then every element of H lifts to a conformal automorphism of the region of discontinuity of the Schottky group (Schottky uniformization) defined, up to conjugation, by these g loops.*

Proposition 3. *Let S be a closed Riemann surface of genus g and let $\{\alpha_k : k = 1, \ldots, t\}$ be a family of disjoint simple loops on S. Let us assume that $S - \bigcup_{k=1}^{t} \alpha_k$ is a disjoint union of surfaces of genus zero with boundaries. Then there exist g loops in this family, say $\alpha_1, \ldots, \alpha_g$, such that $S - \bigcup_{k=1}^{g} \alpha_k$ is a sphere with $2g$ deleted discs. In particular, they are homologically independent and the normalizer in $\Pi_1(S)$ of these g loops is the same as for all the family.*

To show our main theorem, we construct a set of loops satisfying the hypotheses of the above two propositions. It is done in the next section.

3. Proof of the Main Theorem. Let S be a closed Riemann surface and let H be a finite Abelian group of conformal automorphisms of S satisfying the condition (A). Assume H has order N and let us pair the fixed points of the non-trivial elements of H satisfying (A1)–(A4) of condition (A).

Denote by S/H the quotient Riemann surface and by $\pi : S \to S/H$ the natural holomorphic branched covering induced by the action of H on S. The genus of S and S/H will be denoted by g and γ, respectively. The subset of S/H consisting of the branch values of π will be denoted by $\mathscr{B}$. The branch values of π are exactly the projection to S/H of the fixed points of the non-trivial elements of H. In case that $\mathscr{B}$ is not empty, we can write

$$\mathscr{B} = \{P_1, Q_1, P_2, Q_2, \ldots, P_t, Q_t, Z_1, \ldots, Z_m\},$$

where for each $j \in \{1, \ldots, t\}$, there exists a pair as defined above, say (p_j, q_j), such that $\pi(p_j) = P_j$ and $\pi(q_j) = Q_j$, and for each $k \in \{1, \ldots, m\}$, there exists a pair (r_k, s_k) such that $\pi(r_k) = \pi(s_k) = Z_k$.

Condition (A) and the fact that H is abelian imply that the stabilizer of each point over Z_k is the same group in two elements $\mathbb{Z}/2\mathbb{Z}$. In particular, for each pair (r, s) such that $\pi(r) = \pi(s)$ we have that the involution fixing r and s and the involution that permutes them generate the Klein group $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Clearly, if the liftings of Z_k and Z_l have the same involution as stabilizer, then we can change our pairing still satisfying the condition (A) in the following way. Let (r, s) and (t, u) be two pairs such such that $\pi(r) = \pi(s) = Z_k$ and $\pi(t) = \pi(u) = Z_l$. Pair r with t and s with u. Now use the action of H to pair $h(r)$ with $h(t)$ and to pair $h(s)$ with $h(u)$, for all h in H. Now we can assume that the liftings of different Z_k have different involutions as stabilizer. From now on, we assume this is the case.

For each $j \in \{1, \ldots, t\}$ and each $k \in \{1, \ldots, m\}$, let us write

(1) $\pi^{-1}(P_j) = \{p_j^{i_j}; i_j = 1, \ldots, l_j\}$;
(2) $\pi^{-1}(Q_j) = \{q_j^{i_j}; i_j = 1, \ldots, l_j\}$; and
(3) $\pi^{-1}(Z_k) = \{r_k^{v_k}, s_k^{v_k}, v_k = 1, \ldots, N/4\}$,

where l_k divides N, and $(p_j^{i_j}, q_j^{i_j})$ is a pair. Denote by $n_j = N/l_j$.

For each $j \in \{1, \ldots, t\}$ and each $i \in \{1, \ldots, l_j\}$, let $F_j^i \in H$ be such that $0 < \alpha(F_j^i, p_j^i) \leq \|\alpha(h, p_j^i)\|$, over all $h \in H(p_j^i) - \{I\}$. Clearly, $\alpha(F_j^i, p_j^i) = 2\pi/n_j$.

For each $k \in \{1, \ldots, m\}$, denote by h_k the involution which stabilizes one and therefore every point in $\pi^{-1}(Z_k)$.

The Riemann–Hurwitz formula [4] gives us the following relation

$$g = N(\gamma + t + m/4 - 1) + 1 - \sum_{j=1}^{t} l_j.$$

We proceed to construct a set of disjoint simple loops on S such that they are invariant as set of loops under the action of H and they decompose the surface S into spheres with holes.

Let us construct a system of oriented simple loops as follows. If $\gamma \geq 1$, let α_i and β_i, $i = 1, \ldots, \gamma$, be a set of oriented simple loops on $S/H - \mathcal{B}$ satisfying the following:

(1) $\alpha_i \cap \mathcal{B} = \phi$, for all $i = 1, \ldots, \gamma$;
(2) $\beta_i \cap \mathcal{B} = \phi$, for all $i = 1, \ldots, \gamma$;
(3) $\alpha_i \cap \alpha_j = \phi$, for all $i \neq j$;
(4) $\alpha_i \cap \beta_j = \phi$, for all $i \neq j$;
(5) $\beta_i \cap \beta_j = \phi$, for all $i \neq j$;
(6) $\alpha_i \cap \beta_i$ consists of exactly one point, for all $i = 1, \ldots, \gamma$; and
(7) $\alpha_i \,.\, \beta_i = +1$, for all $i = 1, \ldots, \gamma$, where . denotes the intersection number (see Figure 1).

Let us consider a set of simple loops (oriented in the natural way) η_i, $i = 1, \ldots, \gamma$, such that

(8) η_i is homotopic to the commutator $\alpha_i \beta_i \alpha_i^{-1} \beta_i^{-1}$ on $S/H - \mathcal{B}$; and
(9) $\eta_i \cap \mathcal{B} = \phi$ (see Figure 1).

Let us denote by T_k the torus with boundary η_i obtained by cutting S/H along the η_k loops, for $k = 1, \ldots, \gamma$ (see Figure 1).

If $t \geq 1$, we consider a set of disjoint simple loops, δ_j, $j = 1, \ldots, t$, on $S/H - \mathcal{B}$ satisfying the following properties.

(10) If $\gamma \geq 1$, then δ_j is disjoint from the loops α_i $(i = 1, \ldots, \gamma)$ and η_k $(k = 1, \ldots, \gamma)$ constructed above, for all $j = 1, \ldots, t$;
(11) δ_j bounds a conformal disc containing the points P_j and Q_j and no other branch value of π, for all $j = 1, \ldots, t$.

Let us denote by Δ_j the conformal disc bounded by the loop δ_j, for $j = 1, \ldots, t$ (see Figure 1).

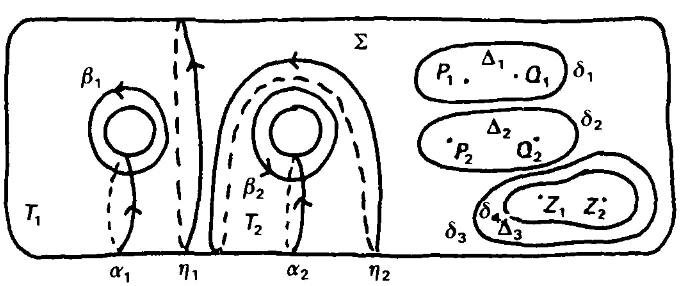

Figure 1: $\gamma = 2$, $t = 2$ and $m = 3$.

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If $m \geq 1$, let δ_{t+1} be a simple loop disjoint from the above ones and bounding a disc Δ_{t+1} containing in its interior the branch values Z_k, for all k (see Figure 1).

If m is odd, we construct a simple loop δ_{t+2} inside the disc Δ_{t+1}, such that this loop bound a disc containing in its interior the branch values Z_k, $k \in \{1, \ldots, m-1\}$. Denote by σ surface bounded by the loops δ_{t+1} and δ_{t+2} (see Figure 1).

In the disc bounded by the loop δ_{t+2} if m is odd, or by the loop δ_{t+1} otherwise, we construct a set of disjoint simple loops ϕ_r, $r = 1, \ldots, T$, satisfying the following properties;

(I) If m is even, say $m = 2M$;
there are M simple loops, say ϕ_k for $k = 1, \ldots, M$, each one bounding a disc Φ_k containing exactly two branch values.

(II) If m, is odd, say $m = 2M + 1$;
there are M simple loops, say ϕ_k for $k = 1, \ldots, M$, each one bounding a disc Φ_k containing exactly two branch values.

(III) The rest of the loops are contained in the common region bounded by the above M loops and the loop δ_{t+1} or δ_{t+2}, and these loops dissect the above region into pants,

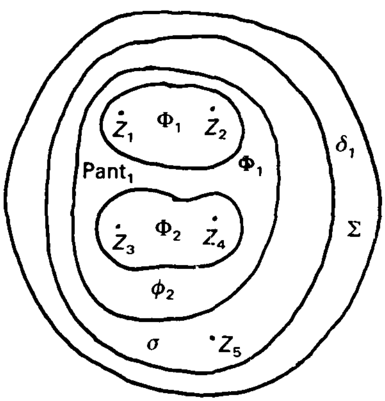

Figure 2: $\gamma = 0$, $t = 0$ and $m = 5$.

denoted by Pant$_s$, for $s = 1, \ldots, K$ (see Figure 2).

We rename the branch values z_k in such a way that the disc Φ_k, bounded by ϕ_k, contains Z_{2k-1} and Z_{2k}. Clearly the group generated by all the involutions h_k is isomorphic to a group of the form $\mathbb{Z}/2\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/2\mathbb{Z} = \bigoplus_{i=1}^{l} \mathbb{Z}/2\mathbb{Z}$, for some l.

Now, the loops α_i, η_i, δ_j and ϕ_r dissect the surface S/H into spheres with holes. The next step is to show that the liftings under π of these loops define a set of disjoint simple loops on S satisfying the desired conditions.

Let us call by Σ the (connected) surface with boundary the loops η_i and δ_j, for $i = 1, \ldots, \gamma$ and $j = 1, \ldots, t+1$. This surface is sphere with $t + 1 + \gamma$ deleted discs.

LEMMA 1. *Each loop η_i and δ_j lifts to a loop on S, for $i = 1, \ldots, \gamma$ and for $j = 1, \ldots, t+1$.*

Proof. For fixed $i \in \{1, \ldots, \gamma\}$, the loop η_i lifts to a loop if and only if the loop

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$\alpha_i\beta_i\alpha_i^{-1}\beta_i^{-1}$ lifts to a loop. Denote by x the intersection of the loops α_i and β_i. If z is any lifting of x, let us lift the loops α_i and β_i at z. The end point of the lifting of α_i is $f(z)$ and the end point of the lifting of β_i is $h(z)$, for some f and h in H. The end point of the lifting of $\alpha_i\beta_i\alpha_i^{-1}\beta_i^{-1}$ at z is $h^{-1}f^{-1}hf(z)$. Since the group H is abelian, this end point is again z, and we are done in this case. For fixed $j \in \{1, \ldots, t\}$, let y be any point in δ_j. Let us consider two disjoint (except at y) simple loops based at y, say d_1 and d_2, such that d_1 bounds a disc containing the point P_j and d_2 bounds a disc containing the point Q_j. Orient these loops in such a way that going in the positive orientation the branch point they bound is at the left side. If we show that the loop d_1d_2 lifts to a loop we will be done. Let ω be a lifting of y and let $f(\omega)$ and $h(\omega)$ the end points of the liftings of d_1 and d_2 at ω respectively. Since P_j and Q_j are projections of a pair, then necessarily $f = h^{-1}$. Now the end point of the lifting of d_1d_2 at ω is $fh(\omega) = ff^{-1}(\omega) = \omega$. Since the loop δ_{t+1} is homotopic to the product of the η_i and δ_j loops, for $i = 1, \ldots, \gamma$ and $j = 1, \ldots, t$, then it must lift to a loop on S.

If m is odd, then clearly the loop δ_{t+2} cannot lift to a loop on S. But δ_{t+2}^2 lifts to a loop on S. This is a consequence of the fact that δ_{t+2} is homotopic to the product of two small simple loops around Z_m and δ_{t+1}, respectively. Each loop ϕ_r either lifts to a loop or its square lifts to a loop. The last is a consequence of the fact that each loop ϕ_r is homotopic to the product of small simple loops around the points Z_k contained in the disc bounded by it.

Definition 7. A *circle domain* is a region on a closed Riemann surface obtained as a component of cutting the closed surface along a finite number of pairwise disjoint rounded simple loops (in the natural hyperbolic, spherical or euclidean structure of the surface). We will use the following theorem due to B. Maskit.

Theorem 5 **[16]**. *Let S be a topologically finite Riemann surface of genus g. Then there exists a closed Riemann surface $\bar{S}$ of genus g, and there is a conformal embedding $f: S \to \bar{S}$ so that $f(S)$ is a circle domain on $\bar{S}$. This representation is unique; that is, if there is another closed Riemann surface $\bar{S}'$, also of genus g, and there is a conformal embedding $f': S \to \bar{S}'$, also of genus g, so that $f'(S)$ is also a circle domain, then there is a conformal homeomorphism $h: \bar{S} \to \bar{S}'$ with $f' = hf$. Moreover, if H is the group of conformal automorphisms of S, then fHf^{-1} can be extended to a group of conformal automorphisms of $\bar{S}$.*

Now we proceed to describe each component of the lifting via π of each surface T_i, Δ_j, Σ, σ, Φ_k and Pant_s.

Lemma 2. $\pi^{-1}(\Sigma) = \coprod_{l=1}^{N} \Sigma_l$, *where each Σ_l is homeomorphic under π to Σ.*

Proof. Let Σ_l be a component of $\pi^{-1}(\Sigma)$ and let us consider the restriction of π to these surfaces

$$\pi: \Sigma_l \to \Sigma.$$

This is a holomorphic unbranched regular covering, with covering group H_l. Let us denote by L_l the order of this group, so the above covering has degree L_l. Since each loop

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η_i and δ_j lifts to a loop on S, we have that Σ_l is a surface of genus g_l with $L_l(t+\gamma+1)$ deleted discs. We can holomorphically embedd the surface Σ in the Riemann sphere, $\hat{\mathbb{C}}$, as the complement of $(t+\gamma+1)$ deleted holomorphic discs, and we can holomorphically embedd the surface Σ_l in a closed Riemann surface, $\tilde{\Sigma}_l$, of genus g_l as the complement of $L_l(t+\gamma+1)$ deleted holomorphic discs. Now, we can extend the covering map π to a regular unbranched covering

$$\pi:\tilde{\Sigma}_l\to\hat{\mathbb{C}}.$$

Let us apply the Riemann–Hurwitz formula to this covering. In this case, we obtain the following equality:

$$g_l=1-L_l,$$

from where we obtain that $L_l=1$ and $g_l=0$.

LEMMA 3. *For each $j=1,\ldots,t$, $\pi^{-1}(\Delta_j)=\coprod_{k=1}^{l_j}\Delta_{j,k}$, where each $\Delta_{j,k}$ is a surface of genus zero with n_j deleted discs, precisely invariant under the cyclic group generated by F_j^k in H.*

Proof. Let $\Delta_{j,k}$ be a component of $\pi^{-1}(\Delta_j)$ and let us consider the restriction of π to these surfaces

$$\pi:\Delta_{j,k}\to\Delta_j.$$

This is a holomorphic regular covering branched at P_j and Q_j, with covering group H_j and with the property that the loop δ_j lifts to a loop. Let us denote by L_j the order of the group H_j, so the above covering has degree L_j. We have that $\Delta_{j,k}$ is a surface of genus g_j with L_j deleted discs. We can holomorphically embedd the surface Δ_j in the Riemann sphere, $\hat{\mathbb{C}}$, as the complement of a holomorphic disc, and we can holomorphically embedd the surface $\Delta_{j,k}$ in a closed Riemann surface, $\tilde{\Delta_{j,k}}$, of genus g_j as the complement of L_j holomorphic discs. Now, we can extend the covering map π to a regular branched covering

$$\pi:\tilde{\Delta_{j,k}}\to\hat{\mathbb{C}},$$

with the same branching as before, that is, we do not add extra branch points to this covering. Let us apply the Riemann–Hurwitz formula to this covering. In this case, we obtain the following equality:

$$g_l=1-L_j/n_j,$$

from where we obtain that $L_j=n_j$ and $g_j=0$. Now it is also clear that H_j must be the cyclic group generated by F_j^k.

LEMMA 4. *$\pi^{-1}(T_i)=\coprod_{r=1}^{R_i}T_{i,r}$, where R_i divides N and $T_{i,r}$ is a torus with N/R_i deleted discs. Moreover, $T_{i,r}-(\pi^{-1}(\alpha_i)\cap T_{i,r})$ is the disjoint union of N_i spheres with (N/N_iR_i+2) deleted discs, where N_iR_i divides N.*

Proof. Let $T_{i,r}$ be a component of $\pi^{-1}(T_i)$ and let us consider the restriction of π to these surfaces

$$\pi:T_{i,r}\to T_i.$$

This is a holomorphic unbranched regular covering with covering group H_i and with the property that the loop η_i lifts to a loop. Let us denote by L_i the order of the group H_i, so the above covering has degree L_i. We have that $T_{i,r}$ is a surface of genus g_i with L_i deleted discs. We can holomorphically embedd the surface T_i in a torus, $\mathcal{T}_\rho$, as the complement of a holomorphic disc, and we can holomorphically embedd the surface $T_{i,r}$ in a closed Riemann surface, $\tilde{T}_{i,r}$, of genus g_i as the complement of L_i holomorphic discs. Now, we can extend the covering map π to a regular unbranched covering

$$\pi : \tilde{T}_{i,r} \to \mathcal{T}_\rho.$$

Let us apply the Riemann–Hurwitz formula to this covering. In this case, we obtain the following equality:

$$g_i = 1,$$

that is, $\tilde{T}_{i,r}$ is a torus. The group H_i is a finite subgroup of conformal automorphisms of $T_{i,r}$ acting without fixed points. This group is known to be of the form $\mathbb{Z}/N_i\mathbb{Z} \oplus \mathbb{Z}/(N/N_iR_i)\mathbb{Z}$, for some N_i and R_i, such that N_iR_i divides N. The lemma follows as consequence of the above.

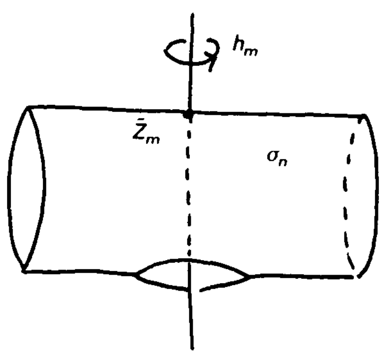

Figure 3.

Lemma 5. $\pi^{-1}(\sigma) = \coprod_{r=1}^{N/2} \sigma_r$, *where* σ_r *is a sphere with* 3 *deleted discs and stabilizer in* H *the group in two elements generated by the involution* h_m *that fixes any lifting of* Z_m. (See Figure 3.)

Proof. Let σ_r be any component of $\pi^{-1}(\sigma)$. Let us restrict our covering π to these parts, that is

$$\pi : \sigma_r \to \sigma.$$

This is a regular covering of degree d with branch value Z_m of order two. The loop δ_{t+1} lifts to a loop and the loop δ_{t+2} lifts to a path, but its square lifts to a loop on σ_r. The surface σ_r is a surface of genus g_r and with $(d + d/2)$ holes. By Maskit's result, we may assume the surface σ to be the Riemann sphere minus a holomorphic disc and the surface

σ_r to be the complement of $(d+d/2)$ disjoint holomorphic discs on a closed Riemann surface $\tilde{\sigma}_r$ of genus g_r. We can extend our regular branch covering to a branch covering

$$\pi:\tilde{\sigma}_r \to \hat{\mathbb{C}},$$

with the same covering group and with two branch values of order two, one of them Z_m and the other in the complement disc of σ on the Riemann sphere $\hat{\mathbb{C}}$. Now, let us apply the Riemann–Hurwitz formula to this covering to obtain the equality

$$g_r = 1 - d/2.$$

As consequence, $g_r = 0$ and $d = 2$. Since the involution h_m must be the covering group of this covering, we must have that this group is generated by the involution h_m.

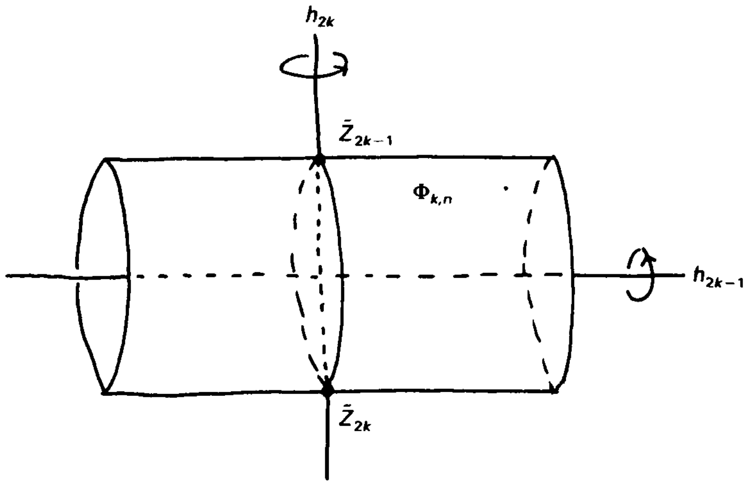

Figure 4.

LEMMA 6. *For each* $k = 1, \ldots, M$, $\pi^{-1}(\Phi_k) = \coprod_{r=1}^{N/4} \Phi_{k,r}$, *where* $\Phi_{k,r}$ *is a sphere with* 2 *holes an invariant under the Klein group generated by* h_{2k-1} *and* h_{2k}. (See Figure 4.)

Proof. Let $\Phi_{k,r}$ be any component of $\pi^{-1}(\Phi_k)$. Let us restrict our covering π to these parts, that is

$$\pi:\Phi_{k,r} \to \Phi_k.$$

This is a regular covering of degree d with branch values Z_{2k-1} and Z_{2k} of order two. The loop ϕ_k lifts to a path, but its square lifts to a loop on $\Phi_{k,r}$. The surface $\Phi_{k,r}$ is a surface of genus g_k and with $d/2$ holes. By Maskit's result, we may assume the surface Φ_k to be the Riemann sphere minus a holomorphic disc and the surface $\Phi_{k,r}$ to be the complement of $d/2$ disjoint holomorphic discs on a closed Riemann surface $\Phi_{k,r}^{\sim}$ of genus g_k. We can extend our regular branch covering to a branch covering

$$\pi:\Phi_{k,r}^{\sim} \to \hat{\mathbb{C}},$$

with the same covering group and with three branch values of order two, two of them Z_{2k-1}, Z_{2k} and the other in the complementary disc of Φ_k on the Riemann sphere $\hat{\mathbb{C}}$. Now, let us apply the Riemann–Hurwitz formula to this covering to obtain the equality

$$g_k = 1 - d/4.$$

As a consequence, $g_k = 0$ and $d = 4$. Since the involutions h_{2k-1} and h_{2k} belong to the covering group and they generate a Klein group, then we must have that this group is exactly all the covering group.

For each r, denote by P the pant Pant_r and by γ_1, γ_2 and γ_3 the boundary loops of $P = \mathrm{Pant}_r$. Since each of the loops γ_i either lifts to a loop or its square lifts to a loop and the loop γ_3 is homotopic to the product of γ_1 and γ_2, we have three possibilities for the liftings of P.

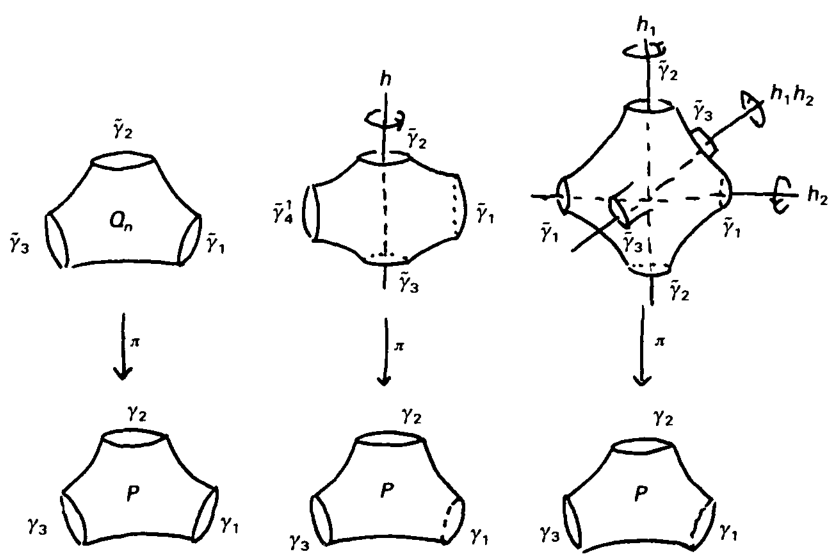

Figure 5.

Lemma 7. (1) *If* γ_1 *and* γ_2 *lift to loops, then* $\pi^{-1}(P) = \coprod_{r=1}^{N} Q_r$, *where* Q_r *is biholomorphic to P under* π. *In particular*, Q_r *is a pant.*

(2) *If* γ_1 *lifts to a loop and* γ_2 *lifts to a path, then* $\pi^{-1}(P) = \coprod_{r=1}^{N/2} Q_r$, *where* Q_r *is a sphere with four holes. The subgroup of H keeping* Q_r *invariant is a group of order* 2.

(3) *If* γ_1 *and* γ_2 *both lift to paths, then either we are in case* (2) *by permuting the loops* γ_1 *and* γ_3 *or* $\pi^{-1}(P) = \coprod_{r=1}^{N/4} Q_r$, *where* Q_r *is a sphere with six holes. In the last case, the subgroup of H keeping invariant* Q_r *is a Klein group.* (See Figure 5.)

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Proof. (1) Let us assume the loops γ_i lift to loops for $i = 1, 2$. Since the loop γ_3 is homotopic to the product of them, then it lifts also to a loop. Let Q_r be a component of $\pi^{-1}(P)$, and consider the regular unbranched covering

$$\pi : Q_r \to P.$$

This is a regular unbranched covering of degree d. The surface Q_r is a surface of genus g with $3d$ deleted discs. Again, using Maskit's result we may assume that P is the complement of three disjoint holomorphic discs in the Riemann sphere and Q_r is the complement of $3d$ disjoint holomorphic discs in a closed Riemann surface $\tilde{Q}_r$ of genus g. We can extend our covering to a regular unbranched one of degree d

$$\pi : \tilde{Q}_r \to \hat{\mathbb{C}}.$$

We apply the Riemann–Hurwitz formula to this covering to obtain the equality

$$g = 1 - d.$$

From this equality we obtain that $g = 0$ and $d = 1$.

(2) Let us assume the loop γ_1 lifts to loop and the loop α_2 lifts to a path. Since the loop γ_3 is homotopic to the product of them, then it lifts also to a path. Let Q_r be a component of $\pi^{-1}(P)$, and consider the regular unbranched covering

$$\pi : Q_r \to P.$$

This is a regular unbranched covering of degree d. The surface Q_r is a surface of genus g with $(d + d/2 + d/2) = 2d$ deleted discs. Using Maskit's result we may assume that P is the complement of three disjoint holomorphic discs in the Riemann sphere and Q_r is the complement of $2d$ disjoint holomorphic discs in a closed Riemann surface $\tilde{Q}_r$ of genus g. We can extend our covering to a regular one of degree d

$$\pi : \tilde{Q}_r \to \hat{\mathbb{C}},$$

With two branch values of order two. We apply the Riemann–Hurwitz formula to this covering to obtain the equality

$$g = 1 - d/2.$$

From this equality we obtain that $g = 0$ and $d = 2$.

(3) Let us assume the loops γ_i lift to paths for $i = 1, 2$. Since the loop γ_3 is homotopic to the product of them, then it lifts either to a loop or to a path. In the first case, we permute the loops γ_1 and γ_3. In this way, we are in the previous case. Let us assume now that the three loops lift to paths. Let f_i be the elements of H which are determined in the following way. Take a point x_i in γ_i and let u_i be a lifting of that point in Q_r. Lift the loop γ_i at u_i and look at its end point. That point has the form $f_i(u_i)$ for a unique f_i in H. Since the group H is abelian, the transformation f_i is well defined. Also, since the loops lift to paths with the property that their square lift to loops, these elements are non-trivial and of order two. Since γ_3 is homotopic to the product of γ_1 and γ_2, then $f_3 = f_1 f_2$. In particular, the group generated by these elements is the Klein group. Let Q_r be a component of $\pi^{-1}(P)$, then we consider the regular unbranched covering

$$\pi : Q_r \to P.$$

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This is a regular unbranched covering of degree d. The surface Q_r is a surface of genus g with $3d/2$ deleted discs. Again, using Maskit's result we may assume that P is the complement of three disjoint holomorphic discs in the Riemann sphere and Q_r is the complement of $3d/2$ disjoint holomorphic discs in a closed Riemann surface $\tilde{Q}_r$ of genus g. We can extend our covering to a regular one of degree d with three branch values of order two

$$\pi : \tilde{Q}_r \to \hat{\mathbb{C}}.$$

We apply the Riemann–Hurwitz formula to this covering to obtain the equality

$$g = 1 - d/4.$$

From this equality we obtain that $g = 0$ and $d = 4$. Since the Klein group has order 4, then the covering group is exactly the one generated by the involutions f_i, for $i = 1, 2$.

As consequence of the above lemmas, we obtain that the liftings of the loops α_i, η_i, δ_j and the loops ϕ_r are a family of disjoint simple loops on S which dissect S into spheres with holes. Now, we can proceed to finish the proof of the Main Theorem as consequence of Propositions 2 and 3 in Section 2. Denote by $\mathcal{M}$ the normalizer in $\Pi_1(S)$ of the liftings of the loops η_i, α_i^{N/N_iR_i}, δ_j and ϕ_r, for $i = 1, \ldots, \gamma$, $j = 1, \ldots, t$ and $r = 1, \ldots, T$. Since S minus the liftings of the above loops is a disjoint union of spheres with deleted discs, then there exists a sub-family $\mathcal{N}$ of the above liftings consisting of g loops, such that they dissect S into a sphere with $2g$ deleted discs. It is clear that the normalizer of $\mathcal{N}$ is also $\mathcal{M}$. In particular, $\mathcal{M}$ ($\mathcal{N}$) defines a Schottky group G, up to isomorphisms, and in particular, a Schottky uniformization $(\Omega(G), G, \pi : \Omega(G) \to S)$, up to equivalence, of the surface S. Since the set of loops obtained, as a consequence of the above results, is invariant under the action of H, its normalizer $\mathcal{M}$ is also invariant under H; in particular, the Schottky uniformization defined by the family $\mathcal{M}$ loops is the desired one. This ends the proof of our theorem.

We must remark that this proof is not so informative as the proof given for the cyclic, $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and dihedral cases done in [**8**, **9**, **10**].

REFERENCES

1. L. V. Ahlfors and L. Sario, *Riemann surfaces* (Princeton Univ. Press, 1960).

2. L. Bers, Automorphic forms for Schottky groups, *Adv. in Math.* **16** (1975), 332–361.

3. V. Chuckrow, On Schottky groups with applications to Kleinian groups. *Ann. of Math.* **88** (1968), 47–61.

4. H. Farkas and I. Kra, *Riemann surfaces* (Springer-Verlag, 1980).

5. R. Hidalgo, On γ-hyperelliptic Riemann surfaces, *Preprintreihe Sonderforschungsbereich* 343 "*Discrete Structuren in der Mathematik*" *91-045.*

6. R. Hidalgo, On γ-hyperelliptic Schottky groups, *Notas Soc. Mat. de Chile* **8** (1989), 27–36.

7. R. Hidalgo, *On Schottky groups with automorphisms* (Thesis Ph.D. in Mathematics, S.U.N.Y. at Stony Brook, 1991).

8. R. Hidalgo, On Schottky groups with automorphisms, *Preprintreihe Sonderforschungsbereich 343* "*Discrete Structuren in der Mathematik*", *91-088.*

9. R. Hidalgo, Schottky uniformizations of closed Riemann surfaces with the Klein groups as conformal group of symmetries, *Preprintreihe Sonderforschungsbereich 343* "*Discrete Structuren in der Mathematik*", *91-087.*

10. R. Hidalgo, Closed Riemann surfaces with Dihedral groups of Conformal Automorphisms, *Preprintreihe Sonderforschungsbereich 343 "Discrete Structuren in der Mathematik", 91-086.*

11. L. Keen, On hyperellptic Schottky groups, *Ann. Acad. Sci. Fenn., series A.I. Mathematica* **5** (1980), 165–174.

12. F. Klein, Neue Beiträge zur Riemann'schen Funktionentheorie, *Math. Ann.* **21** (1883), 141–218.

13. P. Koebe, Über die Uniformisierung der Algebraischen Kurven II, *Math. Ann.* **69** (1910), 1–81.

14. B. Maskit, *Kleinian groups* (Grundlehren der Mathematischen Wissenschaften, vol. 287, Springer-Verlag, 1988).

15. B. Maskit, A characterization of Schottky groups, *J. d'Analyse Math.* **19** (1967), 227–230.

16. B. Maskit, Canonical domains on Riemann surfaces, *Proc. of the Amer. Math. Soc.* **106** (1989), 713–721.

17. B. Maskit, On the classification of Kleinian groups I and II, *Acta Math.* **135** (1975), 249–270 and **138** (1977), 17–42.

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