



The Higher K -Theory of a Complex Surface

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Abstract. Let X be a smooth complex variety of dimension at most two, and let F be its function field. We prove that the K -groups of F are divisible above the dimension of X , and that the K -groups of X are divisible-by-finite. We also describe the torsion in the K -groups of F and X .

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Since Suslin described the K -theory of the complex numbers \mathbb{C} in [Su1], there has been a renewed interest in the structure of the K -theory of fields. In this paper we shall describe the Abelian group structure of the K -groups of a complex variety X and its function field F , assuming that X has dimension at most two, *i.e.*, that X is a curve or a surface. This restriction is forced by our ignorance concerning the higher Chow groups $CH^i(F, n)$ for $2 \leq i < \dim(X)$. In a sequel we shall deal with higher-dimensional varieties, using Voevodsky's solution to the Milnor Conjecture to describe the torsion in the higher Chow groups $CH^i(F, n)$.

The first main result in this paper (Theorem 4.8) describes the structure of the Abelian groups $K_n(F)$, at least when $d = \text{tr. deg.}(F)$ is at most two. We prove that $K_n(F)$ is a divisible group for all $n > d$, and determine its torsion subgroup. Our result builds upon Suslin's result [Su3] that the Milnor K -groups $K_n^M(F)$ are divisible for $n > d$, because $K_n^M(F)/m \cong H_{\text{ct}}^n(F, \mathbb{Z}/m)$ vanishes for $n > d$.

In the range $n \leq 2$, the structure of $K_n(F)$ is well known, albeit rich and complicated, when F is any function field over \mathbb{C} . The structure of $K_1(F) = F^\times$ is classical; it is the product of \mathbb{C}^\times and a free Abelian group (see Example 1.1 below). The group $K_2(F)$ need not be divisible either, but its torsion subgroup is always divisible. This follows from the Merkurjev–Suslin theorem [MS] that $K_2(F)/m$ is isomorphic to the m -torsion in the Brauer group $\text{Br}(F)$, and from Suslin's formula

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for the torsion subgroup of $K_2(F)$ in [Su2, 3.7]: the m -torsion subgroup of $K_2(F)$ is isomorphic to $F^\times \otimes \mu_m$.

Our second main result for curves and surfaces (Theorems 3.2 and 6.6) is that when $n > \dim(X)$ the group $K_n(X)$ is divisible-by-finite: its torsion consists of a specified number of copies of \mathbb{Q}/\mathbb{Z} , together with the finite summands, which are the torsion subgroups of the Betti cohomology $H^*(X, \mathbb{Z})$. This result holds for all $n > 0$ when X is proper. In particular, if X is a curve then $K_n(X)$ is a divisible group for $n \geq 2$ (and $n = 1$ if X is proper).

A related phenomenon for surfaces is our result (see Proposition 6.3 below) that the finite group $H^3(X, \mathbb{Z})_{tors}$ is a summand of $K_1(X)$. When X is any smooth proper variety, this summand was found by Colliot-Thélène and Raskind [CT-R, 2.2].

One way to interpret our results is to consider the comparison map ρ between the algebraic K -theory of the variety X and the topological K -theory $KU^*(X)$ of the underlying topological space. Recall that each $KU^{-n}(X)$ is a finitely generated Abelian group. Define the relative groups $K_n^{rel}(X)$ to fit into an exact sequence

$$\dots K_{n+1}(X) \xrightarrow{\rho} KU^{-n-1}(X) \rightarrow K_n^{rel}(X) \rightarrow K_n(X) \xrightarrow{\rho} KU^{-n}(X) \dots$$

If $\dim(X) \leq 2$, then $K_n^{rel}(X)$ is a uniquely divisible group for all $n \geq 0$, and hence $K_n(X)$ is the direct sum of a divisible group and a finitely generated group $\mathbb{Z}^r \oplus A$, which injects into $KU^{-n}(X)$ and has $A \cong KU^{-n}(X)_{tors}$ (Corollary 2.12). Our third main result computes the torsion in $K_n(X)$ for $n \geq \dim(X)$ (Theorem 6.6). The divisible part of this divisible-by-finite group is the mirror image $(\mathbb{Q}/\mathbb{Z})^s$ of the free part \mathbb{Z}^s of $KU^{-n-1}(X)$.

This paper is organized as follows. In Section 1 we make some elementary observations about the higher Chow groups $CH^i(F, n)$ when $\mathbb{C} \subseteq F$. In Section 2 we describe the K -theory with coefficients \mathbb{Z}/m for F and X ; many of the results in this section were announced by Suslin in [SuM]. In Section 3 we describe the structure of $K_n(Y)$ when Y is a smooth curve.

In the second half of the paper we focus on a smooth surface X and its function field F . In Section 4 we introduce Chern classes with values in Deligne–Beilinson cohomology, and use them to determine the structure of $K_n(F)$: it is divisible for $n > 2$ with prescribed torsion. In Section 5 we use this structure to determine the K -cohomology of X , and we describe the groups $K_n(X)$ in Section 6.

We will use the following notation. If A is an Abelian group and m is a positive integer, A_m will denote the subgroup $\{a \in A : ma = 0\}$, and we will write A/m for A/mA . The torsion subgroup of A will be written as A_{tors} .

If X is a complex variety, we write $X(\mathbb{C})$ for the complex analytic space underlying X . If \mathcal{A} is a sheaf on $X(\mathbb{C})$, such as \mathbb{Z} , \mathbb{C}^\times or $\Omega_{X,an}^p$, we write $H_{an}^*(X, \mathcal{A})$ for the sheaf cohomology of \mathcal{A} on $X(\mathbb{C})$. If \mathcal{A} is a constant sheaf, we drop the subscripts and just write $H^*(X, \mathcal{A})$ because, of course, this equals the classical singular cohomology of X with coefficients in \mathcal{A} . If $\mathcal{A} = \mathbb{Z}/m$ then it is well known that $H^*(X, \mathbb{Z}/m)$ also agrees with the étale cohomology groups $H_{\text{ét}}^*(X, \mathbb{Z}/m)$.

If F is the function field of X , we shall write $H_{\text{an}}^*(F, \mathcal{A})$ for the direct limit of the groups $H_{\text{an}}^*(U, \mathcal{A})$ as U runs through all Zariski open subsets of X ; this is a birational invariant, independent of the choice of X .

We will also use the standard notation $K_n(X; \mathbb{Z}/m)$ for the K -theory of X with coefficients \mathbb{Z}/m , and $CH^i(X, n; \mathbb{Z}/m)$ for the higher Chow groups of X with coefficients \mathbb{Z}/m . The calligraphic $\mathcal{K}_n, \mathcal{K}_n(\mathbb{Z}/m)$ and $\mathcal{H}^n(\mathbb{Z}/m)$ refer to the Zariski sheaves associated to the presheaves sending U to $K_n(U), K_n(U; \mathbb{Z}/m)$ and $H_{\text{ét}}^n(U, \mathbb{Z}/m)$, respectively.

1. Divisibility of Chow Groups of Fields

Let F be a field of finite type over \mathbb{C} . We cannot expect every higher Chow group $CH^i(F, n)$ to be divisible. For example, we have $K_0(F) = CH^0(F, 0) = \mathbb{Z}$ for every field F .

In this section we show that the groups $CH^i(F, n)$ are divisible for sufficiently large n , at least if F has small transcendence degree over \mathbb{C} . Much of this material is implicit in the work of Suslin [SuM] and Kahn [K]. We begin with some low degree calculations.

EXAMPLE 1.1. (a) Assuming $F \neq \mathbb{C}$, the group $K_1(F) = CH^1(F, 1) = F^\times$ is not divisible either. To see this, choose a smooth projective variety X such that F is the function field of X . It is well known that F^\times is the product of \mathbb{C}^\times and the group $PDiv(X)$ of principal divisors on X [Hart, II.6]. Since $PDiv(X)$ is a free Abelian group, the group F^\times cannot be divisible.

(b) The group $K_2(F)$ is known to be divisible when F is the function field of a curve, but not always when F is the function field of a surface. Indeed, $K_2(F)/m$ is the subgroup $Br(F)_m$ of the Brauer group $Br(F)$ [MS], so these are just restatements about the Brauer group of F ; see [Dix, III].

The isomorphisms $K_0(F) \cong CH^0(F, 0)$ and $K_1(F) \cong CH^1(F, 1)$ are degenerate cases of the (third quadrant) Bloch–Lichtenbaum spectral sequence. It converges to the K -theory of F [BL]:

$$E_2^{p,q} = CH^{-q}(F, -p - q) \implies K_{-p-q}(F), \quad p, q \leq 0. \tag{1.2}$$

Note that $CH^0(F, n) = 0$ for $n \neq 0$, and $CH^1(F, n) = 0$ for $n \neq 1$ (because $CH^1(F, 0) = \text{Pic}(F) = 0$) by [BL]. Thus $K_2(F) \cong CH^2(F, 2)$ is another degenerate case. The following result, essentially due to Soulé, gives a criterion for further degeneration of the spectral sequence.

PROPOSITION 1.3. *Set $i = -q$ and $n = -p - q$, so that $E_2^{p,q} = CH^i(F, n)$ in the Bloch–Lichtenbaum spectral sequence (1.2).*

If $CH^i(F, n)$ is uniquely divisible, then $E_\infty^{p,q} = CH^i(F, n)$.

If $CH^i(F, n)$ is divisible, then $E_\infty^{p,q}$ is a quotient of $CH^i(F, n)$.

If $CH^i(F, n)$ is torsion-free, then $E_\infty^{p,q}$ is a subgroup of $CH^i(F, n)$.

Proof. Soulé has proven [Sou2] that the Adams operations ψ^k commute with the differentials d_r in the spectral sequence, and that the ψ^k are multiplication by k^i on $CH^i(F, n)$. Hence, we have $\psi^k = k^i$ on each $E_r^{p,-i}$. Since $\psi^k = k^i$ holds for every i and k , it follows from [Sou1, 2.8] that for each q and r there is an integer N so that $Nd_r^{p,q} = 0$ for every p . The result is now a straightforward induction on r ; if $E_r^{p,q}$ is divisible, $d_r^{p,q} = 0$ and $E_{r+1}^{p,q}$ is a quotient, while if $E_r^{p,q}$ is torsion-free then $d_r^{p-r,q+r-1} = 0$ and $E_{r+1}^{p,q}$ is a subgroup. \square

LEMMA 1.4. Let F be a field of finite type over \mathbb{C} , and set $d = \text{tr. deg.}(F)$. Then for each $i \geq d$:

$CH^i(F, n)$ is uniquely divisible for $n \neq 2i, \dots, 2i - d - 1$;

$CH^i(F, 2i)$ is torsion-free;

$CH^i(F, 2i - d - 1)$ is divisible.

Proof. This follows from the combination of the universal coefficient sequence

$$0 \rightarrow CH^i(F, n)/m \rightarrow CH^i(F, n; \mathbb{Z}/m) \xrightarrow{\partial} CH^i(F, n-1)_m \rightarrow 0,$$

the theorem of Suslin [Su3, 4.3] that $CH^i(F, n; \mathbb{Z}/m) \cong H_{\text{ét}}^{2i-n}(F, \mathbb{Z}/m)$ for $i \geq d$, and the fact that F has étale cohomological dimension d . \square

Clearly the combination of Proposition 1.3 and Lemma 1.4 imply that almost all the differentials vanish, and they all vanish if F has transcendence degree at most 2. In these cases the spectral sequence determines $K_*(F)$ up to the usual extension problem.

When $n = 2i$ and $i \geq d$, Lemma 1.4 states that $CH^i(F, 2i)$ is torsion-free. We will prove it is uniquely divisible. The critical case to consider is $F = \mathbb{C}$.

COROLLARY 1.5. ($F = \mathbb{C}$) For each $n \geq 0$, the group $K_n(\mathbb{C})$ is the direct sum of the groups $CH^i(\mathbb{C}, n)$, $0 \leq i \leq n$. Moreover, each summand $CH^i(\mathbb{C}, n)$ is uniquely divisible, except when $n = 2i - 1$ or $i = 0$.

If $i \geq 1$, the group $CH^i(\mathbb{C}, 2i - 1)$ is divisible, and its torsion subgroup is:

$$CH^i(\mathbb{C}, 2i - 1)_{\text{tors}} \cong K_{2i-1}(\mathbb{C})_{\text{tors}} \cong \mathbb{Q}/\mathbb{Z}.$$

Proof. By Lemma 1.4, $CH^i(\mathbb{C}, n)$ is divisible for $n \neq 2i$, and torsion-free for $n \neq 2i - 1$. By Proposition 1.3, all differentials are zero, i.e., $E_\infty^{p,q} = CH^{-q}(\mathbb{C}, -p - q)$.

Hence the filtration quotients for the abutment $K_n(\mathbb{C})$ are the groups $CH^j(\mathbb{C}, n)$. To solve the extension problem, recall that for each n the groups $CH^j(\mathbb{C}, n)$ are uniquely divisible with one possible exception. If n is odd, the exception is divisible, and it follows easily that if $n = 2i - 1$ then $K_n(\mathbb{C}) = \bigoplus CH^j(\mathbb{C}, n)$, with

$K_n(\mathbb{C})_{tors} \cong \mathbb{Q}/\mathbb{Z}$ equal to $CH^i(\mathbb{C}, n)_{tors}$. Since $CH^i(\mathbb{C}, n)_m \cong \mathbb{Z}/m$ is a quotient of $CH^i(\mathbb{C}, 2i; \mathbb{Z}/m) \cong \mathbb{Z}/m$, the universal coefficient sequence forces each $CH^i(\mathbb{C}, 2i)$ to be divisible. From this we get a splitting of $K_n(\mathbb{C})$ for even $n > 0$. \square

PROPOSITION 1.6. *Let F be a field properly containing \mathbb{C} . Then for $i = 1$ or $i \geq \text{tr. deg.}(F)$:*

- (1) $CH^i(F, 2i)$ is uniquely divisible, and
- (2) the torsion subgroup of $CH^i(F, 2i - 1)$ is isomorphic to \mathbb{Q}/\mathbb{Z} , the torsion subgroup of $CH^i(\mathbb{C}, 2i - 1)$.

The group \mathbb{Q}/\mathbb{Z} is also a canonical summand of $K_{2i}(F; \mathbb{Q}/\mathbb{Z})$ and $K_{2i-1}(F)_{tors}$.

Proof. Let \bar{F} denote the algebraic closure of F . Suslin proved in [Su1] that for each m and i there is an isomorphism $K_{2i}(\mathbb{C}; \mathbb{Z}/m) \cong K_{2i}(\bar{F}; \mathbb{Z}/m)$, and that it factors through $K_{2i}(F; \mathbb{Z}/m)$. The same proof, applied to $T(F) = CH^i(F, 2i - 1)_m$, shows that there is an isomorphism $T(\mathbb{C}) \cong T(\bar{F})$ factoring through $T(F)$. Hence $T(\mathbb{C})$, which is isomorphic to \mathbb{Z}/m by Corollary 1.5, is a summand of $T(F)$. If $i = 1$, $T(F) = T(\mathbb{C}) = \mu_m$. When $i \geq \text{tr. deg.}(F)$, $T(F)$ is a quotient of the finite group

$$CH^i(F, 2i; \mathbb{Z}/m) \cong H_{\text{et}}^0(F, \mathbb{Z}/m(i)) \cong \mathbb{Z}/m.$$

Hence we must have $T(F) \cong T(\mathbb{C}) \cong \mathbb{Z}/m$. Now let m go to infinity. \square

DEFINITION 1.7. The canonical summands \mathbb{Q}/\mathbb{Z} of $K_{2i}(F; \mathbb{Q}/\mathbb{Z})$, $K_{2i-1}(F)_{tors}$ and $CH^i(F, 2i - 1)_{tors}$ will be called the *Bott summands* of these groups. For each m , we will also refer to the canonical summands \mathbb{Z}/m of $K_{2i}(F; \mathbb{Z}/m)$, $K_{2i-1}(F)_m$ and $CH^i(F, 2i - 1)_m$ as the Bott summands.

They are not canonically summands of $K_{2i-1}(F)$ or $CH^i(F, 2i - 1)$. For example, when $i = 1$ we can identify \mathbb{Q}/\mathbb{Z} with the group of roots of unity in $K_1(F) = CH^1(F, 1) = F^\times$.

PROPOSITION 1.8. *Let F be a field containing \mathbb{C} , and suppose $i \geq 2$. Then*

- (a) The quotient of $CH^i(F, 2i - 1)$ by the Bott summand \mathbb{Q}/\mathbb{Z} is uniquely divisible.
- (b) The torsion subgroup of $CH^i(F, 2i - 2)$ is a divisible group, isomorphic to $F^\times \otimes \mathbb{Q}/\mathbb{Z}$, and its m -torsion subgroup is given by:

$$CH^i(F, 2i - 2)_m \cong H^1(F, \mathbb{Z}/m(i)) \cong F^\times / F^{\times m} \quad \text{for all } m.$$

Proof. Since $CH^i(F, 2i - 1; \mathbb{Z}/m) \cong H^1(F, \mathbb{Z}/m(i))$, the universal coefficient sequence shows that it suffices to prove that $CH^i(F, 2i - 1)$ is divisible by each prime ℓ . For this we modify the argument of [Su2, 3.4].

Consider the filtered poset of all subfields F' of F which are finitely generated over \mathbb{Q} . The natural map from $CH^i(F, 2i - 1)$ to $H^1(F, \mathbb{Z}/\ell^v(i))$ factors through the direct limit of the corresponding maps for F' , and by naturality in $m = \ell^v$ each of these

factor through the inverse limit $H^1(F', \mathbb{Z}_\ell(i)) = \varprojlim H^1(F', \mathbb{Z}/\ell^v(i))$. Hence it suffices to show that $\varinjlim H^1(F', \mathbb{Z}_\ell(i))$ vanishes for $i \geq 2$.

The proof of [Su2, 2.4] goes through with $\mathbb{Z}_\ell(1)$ replaced by $\mathbb{Z}_\ell(i - 1)$ for any $i \geq 2$. Given this, the proof of [Su2, 2.7] goes through to show that if F' is a finitely generated subfield of F with ground field F'_0 then $H^1(F'_0, \mathbb{Z}_\ell(i)) = H^1(F', \mathbb{Z}_\ell(i))$. Taking the limit over all such F' yields the desired vanishing:

$$\varinjlim H^1(F', \mathbb{Z}_\ell(i)) = \varinjlim H^1(F'_0, \mathbb{Z}_\ell(i)) = H^1(\bar{\mathbb{Q}}, \mathbb{Z}_\ell(i)) = 0. \quad \square$$

Since $CH^2(\mathbb{C}, 2) = K_2(\mathbb{C})$ is divisible and $CH^1(\mathbb{C}, 1) = \mathbb{C}^\times$, the universal coefficient sequence yields an isomorphism between $CH^1(\mathbb{C}, 2; \mathbb{Z}/m)$ and the group μ_m of m th roots of unity. Fixing a primitive root of unity ζ , we shall refer to the corresponding element β of $CH^1(\mathbb{C}, 2; \mathbb{Z}/m)$ as the *Bott element*.

LEMMA 1.9. *Let F have transcendence degree d over \mathbb{C} . Multiplication by the Bott element β induces isomorphisms $CH^i(F, n; \mathbb{Z}/m) \cong CH^{i+1}(F, n + 2; \mathbb{Z}/m)$ for all $i \geq d$.*

Proof. By [Su3, 4.3] the norm residue map $CH^i(F, n; \mathbb{Z}/m) \rightarrow H_{\text{et}}^{2i-n}(F, \mu_m^{\otimes i})$ is an isomorphism for $i \geq d$ (both vanish unless $0 \leq n \leq 2i$). By [W3, 5.2] this map is compatible with multiplication. Since β maps to the class $[\zeta]$ in $H_{\text{et}}^0(F, \mu_m^{\otimes 1}) = \mu_m$, this means the following diagram commutes (proving the lemma). \square

$$\begin{array}{ccc} CH^i(F, n; \mathbb{Z}/m) & \xrightarrow[\alpha]{\cong} & H_{\text{et}}^{2i-n}(F, \mu_m^{\otimes i}) \\ \downarrow \cup \beta & & \cong \downarrow \cup [\zeta] \\ CH^{i+1}(F, n + 2; \mathbb{Z}/m) & \xrightarrow[\alpha]{\cong} & H_{\text{et}}^{2i-n}(F, \mu_m^{\otimes i+1}). \end{array}$$

2. K-Theory with Coefficients

We turn our attention to K -theory with coefficients \mathbb{Z}/m , and calculate the groups $K_n(X; \mathbb{Z}/m)$. The following result was observed by Suslin in [SuM, p. 350].

PROPOSITION 2.1 (Suslin). *Let F be the function field of a curve or surface over \mathbb{C} . Then there are natural isomorphisms for all $n \geq 1$:*

$$K_n(F; \mathbb{Z}/m) \cong \begin{cases} \mathbb{Z}/m \oplus H^2(F, \mathbb{Z}/m) & \text{if } n \geq 2 \text{ is even;} \\ H^1(F, \mathbb{Z}/m) & \text{if } n \geq 1 \text{ is odd.} \end{cases}$$

In effect, the Bloch–Lichtenbaum spectral sequence (1.2) has an analogue with coefficients (constructed in [RW]), and it degenerates when $\text{tr. deg.}(F) \leq 2$. The extension problem is solved by Corollary 1.6, because the Bott summand is the quotient $H^0(F, \mathbb{Z}/m(i)) \cong \mathbb{Z}/m$ of $K_{2i}(F; \mathbb{Z}/m)$.

Let $K_n^{et}(X; \mathbb{Z}/m)$ denote the étale K -theory of X with coefficients \mathbb{Z}/m . There are natural maps $\rho_n(X): K_n(X; \mathbb{Z}/m) \rightarrow K_n^{et}(X; \mathbb{Z}/m)$, constructed in [Fr2, 1.3].

THEOREM 2.2 (Suslin). *Let X be a smooth complex variety with function field F . If $\dim(X) \leq 2$, there are isomorphisms for all $n \geq 1$ (and injections for $n = 0$):*

$$\begin{aligned} \rho_n(X): K_n(X; \mathbb{Z}/m) &\xrightarrow{\cong} K_n^{et}(X; \mathbb{Z}/m), \\ \rho_n(F): K_n(F; \mathbb{Z}/m) &\xrightarrow{\cong} K_n^{et}(F; \mathbb{Z}/m). \end{aligned}$$

This theorem was announced in [SuM, 4.7], but the proof in loc. cit. has a gap because the multiplicative properties of the Bott element $\beta \in K_2(\mathbb{C}; \mathbb{Z}/m)$ on the Bloch–Lichtenbaum spectral sequence for a general field F are presently unknown.

One of the main purposes of this section is to provide a proof of Theorem 2.2. We will first dispose of the case of curves, stating a slightly sharper result.

PROPOSITION 2.3. *Let Y be a smooth curve over \mathbb{C} . Then multiplication by the Bott element $\beta \in K_2(\mathbb{C}; \mathbb{Z}/m)$ induces isomorphisms $K_n(Y; \mathbb{Z}/m) \cong K_{n+2}(Y; \mathbb{Z}/m)$ for all $n \geq 0$, and there are isomorphisms:*

$$K_n(Y; \mathbb{Z}/m) \cong \begin{cases} \mathbb{Z}/m \oplus H^2(Y, \mathbb{Z}/m) & \text{if } n \geq 0 \text{ is even,} \\ H^1(Y, \mathbb{Z}/m) & \text{if } n > 0 \text{ is odd.} \end{cases} \tag{2.3.1}$$

Moreover, $K_n(Y; \mathbb{Z}/m) \xrightarrow{\cong} K_n^{et}(Y; \mathbb{Z}/m)$ and $K_n(F; \mathbb{Z}/m) \xrightarrow{\cong} K_n^{et}(F; \mathbb{Z}/m)$ for all $n \geq 0$, where $F = \mathbb{C}(Y)$.

Proof. Write F for the function field of Y , and $i: Y \rightarrow \text{Spec}(\mathbb{C})$ for the structure map. Using the abbreviations $K_n^*(Y)$ for $K_n(Y; \mathbb{Z}/m)$ and $H^*(Y)$ for $H^*(Y; \mathbb{Z}/m)$, we have a diagram for all $n \geq 0$ even:

$$\begin{array}{ccccccccccc} & & & & & & & & & K_n''(\mathbb{C}) & = & K_n''(\mathbb{C}) \\ & & & & & & & & & i^* \downarrow & & i^* \downarrow \cong \\ 0 & \longrightarrow & K_{n+1}''(Y) & \longrightarrow & K_{n+1}''(F) & \xrightarrow{\partial} & \bigoplus_{y \in Y(\mathbb{C})} K_n''(\mathbb{C}) & \longrightarrow & K_n''(Y) & \longrightarrow & K_n''(F) & \longrightarrow & 0 \\ & & & & \cong \downarrow & & \downarrow \cong & & & & & & \\ 0 & \longrightarrow & H^1(Y) & \longrightarrow & H^1(F) & \xrightarrow{\partial} & \bigoplus_{y \in Y(\mathbb{C})} H^0(\mathbb{C}) & \longrightarrow & H^2(Y) & \longrightarrow & 0. \end{array}$$

The rows are the exact localization sequences in K -theory and étale cohomology. The vertical isomorphisms are from Proposition 2.1.

By Lemma 2.3.2 below, the middle square commutes up to a natural isomorphism of $H^0(\mathbb{C}) \cong \mathbb{Z}/m$. Given this, we can finish the proof of Proposition 2.3. A diagram chase yields the isomorphisms (2.3.1). Since Y has the étale homotopy type of the complex surface $Y(\mathbb{C})$, we see from [Fr1, 1.2(iv)] that there are similar isomorphisms for $K_n^{et}(Y; \mathbb{Z}/m)$. Hence, the source and target of each $\rho_n(Y)$ are finite groups of the same order. By [Fr2, 2.9], the maps $\rho_n(Y)$ are onto, and are isomorphisms for $n = 0, 1$. Thus each $\rho_n(Y)$ must be an isomorphism. Passing to the limit over all open subsets of Y yields the result for $F = \mathbb{C}(Y)$. \square

LEMMA 2.3.2. *Let y be a closed point of Y and $n \geq 0$ even. Then the square formed by the connecting homomorphisms ∂_y and the isomorphisms of Proposition 2.1 commutes up to a natural isomorphism of $H^0(\mathbb{C}) \cong \mathbb{Z}/m$.*

$$\begin{array}{ccc} K''_{n+1}(F) & \xrightarrow{\partial_y} & K''_n(\mathbb{C}) \\ \cong \downarrow & & \downarrow \cong \\ H^1(F) & \xrightarrow{\partial_y} & H^0(\mathbb{C}) \end{array}$$

Proof. We shall prove this by mimicking the argument of [RW, (6.4)]. Form the henselization R_y of $\mathcal{O}_{Y,y}$ and write F_y for its field of fractions. Because F_y is a direct limit of function fields of curves, Proposition 2.1 applies to F_y too. Hence we can form the following diagram, using the isomorphisms of 2.1 for the vertical maps.

$$\begin{array}{ccccc} K''_{n+1}(F) & \longrightarrow & K''_{n+1}(F_y) & \xrightarrow[\cong]{\partial_y} & K''_n(\mathbb{C}) \\ \cong \downarrow & & \cong \downarrow & & \cong \downarrow \\ H^1(F) & \longrightarrow & H^1(F_y) & \xrightarrow[\cong]{\partial_y} & H^0(\mathbb{C}). \end{array}$$

The left square commutes by naturality of the isomorphism in Proposition 2.1, and the outer square is the square in question. So it suffices to show that the horizontal maps labelled ∂_y are isomorphisms. (We do not care here if the right square commutes.)

By rigidity, we know that $H^i(R_y) = H^i(\mathbb{C}) = 0$ for $i > 0$ so the connecting map $\partial_y: H^1(F_y) \rightarrow H^0(\mathbb{C})$ in the localization sequence is an isomorphism; see [Sou, III.2]. Similarly, $K''_{n+1}(R_y) = K''_{n+1}(\mathbb{C}) = 0$ by Gabber rigidity, so the connecting map $\partial_y: K''_{n+1}(F_y) \rightarrow K''_n(\mathbb{C})$ in the K -theory localization sequence is an injection; since both source and target have order m , it is an isomorphism. (Alternatively, one could argue as in [Sou] or [RW, 3.3].) □

As observed by Suslin in loc. cit., this shows that even if Y is a singular curve then multiplication by the Bott element induces isomorphisms $K'_n(Y; \mathbb{Z}/m) \cong K'_{n+2}(Y; \mathbb{Z}/m)$ for all $n \geq 0$. We also have $K_n(Y; \mathbb{Z}/m) \cong K_{n+2}(Y; \mathbb{Z}/m)$ for all $n \geq 0$ by the following result.

COROLLARY 2.4. *If Y is a singular curve over \mathbb{C} , there are isomorphisms:*

$$\rho_n(Y): K_n(Y; \mathbb{Z}/m) \xrightarrow{\cong} K_n^{et}(Y; \mathbb{Z}/m) \quad \text{for all } n \geq 0.$$

Proof. We may assume that Y is reduced, since replacing Y by Y_{red} doesn't change either K -group (see [W1, 1.4]). Let \tilde{Y} be the normalization of Y , S the singular set, and set $\tilde{S} = S \times_Y \tilde{Y}$. By [W1, 1.3], there is a Mayer–Vietoris exact sequence

$$\dots K_{n+1}(\tilde{S}; \mathbb{Z}/m) \rightarrow K_n(Y; \mathbb{Z}/m) \rightarrow K_n(\tilde{Y}; \mathbb{Z}/m) \oplus K_n(S; \mathbb{Z}/m) \rightarrow K_n(\tilde{S}; \mathbb{Z}/m).$$

Since $\rho(\mathbb{C})$ is an isomorphism, so are $\rho(S)$ and $\rho(\tilde{S})$. Since $\rho(\tilde{Y})$ is an isomorphism by the theorem, the 5-lemma implies that $\rho(Y)$ is also an isomorphism. \square

LEMMA 2.5. *Let F be the function field $\mathbb{C}(s, t)$ of the plane. Multiplication by the Bott element β is an isomorphism $K_n(F; \mathbb{Z}/m) \cong K_{n+2}(F; \mathbb{Z}/m)$ for all $n \geq 1$.*

Proof. We regard $F = \mathbb{C}(s, t)$ as the field of fractions of the polynomial ring $R = \mathbb{C}(s)[t]$. For every residue field E of R , there is a tame symbol $\partial_E: K_n(F; \mathbb{Z}/m) \rightarrow K_{n-1}(E; \mathbb{Z}/m)$, and the direct sum over all such E is one of the maps in the K -theory localization sequence. It is well known (see [Sou, p. 271]) that the K -theory localization sequence breaks up into short exact sequences, and that it is a sequence of modules for the graded ring $K_*(\mathbb{C}; \mathbb{Z}/m)$. This gives us a commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_n(\mathbb{C}(s); \mathbb{Z}/m) & \longrightarrow & K_n(F; \mathbb{Z}/m) & \xrightarrow{\partial} & \bigoplus_E K_{n-1}(E; \mathbb{Z}/m) \longrightarrow 0 \\ & & \cup\beta \downarrow \cong & & \cup\beta \downarrow & & \cup\beta \downarrow \cong \\ 0 & \longrightarrow & K_{n+2}(\mathbb{C}(s); \mathbb{Z}/m) & \longrightarrow & K_{n+2}(F; \mathbb{Z}/m) & \xrightarrow{\partial} & \bigoplus_E K_{n+1}(E; \mathbb{Z}/m) \longrightarrow 0. \end{array}$$

For $n \geq 1$, the outside vertical maps are isomorphisms by Proposition 2.3. The 5-lemma shows that $\cup\beta$ is an isomorphism on $K_n(F; \mathbb{Z}/m)$. \square

Proof of Theorem 2.2. First we show that $\cup\beta: K_n(F; \mathbb{Z}/m) \cong K_{n+2}(F; \mathbb{Z}/m)$ for all $n \geq 1$. For n odd, this follows from Proposition 2.3 and the fact that every element of $K_n(F; \mathbb{Z}/m) \cong F^\times/m$ comes from $K_n(E; \mathbb{Z}/m)$ for some subfield E of transcendence degree 1 over \mathbb{C} . For $n = 2i$ even, we have isomorphisms

$$K_2(F; \mathbb{Z}/m) \xrightarrow{\rho} K_2^{et}(F; \mathbb{Z}/m) \xrightarrow{\cup\beta^i} K_{2i+2}^{et}(F; \mathbb{Z}/m)$$

by [DF, 8.2]. Hence the intermediate map $K_2(F; \mathbb{Z}/m) \xrightarrow{\cup\beta^i} K_{2i+2}(F; \mathbb{Z}/m)$ is an injection. Since the Bott summands are generated by the powers of β , it remains to show that the summand $H^2(F, \mathbb{Z}/m) \cong K_2(F)/m$ of $K_{2i+2}(F; \mathbb{Z}/m)$ is in the image of $\cup\beta^i$. This summand is generated by the symbols $\{s, t\}$, $s, t \in F^\times$. By naturality, the symbol $\sigma = \{s, t\}$ comes from the summand $H^2(\mathbb{C}(s, t); \mathbb{Z}/m)$ of $K_{n+2}(\mathbb{C}(s, t); \mathbb{Z}/m)$. By Lemma 2.5, $\sigma = x \cup \beta^i$ for some $x \in K_2(F)/m$ coming from $K_2(\mathbb{C}(s, t))/m$. Hence the map $\cup\beta^i$ is onto, as desired.

With this, the rest of Suslin’s proof of Theorem 2.2 in loc. cit. goes through for surfaces, using Thomason’s theorem and the long exact sequences

$$\dots K'_{*+1}(F; \mathbb{Z}/m) \rightarrow \varinjlim K'_*(Y; \mathbb{Z}/m) \rightarrow K'_*(X; \mathbb{Z}/m) \rightarrow K'_*(F; \mathbb{Z}/m) \dots \quad \square$$

Now suppose that X is a smooth affine surface. Since $H^3(X) = H^4(X) = 0$, the spectral sequence for étale K -theory [DF, 5.2] degenerates. Using Theorem 2.2, this yields $K_n(X; \mathbb{Z}/m)$ for $n \geq 1$: it is $H^1(X, \mathbb{Z}/m)$ if n is odd, and it is the direct sum of the Bott summand \mathbb{Z}/m and $H^2(X, \mathbb{Z}/m)$ when n is even.

Taking the direct limit over all affine open subsets of X , the affine case immediately yields the following description of the Zariski sheaves $\mathcal{K}_n(\mathbb{Z}/m)$.

LEMMA 2.6. *When X is a smooth surface, we have:*

$$\mathcal{K}_n(\mathbb{Z}/m) \cong \begin{cases} \mathbb{Z}/m \oplus \mathcal{H}^2(\mathbb{Z}/m) & \text{if } n \geq 2 \text{ is even;} \\ \mathcal{H}^1(\mathbb{Z}/m) & \text{if } n \geq 1 \text{ is odd.} \end{cases}$$

Remark 2.6.1. The isomorphism $\mathcal{K}_2(\mathbb{Z}/m) \cong \mathbb{Z}/m \oplus \mathcal{H}^2(\mathbb{Z}/m)$ is induced by the Chern class $c_2: \mathcal{K}_2(\mathbb{Z}/m) \rightarrow \mathcal{H}^2(\mathbb{Z}/m)$; see [CT-R, p.168]. Applying H^1 , we see that c_2 induces an isomorphism between $H^1(X, \mathcal{K}_2(\mathbb{Z}/m))$ and $H^1(X, \mathcal{H}^2(\mathbb{Z}/m))$, which equals $H^3(X, \mathbb{Z}/m)$ when X is a surface by [BO]. This observation is implicit in [Su2, p. 19].

COROLLARY 2.7. *When X is a smooth surface, we have:*

$$\text{If } n \geq 2 \text{ is even, } H^p(X, \mathcal{K}_n(\mathbb{Z}/m)) \cong \begin{cases} \mathbb{Z}/m \oplus \text{Br}(X)_m & \text{if } p = 0; \\ H^3(X, \mathbb{Z}/m) & \text{if } p = 1; \\ H^4(X, \mathbb{Z}/m) & \text{if } p = 2. \end{cases}$$

$$\text{If } n \geq 1 \text{ is odd, } H^p(X, \mathcal{K}_n(\mathbb{Z}/m)) \cong \begin{cases} H^1(X, \mathbb{Z}/m) & \text{if } p = 0; \\ \text{Pic}(X)/m & \text{if } p = 1; \\ 0 & \text{if } p = 2. \end{cases}$$

Proof. Just combine the Bloch–Ogus resolutions of the sheaves $\mathcal{H}^n(\mathbb{Z}/m)$ appearing in Lemma 2.6, together with the Leray spectral sequence:

$$E_2^{p,q} = H^p(X, \mathcal{H}^q(\mathbb{Z}/m)) \implies H^{p+q}(X, \mathbb{Z}/m), \quad q \geq p \geq 0. \tag{2.7.1}$$

For example, $H^0(X, \mathcal{H}^2(\mathbb{Z}/m)) \cong \text{Br}(X)_m$ and $H^1(X, \mathcal{H}^1(\mathbb{Z}/m)) \cong \text{Pic}(X)/m$ follow from the Bloch–Ogus resolutions; see [BO, 7.7]. □

THEOREM 2.8. *Let X be a smooth irreducible surface over \mathbb{C} . Then:*

$$K_n(X; \mathbb{Z}/m) \cong \begin{cases} \mathbb{Z}/m \oplus H^2(X, \mathbb{Z}/m) \oplus H^4(X, \mathbb{Z}/m) & \text{if } n \geq 2 \text{ is even;} \\ H^1(X, \mathbb{Z}/m) \oplus H^3(X, \mathbb{Z}/m) & \text{if } n \geq 1 \text{ is odd} \end{cases}$$

Topological proof. Let $KU^*(X)$ denote the topological K -theory of complex vector bundles on the underlying space $X(\mathbb{C})$, and let $KU^*(X; \mathbb{Z}/m)$ denote the corresponding theory with coefficients \mathbb{Z}/m . We know by [Fr, 1.6] that $K_n^{et}(X; \mathbb{Z}/m) \cong KU^{-n}(X; \mathbb{Z}/m)$. As an exercise, the reader might want to derive Theorem 2.8 from Theorem 2.2, using the Atiyah–Hirzebruch calculation:

$$KU^n(X) \cong \begin{cases} \mathbb{Z} \oplus H_{an}^2(X, \mathbb{Z}) \oplus H_{an}^4(X, \mathbb{Z}) & \text{if } n \text{ is even;} \\ H_{an}^1(X, \mathbb{Z}) \oplus H_{an}^3(X, \mathbb{Z}) & \text{if } n \text{ is odd.} \end{cases} \tag{2.8.1}$$

Algebraic Proof. Consider the Brown–Gersten spectral sequence:

$$E_2^{p,-q} = H^p(X, \mathcal{K}_q(\mathbb{Z}/m)) \implies K_{-p-q}(X; \mathbb{Z}/m). \tag{2.9}$$

We first show that the spectral sequence degenerates at E_2 . Since the only possible nonzero differential is from $H^1(X, \mathbb{Z}/m)$ to $H^4(X, \mathbb{Z}/m)$, it suffices to show that the edge map $\eta_n: H^4(X, \mathbb{Z}/m) \rightarrow K_n(X; \mathbb{Z}/m)$ is injective for all $n \geq 0$. This is trivial when $H^4(X, \mathbb{Z}/m) = 0$, *i.e.*, when X is not projective.

Suppose now that X is projective, so that $H^4(X, \mathbb{Z}/m) = \mathbb{Z}/m$. Choose a point $i: \text{Spec}(\mathbb{C}) \rightarrow X$ and let p denote the structure map $X \rightarrow \text{Spec}(\mathbb{C})$. Then the composite

$$K_n(\mathbb{C}; \mathbb{Z}/m) \xrightarrow{i_*} K_n(X; \mathbb{Z}/m) \xrightarrow{p_*} K_n(\mathbb{C}; \mathbb{Z}/m)$$

is the identity. By Quillen’s construction of the spectral sequence (2.9), i_* is one component of $E_1^{2,n+2} = \coprod_x K_n(k(x); \mathbb{Z}/m)$. Hence the injection i_* factors through both the edge map $E_1^{2,n+2} \rightarrow K_n(X; \mathbb{Z}/m)$ and its quotient η_n . By counting, the map $i_*: K_n(\mathbb{C}; \mathbb{Z}/m) \rightarrow H^4(X, \mathbb{Z}/m) = \mathbb{Z}/m$ is an isomorphism. It follows that the edge map η_n is an injection, as claimed.

It remains to resolve the extension problems. There is no problem for even n , since $K_n(X; \mathbb{Z}/m)$ contains both the Bott summand \mathbb{Z}/m and the summand $i_*(\mathbb{Z}/m)$. When $n \geq 1$ is odd, $K_n(X; \mathbb{Z}/m)$ is isomorphic to $K_1(X; \mathbb{Z}/m)$ by Theorem 2.2 (by repeated multiplication by β). Thus it suffices to show that the extension splits for $K_1(X; \mathbb{Z}/m)$. Now $\text{Pic}(X)$ is a summand of $K_0(X)$, and $H^0(X, \mathcal{K}_1)$ is a summand of $K_1(X)$, so there is a natural splitting map from $H^1(X, \mathbb{Z}/m)$ into $K_1(X; \mathbb{Z}/m)$, as claimed. □

COROLLARY 2.10. $K_n(X; \mathbb{Q}/\mathbb{Z}) \rightarrow H^0(X, \mathcal{K}_n(\mathbb{Q}/\mathbb{Z}))$ is a split surjection.

Proof. Passing to the limit in (2.9) as $m \rightarrow \infty$ yields the Brown–Gersten spectral sequence with coefficients \mathbb{Q}/\mathbb{Z} . The proof of Theorem 2.8 shows that it degenerates at E_2 , and that the extensions split. □

Let $K_n^{rel}(X)$ denote the relative term in the natural sequence

$$\dots K_{n+1}(X) \xrightarrow{\rho} KU^{-n-1}(X) \rightarrow K_n^{rel}(X) \rightarrow K_n(X) \xrightarrow{\rho} KU^{-n}(X) \dots \tag{2.11.0}$$

PROPOSITION 2.11. *Let X be a smooth surface over \mathbb{C} . Then for all $n \geq 0$, the groups $K_n^{rel}(X)$ are uniquely divisible, while $K_{-1}^{rel}(X)$ is torsion-free.*

Proof. The usual homological yoga yields a sequence with coefficients \mathbb{Z}/m :

$$\dots \xrightarrow{\rho} KU^{-n-1}(X; \mathbb{Z}/m) \rightarrow K_n^{rel}(X; \mathbb{Z}/m) \rightarrow K_n(X; \mathbb{Z}/m) \xrightarrow{\rho} KU^{-n}(X; \mathbb{Z}/m).$$

By Theorem 2.2, the group $K_n^{rel}(X; \mathbb{Z}/m)$ vanishes for $n \geq 0$. This implies the result, since $K_n^{rel}(X)/m$ is a subgroup, and the m -torsion in $K_{n-1}^{rel}(X)$ is a quotient. □

COROLLARY 2.12. *Let X be a smooth surface over \mathbb{C} . Then each group $K_n(X)$ is the sum of a divisible group and a finitely generated group. The finitely generated subgroup injects into $KU^{-n}(X)$ and has the same torsion subgroup.*

Proof. Everything follows from Proposition 2.11 because $KU^{-n}(X)$ is a finitely generated Abelian group. \square

3. K -Theory of Curves

At this point, we pause to collect the information about the K -theory of a curve Y of finite type over \mathbb{C} . The first step is to describe the K -theory of its function field E . Recall from Example 1.1(a) that the group $K_1(E) = E^\times$ is never divisible, being the product of \mathbb{C}^\times and an uncountable free Abelian group.

PROPOSITION 3.1. *Let $E = \mathbb{C}(Y)$ be the function field of a curve over \mathbb{C} . Then $K_n(E)$ is a divisible group for every $n \geq 2$, and is the direct sum of the groups $CH^i(E, n)$, $2 \leq i \leq n$. Moreover, the torsion subgroup of $K_n(E)$ is:*

$$K_n(E)_{tors} \cong \begin{cases} \mathbb{Q}/\mathbb{Z} & \text{if } n \geq 1 \text{ is odd;} \\ H^1(E, \mathbb{Q}/\mathbb{Z}(i+1)) & \text{if } n \geq 2 \text{ is even.} \end{cases}$$

Remark. The groups $H^1(E, \mathbb{Q}/\mathbb{Z}(i+1))$ are all isomorphic to $E^\times \otimes \mathbb{Q}/\mathbb{Z}$, which as we have seen is an uncountable direct sum of copies of \mathbb{Q}/\mathbb{Z} . The isomorphism $K_2(E)_{tors} \cong E^\times \otimes \mathbb{Q}/\mathbb{Z}$ is due to Suslin [Su2, 3.7], while divisibility of $K_3(E)$ follows from this and [MS1, 8.4].

Proof. By Lemma 1.4 and Propositions 1.6 and 1.8, $CH^i(E, n)$ is divisible for $n = 2i - 1$ and $n = 2i - 2$ and is uniquely divisible otherwise.

Arguing as in Proposition 1.3, each differential in the Bloch–Lichtenbaum spectral sequence (1.2) vanishes, because it has either a divisible source or a torsion-free image. Hence, each $K_n(E)$ is the direct sum of the divisible groups $CH^i(E, n)$.

This yields $K_n(E)_{tors} \cong K_{n+1}(E; \mathbb{Q}/\mathbb{Z})$, a group described in Proposition 2.1. \square

THEOREM 3.2. *Let Y be a smooth irreducible curve over \mathbb{C} . Then $K_n(Y)$ is divisible for $n \geq 2$, and its torsion subgroup is given by:*

$$K_n(Y)_{tors} = \begin{cases} H^1(Y, \mathbb{Q}/\mathbb{Z}), & \text{if } n \text{ is even,} \\ \mathbb{Q}/\mathbb{Z}, & \text{if } n \text{ is odd and } Y \text{ is affine,} \\ \mathbb{Q}/\mathbb{Z} \oplus \mathbb{Q}/\mathbb{Z} & \text{if } n \text{ is odd and } Y \text{ is projective.} \end{cases}$$

Remarks 3.2.1 (1) When Y is a projective curve of genus g and $n = 2i \geq 2$, this yields $K_{2i}(Y)_{tors} = \text{Pic}(Y)_{tors} \cong (\mathbb{Q}/\mathbb{Z})^{2g}$. However, if $Y = \text{Spec}(A)$ is affine then $K_{2i}(A)_{tors}$ is the direct sum of $\text{Pic}(Y)_{tors}$ and $A^\times \otimes \mathbb{Q}/\mathbb{Z}$.

(2) Let $SK_1(Y)$ denote the kernel of $K_1(Y) \rightarrow K_1(E) = E^\times$. Then:

$$K_1(Y) \cong SK_1(Y) \oplus H^0(X, \mathcal{O}_Y^\times).$$

The proof also shows that $SK_1(Y)$ is divisible, and that its torsion subgroup $SK_1(Y)_{tors}$ is: zero for affine curves, and \mathbb{Q}/\mathbb{Z} for a projective curve. These observations were first made by Gersten [Ger, p. 38].

(3) The fact that $H^1(Y, \mathbb{Q}/\mathbb{Z})$ is the torsion in $K_2(Y)$ is due to Suslin [Su2, 5.2], given the calculation implicit in [MS1, 11,11,1] that $H^1(Y, \mathcal{K}_3)$ is uniquely divisible.

Proof. The K -theory localization sequence for Y is

$$K_{n+1}(E) \rightarrow \coprod K_n(\mathbb{C}) \rightarrow K_n(Y) \rightarrow K_n(E) \rightarrow \coprod K_{n-1}(\mathbb{C})$$

where E is the function field of Y and the coproduct is over all closed points of Y . When $n \geq 2$, the outer four terms are divisible (by Proposition 3.1). If n is odd, the lack of torsion in $K_{n-1}(\mathbb{C})$ forces the middle term $K_n(Y)$ to be divisible.

Fix an even number $n \geq 2$ and a positive integer m . Since $K_{n+1}(Y)$ is divisible, we have $K_n(Y)_m \cong K_{n+1}(Y; \mathbb{Z}/m) \cong H^1(Y, \mathbb{Z}/m)$ by Proposition 2.3. Hence, the localization sequence for K -theory with coefficients \mathbb{Z}/m becomes

$$0 \rightarrow K_n(Y)_m \rightarrow H^1(E, \mathbb{Z}/m) \rightarrow \coprod \mathbb{Z}/m \rightarrow K_{n-1}(Y; \mathbb{Z}/m) \rightarrow \mathbb{Z}/m$$

The final arrow in this sequence is a split surjection, arising from the Bott summand \mathbb{Q}/\mathbb{Z} in $K_{n-1}(\mathbb{C}) \subseteq K_{n-1}(Y)$ by Proposition 1.6.

Comparing with the localization sequence for étale cohomology, we see that $K_n(Y)_{tors}$ is isomorphic to $H^1(Y, \mathbb{Z}/m)$, and that $K_n(Y; \mathbb{Z}/m)$ is the sum of \mathbb{Z}/m and $H^2(Y, \mathbb{Z}/m)$. If Y is affine, the latter group is zero and $K_n(Y; \mathbb{Z}/m)$ equals the Bott summand \mathbb{Z}/m of $K_{n-1}(Y)_m$. This implies that $K_n(Y)/m = 0$, i.e., $K_n(Y)$ is m -divisible.

If Y is projective, choose a point $i: \text{Spec}(\mathbb{C}) \rightarrow Y$ and let p denote the structure map $Y \rightarrow \text{Spec}(\mathbb{C})$. Then the composite

$$K_{n-1}(\mathbb{C}) \xrightarrow{i_*} K_{n-1}(Y) \xrightarrow{p_*} K_{n-1}(\mathbb{C})$$

is the identity, while p_* vanishes on the Bott summand. This provides a subgroup of $K_{n-1}(Y)$ isomorphic to $\mathbb{Q}/\mathbb{Z} \oplus \mathbb{Q}/\mathbb{Z}$. Therefore $K_n(Y; \mathbb{Z}/m) \cong \mathbb{Z}/m \oplus \mathbb{Z}/m$ is isomorphic to $K_{n-1}(Y)_m$, and again this implies that $K_n(Y)/m = 0$, i.e., $K_n(Y)$ is m -divisible. \square

COROLLARY 3.3. *Let Y be a smooth curve over \mathbb{C} . If $n > 0$ is even, the sheaf \mathcal{K}_n is uniquely divisible. If $n > 0$ is odd, the sheaf \mathcal{K}_n is divisible, the direct sum of the constant sheaf \mathbb{Q}/\mathbb{Z} and a uniquely divisible sheaf.*

4. Chern Classes

In this section we will use Chern classes to show that the K -theory of $\mathbb{C}(X)$ is divisible when X is a surface. First, we need to introduce some notation.

Let X be a smooth variety over \mathbb{C} , and let $F = \mathbb{C}(X)$ denote the field of rational functions of X . For each analytic sheaf \mathcal{A} on X , such as $\mathcal{A} = \mathbb{Z}$ or \mathbb{C}^\times , we write $H_{\text{an}}^j(F, \mathcal{A})$ for the direct limit $\lim_{\rightarrow} H_{\text{an}}^j(U, \mathcal{A})$, taken over all U open in X . For example, if \mathcal{A} is \mathbb{Z}/m or \mathbb{Q}/\mathbb{Z} then $H_{\text{et}}^j(F, \mathcal{A}) \cong H_{\text{an}}^j(F, \mathcal{A})$ for all j .

LEMMA 4.0. *Identifying \mathbb{Q}/\mathbb{Z} with the torsion subgroup of \mathbb{C}^\times ,*

$$H^j(X, \mathbb{Q}/\mathbb{Z}) \cong H_{\text{an}}^j(X, \mathbb{C}^\times)_{\text{tors}} \quad \text{and} \quad H^j(F, \mathbb{Q}/\mathbb{Z}) \cong H_{\text{an}}^j(F, \mathbb{C}^\times)_{\text{tors}}$$

for all j . Moreover, for each m there is a (noncanonically) split exact sequence

$$0 \rightarrow A_j/mA_j \rightarrow H^j(X, \mathbb{Z}/m) \rightarrow H_{\text{an}}^j(X, \mathbb{C}^\times)_m \rightarrow 0,$$

where $A_j = A_j(X)$ is the finite group $H_{\text{an}}^j(X, \mathbb{Z})_{\text{tors}}$.

Proof. The finitely generated group $H_{\text{an}}^j(X, \mathbb{Z})$ is isomorphic to $A_j \oplus \mathbb{Z}^{b_j}$, where b_j is the j th Betti number of X . Hence the exponential sequence $\mathbb{Z} \rightarrow \mathbb{C} \rightarrow \mathbb{C}^\times$ of constant sheaves on X yields a (split) exact sequence of groups

$$0 \rightarrow (\mathbb{C}/\mathbb{Z})^{b_j} \rightarrow H_{\text{an}}^j(X, \mathbb{C}^\times) \rightarrow A_{j+1} \rightarrow 0.$$

Tensoring with \mathbb{Z}/m yields $H_{\text{an}}^j(X, \mathbb{C}^\times)/m \cong A_{j+1}/m$. From the Kummer sequence $0 \rightarrow \mathbb{Z}/m \rightarrow \mathbb{C}^\times \xrightarrow{m} \mathbb{C}^\times \rightarrow 0$ we get an extension

$$0 \rightarrow H_{\text{an}}^{j-1}(X, \mathbb{C}^\times)/m \rightarrow H^j(X, \mathbb{Z}/m) \rightarrow H_{\text{an}}^j(X, \mathbb{C}^\times)_m \rightarrow 0. \tag{4.0.1}$$

The first map factors as the split inclusion of A_j/m in $H_{\text{an}}^j(X, \mathbb{Z})/m$ followed by $H_{\text{an}}^j(X, \mathbb{Z})/m \rightarrow H^j(X, \mathbb{Z}/m)$, which is a split inclusion by the Künneth formula. This yields the split exact sequence. The direct limit over m yields $H^j(X, \mathbb{Q}/\mathbb{Z}) \cong H_{\text{an}}^j(X, \mathbb{C}^\times)_{\text{tors}}$ because $A_j \otimes \mathbb{Q}/\mathbb{Z} = 0$. Also, replacing X by open subsets U and taking the limit over U yields $H^j(F, \mathbb{Q}/\mathbb{Z}) \cong H_{\text{an}}^j(F, \mathbb{C}^\times)_{\text{tors}}$. \square

We will need the Deligne–Beilinson cohomology groups $H_{\mathcal{D}}^*$, referring the reader to [EV] for details of their construction. We shall also need the Chern class maps

$$c_i: K_n(X) \rightarrow H_{\mathcal{D}}^{2i-n}(X, \mathbb{Z}(i)), \quad i \geq 1,$$

constructed in [Gi1, Gi2]. In the particular case when $n = 2i - 3$, these maps go from $K_{2i-3}(X)$ to $H_{\mathcal{D}}^3(X, \mathbb{Z}(i))$.

LEMMA 4.1. *Let $F = \mathbb{C}(X)$ be the field of rational functions of a smooth variety X over \mathbb{C} . Then for all m :*

- (1) $H_{\text{an}}^j(F, \mathbb{C}^\times)$ is divisible for $j \leq 2$;
- (2) $H^j(F, \mathbb{Z}/m) \cong H_{\text{an}}^j(F, \mathbb{C}^\times)_m$ for $j \leq 3$;

- (3) $H_{\mathcal{D}}^j(X, \mathbb{Z}(i))_m \cong H_{\text{an}}^{j-1}(X, \mathbb{C}^\times)_m$ for all $i \geq j$;
- (4) $H_{\mathcal{D}}^j(F, \mathbb{Z}(i))_m \cong H_{\text{an}}^{j-1}(F, \mathbb{C}^\times)_m$ for all $i \geq j$;
- (5) The Chern class c_i induces maps between m -torsion subgroups:

$$K_{2i-3}(F)_m \rightarrow H_{\mathcal{D}}^3(F, \mathbb{Z}(i))_m \cong H_{\text{an}}^2(F, \mathbb{C}^\times)_m \cong H^2(F, \mathbb{Z}/m), \quad i \geq 3.$$

Proof. Clearly $H_{\text{an}}^0(F, \mathbb{C}^\times(i)) = \mathbb{C}^\times$ is divisible. Next we use the result of Barbieri–Viale [BV1, 3.2 and 4.3] that $H_{\text{an}}^2(F, \mathbb{Z})$ and $H_{\text{an}}^3(F, \mathbb{Z})$ are torsion-free, and thus inject into $H_{\text{an}}^2(F, \mathbb{C})$ and $H_{\text{an}}^3(F, \mathbb{C})$. From the exponential sequence, it follows that $H_{\text{an}}^1(F, \mathbb{C}^\times)$ is a quotient of $H_{\text{an}}^1(F, \mathbb{C})$, and $H_{\text{an}}^2(F, \mathbb{C}^\times)$ is a quotient of $H_{\text{an}}^2(F, \mathbb{C})$. Hence they are divisible, proving part (1).

Part (2) follows from (1) using the Kummer sequence (4.0.1).

Next we suppose $i \geq j$, so that $F^i H_{\text{an}}^{j-1}(U, \mathbb{C}) = 0$. For each open $U \subset X$ there is an exact sequence (see [EV, 2.10c]):

$$0 \rightarrow H_{\text{an}}^{j-1}(U, \mathbb{C}^\times(i)) \rightarrow H_{\mathcal{D}}^j(U, \mathbb{Z}(i)) \rightarrow F^i H_{\text{an}}^j(U, \mathbb{C}).$$

Since $F^i H^j(U, \mathbb{C})$ is torsion-free, this yields part (3). Part (4) follows from (3) by taking the limit over U . Part (5) follows from (2) and (4). □

Remark. Suppose that X is a surface. Then $H_{\text{an}}^j(F, \mathbb{C}^\times) = 0$ for all $j \geq 3$. In this case, some parts of Lemma 4.1 are trivial.

We will only need the Chern classes which land in $H_{\mathcal{D}}^j$, $j \leq 4$.

COROLLARY 4.2. *Let X be a smooth surface over \mathbb{C} . Then*

$$\begin{aligned} H^2(X, \mathbb{Q}/\mathbb{Z}) &\cong H_{\text{an}}^2(X, \mathbb{C}^\times)_{\text{tors}} \cong H_{\mathcal{D}}^3(X, \mathbb{Z}(i+1))_{\text{tors}}, \quad i \geq 2; \\ H^3(X, \mathbb{Q}/\mathbb{Z}) &\cong H_{\text{an}}^3(X, \mathbb{C}^\times)_{\text{tors}} \cong H_{\mathcal{D}}^4(X, \mathbb{Z}(i+2))_{\text{tors}}, \quad i \geq 1. \end{aligned}$$

If X is proper then the first line also holds for $i = 1$.

Proof. By Lemma 4.0, $H^j(X, \mathbb{Q}/\mathbb{Z}) \cong H_{\text{an}}^j(X, \mathbb{C}^\times)_{\text{tors}}$ for all j . We are done for $i \geq 2$, as the other isomorphisms are Lemma 4.1(3). So suppose that $i = 1$. For X proper, both cases are proven in [BPW, 4.7(iii)]. For the remaining case, it suffices to prove that $H^3(X, \mathbb{C}^\times) \cong H_{\mathcal{D}}^4(X, \mathbb{Z}(3))$. Because $\dim(X) < 3$, this follows from the exact sequence

$$F^3 H^3(X, \mathbb{C}) \rightarrow H^3(X, \mathbb{C}^\times) \rightarrow H_{\mathcal{D}}^4(X, \mathbb{Z}(3)) \rightarrow F^3 H^4(X, \mathbb{C}),$$

because $F^3 H^n(X, \mathbb{C}) = 0$ for all n by [D, 8.2.4]. □

DEFINITIONS 4.3. Let ζ be a primitive m th root of 1 in \mathbb{C} and let $\beta \in K_2(\mathbb{C}; \mathbb{Z}/m)$ be the Bott element corresponding to ζ . Let $\gamma = \gamma_m \in K_{2i-1}(\mathbb{C})_m$ be the generator of $K_{2i-1}(\mathbb{C})$ coming from the power β^i of the Bott element in $K_{2i}(\mathbb{C}; \mathbb{Z}/m)$.

By abuse of notation, we shall write $\zeta(i)$ for the image of the root of unity ζ under the injection of $\mu_m(\mathbb{C}) \cong H^0(\mathbb{C}, \mu_m^{\otimes i})$ into $H_{\mathcal{D}}^1(\mathbb{C}, \mathbb{Z}(i)) \cong \mathbb{C}^\times(i)$.

LEMMA 4.4. *The Chern class $c_i: CH^i(\mathbb{C}, 2i - 1)_m \rightarrow H_{\mathcal{D}}^1(\mathbb{C}, \mathbb{Z}(i))_m \cong \mathbb{Z}/m$ satisfies: $c_i(\gamma) = (-1)^{i-1}(i - 1)! \zeta(i)$.*

Proof. From [BPW, p.163], we know that the following diagram commutes

$$\begin{array}{ccc} K_{2i}(\mathbb{C}; \mathbb{Z}/m) & \longrightarrow & K_{2i-1}(\mathbb{C}) \\ c_i \downarrow & & \downarrow c_i \\ H_{\text{et}}^0(\mathbb{C}, \mu_m^{\otimes i}) & \longrightarrow & H_{\mathcal{D}}^1(\mathbb{C}, \mathbb{Z}(i)) \end{array}$$

and the étale Chern class satisfies the product formula: $c_i(\beta^i) = (-1)^{i-1}(i - 1)! \zeta^{\otimes i}$. (See [W2, 3.3].) Since the bottom map sends $\zeta^{\otimes i}$ to $\zeta(i)$, we are done. \square

So the map c_i induces a nontrivial map on m -torsion subgroups when $m \gg 0$.

LEMMA 4.5. *Let F be the function field of a variety over \mathbb{C} . Then the Chern class*

$$c_2: CH^2(F, 2)/m \cong K_2(F)/m \rightarrow H_{\mathcal{D}}^2(F, \mathbb{Z}(2))/m$$

is an isomorphism.

Proof. By [BV2, 2.2(iii)], there is an isomorphism between Zariski sheaves on X

$$\mathcal{H}_{\mathcal{D}}^2(\mathbb{Z}(2)) \otimes \mathbb{Z}/m \cong \mathcal{H}_{\text{et}}^2(\mathbb{Z}/m),$$

where $\mathcal{H}_{\mathcal{D}}^2(\mathbb{Z}(2))$ and $\mathcal{H}_{\text{et}}^2(\mathbb{Z}/m)$ are the sheaves associated to the Deligne–Beilinson and étale comology groups, respectively. The isomorphism of stalks at the generic point is

$$H_{\mathcal{D}}^2(F, \mathbb{Z}(2))/m \rightarrow H_{\text{et}}^2(F, \mathbb{Z}/m(2)).$$

Now the Merkurjev–Suslin isomorphism $K_2(F)/m \cong H_{\text{et}}^2(F, \mathbb{Z}/m(2))$ is given by the étale Chern class, which factors through the Deligne–Beilinson Chern class c_2 . Since $CH^2(F, 2)/m \cong K_2(F)/m$, the claim immediately follows. \square

We can describe the Chern classes on $K_j(F; \mathbb{Z}/m)$ for $j > 2$ using the product formula. Recall from Theorem 2.2 that multiplication by β^i induces isomorphisms $K_j(F; \mathbb{Z}/m) \rightarrow K_{j+2i}(F; \mathbb{Z}/m)$ for all $j \geq 1$. Composing multiplication by β^i with the boundary ∂ in the universal exact sequence for K -Theory with coefficients amounts to multiplication by $\gamma \in K_{2i-1}(\mathbb{C})$; this is a map from $K_j(F)/m$ to $K_{j+2i-1}(F)_m$. Composing this with the Deligne–Beilinson Chern class c_{i+2} , we obtain maps

$$\begin{array}{ccccccc} K_1(F)/m & \xrightarrow{\cup \beta^i} & K_{2i+1}(F; \mathbb{Z}/m) & \xrightarrow{\partial} & K_{2i}(F)_m & \xrightarrow{c_{i+1}} & H_{\mathcal{D}}^2(F, \mathbb{Z}(i+1))_m, \\ K_2(F)/m & \xrightarrow{\cup \beta^i} & K_{2i+2}(F; \mathbb{Z}/m) & \xrightarrow{\partial} & K_{2i+1}(F)_m & \xrightarrow{c_{i+2}} & H_{\mathcal{D}}^3(F, \mathbb{Z}(i+2))_m. \end{array}$$

Composing with the isomorphisms $H_{\mathcal{D}}^{j+1}(F, \mathbb{Z}(i))_m \cong H_{\text{et}}^j(F, \mathbb{Z}/m(j)) \cong K_j(F)/m$ of

Lemma 4.1 yields maps which depend upon i, j, m , and upon the choice of ζ :

$$\sigma_{ij}(m): K_j(F)/m \rightarrow K_j(F)/m, \quad j = 1, 2. \tag{4.5.1}$$

LEMMA 4.6. *For all i, m and $j = 1, 2$ the map $\sigma_{ij}(m): K_j(F)/m \rightarrow K_j(F)/m$ is multiplication by $(-1)^i i!$ and $(-1)^i (i + 1)!$, respectively.*

Proof. For $j = 1, 2$ we fix elements $a \in K_j(F)$. The Bockstein ∂ applied to $a \cup \beta^i$ is $a \cup \gamma$. The product formula for the Chern class c_n yields:

$$c_{i+1}(a \cup \gamma) = \frac{-i!}{(i-1)!} c_1(a) \cup c_i(\gamma) = (-1)^i i! c_1(a) \cup \zeta(i) \tag{4.6.1}$$

for $a \in K_1(F)/m$, and

$$c_{i+2}(a \cup \gamma) = \frac{-(i+1)!}{(i-1)!} c_2(a) \cup c_i(\gamma) = (-1)^i (i+1)! c_2(a) \cup \zeta(i) \tag{4.6.2}$$

for $a \in K_2(F)$. Now the isomorphisms

$$K_1(F)/m \cong H_{\text{et}}^1(F, \mathbb{Z}/m(1)) \cong H_{\text{et}}^1(F, \mathbb{Z}/m(i+1)) \cong H_{\mathbb{D}}^2(F, \mathbb{Z}(i+1))_m$$

send a to $c_1(a) \cup \zeta(i)$. Similarly, the isomorphisms

$$K_2(F)/m \cong H_{\text{et}}^2(F, \mathbb{Z}/m(2)) \cong H_{\text{et}}^2(F, \mathbb{Z}/m(i+2)) \cong H_{\mathbb{D}}^3(F, \mathbb{Z}(i+2))_m$$

send a to $c_2(a) \cup \zeta(i)$. The result follows. □

Recall that if A is any Abelian group, its *Tate module* $T(A)$ is the inverse limit of the system of groups A_m .

$$T(A) = \varprojlim \{A_m \leftarrow A_{mn} \leftarrow \dots\}$$

It is well known that the Tate module of any Abelian group is torsion-free (see [CT-R, 1.3]).

If A is any Abelian group, we write \hat{A} for its profinite completion $\varprojlim A/mA$. We shall also write $K_n(F; \widehat{\mathbb{Z}})$ for the inverse limit of the groups $K_n(F; \mathbb{Z}/m)$.

LEMMA 4.7. *Let F be the function field of a surface X . Then the inverse limit $K_n(F; \widehat{\mathbb{Z}})$ is a torsion-free group for all n .*

Proof. Suslin's Proposition 2.1 shows that $K_{2i}(F; \widehat{\mathbb{Z}})$ is the sum of $\widehat{\mathbb{Z}} = \varprojlim \mathbb{Z}/m$ and $\varprojlim H^2(F, \mathbb{Z}/m)$. By Lemma 4.1, the latter group is the inverse limit of the groups $H_{\text{an}}^2(\overleftarrow{F}, \mathbb{C}^\times)_m$, so it is the Tate module of $H_{\text{an}}^2(F, \mathbb{C}^\times)$. As such it is torsion-free.

Similarly, we see from Proposition 2.1 and Lemma 4.1(2) that there is a natural isomorphism between $K_{2i-1}(F; \mathbb{Z}/m)$ and $H_{\text{an}}^1(F, \mathbb{C}^\times)_m$. Hence $K_{2i-1}(F; \widehat{\mathbb{Z}})$ is the Tate module of $H_{\text{an}}^1(F, \mathbb{C}^\times)$, and as such is torsion-free. □

THEOREM 4.8. *Let $F = \mathbb{C}(X)$ be the field of rational functions of a complex surface. Then for every $n \geq 2$ there are isomorphisms:*

$$(a) \quad K_n(F)_{tors} \cong \begin{cases} H^2(F, \mathbb{Q}/\mathbb{Z}) \oplus \mathbb{Q}/\mathbb{Z} & \text{if } n = 2i - 1; \\ H^1(F, \mathbb{Q}/\mathbb{Z}) & \text{if } n = 2i \end{cases}$$

(b) $K_n(F)$ is divisible for every $n \geq 3$.

The group \mathbb{Q}/\mathbb{Z} in part a) is the Bott summand. The summand $H^2(F, \mathbb{Q}/\mathbb{Z})$ equals the Brauer group $\text{Br}(F)$ of F , and $H^1(F, \mathbb{Q}/\mathbb{Z}) \cong F^\times \otimes \mathbb{Q}/\mathbb{Z}$.

Remark 4.8.1. Our proof makes the maps explicit. The cup product with $\gamma_i \in K_{2i-1}(\mathbb{C})$ induces the isomorphism from $K_1(F)/m = F^\times/F^{\times m}$ to $K_{2i}(F)_m$, and the injection from $K_2(F)/m$ into $K_{2i+1}(F)_m$ complementary to the Bott summand. The isomorphism $K_2(F)_m \cong H^1(F, \mathbb{Z}/m)$ has been proven in [Su, 3.7]. The first new case is the isomorphism, $K_3(F)_m \cong \mathbb{Z}/m \oplus H^2(F, \mathbb{Z}/m)$, where multiplication by the primitive m th root of unity $\zeta \in F^\times$ induces the injection

$$H^2(F, \mathbb{Z}/m) \cong K_2(F)/m \hookrightarrow K_3(F)_m.$$

This summand $H^2(F, \mathbb{Z}/m)$ is the torsion in the subgroup $K_3^M(F)$, while the complementary summand \mathbb{Z}/m of $K_3(F)_m$ corresponds to the torsion in $K_3(F)^{ind} = K_3(F)/K_3^M(F)$; see [MS1].

Proof. Consider the short exact sequence of towers of groups

$$0 \rightarrow \{K_n(F)/m\} \rightarrow \{K_n(F; \mathbb{Z}/m)\} \rightarrow \{K_{n-1}(F)_m\} \rightarrow 0.$$

Since the maps in the left-hand tower are all surjections, its \varprojlim^1 vanishes, and we have a short exact sequence after taking the inverse limit:

$$0 \rightarrow K_n(F)^\wedge \rightarrow K_n(F; \widehat{\mathbb{Z}}) \rightarrow T(K_{n-1}(F)) \rightarrow 0.$$

Now the Deligne–Beilinson Chern class $c_i: K_{2i-j}(F) \rightarrow H_D^j(F, \mathbb{Z}(i))$ induce a map on completion. By Lemma 4.6, the composition

$$\varprojlim \sigma_{ij}: K_j(F)^\wedge \rightarrow K_{2i+j}(F; \widehat{\mathbb{Z}}) \rightarrow T(K_{2i+j-1}(F)) \rightarrow K_j(F)^\wedge$$

is multiplication by a constant, so it is an injection. When $j = 1$, Theorem 2.2 implies that $K_1(F)^\wedge \cong K_{2i+1}(F; \widehat{\mathbb{Z}})$. Hence, the subgroup $K_{2i+1}(F)^\wedge$ must be zero, because it vanishes in the intermediate group $T(K_{2i}(F))$. When $j = 2$, we need an additional argument. Since $K_2(F)/m$ is the kernel of $K_2(F; \mathbb{Z}/m) \rightarrow K_2(\bar{F}; \mathbb{Z}/m)$, Theorem 2.2 implies that for each m the image of $K_2(F)/m$ in $K_{2i}(F; \mathbb{Z}/m)$ is the kernel of $K_{2i}(F; \mathbb{Z}/m) \rightarrow K_{2i}(\bar{F}; \mathbb{Z}/m)$, where \bar{F} is the algebraic closure of F . As such, it contains the subgroup $K_{2i}(F)/m$. Passing to the limit, we see that the image of $K_2(F)^\wedge$ in $K_{2i}(F; \widehat{\mathbb{Z}})$ contains the subgroup $K_{2i}(F)^\wedge$. This latter group vanishes in $T(K_{2i-1}(F))$, so it is in the kernel of $\varprojlim \sigma_{ij}$. As before, this forces $K_{2i}(F)^\wedge = 0$.

Since each group $K_n(F)/m$ is a quotient of $K_n(F)^\wedge$, and the latter is zero, each $K_n(F)/m$ vanishes. But then each $K_n(F)$ is divisible, as required. \square

5. The K -Cohomology of X

In this section we make some preliminary comments about the divisibility of the K -cohomology groups $H^p(X, \mathcal{K}_q)$ of the sheaves \mathcal{K}_q . In the next section, we will use this information to describe the structure of the K -theory of X using the Brown–Gersten spectral sequence:

$$E_2^{p,-q} = H^p(X, \mathcal{K}_q) \implies K_{-p-q}(X) \tag{5.1}$$

PROPOSITION 5.2. *Let X be a smooth surface over \mathbb{C} . Then*

$$(\mathcal{K}_n)_{tors} \cong \begin{cases} \mathbb{Q}/\mathbb{Z} \oplus \mathcal{H}^2(\mathbb{Q}/\mathbb{Z}) & \text{if } n \geq 3 \text{ is odd;} \\ \mathcal{H}^1(\mathbb{Q}/\mathbb{Z}) & \text{if } n \geq 2 \text{ is even.} \end{cases}$$

For all $n \geq 3$ the sheaf \mathcal{K}_n is divisible, and there are short exact sequence of sheaves:

$$0 \rightarrow \mathcal{K}_{n+1}(\mathbb{Z}/m) \rightarrow \mathcal{K}_n \xrightarrow{m} \mathcal{K}_n \rightarrow 0 \text{ for all } n \geq 3. \tag{5.2.1}$$

Remark 5.2.2. This fails of course for $n = 0, 1$ because we have $\mathcal{K}_0 = \mathbb{Z}$, the sheaves \mathcal{K}_1 and \mathcal{K}_2 are not divisible by Example 1.1, and $(\mathcal{K}_1)_{tors} \cong \mathbb{Q}/\mathbb{Z}$.

Proof. Let F denote the function field $\mathbb{C}(X)$ of X , and let $\zeta: \text{Spec}(F) \rightarrow X$ denote the inclusion of the generic point. From the universal exactness of the Gersten–Quillen resolution for the sheaf \mathcal{K}_n we get a resolution of the sheaf \mathcal{K}_n/m which begins:

$$0 \rightarrow \mathcal{K}_n/m \rightarrow \zeta_*\mathcal{K}_n(F)/m \rightarrow \dots$$

For $n \geq 3$ we have $\mathcal{K}_n(F)/m = 0$ by Theorem 4.8; it follows that $\mathcal{K}_n/m = 0$ and thus the sheaf \mathcal{K}_n is divisible. The sequence (5.2.1) is immediate, and (using Remark 5.2.2) so is the sequence

$$0 \rightarrow \mathcal{K}_3(\mathbb{Z}/m) \rightarrow \mathcal{K}_2 \xrightarrow{m} \mathcal{K}_2 \rightarrow \mathcal{H}^2(\mathbb{Z}/m) \rightarrow 0. \tag{5.2.3}$$

Hence $(\mathcal{K}_n)_m$ is isomorphic to $\mathcal{K}_{n+1}(\mathbb{Z}/m)$ for all $n \geq 2$. The description of $(\mathcal{K}_n)_{tors}$ is just a recasting of Lemma 2.6. □

COROLLARY 5.3. *Let X be a smooth irreducible surface over \mathbb{C} . For all $n \geq 2$ we have isomorphisms*

$$H^0(X, \mathcal{K}_n)_{tors} \cong \begin{cases} \mathbb{Q}/\mathbb{Z} \oplus \text{Br}(X) & \text{if } n \text{ is odd;} \\ H^1(X, \mathbb{Q}/\mathbb{Z}) & \text{if } n \text{ is even.} \end{cases}$$

Proof. Apply H^0 to (5.2.1–3) to get $H^0(X, \mathcal{K}_n)_{tors} \cong H^0(X, \mathcal{K}_{n+1}(\mathbb{Q}/\mathbb{Z}))$. Now use Corollary 2.7, recalling that $\text{Br}(X)$ is a torsion group. □

COROLLARY 5.4. *Let X be a smooth surface over \mathbb{C} . Then for all $n \geq 3$:*

- (1) *The groups $H^2(X, \mathcal{K}_n)$ are divisible, and*
- (2) *We have short exact sequences, natural in X :*

$$\begin{aligned}
 0 &\rightarrow H^0(X, \mathcal{K}_n)/m \rightarrow H^1(X, \mathcal{K}_{n+1}(\mathbb{Z}/m)) \rightarrow H^1(X, \mathcal{K}_n)_m \rightarrow 0 \\
 0 &\rightarrow H^1(X, \mathcal{K}_n)/m \rightarrow H^2(X, \mathcal{K}_{n+1}(\mathbb{Z}/m)) \rightarrow H^2(X, \mathcal{K}_n)_m \rightarrow 0.
 \end{aligned}$$

Proof. This is just a rewriting of the cohomology sequence of (5.2.1). □

Remark 5.4.1 ($n = 2$). These sequences should be contrasted with the sequences of [Su2, 4.4] for \mathcal{K}_2 on a surface X :

$$\begin{aligned}
 0 &\rightarrow H^0(X, \mathcal{K}_2)/m \rightarrow H^2(X, \mathbb{Z}/m) \rightarrow H^1(X, \mathcal{K}_2)_m \rightarrow 0 \\
 0 &\rightarrow H^1(X, \mathcal{K}_2)/m \rightarrow H^3(X, \mathbb{Z}/m) \rightarrow H^2(X, \mathcal{K}_2)_m \rightarrow 0.
 \end{aligned}$$

In effect, passage from \mathcal{K}_2 to \mathcal{K}_4 replaces $H^2(X, \mathbb{Z}/m)$ by the subgroup $\text{Pic}(X)/m$, and $H^3(X, \mathbb{Z}/m)$ by 0, as we see using Corollary 2.7 to describe $H^p(X, \mathcal{K}_3(\mathbb{Z}/m))$. The reader is invited to explain this passage using the sequences (5.2.1) and (5.2.3).

If X is projective, the group $H^2(X, \mathcal{K}_2)$ is not divisible either; it is isomorphic to the Chow group $CH^2(X)$ (Bloch’s formula), which is the direct sum of \mathbb{Z} and the divisible group $A_0(X)$.

THEOREM 5.5. *Let X be a smooth irreducible surface over \mathbb{C} . Then for all $n \geq 3$:*

- (1) *The groups $H^1(X, \mathcal{K}_n)$ and $H^2(X, \mathcal{K}_n)$ are divisible.*
- (2) *If X is not projective then the group $H^2(X, \mathcal{K}_n)$ is uniquely divisible.*
- (3) *If X is projective and $n \geq 4$ is even, $H^2(X, \mathcal{K}_n)$ is uniquely divisible.*
- (4) *If X is projective and $n \geq 3$ is odd, the edge map in the Brown–Gersten spectral sequence (5.1) induces an injection $H^2(X, \mathcal{K}_n)_{tors} \hookrightarrow K_{n-2}(X)$, and:*

$$H^2(X, \mathcal{K}_n)_{tors} \cong K_{n-2}(\mathbb{C})_{tors} \cong \begin{cases} \mathbb{Q}/\mathbb{Z} & \text{if } n \geq 3 \text{ is odd;} \\ 0 & \text{if } n \geq 4 \text{ is even.} \end{cases}$$

Remark 5.5.1 ($n = 2$). The group $H^1(X, \mathcal{K}_2)$ is not divisible in general. If X is projective, then we know by [CT-R, 2.2] that $H^1(X, \mathcal{K}_2)$ is the direct sum of a divisible group and the finite group $H^3(X, \mathbb{Z})_{tors}$.

Proof. If X is not projective, or n is even, then $H^2(X, \mathcal{K}_{n+1}(\mathbb{Z}/m)) = 0$ by Corollary 2.7. In these cases everything follows from Corollary 5.4.

Now suppose that $n \geq 3$ is odd and X is projective. Let $p: X \rightarrow \text{Spec}(\mathbb{C})$ be the structure map, and choose a closed point $i: \text{Spec}(\mathbb{C}) \rightarrow X$. As in the proof of Theorem 2.8, the map $i_*: K_{n-2}(\mathbb{C}) \rightarrow K_{n-2}(X)$ is an injection split by p_* , and it factors through $H^2(X, \mathcal{K}_n)$ because $\dim(X) = 2$ (via the Gersten resolution).

Since the map $K_*(X; \mathbb{Z}/m) \rightarrow K_{*-1}(X)$ induces a map between the respective Brown–Gersten–Quillen spectral sequences (2.9) and (5.1), their edge maps fit into

a commutative diagram:

$$\begin{array}{ccccccc}
 K_{n-1}(\mathbb{C}; \mathbb{Z}/m) & \xrightarrow[\cong]{i_*} & H^2(X, \mathcal{K}_{n+1}(\mathbb{Z}/m)) & \xrightarrow{\text{edge}} & K_{n-1}(X; \mathbb{Z}/m) & \xrightarrow{P_*} & K_{n-1}(\mathbb{C}; \mathbb{Z}/m) \\
 \text{into} \downarrow & & \alpha \downarrow & & \downarrow & & \text{into} \downarrow \\
 K_{n-2}(\mathbb{C}) & \xrightarrow{i_*} & H^2(X, \mathcal{K}_n) & \xrightarrow{\text{edge}} & K_{n-2}(X) & \xrightarrow{P_*} & K_{n-2}(\mathbb{C}).
 \end{array}$$

The outside vertical maps are injections by Corollary 1.5, and the horizontal composites are the identity. We saw in the proof of Theorem 2.8 that the upper left map i_* is an isomorphism. Hence the second vertical map α , which comes from (5.2.1), is an injection. By Corollary 5.4, $H^1(X, \mathcal{K}_n)/m$ is the kernel of α , so it is zero for all m , i.e., $H^1(X, \mathcal{K}_n)$ is divisible. Since the m -torsion in $H^2(X, \mathcal{K}_n)$ is the image of α , again by 5.4, $H^2(X, \mathcal{K}_n)_m \cong \mathbb{Z}/m$. The result now follows. \square

LEMMA 5.6. *Let X be a smooth surface. For even $n \geq 4$, $H^0(X, \mathcal{K}_n) \otimes \mathbb{Q}/\mathbb{Z} = 0$, and the map (induced by the K-theory product $\mathcal{K}_1 \otimes K_{n-1}(\mathbb{C}) \rightarrow \mathcal{K}_n$)*

$$\text{Pic}(X) \otimes K_{n-1}(\mathbb{C})_{\text{tors}} \hookrightarrow H^1(X, \mathcal{K}_1) \otimes K_{n-1}(\mathbb{C}) \xrightarrow{\cup} H^1(X, \mathcal{K}_n)$$

induces an isomorphism of $\text{Pic}(X) \otimes \mathbb{Q}/\mathbb{Z}$ with $H^1(X, \mathcal{K}_n)_{\text{tors}}$.

Remark 5.6.1 ($n = 2$). If X is proper over \mathbb{C} then $H^0(X, \mathcal{K}_2) \otimes \mathbb{Q}/\mathbb{Z} = 0$ as well, and the product induces an injection of $\text{Pic}(X) \otimes \mathbb{Q}/\mathbb{Z} \cong (\mathbb{Q}/\mathbb{Z})^\rho$ into $H^1(X, \mathcal{K}_2)_{\text{tors}}$. This was proven in [CT-R, 2.7]. We will see in Example 6.5.1 below that $H^1(X, \mathcal{K}_2)_{\text{tors}} \cong B \oplus (\mathbb{Q}/\mathbb{Z})^{b_2}$.

Proof. Set $n = 2i, i \geq 2$. We will use the Deligne–Beilinson Chern class

$$c_{i+1}: K_n(X) \rightarrow H^2_{\mathcal{D}}(X, \mathbb{Z}(i+1)).$$

For each m , choose an m th root of unity ζ_m , so $\cup \zeta_m^{\otimes i}: \mathcal{H}^1(\mu_m) \cong \mathcal{H}^1(\mu_m^{\otimes i+1})$. Composing with $c_1^{et}: \text{Pic}(X)/m \cong H^1(X, \mathcal{H}^1(\mu_m))$ yields the isomorphism c'_1 in the following diagram, which commutes by [BPW, p. 163] and Corollary 2.7.

$$\begin{array}{ccccc}
 \text{Pic}(X)/m & \xrightarrow{\cong} & H^1(X, \mathcal{K}_{n+1}(\mathbb{Z}/m)) & \xrightarrow{\partial} & H^1(X, \mathcal{K}_n)_m \\
 & & \downarrow c_{i+1}^{et} & & \downarrow c_{i+1} \\
 \text{Pic}(X)/m & \xrightarrow[\cong]{c'_1} & H^1(X, \mathcal{H}^1(\mu_m^{\otimes i+1})) & \xrightarrow{\gamma} & H^1(X, \mathcal{H}^2_{\mathcal{D}}(\mathbb{Z}(i+1)))_m \\
 & & \text{into} \downarrow & & \text{into} \downarrow \\
 0 \rightarrow A_2/m & \longrightarrow & H^2(X, \mathbb{Z}/m(i+1)) & \xrightarrow{\text{onto}} & H^3_{\mathcal{D}}(X, \mathbb{Z}(i+1))_m \rightarrow 0.
 \end{array}$$

The bottom row is the exact sequence of Lemma 4.0, using Corollary 4.2. The map γ is induced by the inclusion $\mathcal{H}^1(\mu_m^{\otimes i+1}) \subset \mathcal{H}^2_{\mathcal{D}}(\mathbb{Z}(i+1))$ of 4.1. The top composite is the map of the lemma, and the isomorphism $\text{Pic}(X)/m \cong H^1(X, \mathcal{K}_{n+1}(\mathbb{Z}/m))$ comes from Theorem 2.2 and Corollary 2.7. The inclusions of $H^1(X, \mathcal{H}^1(\mu_m^{\otimes i+1}))$ in

$H^2(X, \mathbb{Z}/m(i+1))$ and $H^1(X, \mathcal{H}_D^2)$ in $H_D^3(X)$ follow from the Bloch–Ogus spectral sequences.

Let $\gamma_m(i)$ be the element in $K_{2i-1}(\mathbb{C})_{tors}$ corresponding to ζ_m , as in Definition 4.3. The product of $\gamma_m(i)$ with $\lambda \in \text{Pic}(X)$ is an element $\{\lambda, \gamma_m(i)\}$ in $H^1(X, \mathcal{K}_n)_m$. As in (4.6.1), the product formula yields:

$$c_{i+1}(\{\lambda, \gamma_m(i)\}) = (-1)^i \cdot i! \cdot c_1(\lambda) \cup \zeta_m(i) = (-1)^i \cdot i! \cdot \gamma(c'_1(\lambda))$$

in $H_D^3(X, \mathbb{Z}(i))$, where $c_1(\lambda) \in H^1(X, \mathcal{H}_D^1(1))$ and $\zeta_m(i) \in H^0(X, \mathcal{H}_D^1(i))$.

A diagram chase shows that the kernel of the top map $\text{Pic}(X)/m \rightarrow H^1(X, \mathcal{K}_n)$ has exponent $N = (-1)^i |A_2| \cdot i!$, independent of m . But this kernel is $H^0(X, \mathcal{K}_n)/m$ by Corollary 5.4. Passing to the limit over m , we see that $H^0(X, \mathcal{K}_n) \otimes \mathbb{Q}/\mathbb{Z}$ has exponent N , and is divisible, so it is zero. By Corollary 5.4 again, this implies that $\text{Pic}(X) \otimes \mathbb{Q}/\mathbb{Z} \cong H^1(X, \mathcal{K}_n)_{tors}$, via the indicated map. \square

LEMMA 5.7. *Let X be a smooth surface. For odd $n \geq 3$, $H^0(X, \mathcal{K}_n) \otimes \mathbb{Q}/\mathbb{Z} = 0$*

Proof. Set $n = 2i + 1$ and $N = (-1)^i \cdot (i + 1)!$. Applying H^1 to the Chern classes c_{i+2} from K-Theory to singular and Deligne–Beilinson cohomology yields maps

$$\begin{aligned} H^1(X, \mathcal{K}_{n+1}(\mathbb{Z}/m)) &\rightarrow H^1(X, \mathcal{H}^2(\mathbb{Z}/m)) \cong H^3(X, \mathbb{Z}/m), \\ H^1(X, \mathcal{K}_n) &\rightarrow H^1(X, \mathcal{H}_D^3(\mathbb{Z}(i+2))) \cong H_D^4(X, \mathbb{Z}(i+2)). \end{aligned}$$

We claim that these fit into a commutative diagram for $i \geq 2$:

$$\begin{array}{ccccc} H^1(X, \mathcal{K}_2(\mathbb{Z}/m)) & \xrightarrow[\cong \cup \beta^i]{} & H^1(X, \mathcal{K}_{n+1}(\mathbb{Z}/m)) & \xrightarrow{\partial} & H^1(X, \mathcal{K}_n)_m \\ \cong \downarrow c_2 & & \downarrow c_{i+2} & & \downarrow c_{i+2} \\ H^3(X, \mathbb{Z}/m) & \xrightarrow{N} & H^3(X, \mathbb{Z}/m) & \xrightarrow{\text{onto}} & H_D^4(X, \mathbb{Z}(i+2))_m. \end{array} \tag{5.7.1}$$

The right square commutes by [BPW, p. 168], and the lower right horizontal arrow is a surjection with kernel A_3/m by Lemmas 4.0 and 4.2. The two isomorphisms in the upper left are isomorphisms by Theorem 2.2 and Remark 2.6.1. We must show that the left square in (5.7.1) commutes.

Let ζ be the primitive m th root of 1 in \mathbb{C} corresponding to the Bott element $\beta \in K_2(\mathbb{C}; \mathbb{Z}/m)$. As in (4.6.2), the product formula for c_{i+2} yields the following equalities for all $x \in H^1(X, \mathcal{K}_2(\mathbb{Z}/m))$:

$$c_{i+2}(x \cup \beta^i) = \frac{-(i+1)!}{(i-1)!} \cdot c_2(x) \cdot c_i(\beta^i) = N \cdot c_2(x) \cup \zeta(i).$$

Since $\cup \zeta(i)$ is the natural identification of $H^3(X, \mathbb{Z}/m(2))$ with $H^3(X, \mathbb{Z}/m(i+2))$, this establishes the commutativity of (5.7.1), as claimed.

From (5.7.1) we see that the kernel of $\partial: H^1(X, \mathcal{K}_{n+1}(\mathbb{Z}/m)) \rightarrow H^1(X, \mathcal{K}_n)_m$ has exponent $N'' = |A_3| \cdot N$ for all m . By Corollary 5.4, we have $\ker(\partial) =$

$H^0(X, \mathcal{K}_n)/m$. Letting m go to infinity, we see that the group $H^0(X, \mathcal{K}_n) \otimes \mathbb{Q}/\mathbb{Z}$ has exponent N^n . Since this group is divisible, it must be zero. \square

6. The K -Groups of a Surface

In this section we shall describe the Abelian group structure of $K_n(X)$. For this it will be convenient to introduce some notation.

NOTATION 6.1. Suppose that X is a complex variety. It is known that each cohomology group $H_{\text{an}}^n(X, \mathbb{Z})$ is a finitely generated Abelian group of rank b_n , where $b_n = \dim H_{\text{an}}^n(X, \mathbb{Q})$ is the n th Betti number of X . We set

$$A = H_{\text{an}}^2(X, \mathbb{Z})_{\text{tors}}, \quad B = H_{\text{an}}^3(X, \mathbb{Z})_{\text{tors}}.$$

When X is a surface, the groups $H_{\text{an}}^n(X, \mathbb{Z})$ are torsion free for $n \neq 2, 3$. Therefore the cohomology of a surface X with coefficients \mathbb{Z}/m is:

$$\begin{aligned} H^1(X, \mathbb{Z}/m) &\cong (\mathbb{Z}/m)^{b_1} \oplus A_m, \\ H^2(X, \mathbb{Z}/m) &\cong (\mathbb{Z}/m)^{b_2} \oplus (A/m) \oplus B_m, \\ H^3(X, \mathbb{Z}/m) &\cong (\mathbb{Z}/m)^{b_3} \oplus (B/m). \end{aligned} \tag{6.1.1}$$

We will want to compare these groups with étale cohomology, so we consider the change-of-topology morphism $\pi: X_{\text{an}} \rightarrow X_{\text{ét}}$. Applying $R\pi_*$ to the natural map $\mathcal{O}_{X_{\text{an}}}^\times[-1] \rightarrow \mathbb{Z}$ in the (analytic) exponential sequence yields a morphism $\eta: \mathbb{G}_m[-1] \rightarrow R\pi_*\mathbb{Z}$ in the derived category of étale sheaves on X . We define the groups V^n to be the étale hypercohomology of the cone of η , so that there is a long exact sequence, part of which is:

$$H_{\text{ét}}^{n-1}(X, \mathbb{G}_m) \xrightarrow{\eta} H_{\text{an}}^n(X, \mathbb{Z}) \rightarrow V^n \rightarrow H_{\text{ét}}^n(X, \mathbb{G}_m) \xrightarrow{\eta} H_{\text{an}}^{n+1}(X, \mathbb{Z}) \rightarrow \dots \tag{6.1.2}$$

LEMMA 6.1.3. *Each V^n is a uniquely divisible group, i.e., a \mathbb{Q} -vector space.*

Proof. For each m , consider the Kummer sequence $\mu_m \rightarrow \mathbb{G}_m \xrightarrow{m} \mathbb{G}_m$. Combining with the sequence of analytic sheaves $\mathbb{Z} \xrightarrow{m} \mathbb{Z} \rightarrow \mathbb{Z}/m$, a result of Verdier [WH, Ex. 10.2.6] implies that there is a commutative diagram in the derived category

$$\begin{array}{ccccc} \mathbb{G}_m[-1] & \xrightarrow{\eta} & R\pi_*\mathbb{Z} & \longrightarrow & \text{cone}(\eta) \\ \downarrow m & & \downarrow m & & \downarrow m \\ \mathbb{G}_m[-1] & \xrightarrow{\eta} & R\pi_*\mathbb{Z} & \longrightarrow & \text{cone}(\eta) \\ \downarrow & & \downarrow & & \downarrow \\ \mu_m & \xrightarrow{\cong} & \mathbb{Z}/m & \longrightarrow & 0 \end{array}$$

in which every row and column fit into a triangle. Since the right column fits into a

triangle, multiplication by m is an isomorphism on the cone and, hence, on its hypercohomology groups, the V^n . \square

Remark 6.1.4. In low degrees we can compare (6.1.2) to the exponential sequence to see that $V^0 \cong \mathbb{C}$ for connected X , and that $V^1 \cong \mathbb{R}^{b_1}$ for smooth projective X . In contrast, if X is smooth then the image of $H_{\text{an}}^n(X, \mathbb{Z}) \rightarrow V^n$ is a lattice for all $n \geq 2$, with $V^n \cong \mathbb{Q}^{b_n}$ for all $n \geq 3$. This claim follows from the fact that $H^n(X, \mathbb{G}_m)$ is a torsion group for $n \geq 2$ [Dix, p. 71]. It may be seen by tensoring (6.1.2) with \mathbb{Q} . We are grateful to the referee for pointing this out.

PROPOSITION 6.2. *If X is a variety over \mathbb{C} , then there are integers $\rho_n \leq b_n$ and uniquely divisible groups W^n such that*

$$H_{\text{et}}^n(X, \mathbb{G}_m) \cong \mathbb{Z}^{\rho_{n+1}} \oplus H_{\text{an}}^{n+1}(X, \mathbb{Z})_{\text{tors}} \oplus (\mathbb{Q}/\mathbb{Z})^{b_n - \rho_n} \oplus W^n.$$

If X is smooth then $\rho_n = 0$ for all $n \geq 3$, so in particular

$$H_{\text{et}}^n(X, \mathbb{G}_m) \cong H_{\text{an}}^{n+1}(X, \mathbb{Z})_{\text{tors}} \oplus (\mathbb{Q}/\mathbb{Z})^{b_n}, \quad n \geq 3.$$

Proof. The image of $H_{\text{an}}^n(X, \mathbb{Z}) \rightarrow V^n$ is isomorphic to \mathbb{Z}^{r_n} for some $r_n \leq b_n$; we set $\rho_n = b_n - r_n$. Hence, (6.1.2) breaks up into exact sequences $0 \rightarrow V^n/\mathbb{Z}^{r_n} \rightarrow H_{\text{et}}^n(X, \mathbb{G}_m) \xrightarrow{\eta} \mathbb{Z}^{\rho_{n+1}} \oplus H_{\text{an}}^{n+1}(X, \mathbb{Z})_{\text{tors}} \rightarrow 0$. The result follows, since V^n/\mathbb{Z}^{r_n} may be written as $(\mathbb{Q}/\mathbb{Z})^{b_n - \rho_n} \oplus W^n$. If X is smooth then $\rho_n = 0$ for all $n \geq 3$ by Remark 6.1.4. \square

Let us write $U(X) = H_{\text{et}}^0(X, \mathbb{G}_m) = \mathcal{O}_X^\times(X)$ for the group of global units of X . It is classical that $U(X)$ is the product of \mathbb{C}^\times and a free Abelian group \mathbb{Z}^s , $s \geq 0$.

COROLLARY 6.2.1. *Let X be a smooth variety over \mathbb{C} . Then there is a divisible group $\text{Pic}^0(X)$ and integers $s \leq b_1$, $\rho \leq b_2$ so that $U(X) \cong \mathbb{C}^\times \times \mathbb{Z}^s$, $\text{Pic}^0(X)_{\text{tors}} \cong (\mathbb{Q}/\mathbb{Z})^{b_1 - s}$,*

$$\begin{aligned} \text{Pic}(X) &\cong A \oplus \mathbb{Z}^\rho \oplus \text{Pic}^0(X); \\ \text{Br}(X) &\cong B \oplus (\mathbb{Q}/\mathbb{Z})^{b_2 - \rho}. \end{aligned}$$

Proof. Indeed, because $\text{Pic}(X) = H_{\text{et}}^1(X, \mathbb{G}_m)$ and $\text{Br}(X) = H_{\text{et}}^2(X, \mathbb{G}_m)_{\text{tors}}$, these are just the cases $n \leq 2$, with $s = \rho_1$, $\rho = \rho_2$ and $\text{Pic}^0(X) = V^1/\mathbb{Z}^{r_1}$. \square

EXAMPLE 6.2.2. Suppose that X is a smooth projective surface. Then $b_1 = b_3$, and the finite groups A and B are abstractly isomorphic, being Poincaré dual to each other. Using the exponential sequence, $U(X) = \mathbb{C}^\times$ and the identification $\text{Pic}(X) \cong H_{\text{an}}^1(X, \mathcal{O}_X^\times)$, it is easy to see that $V^0 = \mathbb{C}$ and $V^1 \cong H_{\text{an}}^1(X, \mathcal{O}_X)$, and we recover the usual observations that $\text{Pic}(X) = \text{Pic}^0(X) \oplus NS(X)$, where $\text{Pic}^0(X) \cong H_{\text{an}}^1(X, \mathcal{O}_X)/H_{\text{an}}^1(X, \mathbb{Z})$ and the Néron–Severi group $NS(X)$ is the image

of the map $\eta: \text{Pic}(X) \rightarrow H_{\text{an}}^2(X, \mathbb{Z})$. In this case, it is easy to see directly from the Kummer sequence that $NS(X) \cong A \oplus \mathbb{Z}^\rho$ for some $\rho \leq b_2$. Using this, we can deduce directly from the Kummer sequence that $\text{Br}(X) \cong B \oplus (\mathbb{Q}/\mathbb{Z})^{b_2-\rho}$.

Remark 6.2.3. The invariant subgroups A and B arise in the study of topological vector bundles on $X(\mathbb{C})$. The group of topological line bundles is isomorphic to $H_{\text{an}}^2(X, \mathbb{Z}) = A \oplus \mathbb{Z}^{b_2}$ and, by [Dix, p. 50], the torsion subgroup B of $H_{\text{an}}^3(X, \mathbb{Z})$ equals the ‘topological’ Brauer group, formed from topological bundles of matrix algebras on $X(\mathbb{C})$.

Next, we show that $K_0(X)$ and $K_1(X)$ are divisible-by-finitely generated.

PROPOSITION 6.3. *Let X be a smooth irreducible surface over \mathbb{C} . Then:*

- (1) $K_0(X)$ is the sum of the divisible group $\text{Pic}^0(X) \oplus A_0(X)$ and the finitely generated group $\mathbb{Z} \oplus (A \oplus \mathbb{Z}^\rho) \oplus \mathbb{Z}^{b_4}$;
- (2) $A_0(X)_{\text{tors}} \cong H^2(X, \mathcal{K}_2)_{\text{tors}} \cong (\mathbb{Q}/\mathbb{Z})^{b_3-r_1}$ for some $r_1 \leq b_3$;
- (3) $K_1(X) = U(X) \oplus SK_1(X)$, and $SK_1(X) \cong H^1(X, \mathcal{K}_2) \oplus H^2(X, \mathcal{K}_3)$;
- (4) $H^2(X, \mathcal{K}_3)$ is divisible, with torsion subgroup $(\mathbb{Q}/\mathbb{Z})^{b_4}$;
- (5) Both $SK_1(X)$ and $H^1(X, \mathcal{K}_2)$ are the direct sum of a divisible group and a finitely generated group of the form $B \oplus \mathbb{Z}^{r_1}$.

Proof. Part (1) is classical for surfaces, since the Chern classes split $K_0(X)$ into the sum of \mathbb{Z} , $\text{Pic}(X)$ and $CH^2(X)$. Note that A lies in $\text{Pic}(X)$ by Corollary 6.2.1.

It follows from Corollary 2.12 that $K_1(X)$ is the direct sum of B , a divisible group, and a finitely generated free Abelian group. The same is true for the subgroup $SK_1(X)$, since B vanishes in $U(X)$. Now the Brown–Gersten spectral sequence yields an exact sequence

$$0 \rightarrow T_2 \rightarrow H^2(X, \mathcal{K}_3) \rightarrow SK_1(X) \rightarrow H^1(X, \mathcal{K}_2) \rightarrow 0,$$

where T_2 is the image of the differential $H^0(X, \mathcal{K}_2) \rightarrow H^2(X, \mathcal{K}_3)$. Both part (4) and the fact that T_2 is torsion-free follow from Theorem 5.5(2,4) and this sequence. But it is also known that the group T_2 has exponent 2; see [PW, 1.2(1b)]. Hence, $T_2 = 0$. This gives the decomposition of $SK_1(X)$ and shows that $H^1(X, \mathcal{K}_2)$ is also the sum of a divisible group, B and a free Abelian group \mathbb{Z}^{r_1} . The inequality $r_1 \leq b_3$ and part (2) comes from Suslin’s sequence in Remark 5.4.1 above. \square

EXAMPLE 6.3.1 ($r_1 < b_3$). Let X be a smooth projective surface. Colliot-Thélène and Raskind showed in [CT-R, 2.2] that $H^1(X, \mathcal{K}_2)$ is the direct sum of B and a divisible group. In this case, $r_1 = 0$ and $K_1(X)$ is divisible-by-finite. We will show that $H^1(X, \mathcal{K}_2)_{\text{tors}} \cong B \oplus (\mathbb{Q}/\mathbb{Z})^{b_2}$ in Example 6.5.1 below. This yields a complete description of $K_1(X)$, which we state in Example 6.7 below.

EXAMPLE 6.3.2 ($r_1 = b_3 = 1$). Let $X = Y \times \text{Spec}(\mathbb{C}[t, t^{-1}])$, where Y is a smooth projective curve. The group $K_1(Y) = \mathbb{C}^\times \times SK_1(Y)$ is divisible, and described in

Remark 3.2.1, but the fundamental theorem of K -theory implies that neither $U(X)$ nor $SK_1(X)$ is divisible: $U(X) = \mathbb{C}^\times \times \mathbb{Z}$ and

$$SK_1(X) \cong H^1(X, \mathcal{K}_2) = \mathbb{Z} \oplus \text{Pic}^0(Y) \oplus SK_1(Y).$$

EXAMPLE 6.3.3 ($b_3 = 0$). Let $X = \text{Spec}(R)$ be a smooth affine surface. Although the group $SK_1(X)$ is divisible, because $H_{\text{an}}^3(X, \mathbb{Z}) = 0$, the group $K_1(X)$ need not be divisible because $U(X) = R^\times = \mathbb{C}^\times \times \mathbb{Z}^s$.

In order to show that the groups $K_n(X)$ are divisible-by-finite when $n \geq 3$, but only divisible-by-finitely generated for $n = 2$, we first show that this is true for the groups $H^0(X, \mathcal{K}_n)$. We must proceed indirectly, since we do not even know if these groups are quotients of $K_n(X)$.

PROPOSITION 6.4. *Let X be a smooth surface over \mathbb{C} .*

- (1) *For all even $n \geq 2$, A is a summand of both $K_n(X)$ and $H^0(X, \mathcal{K}_n)$.*
- (2) *For all odd $n \geq 3$, B is a summand of both $K_n(X)$ and $H^0(X, \mathcal{K}_n)$.*

Remark 6.4.1 For $n < \dim(X)$, Proposition 6.3 shows that there is a migration of the finite groups into lower parts of the Brown–Gersten filtration. Indeed, it is clear that the finite group A is not a summand of $H^0(X, \mathcal{K}_0) = \mathbb{Z}$, and the finite group B cannot be a summand of the group of units $H^0(X, \mathcal{K}_1) = U(X)$.

Proof. By Corollary 2.12, the group A (resp. B) is a summand of $K_n(X)$ for every even (resp. odd) $n \geq 0$. Moreover, by Theorem 2.8 and (6.1.1), $K_{n+1}(X; \mathbb{Q}/\mathbb{Z})$ is the direct sum of a divisible group and the finite group A (resp. B). By Corollaries 2.10 and 5.3, the composition $K_{n+1}(X; \mathbb{Q}/\mathbb{Z}) \rightarrow K_n(X) \rightarrow H^0(X, \mathcal{K}_n)$ embeds this finite group (A or B) as a summand of $H^0(X, \mathcal{K}_n)_{\text{tors}}$.

To show that this finite subgroup is a summand of $H^0(X, \mathcal{K}_n)$, we tensor $0 \rightarrow \mathbb{Z}/m \rightarrow \mathbb{Z}/(m^2) \rightarrow \mathbb{Z}/m \rightarrow 0$ with $H^0(X, \mathcal{K}_n)$. If $n \geq 4$ is even and $m \cdot A = 0$, Corollaries 2.7 and 5.4 (and 6.2.1) yield a commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc}
 H^0(X, \mathcal{K}_n)_m & \xrightarrow{\partial} & H^0(X, \mathcal{K}_n)/m & \rightarrow & H^0(X, \mathcal{K}_n)/(m^2) & \rightarrow & H^0(X, \mathcal{K}_n)/m \rightarrow 0 \\
 \text{induced} \downarrow & & \downarrow \text{into} & & \downarrow \text{into} & & \downarrow \text{into} \\
 0 \rightarrow A & \rightarrow & \text{Pic}(X)/m & \rightarrow & \text{Pic}(X)/(m^2) & \rightarrow & \text{Pic}(X)/m \rightarrow 0 \\
 & & \downarrow \text{onto} & & \downarrow \text{onto} & & \downarrow \text{onto} \\
 & & 0 & \rightarrow & H^1(X, \mathcal{K}_n)_m & \rightarrow & H^1(X, \mathcal{K}_n)_{m^2} \rightarrow \dots
 \end{array}$$

By Corollary 5.3, there is a (noncanonically split) surjection $\pi: H^0(X, \mathcal{K}_n)_m \rightarrow A$; from the definition of the top left map ∂ it follows that ∂ is π followed by an injection. By Corollary 6.2.1, the map $A \rightarrow \text{Pic}(X)/m$ is naturally split. It follows that A is a canonical summand of $H^0(X, \mathcal{K}_n)/m$ and, by a diagram chase, a (noncanonical) summand of $H^0(X, \mathcal{K}_n)$.

If $n = 2$ then we replace the middle row of this diagram by the sequence

$$0 \rightarrow A \rightarrow H^2(X, \mathbb{Z}/m) \rightarrow H^2(X, \mathbb{Z}/m^2) \rightarrow H^2(X, \mathbb{Z}/m),$$

which comes from naturality of (6.1.1) in m . By Remark 5.4.1, the columns are still exact, and the same argument works.

The proof for odd $n \geq 3$ is identical, except that (using Corollaries 2.7 and 5.4) the middle row gets replaced by

$$0 \rightarrow B \rightarrow H^3(X, \mathbb{Z}/m) \rightarrow H^3(X, \mathbb{Z}/m^2) \rightarrow H^3(X, \mathbb{Z}/m) \rightarrow 0. \quad \square$$

EXERCISE 6.4.2. A different proof is possible for $n = 2$. First use Theorem 5.5 and the proof of Proposition 6.3 to show that $H^0(X, \mathcal{K}_2)$ is a summand of $K_2(X)$. Then invoke Corollaries 2.12 and 5.3.

THEOREM 6.5. *Let X be a smooth irreducible surface over \mathbb{C} .*

- (1) $H^0(X, \mathcal{K}_2)$ is the direct sum of a uniquely divisible group and a group of the form $(\mathbb{Q}/\mathbb{Z})^{b_1} \oplus A \oplus \mathbb{Z}^{r_2}$ for some $r_2 \leq b_2$, while

$$H^1(X, \mathcal{K}_2)_{tors} \cong B \oplus (\mathbb{Q}/\mathbb{Z})^{b_2-r_2}.$$

- (2) For all even $n \geq 4$, $H^0(X, \mathcal{K}_n)$ is the direct sum of A and a divisible group, while

$$H^1(X, \mathcal{K}_n)_{tors} \cong \text{Pic}(X) \otimes \mathbb{Q}/\mathbb{Z} \cong (\mathbb{Q}/\mathbb{Z})^\rho.$$

- (3) For all odd $n \geq 3$, $H^0(X, \mathcal{K}_n)$ is the direct sum of B and a divisible group, while

$$H^1(X, \mathcal{K}_n)_{tors} \cong H^3(X; \mathbb{Q}/\mathbb{Z}) \cong (\mathbb{Q}/\mathbb{Z})^{b_3}.$$

Proof. In the proof of Proposition 6.3 we saw that the Brown–Gersten differential $H^0(X, \mathcal{K}_2) \rightarrow H^2(X, \mathcal{K}_3)$ vanished, so $H^0(X, \mathcal{K}_2)$ is a quotient of $K_2(X)$. By Corollary 2.12 and (2.8.1), $H^0(X, \mathcal{K}_2)$ is the sum of a divisible group and a finitely generated group of rank r_2 . By Corollary 5.3 and (6.1.1), the torsion subgroup of $H^0(X, \mathcal{K}_2)$ is $(\mathbb{Q}/\mathbb{Z})^{b_1} \oplus A$. Hence $H^0(X, \mathcal{K}_2)$ is the sum of a uniquely divisible group and a group of the form $(\mathbb{Q}/\mathbb{Z})^{b_1} \oplus A \oplus \mathbb{Z}^{r_2}$. The inequality $r_2 \leq b_2$ and the description of $H^1(X, \mathcal{K}_2)_{tors}$ comes from (6.1.1) and Suslin’s sequence in Remark 5.4.1 above.

For $n \geq 3$, Corollaries 2.7 and 5.4 give short exact sequences for each m :

$$0 \rightarrow H^0(X, \mathcal{K}_n)/m \rightarrow \text{Pic}(X)/m \rightarrow H^1(X, \mathcal{K}_n)_m \rightarrow 0, \quad n \geq 4 \text{ even};$$

$$0 \rightarrow H^0(X, \mathcal{K}_n)/m \rightarrow H^3(X, \mathbb{Z}/m) \rightarrow H^1(X, \mathcal{K}_n)_m \rightarrow 0, \quad n \geq 3 \text{ odd}.$$

If n is even then $H^0(X, \mathcal{K}_n) = A \oplus D$ for some group D by Proposition 6.4, so in the first sequence the left group is $A/m \oplus D/m$. The right group contains $(\mathbb{Z}/m)^\rho$ as a subgroup by Lemma 5.6, and the middle group is $A/m \oplus (\mathbb{Z}/m)^\rho$ by

Corollary 6.2.1. It follows that $H^1(X, \mathcal{K}_n)_m = (\mathbb{Z}/m)^\rho$, and that $D/m = 0$ for all m , i.e., D is divisible.

If n is odd then $H^0(X, \mathcal{K}_n) = B \oplus D$ for some group D , so the left group is $B/m \oplus D/m$. The right group contains $(\mathbb{Z}/m)^{b_3}$ as a subgroup by Lemma 5.7, and the middle group is $B/m \oplus (\mathbb{Z}/m)^\rho$ by (6.1.1). It follows that $H^1(X, \mathcal{K}_n)_m = (\mathbb{Z}/m)^{b_3}$, and that $D/m = 0$ for all m , i.e., D is divisible. \square

EXAMPLE 6.5.1. As in Remark 6.4.1, the case $n = 2$ is anomalous because it is B and not A that appears in $H^1(X, \mathcal{K}_2)$. Also both $H^0(X, \mathcal{K}_2)$ and $H^1(X, \mathcal{K}_2)$ can contain a free summand. Example 6.8.1 below shows that the free summand \mathbb{Z}^{r_2} of $K_2(X)$ and $H^0(X, \mathcal{K}_2)$ can be nonzero when X is affine, and Example 6.9 shows that the free summand \mathbb{Z}^{r_1} of $H^1(X, \mathcal{K}_2)$ can be nonzero.

However, if X is a smooth projective surface then $r_2 = 0$, because Colliot-Thélène and Raskind showed in [CT-R, 1.8] that $H^0(X, \mathcal{K}_2)$ is the direct sum of A and a divisible group. It follows from Theorem 6.5 that there is a uniquely divisible group V_{12} so that $H^1(X, \mathcal{K}_2) \cong B \oplus (\mathbb{Q}/\mathbb{Z})^{b_2} \oplus V_{12}$. This result recovers Theorem 2.2 of [CT-R], where it is proven that $H^1(X, \mathcal{K}_2)$ is the direct sum of B and a divisible group.

THEOREM 6.6. *Let X be a smooth irreducible surface over \mathbb{C} . Then there are uniquely divisible groups V_n such that:*

- (1) For some $r_2 \leq b_2$,

$$K_2(X) \cong (A \oplus \mathbb{Z}^{r_2}) \oplus (\mathbb{Q}/\mathbb{Z})^{b_1} \oplus (\mathbb{Q}/\mathbb{Z})^{b_3} \oplus V_2.$$

- (2) For every even $n \geq 4$,

$$K_n(X) \cong A \oplus (\mathbb{Q}/\mathbb{Z})^{b_1} \oplus (\mathbb{Q}/\mathbb{Z})^{b_3} \oplus V_n.$$

- (3) For every odd $n \geq 3$,

$$K_n(X) \cong B \oplus (\mathbb{Q}/\mathbb{Z}) \oplus (\mathbb{Q}/\mathbb{Z})^{b_2} \oplus H^4(X, \mathbb{Q}/\mathbb{Z}) \oplus V_n.$$

The following notation will be useful in the proof of Theorem 6.6. For each n , let $\tilde{K}_n(X)$ denote the kernel of $K_n(X) \rightarrow K_n(F)$, $F = \mathbb{C}(X)$, and let $T = T_n$ denote the image of the differential $H^0(X, \mathcal{K}_n) \rightarrow H^2(X, \mathcal{K}_{n+1})$ in the Brown–Gersten spectral sequence (5.1). Thus we have exact sequences:

$$\begin{aligned} 0 \rightarrow \tilde{K}_n(X) \rightarrow K_n(X) \rightarrow H^0(X, \mathcal{K}_n) \rightarrow T_n \rightarrow 0; \\ 0 \rightarrow T_{n+1} \rightarrow H^2(X, \mathcal{K}_{n+2}) \rightarrow \tilde{K}_n(X) \rightarrow H^1(X, \mathcal{K}_{n+1}) \rightarrow 0. \end{aligned} \tag{6.6.1}$$

Proof. For $n \geq 2$, we see by combining Theorem 6.5 and Proposition 6.4 that the quotient T_{n+1} of $H^0(X, \mathcal{K}_{n+1})$ is divisible, so T_{n+1} is a summand of $H^2(X, \mathcal{K}_{n+2})$. Theorem 5.5 and (6.6.1) imply that T_{n+1} is torsion-free, hence uniquely divisible, and that the groups $H^2(X, \mathcal{K}_{n+2})$ and $H^1(X, \mathcal{K}_{n+1})$ are divisible. From (6.6.1) we

see that $\tilde{K}_n(X)$ is divisible and, again by Theorem 5.5, that $\tilde{K}_n(X)_{tors}$ is either $(\mathbb{Q}/\mathbb{Z})^{b_3}$ or $(\mathbb{Q}/\mathbb{Z})^{\rho+b_4}$, according to the parity of n . The description of $K_n(X)$ follows by piecing this together with the description of $H^0(X, \mathcal{K}_n)$ in Corollary 5.3 and Theorem 6.5. \square

EXAMPLE 6.7. Let X be a smooth projective surface over \mathbb{C} . Then there are uniquely divisible groups V_n so that:

$$K_n(X) \cong \begin{cases} B \oplus (\mathbb{Q}/\mathbb{Z})^{2+b_2} \oplus V_n, & n \geq 1 \text{ odd;} \\ A \oplus (\mathbb{Q}/\mathbb{Z})^{b_1+b_3} \oplus V_n, & n \geq 2 \text{ even.} \end{cases}$$

Indeed, the cases $n = 1, 2$ follow from Example 6.5.1, since $r_2 = 0$, and the cases $n \geq 3$ are part of Theorem 6.6.

We can partially understand the uniquely divisible part as follows. Using the relative sequence (2.11.0) and the calculation of $KU^*(X)$ (see (2.8.1)), we have exact sequences:

$$\begin{aligned} 0 \rightarrow \mathbb{Z}^{b_1+b_3} \rightarrow K_0^{rel}(X) \rightarrow K_0(X) \rightarrow \mathbb{Z} \oplus (A \oplus \mathbb{Z}^\rho) \oplus \mathbb{Z} \rightarrow 0; \\ 0 \rightarrow \mathbb{Z}^{2+b_2} \rightarrow K_n^{rel}(X) \rightarrow K_n(X) \rightarrow B \rightarrow 0, \quad n \geq 1 \text{ odd;} \\ 0 \rightarrow \mathbb{Z}^{b_1+b_3} \rightarrow K_n^{rel}(X) \rightarrow K_n(X) \rightarrow A \rightarrow 0, \quad n \geq 2 \text{ even.} \end{aligned}$$

For $K_0(X)$ this combines the classical description of $\text{Pic}(X)$ in Example 6.2.2 with Roitman’s theorem that the torsion in the Chow group $CH^2(X) = \mathbb{Z} \oplus A_0(X)$ is $(\mathbb{Q}/\mathbb{Z})^{b_3}$. For $K_1(X)$, it shows that in addition to the two standard summands \mathbb{C}^\times and the finite summand B found by Colliot-Thélène and Raskind, there is a torsion summand $(\mathbb{Q}/\mathbb{Z})^{b_2}$, which is the divisible part of $H^1(X, \mathcal{K}_2)_{tors}$. The uniquely divisible part of $K_1(X)$ is the sum of the three uniquely divisible parts: $\mathbb{C}^\times / \exp(2\pi i\mathbb{Q})$; $V_{12} \subset H^1(X, \mathcal{K}_2)$ and $V_{23} \subset H^2(X, \mathcal{K}_3)$. This latter group is isomorphic to $W \oplus \mathbb{C}^\times / \exp(2\pi i\mathbb{Q})$, where W denotes the kernel of the transfer map $H^2(X, \mathcal{K}_3) \rightarrow \mathbb{C}^\times$.

Remark 6.7.1 The results of this section give us the following computations for the groups $H^p(X, \mathcal{K}_q)$ of a smooth projective surface X over \mathbb{C} . In the description below, V_{pq} denotes a uniquely divisible Abelian group, while the finite groups A and B are defined in Notation 6.1.

(i) For $q = 2$ we have

$$\begin{aligned} H^0(X, \mathcal{K}_2) &\cong A \oplus (\mathbb{Q}/\mathbb{Z})^{b_1} \oplus V_{02}; \\ H^1(X, \mathcal{K}_2) &\cong B \oplus (\mathbb{Q}/\mathbb{Z})^{b_2} \oplus V_{12}; \\ H^2(X, \mathcal{K}_2) &\cong CH^2(X) \cong \mathbb{Z} \oplus (\mathbb{Q}/\mathbb{Z})^{b_3} \oplus V_{22}; \end{aligned}$$

(ii) For all odd $n \geq 3$,

$$\begin{aligned} H^0(X, \mathcal{K}_n) &\cong \mathbb{Q}/\mathbb{Z} \oplus B \oplus (\mathbb{Q}/\mathbb{Z})^{b_2-\rho} \oplus V_{0n}; \\ H^1(X, \mathcal{K}_n) &\cong (\mathbb{Q}/\mathbb{Z})^{b_3} \oplus V_{1n}; \\ H^2(X, \mathcal{K}_n) &\cong \mathbb{Q}/\mathbb{Z} \oplus V_{2n}. \end{aligned}$$

(iii) For all even $n \geq 4$,

$$\begin{aligned} H^0(X, \mathcal{K}_n) &\cong A \oplus (\mathbb{Q}/\mathbb{Z})^{b_1} \oplus V_{0n}; \\ H^1(X, \mathcal{K}_n) &\cong (\mathbb{Q}/\mathbb{Z})^\rho \oplus V_{1n}; \\ H^2(X, \mathcal{K}_n) &\cong V_{2n}. \end{aligned}$$

The torsion subgroups of all these groups can be computed by means of the following invariants of X :

$$\{H^2(X, \mathbb{Z})_{tors}; H^3(X, \mathbb{Z})_{tors}; \rho; b_1; b_2; b_3\}$$

EXAMPLE 6.8. Let $X = \text{Spec}(R)$ be a smooth affine surface. Since $H^3_{\text{et}}(X, \mathbb{Z}/m) = 0$, we see that there are uniquely divisible groups V_n such that:

$$K_n(X) \cong \begin{cases} (\mathbb{Q}/\mathbb{Z}) \oplus (\mathbb{Q}/\mathbb{Z})^{b_2} \oplus V_n, & n \geq 3 \text{ odd}; \\ A \oplus (\mathbb{Q}/\mathbb{Z})^{b_1} \oplus V_n, & n \geq 4 \text{ even}. \end{cases}$$

The structure of $K_1(X)$ and $K_2(X)$ can be different from this pattern, as the following two examples show.

6.8.1. Consider the affine surface $X = \text{Spec}(\mathbb{C}[x, 1/x, y, 1/y])$. It is well known (see [Sh, 4.3]) that $H^p(X, \mathcal{K}_q) = 0$ for $p \neq 0$, and that the fundamental theorem of K -theory implies that $K_1(X) = H^0(X, \mathcal{K}_1) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{C}^\times$ and

$$K_2(X) = H^0(X, \mathcal{K}_2) = \mathbb{Z} \oplus \mathbb{C}^\times \oplus \mathbb{C}^\times \oplus K_2(\mathbb{C}).$$

These groups are not divisible.

We claim that for the above affine surface the obvious map

$$H^0(X, \mathcal{K}_2)/m \rightarrow H^0(X, \mathcal{K}_2(\mathbb{Z}/m)) \cong H^0(X, \mathcal{H}^2(\mathbb{Z}/m))$$

is an isomorphism. In particular, it is not the zero map (since all groups are \mathbb{Z}/m). This contrasts with the fact that it is always the zero map when X is projective by [CT-R, 1.7]. The claim follows from the following factorization of the obvious map: $H^0(X, \mathcal{K}_2)/m \rightarrow H^2(X, \mathbb{Z}/m) \rightarrow H^0(X, \mathcal{H}^2(\mathbb{Z}/m))$. Since $H^1(X, \mathcal{K}_2) = 0$, the first map is an isomorphism by Suslin’s sequence (Remark 5.4.1). But since $\text{Pic}(X) = 0$, the second map is also an isomorphism by the Bloch–Ogus sequence:

$$0 \rightarrow \text{Pic}(X)/m \rightarrow H^2(X, \mathbb{Z}/m) \rightarrow H^0(X, \mathcal{H}^2(\mathbb{Z}/m)) \rightarrow 0.$$

6.8.2. Let \bar{Y}_1 and \bar{Y}_2 be smooth projective curves of genus g_1 and g_2 , and let $Y_i = \bar{Y}_i - p_i$ be affine curves obtained by removing one point. The affine surface $X = Y_1 \times Y_2$ has Betti numbers $b_1 = 2(g_1 + g_2)$ and $b_2 = 4g_1g_2$. We claim that $K_n(X)$ is divisible for all $n \geq 1$, with

$$K_n(Y_1 \times Y_2)_{tors} = \begin{cases} \mathbb{Q}/\mathbb{Z} \oplus (\mathbb{Q}/\mathbb{Z})^{b_2}, & n \geq 1 \text{ odd;} \\ (\mathbb{Q}/\mathbb{Z})^{b_1}, & n \geq 2 \text{ even.} \end{cases}$$

To see this, note that the p_i induce closed immersions $p_1 \rightarrow \bar{Y}_1$, $\bar{Y}_1 \rightarrow \bar{Y}_1 \times \bar{Y}_2$ and $Y_2 \rightarrow \bar{Y}_1 \times Y_2$ which are split by proper maps, viz. the projections. Thus these immersions induce split injections on the level of K -theory. The K -theory localization sequences for these immersions induce a decomposition for all n :

$$\begin{aligned} K_n(\bar{Y}_1 \times \bar{Y}_2) &\cong K_n(\bar{Y}_1) \oplus K_n(\bar{Y}_1 \times Y_2) \\ &\cong K_n(k) \oplus K_n(Y_1) \oplus K_n(Y_2) \oplus K_n(Y_1 \times Y_2). \end{aligned}$$

From Example 6.7, $K_n(\bar{Y}_1 \times \bar{Y}_2)$ is divisible for $n \geq 1$, with torsion subgroup either $(\mathbb{Q}/\mathbb{Z})^{4+b_2}$ or $(\mathbb{Q}/\mathbb{Z})^{2b_1}$. (The b_i in this formula are the Betti numbers of X , *not* the Betti numbers of $\bar{Y}_1 \times \bar{Y}_2$.) The description of $K_n(Y_1 \times Y_2)$ follows from this using Theorem 3.2. In this case the difference between $n = 2$ and $n = 4$ is apparent in the K -cohomology; since $0 \leq \rho \leq 2g_1g_2$ we see from Theorem 6.5 and Example 6.5.1 that there are uniquely divisible groups V_n so that:

$$H^1(Y_1 \times Y_2, \mathcal{K}_n) = \begin{cases} V_n, & n \geq 1 \text{ odd;} \\ (\mathbb{Q}/\mathbb{Z})^{4g_1g_2} \oplus V_n, & n = 2; \\ (\mathbb{Q}/\mathbb{Z})^\rho \oplus V_n, & n \geq 4 \text{ even.} \end{cases}$$

EXAMPLE 6.9. Let $U = X - S$, where X is a smooth projective surface and S is a finite set of s closed points. Then $H_{an}^3(U, \mathbb{Z}) = H^3(X, \mathbb{Z}) \oplus \mathbb{Z}^{s-1}$, $H_{an}^4(U, \mathbb{Z}) = 0$ and $H_{an}^i(U, \mathbb{Z}) = H_{an}^i(X, \mathbb{Z})$ for $i \leq 2$. Moreover, the standard resolution of \mathcal{K}_n shows that $H^0(X, \mathcal{K}_n) = H^0(U, \mathcal{K}_n)$ for all n . Referring to Theorem 6.6 and (6.6.1), we see that $K_n(X)_{tors} \cong (\mathbb{Q}/\mathbb{Z}) \oplus K_n(U)_{tors}$ for all odd $n \geq 3$, while for all even $n \geq 2$ we have a split exact sequence

$$0 \rightarrow K_n(X)_{tors} \rightarrow K_n(U)_{tors} \rightarrow K_{n-1}(S)_{tors} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0.$$

We claim that $K_1(X)_{tors} \cong (\mathbb{Q}/\mathbb{Z}) \oplus K_1(U)_{tors}$ as well. Given the decomposition of $K_1(X)$ in Proposition 6.3, the fact that $H^2(X, \mathcal{K}_3)_{tors} \cong (\mathbb{Q}/\mathbb{Z}) \oplus H^2(U, \mathcal{K}_3)_{tors}$ (from Theorem 5.5) implies that we need only focus on $H^1(X, \mathcal{K}_2)$. By [MS, 8.1.4] we can refine the continuation of the localization sequence as the exact sequence:

$$0 \rightarrow H^1(X, \mathcal{K}_2) \rightarrow H^1(U, \mathcal{K}_2) \rightarrow (\mathbb{Z})^s \rightarrow H^2(X, \mathcal{K}_2) \rightarrow H^2(U, \mathcal{K}_2) \rightarrow 0$$

Because $A \mapsto A_{tors}$ is left exact, this yields the claim. In fact,

$$SK_1(U)_{tors} \cong H^1(U, \mathcal{K}_2)_{tors} \cong H^1(X, \mathcal{K}_2)_{tors} \cong B \oplus (\mathbb{Q}/\mathbb{Z})^{b_2}.$$

To study the image of $H^1(X, \mathcal{K}_2)$ in \mathbb{Z}^s , we identify $H^2(X, \mathcal{K}_2)$ with the Chow group $CH^2(X) = \mathbb{Z} \oplus A_0(X)$ of zero-cycles on X , and observe that the image of \mathbb{Z}^s in $CH^2(X)$ is the subgroup generated by the points in S . Picking S so that some differences $[s] - [s']$ are torsion in $A_0(X)$ yields a family of examples where $H^1(U, \mathcal{K}_2)$ contains $B \oplus \mathbb{Z}^{r_1}$ as a summand for any $r_1 \leq b_3$. This shows that the description of $SK_1(U)$ in Proposition 6.3 is best possible. It also illustrates the nontriviality of Suslin's sequence (Remark 5.4.1) for the surface U :

$$0 \rightarrow H^1(U, \mathcal{K}_2)/m \rightarrow H^3(U, \mathbb{Z}/m) \rightarrow H^2(U, \mathcal{K}_2)_m \rightarrow 0.$$

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