10

Singular Integrals

10.1 A Few Words in General

The classical Calderón–Zygmund theory of singular integrals deals with the Lebesgue measure on $\mathbb{R}^m$ and operators T_K:

$$T_K f(x) = \int K(y-x) f(y)\, dy. \tag{10.1}$$

Here K usually is smooth except that it has a singularity of order $|x|^{-m}$ at the origin. Because of this the integral in (10.1) often does not exist. But K is also assumed to possess cancellation, being odd or something less. Then the principal values

$$T_K f(x) = \lim_{\varepsilon \to 0} \int_{|y-x|>\varepsilon} K(y-x) f(y)\, dy \tag{10.2}$$

exist if f is sufficiently nice; usually Lipschitz is enough. This is easily checked, for instance, for the Riesz kernel $|x|^{-m-1}x$.

The core of the theory is the L^2-boundedness; when is $T_K : L^2 \to L^2$ bounded? Very general regularity assumptions on K, saying that the singularity at 0 is not too bad, suffice. For instance, the condition that $|x|^{m+j}|\nabla^j K(x)|$ is bounded for $j = 0, 1, 2, \ldots$, which will appear later, is much more than enough. The L^2-boundedness implies, and often is equivalent to, many other fundamental properties ($L^p, 1 < p < \infty$, BMO, weak L^1, $T(b)$, etc.). In particular, it implies that the convergence of (10.2) takes place for all $f \in L^1$ for almost all $x \in \mathbb{R}^m$. This is because nice functions are dense in L^1.

Our interest is in the case where the Lebesgue measure is replaced by an in some sense m-dimensional measure on $\mathbb{R}^n$. To define the L^2-boundedness without having the pointwise formula (10.2), we define the truncated operators

$T_{K,\mu,\varepsilon}, \varepsilon > 0$:

$$T_{K,\mu,\varepsilon}g(x) = \int_{|x-y|>\varepsilon} K(y-x)g(y)\,d\mu y.$$

We say that $T_{K,\mu}$ is bounded in $L^2(\mu)$ if the $T_{K,\mu,\varepsilon}, \varepsilon > 0$, are uniformly bounded in $L^2(\mu)$. Often this is equivalent to saying that the maximal transform $T^*_{K,\mu}$,

$$T^*_{K,\mu}g(x) = \sup_{\varepsilon>0} \left|\int_{|y-x|>\varepsilon} K(y-x)g(y)\,d\mu y\right|,$$

is bounded in $L^2(\mu)$. In case $\mu = \mathcal{H}^m \llcorner E$, we say that $T_{K,E}$ is bounded in $L^2(E)$.

Coifman and Weiss extended in [116] most of the basic theory from the Lebesgue measure in $\mathbb{R}^m$ to doubling measures μ in metric spaces E, in particular to AD-m-regular sets in $\mathbb{R}^n$. Again a basic question is: when is T_K: $L^2(\mu) \to L^2(\mu)$ bounded? But now it is not only about the kernel but also, and usually mainly, about E and μ. Still L^2-boundedness implies a lot of other things, but not any more automatically the existence of principal values. The cancellation properties of the kernel do not help, even for constant functions, if the measure does not have symmetry properties. But we showed with Verdera in [331] that under very general conditions L^2-boundedness implies weak convergence of the truncated operators.

Is the doubling condition necessary for the general theory? It seems that this question was seriously considered only when it was needed in connection with the Cauchy integral and analytic capacity, recall the $T(b)$-theorems of David and of Nazarov, Treil and Volberg from Chapter 9. In addition to $T(b)$, Nazarov, Treil and Volberg, and later also Tolsa, developed the non-homogeneous (that is, non-doubling) Calderón–Zygmund theory with surprising success in many papers.

10.2 L^2-Boundedness and Uniform Rectifiability

This topic has its origins in David's work in the 1980s, see [133, 134]. Most of the basics were developed by David and Semmes in [146, 147]. The main problem related to rectifiability is the following conjecture of David and Semmes:

Conjecture 10.1 *Let $0 < m < n$ be integers and let $E \subset \mathbb{R}^n$ be AD-m-regular. Then the* Riesz transform *R^m_E is bounded in $L^2(E)$ if and only if E is uniformly m-rectifiable.*

The kernel of R^m_E is the *Riesz kernel* $R_m(x) = |x|^{-m-1}x, x \in \mathbb{R}^n \setminus \{0\}$, so

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$R^m_E = T_{R_m,E}$ according to the above notation. For a measure μ we shall also set $R^m_\mu = T_{R_m,\mu}$. The L^2-boundedness again means that the truncated operators $R^m_{E,\varepsilon}, \varepsilon > 0$,

$$R^m_{E,\varepsilon}g(x) = \int_{\{y\in E:\ |y-x|>\varepsilon\}} R_m(y-x)g(y)\,d\mathcal{H}^m y,$$

are uniformly bounded in $L^2(E)$.

The boundedness of much more general singular integral operators on uniformly rectifiable sets was proved in [146]. We shall return to this soon. The problem is the converse. It is known for $m = 1, n-1$, and only then:

Theorem 10.2 *Let $m = 1$ or $m = n-1$ and let $E \subset \mathbb{R}^n$ be AD-m-regular. Then the Riesz transform R^m_E is bounded in $L^2(E)$ if and only if E is uniformly m-rectifiable.*

For $m = 1$ the proof of Theorem 9.5 gives this too. So it is based on symmetrization as in (9.2). For $m > 1$, this method does not work because the corresponding sum takes both positive and negative values as was shown, more generally, by Farag in [191, 193].

The case $m = n-1$ is due to Nazarov, Tolsa and Volberg in [363]. Their proof is very long and complicated and contains many brilliant ideas. Before saying a few words about it, let us look at a much weaker and simpler result which holds for all m, see [323]:

Proposition 10.3 *Let $0 < m < n$ and let $\mu \in \mathcal{M}(\mathbb{R}^n)$ be AD-m-regular. If the Riesz transform R^m_μ is bounded in $L^2(\mu)$, then for μ almost all $a \in \mathbb{R}^n$, μ has some m-flat tangent measures at a.*

The proof is easy. A duality argument, recall the discussion around (9.1), gives a bounded function g such that the maximal function $R^{m*}_\mu g$ is bounded. Then we can find an AD-m-regular tangent measure ν such that

$$\left|\int_{r<|y-x|<R} |y-x|^{-m-1}(y-x)\,d\nu y\right| \le C \text{ for } x \in \operatorname{spt}\nu, 0 < r < R. \tag{10.3}$$

Since $\operatorname{spt}\nu$ cannot be the whole space, we can find an open ball U disjoint from $\operatorname{spt}\nu$ such that there is $x \in \operatorname{spt}\nu \cap \partial U$. Any tangent measure π of ν at x is again AD-m-regular and satisfies (10.3). It has support in a half-space H with $0 \in \operatorname{spt}\pi \cap \partial H$, so due to (10.3) $\operatorname{spt}\pi$ must be tangential to ∂H at 0. The next tangent measure will have support in ∂H, perhaps with a non-constant density, but the last tangent measure will be flat, and it is also a tangent measure of μ by Preiss's 'tangent measures to tangent measures are tangent measures' principle, see [321, Theorem 14.16].

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Proposition 10.3 gives approximation by planes at *some* arbitrarily small scales, but even for ordinary rectifiability we would need it at *all* arbitrarily small scales. As such it is useless for the proof of Theorem 10.2, but in [363] Nazarov, Tolsa and Volberg used tangent measures to prove a quantitative form of this: sufficiently good pointwise boundedness of the Riesz transform at a range of scales gives a good approximation at some cube in the same range. Then the counter-assumption that the set in Theorem 5.8 is not Carleson allows us to build a Cantor structure based on the generalized dyadic cubes, mentioned in Section 5.5, where the cubes of nearly flatness of $\operatorname{spt}\mu$ (or rather of μ) and cubes of non-flatness are alternating. The proof ends with clever applications of an extremal problem and a maximum principle. These have their origins in [186]. The maximum principle requires harmonicity of the kernel, and this is the main reason (or maybe the only reason) why the proof does not extend to $m < n-1$. The end result is that the L^2-boundedness implies the property of approximation with unions of planes as in Theorem 5.8 and hence uniform rectifiability. This is not a sketch of the proof, only a few tiles from a magnificent structure.

For sets of finite measure, we have the analogue of Theorem 9.2:

Theorem 10.4 *Let $E \subset \mathbb{R}^n$ be $\mathcal{H}^{n-1}$ measurable with $\mathcal{H}^{n-1}(E) < \infty$. If R_E^{n-1} is bounded in $L^2(E)$, then E is $(n-1)$-rectifiable.*

This was proved by Nazarov, Tolsa and Volberg in [364]. They used a result of Eiderman, Nazarov and Volberg from [186] according to which R_E^{n-1} is unbounded in $L^2(E)$ if E has zero lower density. Thus they could assume that $\Theta_*^{n-1}(E,x) > 0$ for $\mathcal{H}^{n-1}$ almost all $x \in E$. Then they used an argument of Pajot from [376] to find AD-$(n-1)$-regular measures μ_j such that $R_{\mu_j}^{n-1}$ is bounded in $L^2(\mu_j)$ and $\mathcal{H}^{n-1} \llcorner E \leq \sum_j \mu_j$. This allowed them to conclude the proof by Theorem 10.2.

As in Theorem 9.6 for $m = 1$, there is a generalization of Theorem 10.2:

Theorem 10.5 *Let $\mu \in \mathcal{M}(\mathbb{R}^n)$ with $\mu(B(x,r)) \leq r^{n-1}$ for $x \in \mathbb{R}^n, r > 0$. Then*

$$||R_\mu^{n-1}||_{L^2(\mu)}^2 + \mu(\mathbb{R}^n) \sim \int_0^\infty \int \beta_\mu^{n-1,2}(x,r)^2 \frac{\mu(B(x,r))}{r^{n-1}}\, d\mu x \frac{1}{r}\, dr + \mu(\mathbb{R}^n). \quad (10.4)$$

The estimate $' \lesssim'$ was proved by Girela-Sarrión [225] for general m and general kernels. The converse was proved by Tolsa [420] based on his paper with Dabrowski [128].

The validity of $' \gtrsim'$ in Theorem 10.5 for $1 < m < n-1$ is an open question, as it should be, since the David–Semmes conjecture is then open. However, Jaye, Nazarov and Tolsa proved in [259] that the L^2-boundedness for all radial-type

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kernels $K \in \mathcal{K}_m(\mathbb{R}^n)$, see below, implies that the right-hand side of (10.4) is finite.

Prat, Puliatti and Tolsa [381] extended Theorems 10.2 and 10.4 to kernels that are gradients of the fundamental solutions of more general elliptic equations; cR_{n-1} is the gradient of the fundamental solution of the Laplace equation. Let $A(x)$, $x \in \mathbb{R}^n$, be an $n \times n$ matrix with Hölder continuous entries satisfying the ellipticity conditions

$$|\xi|^2 \lesssim A(x)\xi \cdot \xi \text{ for all } \xi, x \in \mathbb{R}^n, \tag{10.5}$$

$$A(x)\xi \cdot \eta \lesssim |\xi||\eta| \text{ for all } \xi, \eta, x \in \mathbb{R}^n. \tag{10.6}$$

The equation

$$L_A u(x) := -\mathrm{div}(A\nabla u)(x) = 0 \tag{10.7}$$

has a fundamental solution $\Gamma_A(x, y)$. The kernel then is its gradient, $K_A(x, y) = \nabla_x \Gamma_A(x, y)$. Technically the situation now is more complicated than for the Riesz kernel, but the authors of [381] managed with several modifications to follow the same strategy to prove the analogues of Theorems 10.2 and 10.4. In [348] Molero, Mourgoglou, Puliatti and Tolsa proved generalizations to possibly discontinuous coefficients satisfying Dini-type conditions.

Mas and Tolsa proved in [314] a characterization of uniform rectifiability in terms of the L^2-boundedness of variations of the Riesz transform. They are defined by maximizing $\sum_{m\in\mathbb{Z}} |R^m_{\mu,\varepsilon_{m+1}} f(x) - R^m_{\mu,\varepsilon_m} f(x)|^2$ over decreasing sequences of positive numbers ε_m.

Except for the Riesz kernels and some other particular cases, recall the discussion in Section 9.5, the question for which kernels Theorem 10.2 holds is pretty much open. However, if we consider a large class of kernels, the characterization of uniform rectifiability in all dimensions was already obtained by David and Semmes in [146]. Let us denote by $\mathcal{K}_m(\mathbb{R}^n)$ the set of smooth real-valued odd functions K on $\mathbb{R}^n \setminus \{0\}$ such that $|x|^{m+j}|\nabla^j K(x)|$ is bounded for $j = 0, 1, 2, \ldots$.

Theorem 10.6 *Let E be AD-m-regular. Then E is uniformly rectifiable if and only if $T_{K,E}$ is bounded in $L^2(E)$ for all kernels $K \in \mathcal{K}_m(\mathbb{R}^n)$.*

Already the boundedness with the kernels $K \in \mathcal{K}_m(\mathbb{R}^n)$ which are of the form $K(x) = x_j k(x)$, $j = 1, \ldots, n$, where k is radial, is enough for uniform rectifiability, see [147, Theorem I.2.59]; for rectifiability this was proved in [329]. Related results are in [413].

Jaye and Nazarov studied *reflectionless* measures in several papers. They have some resemblance to the symmetric measures of Section 9.3. A measure μ is reflectionless with respect to a kernel K if $T_{K,\mu}1$ vanishes, in a weak sense, on the support of μ. In [258], they showed that if K is a kernel with respect to which the AD-m-regular reflectionless measures are flat, then the L^2-boundedness on any AD-m-regular set E implies the uniform rectifiability of E. This condition is known to hold for the Cauchy kernel and R_1 in the plane but it is unknown for higher-dimensional Riesz kernels. With this property of the Cauchy kernel, Jaye and Nazarov gave in [256] the earlier mentioned new proof of Theorem 9.5. The example of [257] mentioned near the end of Section 9.5 relies on the fact that the Lebesgue measure on $\mathbb{C}$ is reflectionless with respect to the kernel $\bar{z}/z^2$.

10.3 Principal Values

We have the analogue of Theorem 9.7 for the Riesz kernels:

Theorem 10.7 *Let $E \subset \mathbb{R}^n$ be $\mathcal{H}^m$ measurable with $\mathcal{H}^m(E) < \infty$. Then E is m-rectifiable if and only if the finite limit*

$$\lim_{\varepsilon \to 0} \int_{\{y \in E:\, |y-x|>\varepsilon\}} R_m(y-x)\, d\mathcal{H}^m y$$

exists for $\mathcal{H}^m$ almost all $x \in E$.

Verdera's Hahn–Banach proof [425] for the Cauchy transform generalizes to give that the principal values exist for rectifiable sets. The converse was proved by Tolsa in [412]. His proof for Theorem 9.7 does not work now, as he used Menger curvature for that. It is replaced by L^2 estimates on Lipschitz graphs. That is, Tolsa showed that the L^2 norm of the Riesz transform on Lipschitz graphs is quantitatively bounded by the L^2 norm of the gradient of the function, not only from above but also from below. These estimates are then used to construct Lipschitz graphs containing a positive measure of E. Very roughly, the existence of principal values implies some approximation with a graph of a Lipschitz function, whose gradient cannot have too big L^2 norm by the bounds on the Riesz transform. Arguments similar to those of Legér in the proof of Theorem 3.18 are essential.

The analogue of Theorem 9.8 was proved in [329], but in addition to positive lower density we had to assume that it also is finite:

Theorem 10.8 *Let $\mu \in \mathcal{M}(\mathbb{R}^n)$. If $0 < \Theta^m_*(\mu, x) < \infty$ and the finite limit $\lim_{\varepsilon\to 0} \int_{\{|y-x|>\varepsilon\}} R_m(y-x)\, d\mu y$ exists for μ almost all $x \in \mathbb{R}^n$, then μ is m-rectifiable.*

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Again, we showed that tangent measures ν are symmetric:

$$\int_{B(x,r)} (y - x)\, d\nu y = 0 \text{ for } x \in \operatorname{spt}\nu, r > 0.$$

We could not say very much about them, except that the AD-m-regular symmetric measures are flat. But we could show that if almost everywhere μ has positive lower density and the tangent measures are symmetric, then they are flat.

Combining Theorems 10.4 and 10.7, we see that if $m = 1$ or $m = n - 1$, $E \subset \mathbb{R}^n$ is $\mathcal{H}^m$ measurable with $\mathcal{H}^m(E) < \infty$ and the Riesz transform is bounded in $L^2(E)$, then the principal values exist $\mathcal{H}^m$ almost everywhere in E. But we only know this going through rectifiability. It would be interesting to have a direct proof, or any proof when $1 < m < n-1$. Combining with Theorem 10.7, this would give a proof that the L^2-boundedness implies rectifiability.

The more general question for which kernels L^2-boundedness implies almost everywhere convergence is open, even for AD-regular sets. We know from Section 9.5 that there are reasonable kernels for which this fails, but the Jaye–Nazarov example is not AD-regular. In [331], we proved with Verdera under very general conditions that the L^2-boundedness together with zero density implies the existence of principal values. Jaye and Merchán [253] strengthened this by replacing zero density with the condition that modifications of Tolsa's α's (recall Section 6.4) tend to zero. See also [254] for related results and recall the discussion in 9.5 on [255].

Let $\Omega \subset \mathbb{R}^n$ be a domain with compact AD-$(n-1)$-regular (and a little more) boundary. D. and M. Mitrea and Verdera [347] proved that then Ω is a $C^{1+\alpha}$, $0 < \alpha < 1$, domain if and only if the Riesz transform $R^{n-1}_{\partial\Omega}$ maps $C^\alpha(\partial\Omega)$ into itself. The second half actually means several equivalent conditions, some of them involving uniform rectifiability, corresponding to different definitions of the Riesz transform, for example, principal value and distributional definitions. The proof uses Clifford algebras and an interesting formula expressing the unit normal in terms of the Riesz transform and the Cauchy–Clifford transform.

10.4 Lipschitz Harmonic Functions

The kernel R_{n-1} is a constant multiple of the gradient of the fundamental solution of the Laplacian in $\mathbb{R}^n$, which is $c_n|x|^{2-n}$ for $n \geq 3$, $c_2 \log|x|$ for $n = 2$. Hence the codimension 1 Riesz transform is related to the removability of Lipschitz harmonic functions in the same way as the Cauchy transform is related to the removable sets of bounded analytic functions.

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Let us say that a compact set $E \subset \mathbb{R}^n$ is *removable for Lipschitz harmonic functions*, abbreviated RLH, if whenever U is an open set containing E, every Lipschitz function $u\colon U \to \mathbb{R}$ which is harmonic in $U \setminus E$ is harmonic in U. Since Lipschitz functions always have Lipschitz extensions, we can start with u defined in all of U.

Theorem 10.9 *Let $E \subset \mathbb{R}^n$ be compact with $\mathcal{H}^{n-1}(E) < \infty$. Then E is RLH if and only if E is purely $(n-1)$-unrectifiable.*

The non-removability of rectifiable sets was proved in [328] by methods similar to those used for the analytic functions and discussed in Chapter 9. The converse was proved for $n = 2$ in [143] based on Menger curvature (recall the discussion after Theorem 9.2) and the David–Léger Theorem 3.18. The case of general n is due to Nazarov, Tolsa and Volberg in [364]. The proof is reduced to Theorem 10.2 via a $T(b)$-theorem from [433] in a similar manner as was argued for analytic functions.

In the plane Tolsa's characterization, Theorem 9.3, is valid also for Lipschitz harmonic functions. Hence

Theorem 10.10 *Let $E \subset \mathbb{C}$ be compact. Then the following are equivalent:*

(1) *E is removable for bounded analytic functions.*
(2) *E is removable for Lipschitz harmonic functions.*
(3) *If $\mu \in \mathcal{M}(E)$ is such that $\mu(B(z,r)) \le r$ for $z \in \mathbb{C}$ and $r > 0$, then $c^2(\mu) = \infty$.*

The equivalence of (1) and (2) is only known passing through (3), and hence by a very complicated proof. It is easy to see that (1) implies (2): if u is Lipschitz harmonic, then $\partial_{\bar z}\partial_z u = \Delta u = 0$, so $f = \partial_z u$ is bounded analytic. The converse, to get u from f, would require some kind of integration, which is possible in some special cases but maybe not always.

Tolsa [420] characterized also the removable sets for Lipschitz harmonic functions in all dimensions:

Theorem 10.11 *Let $E \subset \mathbb{R}^n$ be compact. Then E is not removable for Lipschitz harmonic functions if and only if there exists $\mu \in \mathcal{M}(E)$ such that $\mu(B(x,r)) \le r^{n-1}$ for $x \in \mathbb{R}^n$ and $r > 0$ and*

$$\int_0^\infty \int \beta_\mu^{n-1,2}(x,r)^2 \frac{\mu(B(x,r))}{r^{n-1}}\, d\mu x \frac{1}{r}\, dr < \infty.$$

This is a consequence of Theorem 10.5.

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10.5 Parabolic Singular Integrals

In [75,77], it is shown that a large class of parabolic singular integral operators are L^2-bounded on parabolic uniformly rectifiable sets. Recall Section 5.6; as mentioned there the motivation for this theory comes from the heat equation $\Delta_x u(x,t) = \partial_t u(x,t)$. Its fundamental solution W is given for $x \in \mathbb{R}^n, t > 0$, by

$$W(x,t) = ct^{-n/2}e^{-|x|^2/(4t)}.$$

The kernel K, which now replaces the Riesz kernel R_{n-1}, is the heat kernel

$$K(x,t) = \nabla_x W(x,t) = -(c/2)t^{-n/2-1}xe^{-|x|^2/(4t)},$$

with $K(x,t) = 0$, when $t \leq 0$. Notice that it is antisymmetric only in the x variable, but this is good enough for the L^2-boundedness on uniformly rectifiable sets by [75]. The converse, the analogues of Theorems 10.2 and 10.4, is unknown. These would be needed to get a removability result such as Theorem 10.9 for parabolic regular Lipschitz (again BMO in t variable, recall Section 5.6) solutions of the heat equation. Mateu, Prat and Tolsa have done in [317] some preliminary work in this direction. For instance, they showed that positive measure subsets of parabolic regular Lipschitz graphs are not removable. They also constructed Cantor sets with positive measure that are removable.

We shall discuss harmonic measure, induced by the Laplace equation, in Chapter 11. In the same way, the heat equation leads to the caloric measure. To get something like Theorem 11.2 in the parabolic case would also seem to require information about the consequences of the L^2-boundedness of T_K.

10.6 Heisenberg Groups

As compared to the Euclidean theories, rather little is known about the singular integrals

$$T_K f(p) = \int K(p^{-1} \cdot q) f(q)\, d\mu q$$

on lower than full-dimensional subsets of the Heisenberg groups. As was seen in Chapter 8, rectifiability is much better understood than uniform rectifiability. Anyway, intrinsic Lipschitz graphs are good candidates for basic uniformly rectifiable sets. So it makes sense to ask the following questions. For what kernels are the singular integral operators L^2-bounded on intrinsic Lipschitz graphs? When does L^2-boundedness or existence of principal values imply rectifiability?

What kernels should replace the Riesz kernels? If we just look at the expression and the scaling property of R_m in $\mathbb{R}^n$, a similar kernel in $\mathbb{H}^n$ is $R_m =$

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$(R_{m,1}, R_{m,2})$, where for $p = (z,t) \in \mathbb{H}^n$ and $\|p\| = (|z|^4 + t^2)^{1/4}$,

$$R_{m,1}(p) = \|p\|^{-m-1} z \text{ and } R_{m,2}(p) = \|p\|^{-m-2} t.$$

In [106], an analogue of Proposition 10.3 was proved for these kernels. However, if we want a connection to harmonicity, we should start from a Laplacian. Let $X_i, Y_i, i = 1, \ldots, n$ be the vector fields as in Section 8.2. The sub-Riemannian or Kohn Laplacian in $\mathbb{H}^n$ is defined by

$$\Delta_H = \sum_{i=1}^{n} (X_i^2 + Y_i^2).$$

For the potential theory related to it, see [71]. The fundamental solution of $\Delta_H u = 0$ is $\Gamma(p) = c\|p\|^{-2n}$. Note that $2n = \dim \mathbb{H}^n - 2$, so it has the same form as in $\mathbb{R}^n$. The kernel related to the Lipschitz harmonic functions is $K = \nabla_H \Gamma \colon \mathbb{H}^n \to \mathbb{R}^{2n}$. It looks a bit complicated, since its coordinate functions are for $i = 1, \ldots, n$, and $z = (x,y) \in \mathbb{R}^{2n}$,

$$K_i(z,t) = \frac{x_i|z|^2 + y_i t}{\|p\|^{2n+4}}, \quad K_{i+n}(z,t) = \frac{y_i|z|^2 - x_i t}{\|p\|^{2n+4}}.$$

However, it is a reasonable Calderón–Zygmund kernel. It is not odd, but it is horizontally antisymmetric: $K(z,t) = -K(-z,t)$. Due to this, Chousionis, Fässler and Orponen [98] were able to prove in $\mathbb{H}^1$ that T_K, and more general singular integrals, are L^2-bounded on $C^{1,\alpha}$ intrinsic graphs, see also [195] for related results.

Orponen [372] proved that for AD-3-regular subsets of $\mathbb{H}^1$ the L^2-boundedness of the three-dimensional singular integrals with horizontally antisymmetric kernels implies local symmetry of the type in Theorem 5.9 and that this implies weak geometric lemma.

In [107], a class of self-similar purely $(2n+1, \mathbb{H})$-unrectifiable subsets of $\mathbb{H}^n$ with positive $\mathcal{H}^{2n+1}$ measure was introduced on which T_K is L^2-unbounded and which are removable for Lipschitz harmonic functions. Further results in this direction were proven by Chousionis and Urbanski in [109].

Chousionis and Li [100] introduced a class of non-negative 1-homogeneous kernels in $\mathbb{H}^1$ which vanish on the vertical axis $\{z = 0\}$. For some of them, the operator is L^2-bounded on regular curves and for some the L^2-boundedness on an AD-1-regular set implies that it is contained in a regular curve. An extension of the first statement to general Carnot groups is given by Chousionis, Li and Zimmerman [103].

Fässler and Orponen [197] proved in $\mathbb{H}^1$ the L^2-boundedness of many singular integral operators on AD-regular curves and on a class of vertical graphs called Lipschitz flags.