ANZIAM J. 57(2016), 445–460
doi:10.1017/S1446181116000055

BIFURCATION ANALYSIS OF A LOGISTIC PREDATOR–PREY SYSTEM WITH DELAY

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(Received 20 March, 2015; accepted 18 November, 2015; first published online 12 May 2016)

Abstract

We consider a coupled, logistic predator–prey system with delay. Mainly, by choosing the delay time τ as a bifurcation parameter, we show that Hopf bifurcation can occur as the delay time τ passes some critical values. Based on the normal-form theory and the centre manifold theorem, we also derive formulae to obtain the direction, stability and the period of the bifurcating periodic solution at critical values of τ. Finally, numerical simulations are investigated to support our theoretical results.

2010 *Mathematics subject classification*: 92D25.

Keywords and phrases: predator–prey system, delayed logistic differential equation, Hopf bifurcation, stability.

1. Introduction

In the last few decades, the study of dynamical systems of population models has received much attention by theoreticians and experimentalists. In order to observe the effect of the past information on the system, a time delay is incorporated into the population models. Among these models, the delayed predator–prey system has an important role. Especially, the Hopf bifurcation of periodic solutions of delayed systems has received great attention. In particular, the properties of periodic solutions appearing through the Hopf bifurcations in delayed systems are of great interest (see [1, 4, 7–18, 22] and the references therein). In 1973, May [15] first proposed and briefly discussed the following delayed predator–prey system:

$$\begin{aligned} \dot{x}(t) &= x(t)[r_1 - a_{11}x(t-\tau) - a_{12}y(t)], \\ \dot{y}(t) &= y(t)[-r_2 + a_{21}x(t) - a_{22}y(t)], \end{aligned} \tag{1.1}$$

where $x(t)$ and $y(t)$ are the population densities of prey and predator at time t, respectively; $\tau \geqslant 0$ is the time delay of the prey to the growth of the species itself;

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$r_1 > 0$ denotes the intrinsic growth rate of the prey and $r_2 > 0$ denotes the death rate of the predator; the parameters a_{ij} $(i, j = 1, 2)$ are all positive constants. System (1.1) shows that in the absence of the predator species, the prey species is governed by the well-known delay logistic equation $\dot{x}(t) = x(t)[r_1 - a_{11}x(t - \tau)]$ and the predator species decreases in the absence of the prey species.

In many of these studies, the authors have mainly considered the boundedness of the solutions, persistence, local and global stabilities of equilibria, and existence of nonconstant periodic solutions.

In 2005, Song and Wei [19] worked on the dynamics of the system (1.1) by considering the delay time τ as the bifurcation parameter, and obtained that the positive equilibrium is asymptotically stable under certain conditions. However, it is conditionally stable under some other conditions and Yan and Li [20] considered the same delay time τ in the population density of the predator in the second equation of system (1.1), namely,

$$\begin{aligned} \dot{x}(t) &= x(t)[r_1 - a_{11}x(t - \tau) - a_{12}y(t)], \\ \dot{y}(t) &= y(t)[-r_2 + a_{21}x(t) - a_{22}y(t - \tau)]. \end{aligned} \tag{1.2}$$

They found that the unique positive equilibrium of system (1.2) is no longer absolutely stable and the switches from stability to instability and again back to stability disappear. Moreover, by using the normal-form theory and the centre manifold theorem, they obtained the properties of bifurcating periodic solutions.

In addition, Faria [3] studied the following system with two different discrete delays:

$$\begin{aligned} \dot{x}(t) &= x(t)[r_1 - a_{11}x(t) - a_{12}y(t - \tau_1)], \\ \dot{y}(t) &= y(t)[-r_2 + a_{21}x(t - \tau_2) - a_{22}y(t)], \end{aligned} \tag{1.3}$$

where $\tau_1 \geq 0$ and $\tau_2 > 0$. Mainly, the author took τ_2 as the bifurcation parameter to analyze the stability of the interior positive equilibrium, and also obtained the existence of the local Hopf bifurcation and the direction of the stability of bifurcating periodic solutions from the Hopf bifurcation.

Furthermore, Yan and Zhang [21] combined the models (1.2) and (1.3) and considered the following delayed predator–prey model with a single delay:

$$\begin{aligned} \dot{x}(t) &= x(t)[r_1 - a_{11}x(t - \tau) - a_{12}y(t - \tau)], \\ \dot{y}(t) &= y(t)[-r_2 + a_{21}x(t - \tau) - a_{22}y(t - \tau)]. \end{aligned} \tag{1.4}$$

In system (1.1), if we consider the time delay of the predator species to the growth of the species itself and also the delay τ, then system (1.1) should be modified as the following delayed predator–prey system:

$$\begin{aligned} \dot{x}(t) &= x(t)[r_1 - a_{11}x(t) - a_{12}y(t - \tau)], \\ \dot{y}(t) &= y(t)[-r_2 + a_{21}x(t) - a_{22}y(t - \tau)], \end{aligned} \tag{1.5}$$

where $\tau > 0$ is the feedback time delay of the predator species to the growth of the species itself, $r_1 > 0$ denotes the intrinsic growth rate of the prey and $r_2 > 0$ denotes the death rate of the predator.

Our aim is to investigate the stability of the delayed predator–prey system (1.5) and investigate how the delay time τ affects the dynamics of this system. To analyze the system, first we study the local stability of the equilibrium point of the corresponding characteristic equation of the system and obtain the general stability criteria involving the time delay. Second, by choosing the delay τ as bifurcation parameter, we show that the positive equilibrium loses its stability and the equation exhibits Hopf bifurcation. Then, based on the approach of normal-form and centre manifold theory introduced by Hassard et al. [6], we derive the formula for determining the properties of Hopf bifurcation of the model. More specifically, it is shown that the Hopf bifurcation is subcritical and the bifurcating periodic solutions are unstable under certain conditions. Finally, to support these theoretical results, we illustrate by numerical simulations.

This paper is organized as follows. In Section 2, we first focus on the stability and Hopf bifurcation of the positive equilibrium and, in Section 3, we determine the direction and stability of Hopf bifurcation by using normal-form and central manifold theory. In Section 4, numerical simulations are performed to support the stability results. Finally, concluding remarks are presented in Section 5.

2. Stability analysis and Hopf bifurcation

Note that the system (1.5) has equilibria $E_1 = (0, 0)$, $E_2 = (r_1/a_{11}, 0)$, $E_3 = (0, -r_2/a_{22})$ and always has a unique positive equilibrium $E^* = (x^*, y^*)$, provided that the condition

$$(H) \quad r_1a_{21} - r_2a_{11} > 0$$

holds, where

$$x^* = \frac{r_1a_{22} + r_2a_{12}}{a_{11}a_{22} + a_{12}a_{21}}, \quad y^* = \frac{r_1a_{21} - r_2a_{11}}{a_{11}a_{22} + a_{12}a_{21}}.$$

Under the hypothesis (H), and assuming $u_1(t) = x(t) - x^*$ and $u_2(t) = y(t) - y^*$, we can rewrite (1.5) as the following equivalent system:

$$\begin{aligned} \dot{u}_1(t) &= (u_1(t) + x^*)[-a_{11}u_1(t) - a_{12}u_2(t - \tau)], \\ \dot{u}_2(t) &= (u_2(t) + y^*)[a_{21}u_1(t) - a_{22}u_2(t - \tau)]. \end{aligned} \tag{2.1}$$

The linearization of (2.1) at $(0, 0)$ is

$$\begin{aligned} \dot{u}_1(t) &= -a_{11}x^*u_1(t) - a_{12}x^*u_2(t - \tau), \\ \dot{u}_2(t) &= a_{21}y^*u_1(t) - a_{22}y^*u_2(t - \tau). \end{aligned} \tag{2.2}$$

Its characteristic equation is

$$\lambda^2 + p\lambda + (q\lambda + s)e^{-\lambda\tau} = 0, \tag{2.3}$$

where $p = a_{11}x^*$, $q = a_{22}y^*$ and $s = (a_{11}a_{22} + a_{12}a_{21})x^*y^*$.

When there is no delay, that is, $\tau = 0$, the corresponding characteristic equation (2.3) reduces to

$$\lambda^2 + p\lambda + q\lambda + s = 0,$$

https://doi.org/10.1017/S1446181116000055 Published online by Cambridge University Press

and the corresponding eigenvalues are $\lambda_{1,2} = \{-p - q \pm \sqrt{(p+q)^2 - 4s}\}/2$. Since $p = a_{11}x^*$, $q = a_{22}y^*$ and $s = (a_{11}a_{22} + a_{12}a_{21})x^*y^*$ are positive constants, we first have the following result for equation (2.3).

LEMMA 2.1. *The two roots* $\lambda_{1,2} = \{-p - q \pm \sqrt{(p+q)^2 - 4s}\}/2$ *of equation* (2.3) *with* $\tau = 0$ *have always negative real parts, that is, the equilibrium point* $(0, 0)$ *for the linearized system* (2.2) *with* $\tau = 0$ *is asymptotically stable.*

By the Hartman–Grobman theorem [5], since eigenvalues of the linearized system (2.2) have nonzero real parts, the qualitative behaviour of solutions for the nonlinear system (1.5) is the same as the linearized system (2.2) in a neighbourhood of the equilibrium point $E^* = (x^*, y^*)$.

Now we investigate the distribution of roots of the transcendental equation (2.3), since the stability of the point $(0, 0)$ of the linear system (2.2) depends on the locations of the roots of the characteristic equation (2.3). By the roots of $\lambda^2 + p\lambda + (q\lambda + s)e^{-\lambda\tau} = 0$ and Lemma 2.1, there exists $\tau_0 > 0$ such that $\mathrm{Re}\lambda(\tau) < 0$ for $\tau \epsilon\ [0, \tau_0)$. Since a loss of asymptotic stability of (x^*, y^*) arises when $\mathrm{Re}\lambda(\tau) = 0$, we examine whether there exists a $\tau^* > 0$ for which $\mathrm{Re}\lambda(\tau^*) = 0$, that is, we would like to know when equation (2.3) has purely imaginary roots. In this section, we first obtain the conditions of local stability of the equilibrium point.

Suppose that for $\tau = \tau^*$, we have $\lambda = i\omega$ with $\omega > 0$; then we have the following result.

LEMMA 2.2. *For the system* (2.1), *transcendental equation* (2.3) *has one purely imaginary root.*

PROOF. For $\tau = \tau^*$, let $\lambda = i\omega$ be a root of equation (2.3) with ω real and without loss of generality $\omega > 0$. Then

$$(i\omega)^2 + p(i\omega) + (q(i\omega) + s)e^{-(i\omega)\tau} = 0,$$

that is,

$$-\omega^2 + [i(q\omega) + s][\cos(\omega\tau) - i\sin(\omega\tau)] + ip\omega = 0.$$

Separating real and imaginary parts,

$$\omega^2 = q\omega\sin(\omega\tau) + s\cos(\omega\tau) \quad \text{and} \quad -p\omega = q\omega\cos(\omega\tau) - s\sin(\omega\tau),$$

which is equivalent to

$$\omega^4 + (p^2 - q^2)\omega^2 + s^2 = 0.$$

Let $\omega^2 = t$; then

$$t^2 + (p^2 - q^2)t + s^2 = 0, \tag{2.4}$$

which implies that this equation governs the possible values of τ and ω for which $\lambda^2 + p\lambda + (q\lambda + s)e^{-\lambda\tau} = 0$ can have purely imaginary roots.

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Without loss of generality, we denote $\omega = \sqrt{t}$ and, solving the equation (2.4) for ω,

$$\omega = \left[\frac{-(p^2 - q^2) + \sqrt{(p^2 - q^2)^2 + 4s^2}}{2}\right]^{1/2}$$

and

$$\tau_k = \frac{1}{\omega}\arctan\left(\frac{ps + q\omega^2}{\omega s(1 - pq)}\right) + \frac{\pi k}{\omega}, \quad k = 0, 1, 2, \ldots,$$

which completes the proof of the lemma. □

We denote

$$\omega_0 = \left[\frac{-(p^2 - q^2) + \sqrt{(p^2 - q^2)^2 + 4s^2}}{2}\right]^{1/2}$$

and suppose that $\lambda_k(\tau) = \alpha_k(\tau) + i\omega_k(\tau)$ denotes the root of (2.3) near $\tau = \tau_k$ satisfying $\alpha_k(\tau_k) = 0$ and $\omega_k(\tau_k) = \omega_0$, $k = 0, 1, 2, \ldots$. Then we have the following transversality conditions.

Lemma 2.3. *The following transversality conditions are satisfied.*

$$\frac{d\,\mathrm{Re}\lambda_k(\tau_k)}{d\tau} > 0 \quad \textit{for } k = 0, 1, 2, \ldots,$$

that is, system (1.5) undergoes Hopf bifurcation at the positive equilibrium point (x^*, y^*) for $\tau = \tau_k$, $k = 0, 1, 2, \ldots$.

Proof. Differentiating the characteristic equation (2.3) with respect to τ,

$$2\lambda\frac{d\lambda}{d\tau} + q\left[\frac{d\lambda}{d\tau}e^{-\lambda\tau} - \lambda e^{-\lambda\tau}\left(\frac{d\lambda}{d\tau}\tau + \lambda\right)\right] + p\frac{d\lambda}{d\tau} - se^{-\lambda\tau}\left(\frac{d\lambda}{d\tau}\tau + \lambda\right) = 0,$$

that is,

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = -\frac{\tau}{\lambda} + \frac{2\lambda + qe^{-\lambda\tau} + p}{\lambda e^{-\lambda\tau}(q\lambda + s)}.$$

Thus,

$$\begin{aligned}\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\Bigg|_{\lambda=i\omega_0} &= \mathrm{Re}\left[-\frac{\tau_k}{i\omega_0}\right] + \mathrm{Re}\left[\frac{2i\omega_0 + qe^{-i\omega_0\tau_k} + p}{i\omega_0 e^{-i\omega_0\tau_k}(qi\omega_0 + s)}\right] \\ &= \frac{sp\sin(\omega_0\tau_k)}{\omega_0(q^2\omega_0^2 + s^2)} > 0.\end{aligned}$$

By using Rouche's theorem [8], we observe that the transversality condition holds and the conditions for Hopf bifurcation are satisfied at $\tau = \tau_k$, $k = 0, 1, 2, \ldots$, which completes the proof of the lemma. □

Summarizing the results above, we have the following theorem on stability and Hopf bifurcation of system (1.5).

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THEOREM 2.4. *For system* (1.5)*, the following hold:*

(i) *if* $\tau\epsilon[0,\tau_0)$, *then the equilibrium point of system* (1.5) *is asymptotically stable;*
(ii) *if* $\tau > \tau_0$, *then the equilibrium point* (x^*, y^*) *of system* (1.5) *is unstable;*
(iii) *if* $\tau = \tau_k$ $(k = 0, 1, 2, \ldots)$*, then system* (1.5) *undergoes a Hopf bifurcation at the equilibrium point* (x^*, y^*).

REMARK 2.5. It must be pointed out that Theorem 2.4 cannot determine the stability and the direction of bifurcating periodic solutions. In the following section, we investigate the stability of bifurcating periodic solutions by using the normal-form theory and the centre manifold theorem due to Hassard et al. [6], and prove that the Hopf bifurcation is subcritical and bifurcating periodic solutions are unstable.

3. Direction and stability of Hopf bifurcation

We study the direction and stability of Hopf bifurcation for which we obtain the necessary conditions for bifurcating periodic solutions in Section 2. For determining the direction and stability of bifurcating periodic solutions, we apply the normal-form theory and the centre manifold theorem by Hassard et al. [6].

Throughout this section, we suppose that the system (1.5) undergoes Hopf bifurcation at the positive equilibrium point (x^*, y^*) for $\tau = \tau_k$, and $i\omega_0$ is the corresponding purely imaginary root of the characteristic equation at the positive equilibrium point (x^*, y^*). For the sake of simplicity, we use the notation $i\omega$ for $i\omega_0$ and t/τ for t.

We first consider the system (1.5) by the transformation

$$u_1(t) = x(\tau t) - x^*, \quad u_2(t) = y(\tau t) - y^*, \quad \tau = \tau_k + \mu,$$

which is equivalent to the following functional differential equation (FDE) system in $C = C([-1, 0], \mathbb{R}^2)$:

$$\begin{aligned} \dot{u}_1(t) &= \tau(u_1(t) + x^*)[-a_{11}u_1(t) - a_{12}u_2(t-1)],\\ \dot{u}_2(t) &= \tau(u_2(t) + y^*)[a_{21}u_1(t) - a_{22}u_2(t-1)]. \end{aligned} \tag{3.1}$$

For $\phi = (\phi_1, \phi_2) \in C$, the functions $L_\mu : C \to \mathbb{R}$, $f : \mathbb{R} \times C \to \mathbb{R}$ are defined as

$$L_\mu\phi = (\tau_k + \mu)\begin{bmatrix} -a_{11}x^*\phi_1(0) - a_{12}x^*\phi_2(-1)\\ a_{21}y^*\phi_1(0) - a_{22}y^*\phi_2(-1) \end{bmatrix}$$

and

$$f(\mu, \phi) = \begin{bmatrix} -a_{11}\phi_1(0)\phi_1(0) - a_{12}\phi_1(0)\phi_2(-1)\\ a_{21}\phi_2(0)\phi_1(0) - a_{22}\phi_2(0)\phi_2(-1) \end{bmatrix},$$

respectively. By the Riesz representation theorem [2], there exists a function $\eta(\theta, \mu)$ of bounded variation for $\theta \in [-1, 0]$, such that

$$L_\mu\phi = \int_{-1}^{0} d\eta(\theta, 0)\phi(\theta) \quad \text{for } \phi \in C,$$

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where the bounded variation function $\eta(\theta,\mu)$ can be chosen as

$$\eta(\theta,\mu) = (\tau_k+\mu)\begin{bmatrix}-a_{11}x^* & 0\\ a_{21}y^* & 0\end{bmatrix} + (\tau_k+\mu)\begin{bmatrix}0 & -a_{12}x^*\\ 0 & -a_{22}y^*\end{bmatrix}.$$

For $\phi \in C^1([-1,0],\mathbb{R}^2)$, we define

$$A(\mu)\phi = \begin{cases} -\dfrac{d\phi(\theta)}{d\theta}, & \theta\in[-1,0),\\ \displaystyle\int_{-1}^{0} d\eta(\mu,s)\phi(s), & \theta=0\end{cases}$$

and

$$R(\mu)\phi = \begin{cases} 0, & \theta\in[-1,0),\\ f(\mu,\phi), & \theta=0.\end{cases}$$

Then the system (3.1) is equivalent to

$$\dot u_t = A(\mu)u_t + R(\mu)u_t, \tag{3.2}$$

where $u_t(\theta)=u(t+\theta)$ for $\theta\in[-1,0)$. For $\psi\in C^1([-1,0],(\mathbb{R}^2)^*)$, we define

$$A^*\psi(s) = \begin{cases} -\dfrac{d\psi(s)}{ds}, & s\in(0,1],\\ \displaystyle\int_{-1}^{0} d\eta^T(t,0)\psi(-t), & s=0\end{cases}$$

and a bilinear inner product

$$\langle\psi(s),\phi(\theta)\rangle = \bar\psi(0)\phi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta}\bar\psi(\xi-\theta)\,d\eta(\theta)\phi(\xi)\,d\xi,$$

where $\eta(\theta)=\eta(\theta,0)$. Then $A(0)$ and A^* are adjoint operators. Suppose that $q(\theta)$ and $q^*(s)$ are eigenvectors of A and A^* corresponding to $i\omega\tau_k$ and $-i\omega\tau_k$, respectively. Then suppose that $q(\theta)=(1,\alpha)^T e^{i\omega\tau_k\theta}$ is the eigenvector of $A(0)$ corresponding to $i\omega\tau_k$; then $A(0)q(\theta)=i\omega\tau_k q(\theta)$. It follows from the definition of $A(0)$, $L_\mu\phi$ and $\eta(\theta,\mu)$ that

$$q(\theta) = (1,\alpha)^T e^{i\omega\tau_k\theta},$$

where $\alpha=-i\omega/a_{11}x^*$ and $q(0)=(1,\alpha)^T$.

Similarly, let $q^*(s)=D(\beta,1)e^{i\omega\tau_k s}$ be the eigenvector of A^* corresponding to $-i\omega\tau_k$. By definition of A^*,

$$q^*(s) = D(\beta,1)e^{i\omega\tau_k s} = D\Big(-\frac{a_{22}y^*e^{-i\omega\tau_k}-i\omega}{-a_{12}x^*}e^{-i\omega\tau_k},1\Big)e^{i\omega\tau_k s}.$$

To satisfy $\langle q^*(s),q(\theta)\rangle=1$, we need to evaluate the value of D. From the definition of the bilinear inner product,

$$\begin{aligned}\langle q^*(s),q(\theta)\rangle &= \bar D(\bar\beta,1)(1,\alpha)^T - \int_{-1}^{0}\int_{\xi=0}^{\theta}\bar D(\bar\beta,1)e^{-i\omega\tau_k(\xi-\theta)}\,d\eta(\theta)(1,\alpha)^T e^{i\omega\tau_k\xi}\,d\xi\\ &= \bar D\Big\{\alpha+\bar\beta - \int_{-1}^{0}(\bar\beta,1)\theta e^{i\omega\tau_k\theta}\,d\eta(\theta)(1,\alpha)^T\Big\}\\ &= \bar D\{\alpha+\bar\beta+\tau_k(-a_{22}\alpha y^*)e^{-i\omega\tau_k}\}.\end{aligned}$$

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Thus, we can choose $\bar{D}$ as

$$\bar{D} = \frac{1}{\alpha + \overline{\beta} + \tau_k(-a_{22}\alpha y^*)e^{-i\omega\tau_k}},$$

such that $\langle q^*(s), q(\theta)\rangle = 1$ and $\langle q^*(s), \bar{q}(\theta)\rangle = 0$.

In the following, we use the theory by Hassard et al. [6] to compute the coordinates describing the centre manifold C_0 at $\mu = 0$. We define

$$z(t) = \langle q^*, u_t\rangle, \quad W(t,\theta) = u_t - 2\mathrm{Re}z(t)q(\theta). \tag{3.3}$$

On C_0,

$$W(t,\theta) = W(z(t), \bar{z}(t), \theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots,$$

where z and $\bar{z}$ are local coordinates for the centre manifold C_0 in the directions of q and $\bar{q}^*$, respectively. Note that W is real if u_t is real. We consider only real solutions. For the solution $u_t \in C_0$, since $\mu = 0$ and from (3.2),

$$\begin{aligned}
\dot{z} &= i\omega\tau_k z + \langle q^*(\theta), f(0, W(z,\bar{z},\theta) + 2\mathrm{Re}zq(\theta))\rangle \\
&= i\omega\tau_k z + \bar{q}^*(0)f_0(z,\bar{z}) \\
&= i\omega\tau_k z + g(z,\bar{z}),
\end{aligned}$$

where

$$g(z,\bar{z}) = \bar{q}^*(0)f_0(z,\bar{z}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots. \tag{3.4}$$

By using (3.3), we have $u_t(u_{1t}(\theta), u_{2t}(\theta)) = W(t,\theta) + zq(\theta) + \overline{zq}(\theta)$ and $q(\theta) = (1,\alpha)^T e^{i\omega\tau_k\theta}$, and then

$$\begin{aligned}
u_{1t}(0) &= z + \bar{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\bar{z} + W_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + O(|(z,\bar{z})|^3), \\
u_{2t}(0) &= z\alpha + \overline{z\alpha} + W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0)z\bar{z} + W_{02}^{(2)}(0)\frac{\bar{z}^2}{2} + O(|(z,\bar{z})|^3), \\
u_{1t}(-1) &= ze^{-i\omega\tau_k\theta} + \bar{z}e^{i\omega\tau_k\theta} + W_{20}^{(1)}(-1)\frac{z^2}{2} + W_{11}^{(1)}(-1)z\bar{z} + W_{02}^{(1)}(-1)\frac{\bar{z}^2}{2} + O(|(z,\bar{z})|^3), \\
u_{2t}(-1) &= z\alpha e^{-i\omega\tau_k\theta} + \bar{z}\bar{\alpha}e^{i\omega\tau_k\theta} + W_{20}^{(2)}(-1)\frac{z^2}{2} + W_{11}^{(2)}(-1)z\bar{z} + W_{02}^{(2)}(-1)\frac{\bar{z}^2}{2} + O(|(z,\bar{z})|^3).
\end{aligned}$$

To simplify the notation, let

$$M = -\frac{a_{11}x^* + i\omega}{a_{12}x^*}e^{i\omega\tau_k} \quad \text{and} \quad N = -\frac{a_{22}y^*e^{-i\omega\tau_k} - i\omega}{-a_{12}x^*}e^{-i\omega\tau_k}.$$

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Thus,

$$
\begin{aligned}
g(z,\bar{z}) = \overline{DN}\tau_k\Big\{&2(-a_{11}-a_{12}Me^{-i\omega\tau_k})\frac{z^2}{2}+2(-a_{11}-a_{12}\overline{M}e^{i\omega\tau_k})\frac{\bar{z}^2}{2}\\
&+2(-2a_{11}-a_{12}\mathrm{Re}\{\overline{M}e^{i\omega\tau_k}\})z\bar{z}\\
&+[W_{20}^{(1)}(0)(-2a_{11}-a_{12}\overline{M}e^{i\omega\tau_k})-a_{12}[W_{20}^{(2)}(-1)+2W_{11}^{(2)}(-1)]\\
&+W_{11}^{(1)}(0)(-2a_{12}Me^{-i\omega\tau_k}-4a_{11})\}\frac{z\bar{z}^2}{2}+\cdots\Big\}\\
&+\overline{D}\tau_k\Big\{2(a_{21}-a_{22}Me^{-i\omega\tau_k})M\frac{z^2}{2}+2(a_{21}-a_{22}\overline{M}e^{i\omega\tau_k})\overline{M}\frac{\bar{z}^2}{2}\\
&+2(a_{21}(M+\overline{M})-a_{22}|M|^2\mathrm{Re}\{e^{i\omega\tau_k}\})z\bar{z}\\
&+[W_{20}^{(2)}(0)(a_{21}-a_{22}\overline{M}e^{i\omega\tau_k})+2W_{11}^{(2)}(0)(a_{21}-a_{22}Me^{-i\omega\tau_k})\\
&+a_{21}[W_{20}^{(1)}(0)\overline{M}+2W_{11}^{(1)}(0)M]\\
&-a_{22}(W_{20}^{(2)}(-1)\overline{M}+2W_{11}^{(2)}(-1)M)]z^2\bar{z}+\cdots\Big\}.
\end{aligned}
$$

By comparing the coefficients with the equation (3.4),

$$
\begin{aligned}
g_{20} &= -2\overline{DN}\tau_k(a_{11}+a_{12}Me^{-i\omega\tau_k})+2\overline{D}\tau_k(a_{21}-a_{22}Me^{-i\omega\tau_k})M,\\
g_{11} &= -2\overline{DN}\tau_k(2a_{11}+a_{12}\overline{M}e^{i\omega\tau_k})+2\overline{D}\tau_k(2a_{21}\mathrm{Re}\{M\}-a_{22}|M|^2\mathrm{Re}\{e^{i\omega\tau_k}\}),\\
g_{02} &= -2\overline{DN}\tau_k(a_{11}+a_{12}\overline{M}e^{i\omega\tau_k})+2\overline{D}\tau_k(a_{21}-a_{22}\overline{M}e^{i\omega\tau_k})\overline{M},\\
g_{21} &= -\overline{DN}\tau_k[W_{20}^{(1)}(0)(2a_{11}+a_{12}\overline{M}e^{i\omega\tau_k})+a_{12}(W_{20}^{(2)}(-1)+2W_{11}^{(2)}(-1))\\
&\quad+2W_{11}^{(1)}(0)(a_{12}Me^{-i\omega\tau_k}+2a_{11})]+\overline{D}\tau_k[W_{20}^{(2)}(0)(a_{21}-a_{22}\overline{M}e^{i\omega\tau_k})\\
&\quad+2W_{11}^{(2)}(0)(a_{21}-a_{22}Me^{-i\omega\tau_k})+a_{21}(W_{20}^{(1)}(0)\overline{M}+2W_{11}^{(1)}(0)M)\\
&\quad-a_{22}(W_{20}^{(2)}(-1)\overline{M}+2W_{11}^{(2)}(-1)M)].
\end{aligned}
$$

Here

$$
\begin{aligned}
W_{20}(\theta) &= \frac{ig_{20}}{\tau_k\omega}q(0)e^{i\omega\tau_k\theta}+\frac{i\bar{g}_{02}}{3\tau_k\omega}\bar{q}(0)e^{-i\omega\tau_k\theta}+E_1e^{2i\omega\tau_k\theta},\\
W_{11}(\theta) &= -\frac{ig_{11}}{\tau_k\omega}q(0)e^{i\omega\tau_k\theta}+\frac{i\bar{g}_{11}}{\tau_k\omega}\bar{q}(0)e^{-i\omega\tau_k\theta}+E_2,
\end{aligned}
$$

and $E_1=(E_1^{(1)},E_1^{(2)})\in\mathbb{R}^2$ and $E_2=(E_2^{(1)},E_2^{(2)})\in\mathbb{R}^2$ are constant vectors with

$$
\begin{aligned}
E_1^{(1)} &= \frac{1}{A_1}\begin{vmatrix}2(-a_{11}-a_{12}Me^{-i\omega\tau_k}) & a_{12}x^*e^{-2i\omega\tau_k}\\ 2M(a_{21}-a_{22}Me^{-i\omega\tau_k}) & 2i\omega+a_{22}y^*e^{-2i\omega\tau_k}\end{vmatrix},\\
E_1^{(2)} &= \frac{1}{A_1}\begin{vmatrix}2i\omega+a_{11}x^* & a_{12}x^*e^{-2i\omega\tau_k}\\ -a_{21}y^* & 2i\omega+a_{22}y^*e^{-2i\omega\tau_k}\end{vmatrix},
\end{aligned}
$$

where

$$
A_1=\begin{vmatrix}2wi+a_{11}x^* & a_{12}x^*e^{-2iw\tau_k}\\ -a_{21}y^* & 2wi+a_{22}y^*e^{-2iw\tau_k}\end{vmatrix}
$$

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and

$$E_2^{(1)} = -a_{22}\frac{4a_{11} + 2a_{12}\mathrm{Re}(Me^{iw\tau_k})}{a_{11}a_{22}x^* + a_{12}a_{21}x^*} - a_{12}\frac{4a_{21}\mathrm{Re}(M) - 2a_{22}\mathrm{Re}(e^{iw\tau_k})|M|^2}{a_{11}a_{22}y^* + a_{12}a_{21}y^*},$$
$$E_2^{(2)} = a_{11}\frac{x^*}{y^*}\frac{4a_{21}\mathrm{Re}(M) - 2a_{22}\mathrm{Re}(e^{iw\tau_k})|M|^2}{a_{11}a_{22}x^* + a_{12}a_{21}x^*} - a_{21}\frac{4a_{11} + 2a_{12}\mathrm{Re}(Me^{iw\tau_k})}{a_{11}a_{22}x^* + a_{12}a_{21}x^*}.$$

We determine the following values to investigate the quantities of the bifurcating periodic solution in the centre manifold at the critical value τ_k. For this purpose, we express the g_{ij} in terms of the parameters and delay, and then we evaluate the following values:

$$c_1(0) = \frac{i}{2\omega\tau_k}\Big(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\Big) + \frac{g_{21}}{2},$$
$$\mu_2 = -\frac{\mathrm{Re}\{c_1(0)\}}{\mathrm{Re}\{\lambda'(\tau_k)\}},$$
$$\beta_2 = 2\mathrm{Re}\{c_1(0)\},$$
$$T_2 = -\frac{\mathrm{Im}\{c_1(0)\} + \mu_2\mathrm{Im}\{\lambda'(\tau_k)\}}{\omega\tau_k},$$

which are the quantities for determining the bifurcating periodic solutions in the centre manifold at τ_k, so that μ_2 determines the direction of Hopf bifurcation, and β_2 and T_2 state the stability of the bifurcating periodic solution and the period of the bifurcating solution, respectively. Hence, we have the following result.

Theorem 3.1. *The ratio μ_2 determines the direction of Hopf bifurcation; if $\mu_2 > 0$, then the Hopf bifurcation is supercritical and the bifurcating periodic solution exists for $\tau > \tau_0$ and, if $\mu_2 < 0$, then the Hopf bifurcation is subcritical and the bifurcating periodic solution exists for $\tau < \tau_0$. The value of β_2 determines the stability of the bifurcating periodic solution; the bifurcating periodic solution is stable if $\beta_2 < 0$ and unstable if $\beta_2 > 0$. The ratio T_2 determines the period of the bifurcating solution; the period increases if $T_2 > 0$ and decreases if $T_2 < 0$.*

4. Numerical simulations

We illustrate the numerical simulations to support our theorems that we obtained in previous sections by using the MATLAB-DDE (delay differential equation) solver. As a numerical example, we consider the following logistic predator–prey system (4.1) with the parameters $r_1 = 1.5$, $r_2 = 0.6$, $a_{11} = 0.7$, $a_{12} = 0.8$, $a_{21} = 0.45$ and $a_{22} = 0.006$, that is,

$$\begin{aligned}\dot{x}(t) &= x(t)[1.5 - 0.7x(t) - 0.8y(t-\tau)],\\ \dot{y}(t) &= y(t)[-0.6 + 0.45x(t) - 0.006y(t-\tau)],\end{aligned} \tag{4.1}$$

which has a positive equilibrium $E^* = (x^*, y^*) = (1.3427, 0.7001)$. Following the discussions from Section 2, we derive the formulae determining the direction of a Hopf bifurcation and the stability of the bifurcating periodic solution at the critical value τ.

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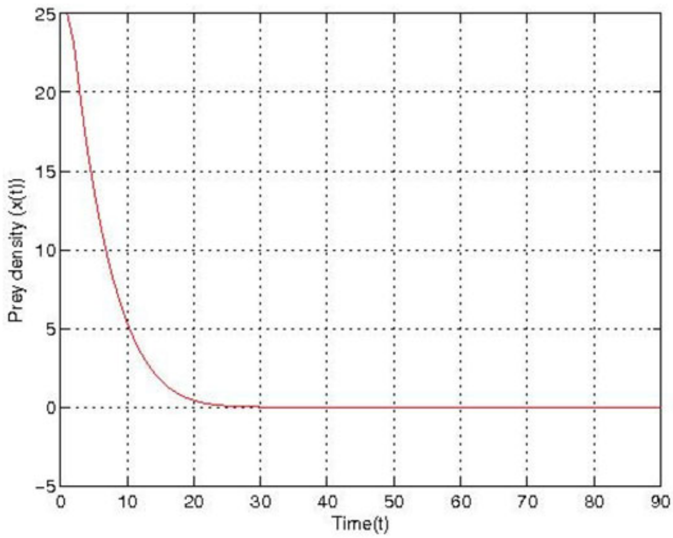

FIGURE 1. The trajectory of prey density $x(t)$ versus time with the initial conditions $x_0 = 50$, $y_0 = 25$ and $\tau = 3 < \tau_0$.

We evaluate that

$$\omega_0 = \left[\frac{-(p^2 - q^2) + \sqrt{(p^2 - q^2)^2 + 4s^2}}{2}\right]^{1/2} = 0.34227$$

and, hence, $\tau_0 = 3.5813$. So, by Theorem 2.4, the equilibrium point E^* is asymptotically stable when $\tau \in [0, \tau_0) = [0, 3.5813)$ and unstable when $\tau > 3.5813$, and also the Hopf bifurcation occurs at $\tau = \tau_0 = 3.5813$, as is illustrated in the graphs below.

By the theory of Hassard et al. [6], we may also determine the direction of the Hopf bifurcation and the other properties of bifurcating periodic solutions for this numerical example, which implies that

$$\mu_2 < 0, \quad \beta_2 > 0, \quad T_2 > 0.$$

Hence, the Hopf bifurcation of system (4.1) occurring at $\tau_0 = 3.5813$ is subcritical, and the bifurcating periodic solution exists when τ crosses τ_0 to the left; also, the bifurcating periodic solution is unstable, which is illustrated in the figures below.

In the numerical simulations, the initial conditions are taken as $(x_0, y_0) = (25, 10)$ and the MATLAB-DDE solver is used to simulate the system (4.1). Figures 1–3 clearly show that the equilibrium point E^* is asymptotically stable when $\tau \in [0, 3.5813)$. We first take $\tau = 3 < \tau_0$ and, by graphing the density functions $x(t)$ and $y(t)$ in Figures 1 and 2, we verify that the equilibrium point E^* is asymptotically stable for $\tau < \tau_0$.

In Figure 3, again taking the delay parameter $\tau = 3 < \tau_0$, we demonstrate the phase portrait of prey density $x(t)$ versus predator density $y(t)$, which also illustrates the asymptotic stability of the equilibrium point E^* in two dimensions.

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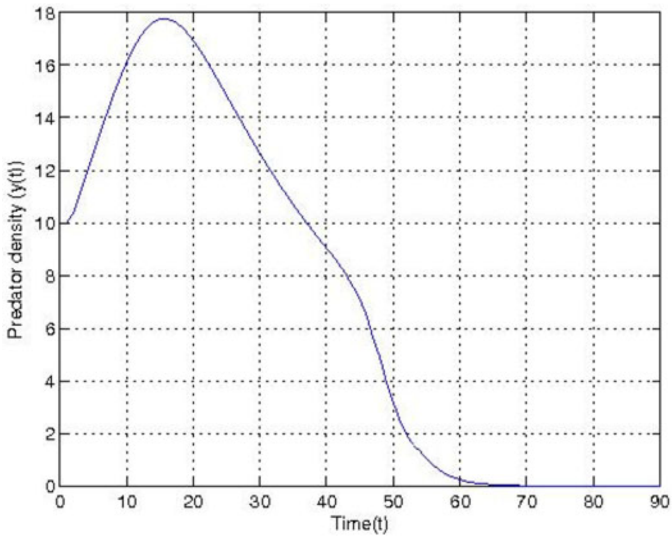

FIGURE 2. The trajectory of predator density $y(t)$ versus time with the initial conditions $x_0 = 50$, $y_0 = 25$ and $\tau = 3 < \tau_0$.

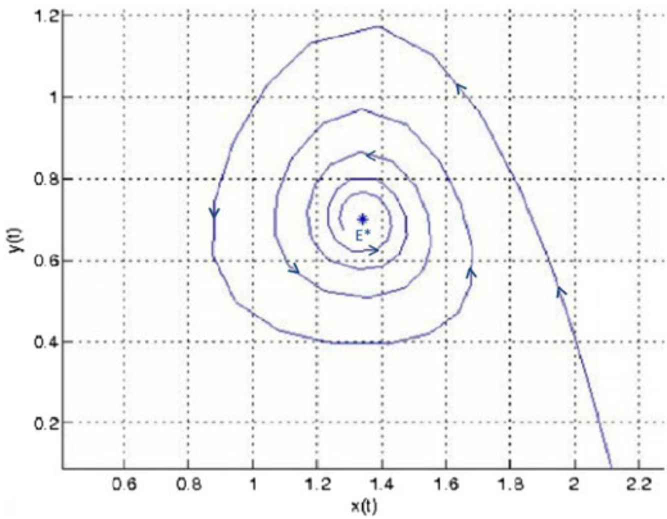

FIGURE 3. The phase portrait of predator $y(t)$ and prey $x(t)$ densities with initial conditions $x_0 = 50$ and $y_0 = 25$. The graph of the solution of the model (4.1) when $\tau = 3 < \tau_0$, which shows that the equilibrium point E^* is asymptotically stable.

The numerical simulations for $\tau = 3.8 > \tau_0$ that is sufficiently close to τ_0 are shown in Figures 4–6, and these graphs show that the bifurcating periodic solutions from the equilibrium point E^* occur and are unstable.

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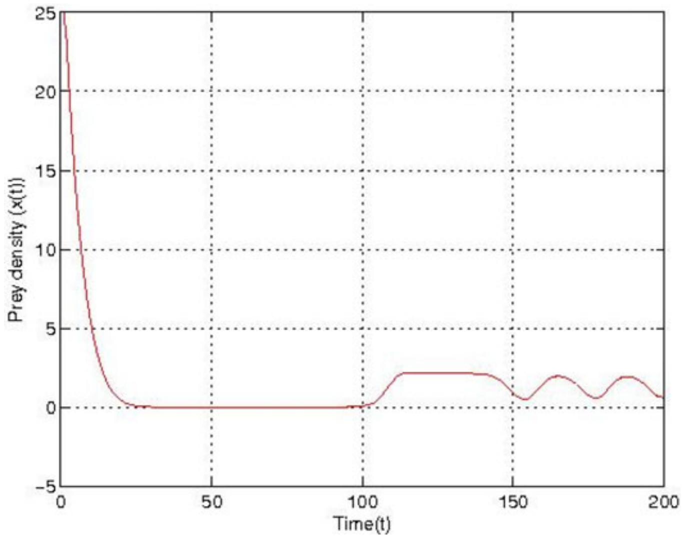

FIGURE 4. The trajectory of prey density $x(t)$ versus time t with the initial conditions $x_0 = 50$, $y_0 = 25$ and $\tau = 3.8 > \tau_0$.

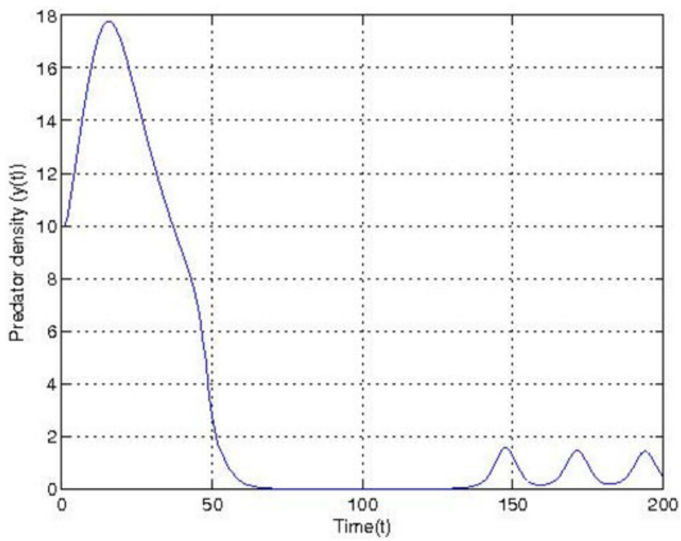

FIGURE 5. The trajectory of predator density $y(t)$ versus time t with the initial conditions $x_0 = 50$, $y_0 = 25$ and $\tau = 3.8 > \tau_0$.

More importantly, in Figures 7 and 8, we construct the bifurcation diagrams for prey and predator densities, respectively. We observe that the bifurcation diagrams allow us to visualize changes in the behaviour of the system as the bifurcation parameter

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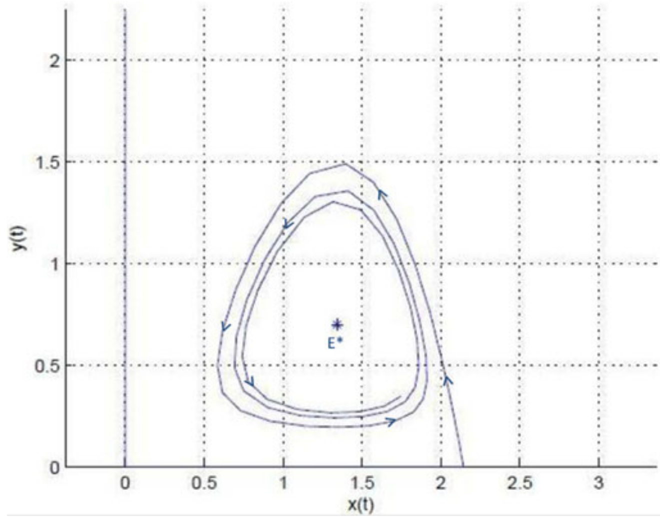

FIGURE 6. The phase portrait of predator density $y(t)$ versus prey density $x(t)$ for the same parameters when $\tau = 3.8 > \tau_0$, which shows that the periodic solutions are bifurcating from E^*.

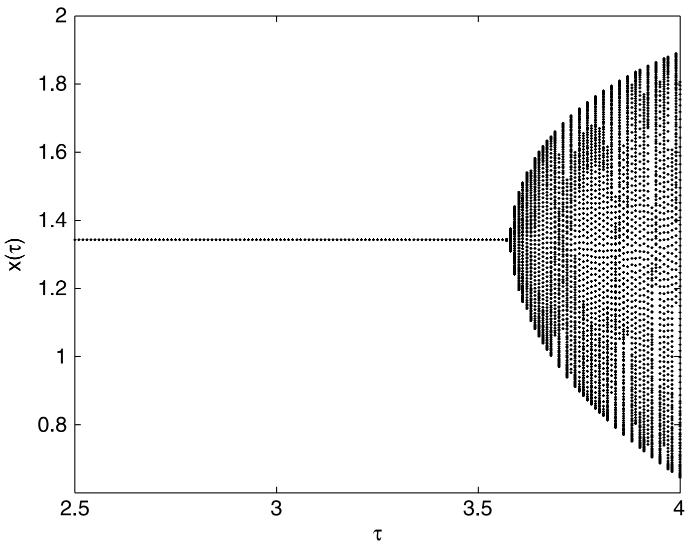

FIGURE 7. Bifurcation diagram for prey density $x(t)$ when the value of τ is from 2.5 to 4.

τ passes through the first critical value τ_0. So, we can conclude that the numerical simulations agree with analytical results on the impact of time delay for the stability of the equilibrium point.

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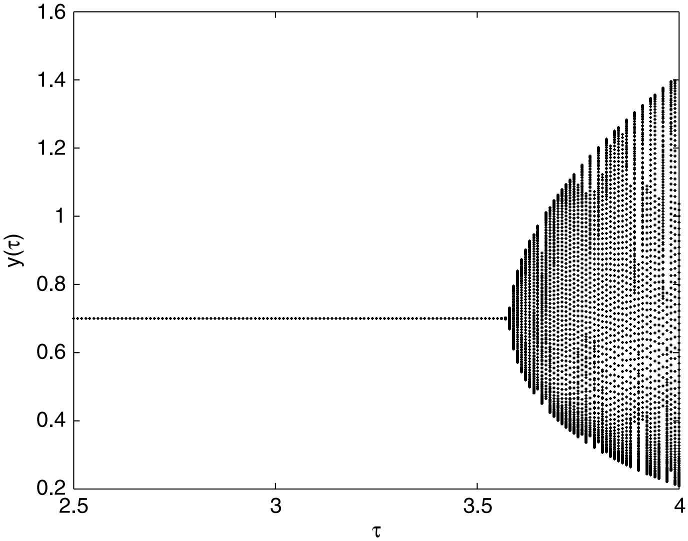

FIGURE 8. Bifurcation diagram for predator density $y(t)$ when the value of τ is from 2.5 to 4.

5. Conclusion

In this paper, we have investigated the local stability of the positive equilibrium point (x^*, y^*) and Hopf bifurcation of a nonlinear predator–prey system with discrete time delay τ. We have shown that if the condition (H) holds and $\tau \in [0, \tau_0)$, then the positive equilibrium point (x^*, y^*) is asymptotically stable. When $\tau > \tau_0$, the positive equilibrium point (x^*, y^*) loses its stability, and the sequence of Hopf bifurcations occurs at the positive equilibrium point if $\tau = \tau_k$ $(k = 0, 1, 2 \ldots)$, that is, a family of periodic orbits bifurcates from the positive equilibrium point (x^*, y^*). Also, the direction of Hopf bifurcation and the stability of the bifurcating periodic orbits are discussed by applying the normal-form theory and the centre manifold theorem introduced by Hassard et al. [6]. Finally, the numerical examples verifying our theoretical results are presented. For future studies, we would like to consider the same model under the diffusion effect, which will be a PDE (partial differential equation) model by incorporating the time delay into the system.

Acknowledgements

We would like to thank the referee for providing useful comments which served to improve the paper. We would also like to thank Dr. Maksat Ashyraliyev for his help and time to improve the quality of software programming.

References

[1] X. Chen, "Periodicity in a nonlinear discrete predator–prey system with state dependent delays", *Nonlinear Anal. Real World Appl.* **8** (2007) 435–446; doi:10.1016/j.nonrwa.2005.12.005.

[2] L. C. Evans, *Partial differential equations* (American Mathematical Society, Providence, RI, USA, 2010).

[3] T. Faria, "Stability and bifurcation for a delayed predator–prey model and the effect of diffusion", *J. Math. Anal. Appl.* **254** (2001) 433–463; doi:10.1006/jmaa.2000.7182.

[4] K. Gopalsamy, "Time lags and global stability in two species competition", *Bull. Math. Biol.* **42** (1980) 728–737; doi:10.1007/BF02460990.

[5] J. K. Hale, *Theory of functional differential equations* (Springer, New York, 1977).

[6] B. D. Hassard, N. D. Kazarinoff and Y.-H. Wan, *Theory and applications of Hopf bifurcation* (Cambridge University Press, Cambridge, 1981).

[7] S. Krise and S. R. Choudhury, "Bifurcations and chaos in a predator–prey model with delay and a laser-diode system with self-sustained pulsations", *Chaos Solitons Fractals* **16** (2003) 59–77; doi:10.1016/S0960-0779(02)00199-6.

[8] Y. Kuang, *Delay differential equations with applications in population dynamics* (Academic Press, Boston, 1993).

[9] Yu. A. Kuznetsov, *Elements of applied bifurcation theory* (Springer, New York, 2004).

[10] A. Leung, "Periodic solutions for a prey–predator differential delay equation", *J. Differential Equations* **26** (1977) 391–403; doi:10.1016/0022-0396(77)90087-0.

[11] B. Liu, Z. Teng and L. Chen, "Analysis of a predator–prey model with Holling II functional response concerning impulsive control strategy", *J. Comput. Appl. Math.* **193** (2006) 347–362; doi:10.1016/j.cam.2005.06.023.

[12] X. Liu and D. Xiao, "Complex dynamic behaviors of a discrete-time predator–prey system", *Chaos Solitons Fractals* **32** (2007) 80–94; doi:10.1016/j.chaos.2005.10.081.

[13] Z. Liu and R. Yuan, "Stability and bifurcation in a harvested one-predator–two-prey model with delays", *Chaos Solitons Fractals* **27** (2006) 1395–1407; doi:10.1016/j.chaos.2005.05.014.

[14] W. Ma and Y. Takeuchi, "Stability analysis on a predator–prey system with distributed delays", *J. Comput. Appl. Math.* **88** (1998) 79–94; doi:10.1016/S0377-0427(97)00203-3.

[15] R. M. May, "Time delay versus stability in population models with two and three trophic levels", *Ecology* **4** (1973) 315–325; doi:10.2307/1934339.

[16] J. D. Murray, *Mathematical biology* (Springer, New York, 1993).

[17] S. Ruan, "Absolute stability, conditional stability and bifurcation in Kolmogorov-type predator–prey systems with discrete delays", *Quart. Appl. Math.* **59** (2001) 159–173; http://www.math.miami.edu/~ruan/MyPapers/Ruan-QAM2001.pdf.

[18] S. Ruan and J. Wei, "Periodic solutions of planar systems with two delays", *Proc. Roy. Soc. Edinburgh Sect. A* **129** (1999) 1017–1032; doi:10.1017/S0308210500031061.

[19] Y. L. Song and J. J. Wei, "Local Hopf bifurcation and global periodic solutions in a delayed predator–prey system", *J. Math. Anal. Appl.* **301** (2005) 1–21; doi:10.1016/j.jmaa.2004.06.056.

[20] X. P. Yan and W. T. Li, "Hopf bifurcation and global periodic solutions in a delayed predator–prey system", *Appl. Math. Comput.* **177** (2006) 427–445; doi:10.1016/j.amc.2005.11.020.

[21] X. P. Yan and C. H. Zhang, "Hopf bifurcation in a delayed Lotka–Volterra predator–prey system", *Nonlinear Anal. Real World Appl.* **9** (2008) 114–127; doi:10.1016/j.nonrwa.2006.09.007.

[22] L. Zhou and Y. Tang, "Stability and Hopf bifurcation for a delay competition diffusion system", *Chaos Solitons Fractals* **14** (2002) 1201–1225; doi:10.1016/S0960-0779(02)00068-1.