Estimates of Henstock-Kurzweil Poisson Integrals

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Abstract. If f is a real-valued function on $[-\pi, \pi]$ that is Henstock-Kurzweil integrable, let $u_r(\theta)$ be its Poisson integral. It is shown that $\|u_r\|_p = o(1/(1-r))$ as $r \to 1$ and this estimate is sharp for $1 \leq p \leq \infty$. If μ is a finite Borel measure and $u_r(\theta)$ is its Poisson integral then for each $1 \leq p \leq \infty$ the estimate $\|u_r\|_p = O((1-r)^{1/p-1})$ as $r \to 1$ is sharp. The Alexiewicz norm estimates $\|u_r\| \leq \|f\|$ ($0 \leq r < 1$) and $\|u_r - f\| \to 0$ ($r \to 1$) hold. These estimates lead to two uniqueness theorems for the Dirichlet problem in the unit disc with Henstock-Kurzweil integrable boundary data. There are similar growth estimates when u is in the harmonic Hardy space associated with the Alexiewicz norm and when f is of bounded variation.

1 Introduction

In this paper we consider estimates of Poisson integrals on the unit circle with respect to Alexiewicz and L^p norms. Define the open disk in $\mathbb{R}^2$ as $D := \{re^{i\theta} \mid 0 \leq r < 1, -\pi \leq \theta < \pi\}$. The Poisson kernel is

$$\Phi_r(\theta) := (1 - r^2)/[2\pi(1 - 2r\cos\theta + r^2)] = [1 + 2\sum_{n=1}^{\infty} r^n \cos(n\theta)]/(2\pi).$$

Let $f\colon \mathbb{R} \to \mathbb{R}$ be 2π-periodic. The Poisson integral of f is its convolution with the Poisson kernel

$$P[f](re^{i\theta}) := f * \Phi_r(\theta) = \int_{-\pi}^{\pi} f(\phi)\Phi_r(\phi - \theta)\, d\phi.$$

Since ∂D has no end points, an appropriate form of the Alexiewicz norm of f is $\|f\| := \sup_{I \subset \mathbb{R}} |\int_I f|$ where I is an interval in $\mathbb{R}$ of length not exceeding 2π. Let $\mathcal{HK}$ denote the 2π-periodic functions $f\colon \mathbb{R} \to \mathbb{R}$ with finite Alexiewicz norm. Of course, with the same periodicity convention, $L^p \subsetneq \mathcal{HK}$ for all $1 \leq p \leq \infty$. Write $\|f\|_A$ for the Alexiewicz norm over set A. The Alexiewicz norm is discussed in [8]. The variation of f over one period is denoted Vf. The set of 2π-periodic functions with finite variation over one period is denoted $\mathcal{BV}$. For a function $u\colon D \to \mathbb{R}$ we write $u_r(\theta) = u(re^{i\theta})$.

The Dirichlet problem, of finding a function harmonic in the disc with prescribed boundary values, is one of the foundational problems in elliptic partial differential equations. An understanding of its solution has been a stepping stone to the study of analytic functions in the complex plane and of the solutions of more general elliptic equations. Owing to the simple geometry of the disc there is an explicit integral representation for solutions through (1). As a Lebesgue integral, the Poisson integral has been studied intensively. For the major results, see [1, 11].

The following results are well known [1]. Suppose that $1 \leq p \leq \infty$ and $f \in L^p$. If $|\theta_0| \leq \pi$ and $z \in D$, we say that $z \to e^{i\theta_0}$ nontangentially if there is $0 \leq \alpha < \pi/2$ such that $z \to e^{i\theta_0}$ within the sector $\{\zeta \in D : |\arg(\zeta - e^{i\theta_0})| < \alpha\}$. Write $u_r(\theta) = P[f](re^{i\theta})$. Then

(1) u_r is harmonic in D.

(2) $\|u_r\|_p \leq \|f\|_p$ for all $0 \leq r < 1$.

(3)   If $1 \leq p < \infty$ then $\|u_r - f\|_p \to 0$ as $r \to 1$.

(4) $u(re^{i\theta}) \to f(\theta_0)$ for almost all θ_0 as $z \to e^{i\theta_0}$ nontangentially in D.

We examine analogues of these results when f is Henstock-Kurzweil integrable (Theorem 6). We also prove that the growth estimate $\|u_r\|_p = o(1/(1-r))$ is sharp for $f \in \mathcal{HK}$ and $1 \leq p \leq \infty$ (Theorem 1). If μ is a finite Borel measure and $u_r(\theta)$ is its Poisson integral then for each $1 \leq p \leq \infty$ the estimate $\|u_r\|_p = O((1-r)^{1/p-1})$ as $r \to 1$ is sharp (Remarks 2). The Poisson integral of a function in $\mathcal{HK}$ need not be the difference of two positive harmonic functions (Remarks 4). There are similar growth estimates when u is in $h^{\mathcal{HK}}$, the harmonic Hardy space associated with the Alexiewicz norm (Theorem 5). The Poisson integral provides an isometry from $\mathcal{HK}$ into (but not onto) $h^{\mathcal{HK}}$ (Theorem 8). In Theorem 9 we consider the above results for functions of bounded variation. Theorem 10 and Theorem 11 establish uniqueness conditions for the Dirichlet problem using the Alexiewicz norm. Example 12 shows the applicability of the uniqueness theorems. All the results also hold when we use the wide Denjoy integral [3].

Since Φ_r and $1/\Phi_r$ are of bounded variation on ∂D, necessary and sufficient for the existence of $P[f]$ in D is that f be integrable, *i.e.,* the Henstock-Kurzweil integral $\int_{-\pi}^{\pi} f$ is finite. In [2], integration by parts was used to show that we can differentiate under the integral sign. This in turn shows that $P[f]$ is harmonic in D and that $P[f] \to f$ nontangentially, almost everywhere on ∂D. In [3, Theorem 4, p. 238], necessary and sufficient conditions were given for determining when a function that is harmonic in D is the Poisson integral of an $\mathcal{HK}$ function. Corresponding results when $\|u_r\|_p$ are uniformly bounded have been known for some time (for example, [1, Theorem 6.13]).

2   Growth Estimates

Our first result is to show that for $1 \leq p \leq \infty$, we have $\|u_r\|_p = o(1/(1-r))$ and this estimate is sharp. That is, $(1-r)\|u_r\|_p \to 0$ as $r \to 1$ ($1 \leq p < \infty$) and

$\sup_{\theta \in [-\pi,\pi]} (1-r)|P[f](re^{i\theta})| \to 0$ as $r \to 1$ $(p = \infty)$. Thus, for $p = \infty$, the manner of approach to the boundary is unrestricted. This same estimate for $p = \infty$ was obtained for L^1 functions in [11]. We show these estimates are the best possible under our minimal existence hypothesis. The proof uses the inequality

$$(5) \qquad \Big| \int_{-\pi}^{\pi} fg \Big| \leq \|f\| \Big(\inf_{[-\pi,\pi]} |g| + Vg \Big),$$

which is valid for all $f \in \mathcal{HK}$ and g of bounded variation on $[-\pi, \pi]$. This was proved in [9, Lemma 24].

Theorem 1   *Let $f \in \mathcal{HK}$. For $re^{i\theta} \in D$ let $u_r(\theta) = P[f](re^{i\theta})$.*

(a) *We have $\sup_{\theta \in [-\pi,\pi]} |P[f](re^{i\theta})| = o(1/(1-r))$ as $r \to 1$, and this estimate is sharp in the sense that if $\psi \colon D \to (0, \infty)$ and $\psi(re^{i\theta}) = o(1/(1-r))$ as $r \to 1$, then there is a function $f \in \mathcal{HK}$ such that $P[f] \neq o(\psi)$ as $r \to 1$.*

(b) *Let $1 \leq p < \infty$. Then $\|u_r\|_p = o(1/(1-r))$ as $r \to 1$, and this estimate is sharp in the sense that if $\psi \colon [0, 1) \to (0, \infty)$ and $\psi(r) = o(1/(1-r))$ as $r \to 1$, then there is a function $f \in \mathcal{HK}$ such that $\|u_r\|_p \neq o(\psi(r))$ as $r \to 1$.*

Proof   (a) Let $\Psi_r(\phi) := (1-r)^2/(1 - 2r\cos\phi + r^2)$ with $\Psi_1(0) := 1$. Let $0 < \delta < \pi$. Then

$$\frac{2\pi(1-r)P[f](re^{i\theta})}{1+r} = \int_{|\phi-\theta|<\delta} f(\phi)\Psi_r(\phi-\theta)\,d\phi + \int_{\delta<|\phi-\theta|<\pi} f(\phi)\Psi_r(\phi-\theta)\,d\phi.$$

Given $\epsilon > 0$, take δ small enough so that $\|f\|_{[\theta-\delta,\theta+\delta]} < \epsilon$ for all θ. Using (5),

$$\Big| \int_{|\phi-\theta|<\delta} f(\phi)\Psi_r(\phi-\theta)\,d\phi \Big| \leq 2\|f\|_{[\theta-\delta,\theta+\delta]}.$$

And

$$\Big| \int_{\theta+\delta}^{\theta-\delta+2\pi} f(\phi)\Psi_r(\phi-\theta)\,d\phi \Big| \leq \|f\| \Big[\frac{2(1-r)^2}{1 - 2r\cos\delta + r^2} - \frac{(1-r)^2}{(1+r)^2} \Big] \to 0 \text{ as } r \to 1.$$

To prove this estimate is sharp, suppose $\psi \colon D \to (0, \infty)$ is given. It suffices to show that $P[f](r_n e^{i\theta_n}) \neq o(\psi(r_n e^{i\theta_n}))$ for some sequence $\{r_n e^{i\theta_n}\} \in D$ with $r_n \to 1$. Take $0 < \theta_n < \pi/2$ and decreasing to 0. Let $a_n = \psi(r_n e^{i\theta_n})$. Take $0 < \alpha_n \leq \min(\pi/2, (\theta_{n-1} - \theta_n)/2, (\theta_n - \theta_{n+1})/2, 1 - r_n)$ with $\theta_0 := \pi$. Then the intervals $(\theta_n - \alpha_n, \theta_n + \alpha_n)$ are disjoint and $\cos(\alpha_n) \geq 1 - \alpha_n^2/2$. Let $f_n = \pi(1-r_n)a_n/\alpha_n$. Define

$$f(\phi) = \begin{cases} f_n, & |\phi - \theta_n| < \alpha_n \quad \text{for some } n \\ 0, & \text{otherwise.} \end{cases}$$

Now,

$$2\pi P[f](r_n e^{i\theta_n}) = (1 - r_n^2) \sum_{k=1}^{\infty} f_k \int_{\theta_k - \alpha_k}^{\theta_k + \alpha_k} \frac{d\phi}{r_n^2 - 2r_n \cos(\theta_n - \phi) + 1}$$

$$\geq \frac{2(1 - r_n^2) f_n \alpha_n}{r_n^2 - 2r_n \cos(\alpha_n) + 1} \geq \frac{2(1 + r_n)(1 - r_n) f_n \alpha_n}{(1 - r_n)^2 + r_n \alpha_n^2}.$$

Hence, $P[f](r_n e^{i\theta_n}) \geq a_n$, and $f \in L^1$ if $\sum f_k \alpha_k = \pi \sum (1 - r_k) a_k < \infty$. Since $(1 - r_k)a_k \to 0$, there is a subsequence $\{(1 - r_n)a_n\}_{n \in I}$ defined by an unbounded index set $I \subset \mathbb{N}$ such that $\sum_{k \in I} (1 - r_k)a_k < \infty$. Now take $f(\phi) = f_n$ when $|\phi - \theta_n| < \alpha_n$ for some $n \in I$, and $f(\phi) = 0$, otherwise. Then $f \in L^1$ and $P[f](r_n e^{i\theta_n}) \geq \psi(r_n e^{i\theta_n})$ for all $n \in I$.

(b) Suppose $1 \leq p < \infty$. From part (a), we can write $u_r(\theta) = w_r(\theta)/(1-r)$ where $\sup_{\theta \in [-\pi, \pi]} |w_r(\theta)| \to 0$ as $r \to 1$. Furthermore, w_r is periodic and real analytic on $\mathbb{R}$ for each $0 \leq r < 1$. Let $1 \leq p < \infty$. Then

$$\|u_r\|_p = \frac{1}{1 - r} \Big[\int_{-\pi}^{\pi} |w_r(\theta)|^p \, d\theta \Big]^{1/p}$$

$$\leq \frac{(2\pi)^{1/p}}{1 - r} \sup_{\theta \in [-\pi, \pi]} |w_r(\theta)|.$$

Hence, $\|u_r\|_p = o(1/(1 - r))$ as $r \to 1$.

To prove this estimate is sharp, first consider $p = 1$. Let $\psi \colon [0, 1) \to (0, \infty)$ with $\psi(r) = o(1/(1 - r))$ be given. Although $\mathcal{HK}$ is not complete, it is barrelled [8]. The Uniform Boundedness Principle [7] applies and this shows the existence of $f \in \mathcal{HK}$ such that $\|u_r\|_1 \neq o(\psi(r))$. We can see this as follows.

Define $r_n = 1 - 1/n$ for $n \in \mathbb{N}$. Let $f_n(\theta) = \psi(r_n) \sin(n\theta)$. Then

$$\|f_n\| = \psi(r_n) \int_0^{\pi/n} \sin(n\theta) \, d\theta$$

$$= 2 \, \psi(r_n)/n$$

$$= 2(1 - r_n)\psi(r_n) \to 0 \text{ as } n \to \infty.$$

For $0 \leq r < 1$, define $S_r \colon \mathcal{HK} \to L^1$ by $S_r[f](\theta) = P[f](re^{i\theta})/\psi(r)$ for each $f \in \mathcal{HK}$. Write $u_r(\theta) = P[f](re^{i\theta})$. Using (5),

$$\text{(6)} \qquad \|u_r\|_1 = \int_{-\pi}^{\pi} \Big| \int_{-\pi}^{\pi} f(\phi) \Phi_r(\phi - \theta) \, d\phi \Big| d\theta$$

$$\leq 2\pi \|f\| \left[\inf \Phi_r + V \Phi_r\right]$$

$$= \|f\| \Big(\frac{1 + 6r + r^2}{1 - r^2} \Big).$$

Therefore, $\|S_r\| \leq \frac{1+6r+r^2}{\psi(r)(1-r^2)}$ and, for each $0 \leq r < 1$, S_r is a bounded linear operator from $\mathcal{HK}$ to L^1.

We have $S_r[f_n](\theta) = \psi(r_n)r^n \sin(n\theta)/\psi(r)$ so that

$$(7) \qquad \|S_{r_n}[f_n]\|_1 = r_n^n \int_{-\pi}^{\pi} |\sin(n\theta)| d\theta$$

$$= 4\left(1 - 1/n\right)^n \to 4/e \text{ as } n \to \infty.$$

It follows that $\{S_{r_n}\}$ is not equicontinuous. The Uniform Boundedness Principle [7, Theorem 11, p. 299] now shows that $\{S_{r_n}\}$ is not pointwise bounded on $\mathcal{HK}$. Hence, there exists $f \in \mathcal{HK}$ such that $\sup_n \|u_{r_n}\|_1/\psi(r_n) = \infty$ and hence $\|u_r\|_1 \neq o(\psi(r))$ as $r \to 1$.

The case $p > 1$ is similar. In place of (6), we have

$$\|u_r\|_p \leq (2\pi)^{1/p-1} \|f\| (1 + 6r + r^2)/(1 - r^2).$$

And, in place of (7),

$$\|S_{r_n}[f_n]\|_p = \left(1 - 1/n\right)^n \left[\frac{2\sqrt{\pi}\,\Gamma((1+p)/2)}{\Gamma(1+p/2)}\right]^{1/p}. \qquad \blacksquare$$

Remarks 2 The "little oh" order relation in Theorem 1(a) is false for measures. If μ is a finite Borel measure on $[-\pi, \pi)$, write $u_r(\theta) = P[\mu](re^{i\theta})$. Then $\|u_r\|_\infty \leq \Phi_r(0)\mu([-\pi, \pi)) = O(1/(1-r))$. The Dirac measure shows this estimate is sharp.

For $1 \leq p < \infty$, let $u_r(\theta) = P[\mu](re^{i\theta})$. The Minkowski inequality for integrals [4, Theorem 6.19] gives

$$\|u_r\|_p = \left\| \int_{-\pi}^{\pi} \Phi_r(\phi - \cdot)\, d\mu(\phi) \right\|_p$$

$$\leq \int_{-\pi}^{\pi} \|\Phi_r(\phi - \cdot)\|_p \, d\mu(\phi)$$

$$= \|\Phi_r\|_p \, \mu([-\pi, \pi)).$$

And for $\mu = \delta$, the Dirac measure, let $v_r(\theta) = P[\delta](re^{i\theta})$. Then

$$(8) \qquad \|v_r\|_p = \|\Phi_r\|_p$$

$$= \frac{1-r^2}{2\pi} \left(\int_{-\pi}^{\pi} \frac{d\phi}{(1 - 2r\cos\phi + r^2)^p} \right)^{1/p}$$

$$= (2\pi)^{1/p-1}(1-r^2)^{1/p-1} \left[{}_2F_1(1-p, 1-p; 1; r^2) \right]^{1/p}.$$

Equation (8) is from integral 3.665.2 in [5] and the hypergeometric linear transformation [5, 9.131.1]. For these values of the parameters, the hypergeometric function is bounded for $0 \leq r \leq 1$ and ${}_2F_1(1-p, 1-p; 1; 1) = \Gamma(2p-1)/\Gamma^2(p) \neq 0$. It follows that $\|u_r\|_p = O((1-r)^{1/p-1})$ as $r \to 1$. The Dirac measure shows this estimate is sharp.

The estimate for $p = 1$ appears as Theorem 6.4(a) in [1].

Several results follow immediately from these estimates. For $1 \le p < \infty$, denote the harmonic Hardy spaces by

$$h^p := \{u\colon D \to \mathbb{R} \mid \Delta u = 0 \text{ in } D, \ \sup_{0 \le r < 1} \|u_r\|_p < \infty\}.$$

And h^∞ is the set of bounded functions that are harmonic in D. The harmonic Hardy space associated with the Alexiewicz norm is defined as

$$h^{\mathcal{HK}} := \{u\colon D \to \mathbb{R} \mid \Delta u = 0, \ \sup_{0 \le r < 1} \|u_r\| < \infty\}.$$

This is a normed linear space under the norm $\|u\|_{\mathcal{HK}} := \sup_{0 \le r < 1} \|u_r\|$.

Corollary 3   *For $1 \le p \le \infty$ we have $h^p \nsubseteq h^{\mathcal{HK}}$.*

Proof   We have $h^q \subset h^p \subset h^{\mathcal{HK}}$ for all $1 \le p < q \le \infty$. And, by Theorem 1(b), there is $f \in \mathcal{HK}$ with $u_r(\theta) := P[f](re^{i\theta})$ and $\|u_r\|_1 \ne O(1)$. ∎

Remarks 4   There is a function $f \in \mathcal{HK}$ such that $P[f]$ is not the difference of two positive harmonic functions. This follows since functions in h^1 are characterised as being the difference of two positive harmonic functions. See [1, Exercise 6.9].

When $u \in h^{\mathcal{HK}}$ we can get slightly different estimates than in Theorem 1. (*cf.* [1, Proposition 6.16 and Exercise 6.11]).

Theorem 5   *Let $1 \le p \le \infty$. If $u \in h^{\mathcal{HK}}$ then $\|u_r\|_p \le (2\pi)^{1/p} \frac{2r\|u\|_{\mathcal{HK}}}{\pi(1-r)}$ for $1/2 \le r < 1$ and $\|u_r\|_p \le (2\pi)^{1/p} \frac{2\|u\|_{\mathcal{HK}}}{\pi}$ for $0 \le r \le 1/2$. (Replace the term $(2\pi)^{1/p}$ by 1 when $p = \infty$.) The order relations are sharp as $r \to 1$.*

Proof   Fix $z = re^{i\theta} \in D$ and $0 < t < 1 - r$. If $0 < t \le r$ then, by the Mean Value Property for harmonic functions, $u(z) = (\pi t^2)^{-1} \int_{r-t}^{r+t} \int_{\theta-\theta_0}^{\theta+\theta_0} u(\rho e^{i\phi}) \, d\phi \, \rho \, d\rho$, where $\theta_0 = \arccos[(r^2 + \rho^2 - t^2)/(2r\rho)]$ and $0 \le \theta_0 \le \pi/2$. Hence,

$$|u(z)| \le \frac{1}{\pi t^2} \int_{r-t}^{r+t} \rho \, d\rho \sup_{|\rho - r| < t} \left| \int_{\theta-\theta_0}^{\theta+\theta_0} u(\rho e^{i\phi}) \, d\phi \right| \le \frac{2r}{\pi t} \|u\|_{\mathcal{HK}}.$$

Now let $t \to 1 - r$ when $1/2 \le r < 1$ and let $t \to r$ when $0 \le r \le 1/2$. This establishes the estimates for $p = \infty$. The estimates for $1 \le p < \infty$ follow from this. The case $r = 0$ is similar.

Note that if $u(re^{i\theta}) = \Phi_r(\theta)$ then $\|u\|_{\mathcal{HK}} = 1$ and $\|\Phi_r\|_\infty = (1 + r)/[2\pi(1 - r)]$. So, the order relation for $\|u_r\|_\infty$ is sharp as $r \to 1$. For $1 \le p < \infty$, the implied order relation $O(1/(1 - r))$ is sharp as $r \to 1$ owing to the example in the proof of Theorem 1(b). For, suppose we are given $\psi \colon [0, 1) \to (0, \infty)$ with $\psi(r) = o((1-r)^{-1})$ as $r \to 1$. From Theorem 1(b) we know there is a function $f \in \mathcal{HK}$ such that if $u_r(\theta) = P[f](re^{i\theta})$ then $\limsup_{r \to 1} \|u_r\|_p / \psi(r) = \infty$. And, by the following Theorem 6(a), $\|u_r\| \le \|f\|$ so $u \in h^{\mathcal{HK}}$. ∎

Now consider the analogues of (2) and (3) for the Alexiewicz norm.

Theorem 6 *Let $f \in \mathcal{HK}$. For $re^{i\theta} \in D$ define $u_r(\theta) := P[f](re^{i\theta})$. Then*

(a) $\|u_r\| \leq \|f\|$ *for all $0 \leq r < 1$, i.e., $\|u\|_{\mathcal{HK}} \leq \|f\|$.*
(b) $\|u_r - f\| \to 0$ *as $r \to 1$*
(c) *In (b), the decay of $\|u_r - f\|$ to 0 can be arbitrarily slow.*

Proof (a) Let $\alpha \in \mathbb{R}$ and $0 < \beta - \alpha \leq 2\pi$. Then, by [3, Theorem 57, p. 58], we can interchange the orders of integration to compute

$$\int_\alpha^\beta u_r = \int_{-\pi}^\pi f(\phi) v_r(\phi) \, d\phi,$$

where $v_r(\theta) = P[\chi_{[\alpha,\beta]}](re^{i\theta})$.

If $\beta - \alpha = 2\pi$ then $v_r = 1$ and the result is immediate. Now assume $0 < \beta - \alpha < 2\pi$. For fixed r, the function v_r has a maximum at $\phi_1 := (\alpha + \beta)/2$ and a minimum at $\phi_2 := \phi_1 + \pi$. Use the Bonnet form of the Second Mean Value Theorem for integrals [3, p. 34] to write

$$\int_\alpha^\beta u_r = \int_{\phi_1}^{\phi_2} f(\phi) v_r(\phi) \, d\phi + \int_{\phi_2}^{\phi_1+2\pi} f(\phi) v_r(\phi) \, d\phi$$

$$= v_r(\phi_1) \int_{\phi_1}^{\xi_1} f(\phi) \, d\phi + v_r(\phi_1) \int_{\xi_2}^{\phi_1+2\pi} f(\phi) \, d\phi$$

$$= v_r(\phi_1) \int_{\xi_2-2\pi}^{\xi_1} f(\phi) \, d\phi$$

where $\phi_1 < \xi_1 < \phi_2$ and $\phi_2 < \xi_2 < \phi_1 + 2\pi$. And

$$\left| \int_\alpha^\beta u_r \right| \leq \max_{\theta \in [-\pi,\pi]} v_r(\theta) \left| \int_{\xi_2-2\pi}^{\xi_1} f \right|$$

$$\leq \|f\|.$$

It follows that $\|u_r\| \leq \|f\|$.

(b) Let $\alpha \in \mathbb{R}$ and $0 < \beta - \alpha \leq 2\pi$. We have

(9)
$$\int_\alpha^\beta \left[u_r(\theta) - f(\theta) \right] d\theta = \int_\alpha^\beta \left[\int_{-\pi}^\pi \Phi_r(\phi - \theta) f(\phi) \, d\phi - f(\theta) \int_{-\pi}^\pi \Phi_r(\phi) \, d\phi \right] d\theta$$

$$= \int_{-\pi}^\pi \Phi_r(\phi) \int_\alpha^\beta \left[f(\theta + \phi) - f(\theta) \right] d\theta \, d\phi.$$

The reversal of integrals in (9) is justified by [3, Theorem 58, p. 60]. We now have

$$\|u_r - f\| \leq \sup_{0 \leq \beta - \alpha \leq 2\pi} \left| \int_{-\pi}^\pi \Phi_r(\phi) \int_\alpha^\beta \left[f(\theta + \phi) - f(\theta) \right] d\theta \, d\phi \right|$$

$$\leq P[g](r) \quad \text{where } g(\phi) = \|f(\phi + \cdot) - f(\cdot)\|.$$

But if $f \in \mathcal{HK}$ then f is continuous in the Alexiewicz norm, *i.e.*, $\|f(\phi + \cdot) - f(\cdot)\|$ $\to 0$ as $\phi \to 0$. See [10]. Hence, g is continuous at 0 so $P[g](r) \to 0$ as $r \to 1$.

(c) Let f be positive on $(0, 1)$ and vanish elsewhere. Then u_r is positive for $0 \leq r < 1$. We then have

$$\|u_r - f\| \geq \int_{-\pi}^{0} u_r(\phi)\, d\phi$$

$$= \int_{0}^{1} f(\theta) P[\chi_{[-\pi,0]}](re^{i\theta})\, d\theta.$$

Now, as $r \to 1$

$$P[\chi_{[-\pi,0]}](re^{i\theta}) \to \begin{cases} 0, & 0 < \theta < \pi \\ 1/2, & \theta = -\pi, 0, \pi \\ 1, & -\pi < \theta < 0. \end{cases}$$

But, the convergence is not uniform. Let a decay rate be given by $A\colon [0,1] \to (0, 1/2)$, where $A(r)$ decreases to 0 as r increases to 1. It is easy to show, for example, using a cubic spline, that A has a decreasing C^1 majorant with limit 0 as $r \to 1$. So, we can assume $A \in C^1([0,1))$. By keeping θ close enough to 0 we can keep $P[\chi_{[-\pi,0]}](re^{i\theta})$ bounded away from 0 for all r. To see this, write $\rho := (1+r)/(1-r)$. Then

$$\|u_r - f\| \geq \int_{0}^{1-r} f(\theta) P[\chi_{[-\pi,0]}](re^{i\theta})\, d\theta$$

$$= \frac{1}{\pi} \int_{0}^{1-r} f(\theta) \left\{ \frac{\pi}{2} - \arctan\left[\rho \tan\left(\frac{\theta}{2} \right) \right] + \arctan\left[\frac{1}{\rho} \tan\left(\frac{\theta}{2} \right) \right] \right\} d\theta$$

$$\geq \int_{0}^{1-r} f(\theta) \left\{ \frac{1}{2} - \frac{1}{\pi} \arctan\left[\rho \tan\left(\frac{\theta}{2} \right) \right] \right\} d\theta$$

$$\geq \int_{0}^{1-r} f(\theta) \left\{ \frac{1}{2} - \frac{\rho\, \theta}{2\pi \cos(\theta/2)} \right\} d\theta$$

$$\geq \left(\frac{1}{2} - \frac{1}{\pi \cos(1/2)} \right) \int_{0}^{1-r} f(\theta)\, d\theta.$$

We can now let

$$f(\theta) := \begin{cases} -\left(\frac{1}{2} - \frac{1}{\pi \cos(1/2)} \right)^{-1} A'(1 - \theta), & 0 < \theta < 1 \\ 0, & \text{otherwise.} \end{cases}$$

And

$$\|u_r - f\| \geq -\int_{0}^{1-r} A'(1 - \theta)\, d\theta = A(r). \qquad \blacksquare$$

Remarks 7

1. We have equality in (a) when f is of one sign.
2. Part (a) and dilation show that if $0 \le r \le s < 1$ then $\|u_r\| = \|P[u_s]_{r/s}\| \le \|u_s\|$ (cf. [1, Corollary 6.6]).
3. The triangle inequality and (b) show that $\|u_r\| \to \|f\|$ as $r \to 1$.
4. In (c), $\|u_r - f\|$ can decay to 0 arbitrarily fast. Take f to be constant!
5. The same proof shows that we can choose $f \in L^p$ to make $\|u_r - f\|_p$ tend to 0 arbitrarily slowly. For $1 \le p < \infty$, let

$$
f(\theta) := \begin{cases} \left(\frac{1}{2} - \frac{1}{\pi \cos(1/2)}\right)^{-1} p^{1/p} \left[A(1-\theta)\right]^{1-1/p} \left[-A'(1-\theta)\right]^{1/p}, & 0 < \theta < 1 \\ 0, & \text{otherwise.} \end{cases}
$$

and then $\|u_r - f\|_p \ge A(r)$.

Theorem 8 *The mapping $P\colon \mathcal{HK} \to h^{\mathcal{HK}}$, $f \mapsto P[f]$, is an isometry into, but not onto, $h^{\mathcal{HK}}$.*

Proof Let $f \in \mathcal{HK}$ and $u = P[f]$. From Remarks 7.2 and 7.3,

$$
\|u\|_{\mathcal{HK}} = \sup_{0 \le r < 1} \|u_r\| = \lim_{r \to 1} \|u_r\| = \|f\|.
$$

Hence, P is an isometry.

However, P is not onto $h^{\mathcal{HK}}$. Let F be continuous on $[-\pi, \pi]$ such that $F(-\pi) = 0$, F is 2π-periodic and F is not in ACG^*, *i.e.*, F is not an indefinite Henstock-Kurzweil integral. See [3] for the definition of ACG^*. The function

$$
(10) \qquad v_r(\theta) := F(\pi)\Phi_r(\pi - \theta) - \int_{-\pi}^{\pi} \Phi_r'(\phi - \theta)F(\phi) \, d\phi
$$

is harmonic in D (using dominated convergence). Let $\alpha \in \mathbb{R}$ and $0 < \beta - \alpha \le 2\pi$. Then

$$
\int_{\alpha}^{\beta} v_r(\theta) \, d\theta = F(\pi) \int_{\alpha}^{\beta} \Phi_r(\pi - \theta) \, d\theta - \int_{-\pi}^{\pi} F(\phi) \int_{\alpha}^{\beta} \Phi_r'(\phi - \theta) \, d\theta \, d\phi
$$

$$
= F(\pi)P[\chi_{[\alpha,\beta]}](-r) + P[F](re^{i\alpha}) - P[F](re^{i\beta}).
$$

So, $\|v_r\| \le 3 \max |F|$ and $v \in h^{\mathcal{HK}}$. If there was $f \in \mathcal{HK}$ such that $v = P[f]$ then write $G(\theta) := \int_{-\pi}^{\theta} f$. Since $G \in ACG^*$, we have

$$
(11) \qquad v(re^{i\theta}) = G(\pi)\Phi_r(\pi - \theta) - \int_{-\pi}^{\pi} \Phi_r'(\phi - \theta)G(\phi) \, d\phi.
$$

Comparing (10) and (11), letting $r \to 0$ shows $G(\pi) = F(\pi)$. Write $H := F - G$. Expand $\Phi'_r(\theta) = (-1/\pi) \sum_{n=1}^{\infty} nr^n \sin(n\theta)$. The series converges uniformly and absolutely on compact subsets of D. Then for all $re^{i\theta} \in D$,

$$0 = \int_{-\pi}^{\pi} H(\phi) \sum_{n=1}^{\infty} nr^n \sin[n(\phi - \theta)] \, d\phi$$

$$= \sum_{n=1}^{\infty} nr^n \int_{-\pi}^{\pi} H(\phi) \sin[n(\phi - \theta)] \, d\phi.$$

For all $n \geq 1$ and all $\theta \in \mathbb{R}$ we have $\int_{-\pi}^{\pi} H(\phi) \sin[n(\phi - \theta)] \, d\phi = 0$. Since H is continuous, it is constant. But then F differs from G by a constant. This contradicts the assumption that $F \notin ACG^*$. Thus, no such F exists and P is not onto $h^{\mathcal{HK}}$. ∎

3 Bounded Variation

Define the 2π-periodic functions of normalised bounded variation by $\mathcal{NBV} := \{g \colon \mathbb{R} \to \mathbb{R} \mid g \text{ is } 2\pi\text{-periodic}, Vg < \infty, g(-\pi) = 0, g \text{ is right continuous}\}$. Using the variation as a norm, $\mathcal{NBV}$ is a Banach space that is the dual of $\mathcal{HK}$ [8]. Analogues of Theorems 1 and 6 now take the following form.

Theorem 9 *Let $g \in \mathcal{BV}$ and $v = P[g]$.*

(a) *If $g \in \mathcal{NBV}$ then $v_r \to g$ weak* in $\mathcal{NBV}$ as $r \to 1$.*
(b) *For all $0 \leq r < 1$, $\|v_r\|_\infty \leq \inf |g| + Vg$.*
(c) *If $g \in \mathcal{NBV}$ then $\|v_r\|_\infty \leq Vg$ for all $0 \leq r < 1$.*
(d) *$Vv_r \leq Vg$ for all $0 \leq r < 1$.*
(e) *There is $\sigma \in \mathcal{NBV}$ such that if $w_r(\theta) = P[\sigma](re^{i\theta})$ then $V[w_r - \sigma] \not\to 0$ as $r \to 1$. And there is $\tau \in \mathcal{BV}$ such that if $w_r(\theta) = P[\tau](re^{i\theta})$ and $\tau(\theta) = [\tau(\theta+) + \tau(\theta-)]/2$ for all $\theta \in [-\pi, \pi]$ then $V(w_r - \tau) \not\to 0$ as $r \to 1$.*
(f) *Let*

$$h^{\mathcal{BV}} := \{u \colon D \to \mathbb{R} \mid \Delta u = 0, \|u\|_{\mathcal{BV}} < \infty\},$$

where $\|u\|_{\mathcal{BV}} := \sup_{0 \leq r < 1} Vu_r$. The mapping $P \colon \mathcal{NBV} \to h^{\mathcal{BV}}, g \mapsto P[g]$, is an isometric isomorphism between the Banach spaces $\mathcal{NBV}$ and $h^{\mathcal{BV}}$.

Proof (a) Let $f \in \mathcal{HK}$. Write $u = P[f]$. Then, using (5) and Theorem 6(b),

$$(12) \qquad \left| \int_{-\pi}^{\pi} f(v_r - g) \right| = \left| \int_{-\pi}^{\pi} (u_r - f)g \right|$$

$$(13) \qquad \leq \|u_r - f\| Vg$$

$$(14) \qquad \to 0 \text{ as } r \to 1.$$

The interchange of orders of integration in (12) is valid by [3, p. 58, Theorem 57].

(b),(c)   These follow immediately from (5).

(d)   Let $\{(s_n, t_n)\}$ be a sequence of disjoint intervals in $(-\pi, \pi)$. Then

$$\sum |v_r(s_n) - v_r(t_n)| = \sum \left| \int_{-\pi}^{\pi} \Phi_r(\phi) \left[g(\phi + s_n) - g(\phi + t_n) \right] d\phi \right|$$

$$\leq P[1](r) \, Vg = Vg.$$

(e)   Let $-\pi < a < b < \pi$, $\sigma = \chi_{[a,b)}$ and $w_r(\theta) = P[\sigma](re^{i\theta})$. Then $\sigma \in \mathcal{NBV}$ and

$$|w_r(b) - \sigma(b) - w_r(-\pi) + \sigma(-\pi)| = w_r(b) - w_r(-\pi) \to 1/2 \quad \text{as } r \to 1.$$

So, $V(w_r - \sigma) \nrightarrow 0$.

Note that if we replace $\tau(\theta)$ by $[\sigma(\theta+) + \sigma(\theta-)]/2$ and now let $w_r(\theta) = P[\tau](re^{i\theta})$ then $w_r(\theta) \to \tau(\theta)$ for all $\theta \in [-\pi, \pi]$. But, $V(w_r - \tau) \to 2$ as $r \to 1$. (Since $w_r(a)$ and $w_r(b) \to 1/2$ as $r \to 1$.)

(f)   Let $\sigma \in \mathcal{NBV}$ and $w_r(\theta) = P[\sigma](re^{i\theta})$. By (d), $\|w\|_{\mathcal{BV}} \leq V\sigma$. From (a), $w_r \to \sigma$ weak* in $\mathcal{NBV}$, hence (*cf.* [1, 6.8]),

$$V\sigma \leq \liminf_{r \to 1} V w_r \leq \liminf_{r \to 1} \|w\|_{\mathcal{BV}} = \|w\|_{\mathcal{BV}}.$$

And P is an isometry.

To show P is onto $h^{\mathcal{BV}}$, let $w \in h^{\mathcal{BV}}$. Since $\mathcal{HK}$ is separable [8], every norm-bounded sequence in $\mathcal{HK}^*$ contains a weak* convergent subsequence [1, Theorem 6.12]. But $\{w_r\}$ is norm-bounded in $\mathcal{NBV}$, so there is a subsequence $\{w_{r_j}\}$ and $\sigma \in \mathcal{NBV}$ such that for all $f \in \mathcal{HK}$ we have $\int_{-\pi}^{\pi} f w_{r_j} \to \int_{-\pi}^{\pi} f \sigma$ as $r_j \to 1$. To show $w = P[\sigma]$, fix $re^{i\theta} \in D$. Then, since each function w_{r_j} is continuous on $\overline{D}$ and harmonic in D, it is the Poisson integral of its boundary values, *i.e.*,

$$(15) \qquad w(r_j re^{i\theta}) = \int_{-\pi}^{\pi} \Phi_r(\phi - \theta) w_{r_j}(\phi) \, d\phi.$$

Now, w is continuous on D, $\Phi_r(\cdot - \theta) \in \mathcal{HK}$ and w_{r_j} is of bounded variation on ∂D, uniformly for $j \geq 1$. Using weak* convergence, taking the limit $r_j \to 1$ in (15) yields $w(re^{i\theta}) = P[\sigma](re^{i\theta})$. Thus, $\mathcal{NBV}$ and $h^{\mathcal{BV}}$ are isomorphic. Since $\mathcal{NBV}$ is a Banach space, $h^{\mathcal{BV}}$ is as well. ∎

4   The Dirichlet Problem

Under an Alexiewicz norm boundary condition, we can prove uniqueness for the Dirichlet problem.

Theorem 10   *Let* $f \in \mathcal{HK}$. *The Dirichlet problem*

$$(16) \qquad u \in C^2(D)$$

$$(17) \qquad \Delta u = 0 \quad \text{in } D$$

$$(18) \qquad \|u_r - f\| \to 0 \text{ as } r \to 1$$

has the unique solution $u = P[f]$.

Proof   First note that from Theorem 6(b) and [2, Proposition 1], $u = P[f]$ is certainly a solution of (16), (17) and (18).

Suppose there were two solutions u and v. Write $w = u - v$. Then w satisfies (16) and (17). And $\|w_r\| \leq \|u_r - f\| + \|v_r - f\|$, which has limit 0 as $r \to 1$. Since w is harmonic in D it has the trigonometric expansion

$$(19) \qquad w(re^{i\theta}) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n \left[a_n \cos(n\theta) + b_n \sin(n\theta) \right],$$

the series converging uniformly and absolutely on compact subsets of D. Fix $0 \leq r < 1$. We have $\|w_r\| \geq |\int_{-\pi}^{\pi} w_r| = \pi|a_0|$. Letting $r \to 1$ shows $a_0 = 0$. And, for $n \geq 1$, we have $\|w_r \cos(n\cdot)\| \geq |\int_{-\pi}^{\pi} w_r(\theta) \cos(n\theta)\,d\theta| = \pi r^n |a_n|$. As well,

$$\|w_r \cos(n\cdot)\| \leq \|w_r\| \left\{ \inf_{|\theta| \leq \pi} |\cos(n\theta)| + V[\theta \mapsto \cos(n\theta)] \right\}$$

$$= 4n\|w_r\|.$$

Therefore, $4n\|w_r\| \geq \pi r^n |a_n|$. Letting $r \to 1$ shows $a_n = 0$. Similarly, $b_n = 0$. It follows that $w = 0$ and we have uniqueness. ∎

In [6], Shapiro gave a uniqueness theorem that combined a pointwise limit with an L^p condition. There is an analogue for the Alexiewicz norm.

Theorem 11   *Suppose $\Delta u = 0$ in D and there exists $f \in \mathcal{HK}$ such that*

$$(20) \qquad u_r(\theta) \to f(\theta) \quad \text{for each } \theta \in [-\pi, \pi)$$

$$(21) \qquad \|u_r\| = o(1/(1-r)) \quad \text{as } r \to 1.$$

Then $u = P[f]$.

Proof   As in Theorem 10, suppose w is a solution of the corresponding homogeneous problem ($f = 0$). Let $\alpha, \beta \in \mathbb{R}$ with $0 < \beta - \alpha \leq 2\pi$. Following the proof of Theorem 3 in [6] and using (5),

$$\left| w(r^2 e^{i\theta}) \right| = \left| P[w_r](re^{i\theta}) \right|$$

$$\leq \frac{\|w_r\| g(r)}{2\pi(1-r)},$$

where $g(r) := (1 + 6r + r^2)/(1 + r)$. But, $g(r) \leq g(1) = 4$. Hence, by (21), $\|w_{r^2}\|_\infty = o(1/(1-r)^2)$ and so $\|w_r\|_\infty = o(1/(1-r)^2)$ as $r \to 1$. It follows from [6, Theorem 1] that $w = 0$. ∎

As pointed out in [6], neither (20) nor (21) can be relaxed. If $u_r \to f$ except for one value $\theta_0 \in [-\pi, \pi)$ then we can add a multiple of $\Phi_r(\theta - \theta_0)$ to $u(re^{i\theta})$. If in place of (21) we have $\|u_r\| = O(1/(1-r))$ then we can add a multiple of Φ_r' to u_r, since for each $\theta \in \mathbb{R}$, $\Phi_r'(\theta) \to 0$ as $r \to 1$.

Example 12 (a) Let $f \in \mathcal{HK} \setminus L^1$. Then the unique solution to (16)–(18) is $u = P[f]$. In this case, the L^p norms of u_r need not be bounded as $r \to 1$. If we are given a harmonic function v such that the Alexiewicz norms $\|v_r\|$ are uniformly bounded for $0 \le r < 1$, then we cannot infer the existence of $g \in \mathcal{HK}$ such that $v = P[g]$. This is because $\mathcal{HK}$ is not complete.

(b) Let $v(z) = (1+z)/(1-z)$ and $w(z) = v(z)e^{-v(z)}$. Define

$$u(re^{i\theta}) = Re(w(re^{i\theta}))$$

$$= \frac{(1-r^2)\cos\left(\frac{2r\sin\theta}{1-2r\cos\theta+r^2}\right) + 2r\sin\theta\sin\left(\frac{2r\sin\theta}{1-2r\cos\theta+r^2}\right)}{\exp(2\pi\Phi_r(\theta))(1-2r\cos\theta+r^2)}.$$

Let

$$f(\theta) := \lim_{r\to 1} u_r(\theta) = \begin{cases} \left(\frac{\sin\theta}{1-\cos\theta}\right)\sin\left(\frac{\sin\theta}{1-\cos\theta}\right), & 0 < |\theta| < \pi \\ 0, & |\theta| = 0, \pi. \end{cases}$$

Note that $f \notin L^p$ for any $1 \le p \le \infty$. The set function μ defined by $\mu(A) = \int_A f$ is not a signed Borel measure. Thus, u is not the Lebesgue-Poisson integral of any L^p function or measure. Since $f(\theta) \sim (2/\theta)\sin(2/\theta)$ as $\theta \to 0$ we have $f \in \mathcal{HK}$. And

$$|(1-r)u_r(\theta)| \le (1-r)e^{-1} + \frac{2r\,e^{-1}}{1+r} \le 1/2.$$

By dominated convergence, $\|(1-r)u_r\| \to 0$ as $r \to 1$. And, by Theorem 11, $u = P[f]$. There is a similar result for the imaginary part of w.

(c) Let $w(z) = [1/(1-z)]e^{[1/(1-z)]}$ and define

$$u(re^{i\theta}) = Re(w(re^{i\theta}))$$

$$= \frac{(1-r\cos\theta)\cos\left(\frac{r\sin\theta}{1-2r\cos\theta+r^2}\right) - r\sin\theta\sin\left(\frac{r\sin\theta}{1-2r\cos\theta+r^2}\right)}{\exp\left(\frac{r\cos\theta-1}{1-2r\cos\theta+r^2}\right)(1-2r\cos\theta+r^2)}.$$

Let

$$f(\theta) := \lim_{r\to 1} u_r(\theta) = \begin{cases} \sqrt{e}\left[\frac{(1-\cos\theta)\cos\left(\frac{\sin\theta}{2(1-\cos\theta)}\right) - \sin\theta\sin\left(\frac{\sin\theta}{2(1-\cos\theta)}\right)}{2(1-\cos\theta)}\right], & 0 < |\theta| \le \pi \\ \infty, & \theta = 0. \end{cases}$$

Although $f \in \mathcal{HK}$, Theorem 11 does not apply since f is not a real-valued function. Indeed, $(1-r)u_r(0) = \exp(1/(1-r)) \to \infty$ as $r \to 1$. From Theorem 1, u is not the Poisson integral of any function in $\mathcal{HK}$ (nor L^p function nor measure). In particular, $u \ne P[f]$.

In examples (b) and (c), the origin is the only point of nonabsolute summability of f. For each $0 \le \lambda < 2\pi$, an example is given in [2] of the Poisson integral of a function in $\mathcal{HK}$ whose set of points of nonabsolute summability in $(-\pi, \pi)$ has measure λ.

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